Geilo Winter School 2012

Lecture 6: Static hyperelasticity in FEniCS

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Course outline

Sunday

L1 Introduction to FEM

Monday

L2 Fundamentals of continuum mechanics (I)

L3 Fundamentals of continuum mechanics (II)

L4 Introduction to FEniCS

Tuesday

L5 Solid mechanics

L6 Static hyperelasticity in FEniCS

L7 Dynamic hyperelasticity in FEniCS

Wednesday

L8 Fluid mechanics

L9 Navier–Stokes in FEniCS

Static hyperelasticity

$$\begin{aligned} