Geilo Winter School 2012

Lecture 1: Introduction to FEM

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Course outline

Sunday

L1 Introduction to FEM

Monday

L2 Fundamentals of continuum mechanics (I)

L3 Fundamentals of continuum mechanics (II)

L4 Introduction to FEniCS

Tuesday

L5 Solid mechanics

L6 Static hyperelasticity in FEniCS

L7 Dynamic hyperelasticity in FEniCS

Wednesday

L8 Fluid mechanics

L9 Navier–Stokes in FEniCS

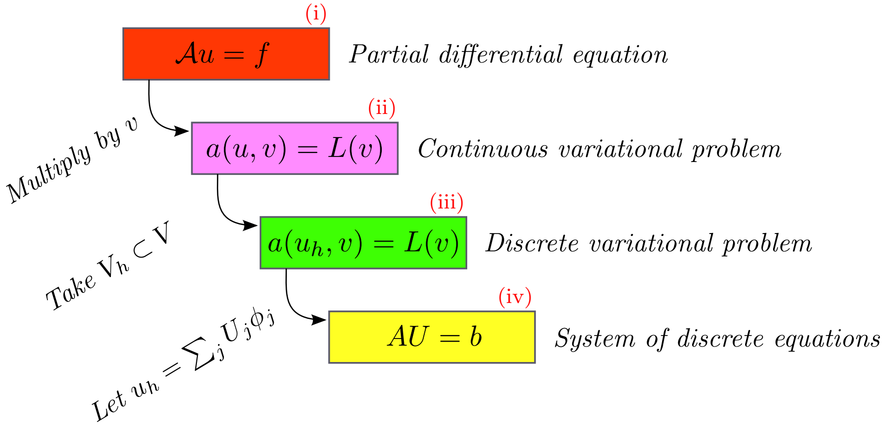
What is FEM?

The finite element method is a framework and a recipe for discretization of differential equations

- Ordinary differential equations
- Partial differential equations
- Integral equations

- A recipe for discretization of PDE
- PDE $\rightarrow Ax = b$
- Different bases, stabilization, error control, adaptivity

The FEM cookbook



The PDE (i)

Consider Poisson's equation, the Hello World of partial differential equations:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= u_0 \quad \text{on } \partial\Omega \end{aligned}$$

Poisson's equation arises in numerous applications:

- heat conduction, electrostatics, diffusion of substances, twisting of elastic rods, inviscid fluid flow, water waves, magnetostatics, ...
- as part of numerical splitting strategies for more complicated systems of PDEs, in particular the Navier–Stokes equations

From PDE (i) to variational problem (ii)

The simple recipe is: multiply the PDE by a test function v and integrate over Ω:

$$-\int_\Omega (\Delta u) v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Then integrate by parts and set $v = 0$ on the Dirichlet boundary:

$$-\int_\Omega (\Delta u) v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}s}_{=0}$$

We find that:

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

The variational problem (ii)

Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x$$

for all $v \in \hat{V}$

The trial space V and the test space $\hat{V}$ are (here) given by

$$V = \{v \in H^1(\Omega) : v = u_0 \text{ on } \partial\Omega\}$$
$$\hat{V} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$$

From continuous (ii) to discrete (iii) variational problem

We approximate the continuous variational problem with a discrete variational problem posed on finite dimensional subspaces of V and $\hat{V}$:

$$
\begin{aligned}
V_h &\subset V \\
\hat{V}_h &\subset \hat{V}
\end{aligned}
$$

Find $u_h \in V_h \subset V$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x$$

for all $v \in \hat{V}_h \subset \hat{V}$

From discrete variational problem (iii) to discrete system of equations (iv)

Choose a basis for the discrete function space:

$$V_h = \operatorname{span}\{\phi_j\}_{j=1}^{N}$$

Make an ansatz for the discrete solution:

$$u_h = \sum_{j=1}^{N} U_j \phi_j$$

Test against the basis functions:

$$\int_\Omega \nabla(\underbrace{\sum_{j=1}^{N} U_j \phi_j}_{u_h}) \cdot \nabla \phi_i \, \mathrm{d}x = \int_\Omega f \phi_i \, \mathrm{d}x$$

From discrete variational problem (iii) to discrete system of equations (iv), contd.

Rearrange to get:

$$\sum_{j=1}^{N} U_j \underbrace{\int_{\Omega} \nabla\phi_j \cdot \nabla\phi_i \,\mathrm{d}x}_{A_{ij}} = \underbrace{\int_{\Omega} f\phi_i \,\mathrm{d}x}_{b_i}$$

A linear system of equations:

$$AU = b$$

where

$$A_{ij} = \int_{\Omega} \nabla\phi_j \cdot \nabla\phi_i \,\mathrm{d}x \tag{1}$$

$$b_i = \int_{\Omega} f\phi_i \,\mathrm{d}x \tag{2}$$

The canonical abstract problem

(i) Partial differential equation:

$$\mathcal{A}u = f \quad \text{in } \Omega$$

(ii) Continuous variational problem: find $u \in V$ such that

$$a(u, v) = L(v) \quad \text{for all } v \in \hat{V}$$

(iii) Discrete variational problem: find $u_h \in V_h \subset V$ such that

$$a(u_h, v) = L(v) \quad \text{for all } v \in \hat{V}_h$$

(iv) Discrete system of equations for $u_h = \sum_{j=1}^{N} U_j \phi_j$:

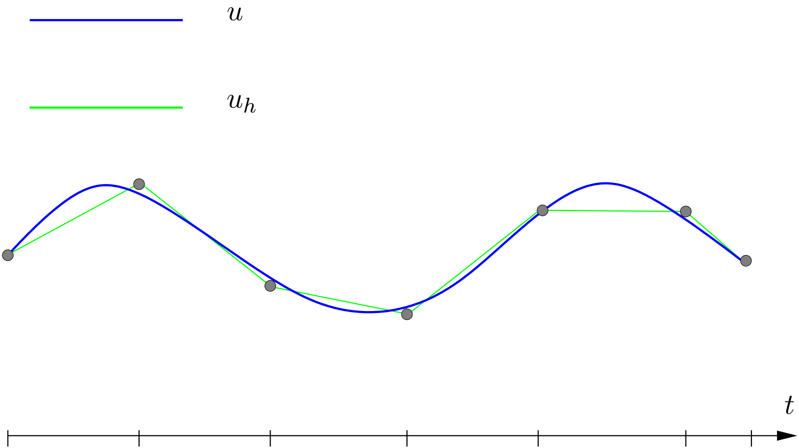
$$\begin{aligned} AU &= b \\ A_{ij} &= a(\phi_j, \phi_i) \\ b_i &= L(\phi_i) \end{aligned}$$

Important topics

- *How to choose* V_h*?*
- *How to compute* A *and* b
- *How to solve* $AU = b$*?*
- *How large is the error* $e = u - u_h$*?*
- Extensions to nonlinear problems

How to choose V_h

Finite element function spaces



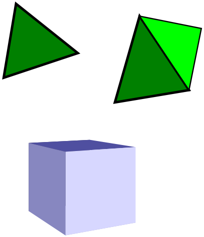
The finite element definition (Ciarlet 1975)

A finite element is a triple $(T, \mathcal{V}, \mathcal{L})$, where

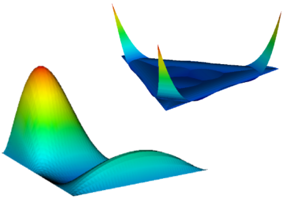
- the domain T is a bounded, closed subset of $\mathbb{R}^d$ (for $d = 1, 2, 3, \dots$) with nonempty interior and piecewise smooth boundary
- the space $\mathcal{V} = \mathcal{V}(T)$ is a finite dimensional function space on T of dimension n
- the set of degrees of freedom (nodes) $\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_n\}$ is a basis for the dual space $\mathcal{V}'$; that is, the space of bounded linear functionals on $\mathcal{V}$

The finite element definition (Ciarlet 1975)

T



$\mathcal{V}$



$\mathcal{L}$

$v(\bar{x})$

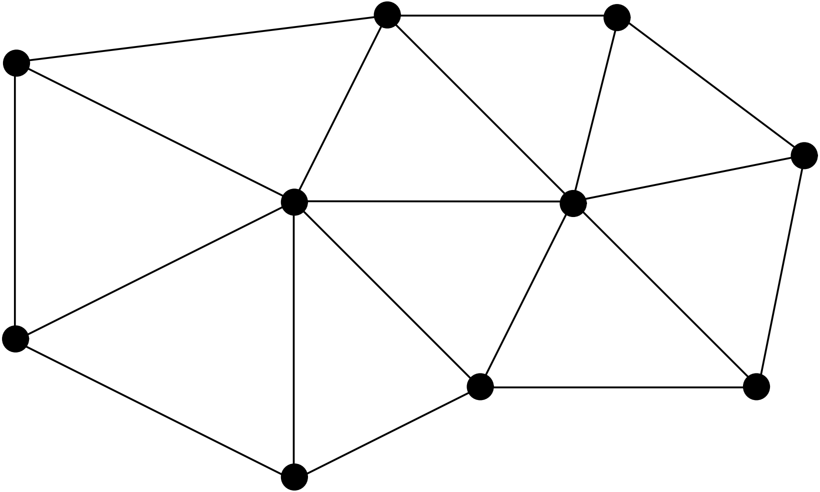
$v(\bar{x}) \cdot n$

$\int_T v(x)w(x)\,\mathrm{d}x$

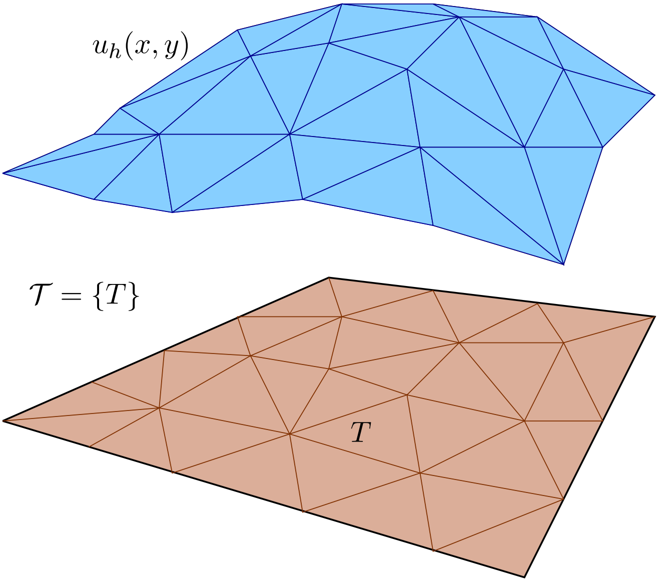
The linear Lagrange element: $(T, \mathcal{V}, \mathcal{L})$

- T is a line, triangle or tetrahedron
- $\mathcal{V}$ is the first-degree polynomials on T
- $\mathcal{L}$ is point evaluation at the vertices

The linear Lagrange element: $\mathcal{L}$



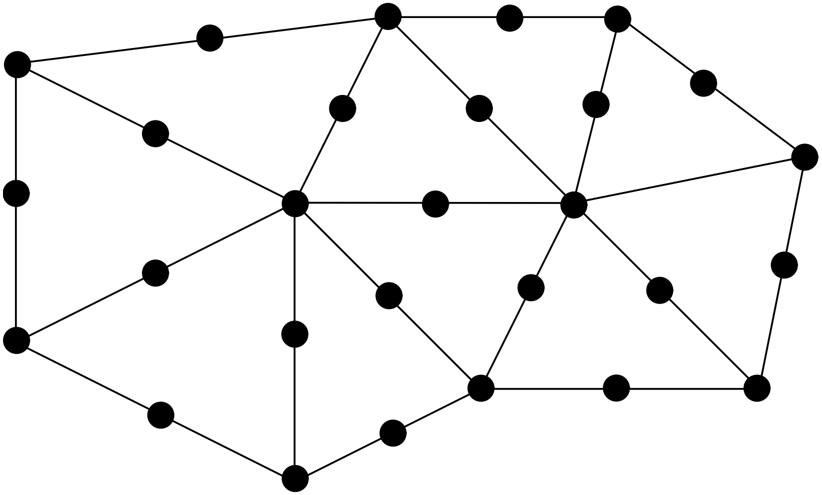
The linear Lagrange element: $\mathcal{V}_h$



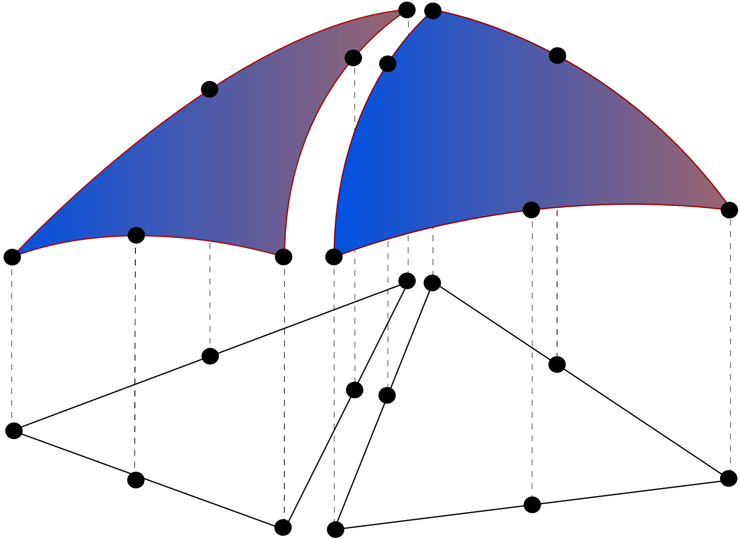
The quadratic Lagrange element: $(T, \mathcal{V}, \mathcal{L})$

- T is a line, triangle or tetrahedron
- $\mathcal{V}$ is the second-degree polynomials on T
- $\mathcal{L}$ is point evaluation at the vertices and edge midpoints

The quadratic Lagrange element: $\mathcal{L}$

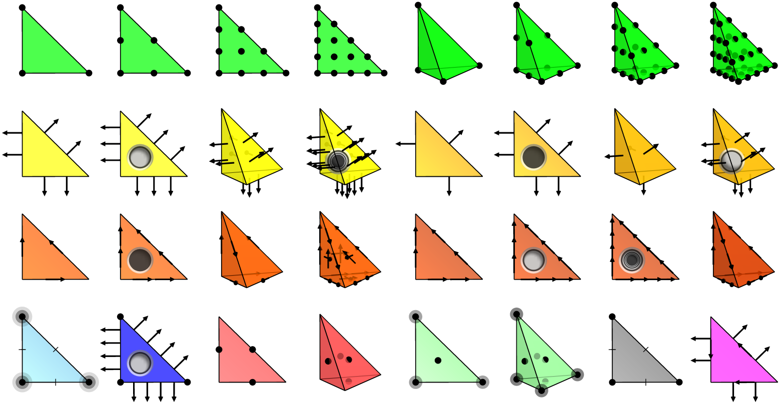


The quadratic Lagrange element: $\mathcal{V}_h$



Families of elements

Families of elements



Computing the sparse matrix A

Naive assembly algorithm

$A = 0$

for $i = 1, \ldots, N$

  for $j = 1, \ldots, N$

    $A_{ij} = a(\phi_j, \phi_i)$

  end for

end for

The element matrix

The global matrix A is defined by

$$A_{ij} = a(\phi_j, \phi_i)$$

The *element matrix* A_T is defined by

$$A_{T,ij} = a_T(\phi_j^T, \phi_i^T)$$

The assembly algorithm

$A = 0$

for $T \in \mathcal{T}$

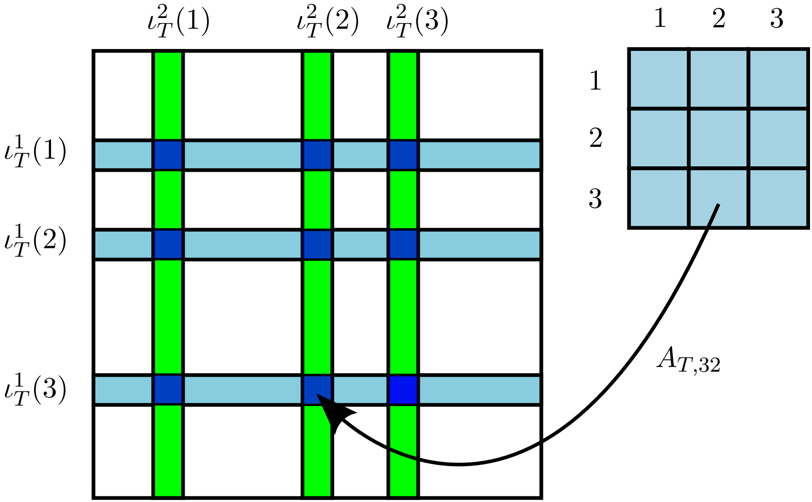
Compute the element matrix A_T

Compute the local-to-global mapping ι_T

Add A_T to A according to ι_T

end for

Adding the element matrix A_T



Solving $AU = b$

Direct methods

- Gaussian elimination
  - Requires $\sim \frac{2}{3}N^3$ operations
- LU factorization: $A = LU$
  - Solve requires $\sim \frac{2}{3}N^3$ operations
  - Reuse L and U for repeated solves
- Cholesky factorization: $A = LL^\top$
  - Works if A is symmetric and positive definite
  - Solve requires $\sim \frac{1}{3}N^3$ operations
  - Reuse L for repeated solves

Iterative methods

Krylov subspace methods

- GMRES (Generalized Minimal RESidual method)
- CG (Conjugate Gradient method)
  - Works if A is symmetric and positive definite
- BiCGSTAB, MINRES, TFQMR, ...

Multigrid methods

- GMG (Geometric MultiGrid)
- AMG (Algebraic MultiGrid)

Preconditioners

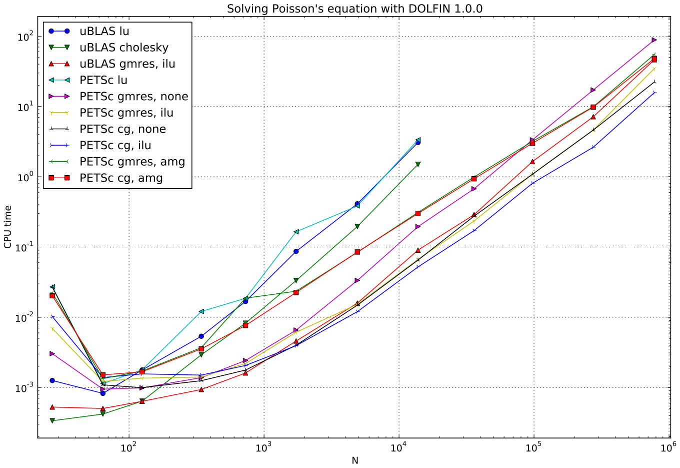
- ILU, ICC, SOR, AMG, Jacobi, block-Jacobi, additive Schwarz, ...

Which method should I use?

Rules of thumb

- Direct methods for small systems
- Iterative methods for large systems
- Break-even at ca 100–1000 degrees of freedom
- Use a symmetric method for a symmetric system
  - Cholesky factorization (direct)
  - CG (iterative)
- Use a multigrid preconditioner for Poisson-like systems
- GMRES with ILU preconditioning is a good default choice

Current timings (2012–01–20)



Homework!

- Install FEniCS 1.0.0!
- Download the FEniCS book!
- Visit the course web page!

`http://fenicsproject.org/`

`http://home.simula.no/~logg/teaching/geilo2012/`

PS: Be alert and ready for the FEniCS challenge(s)...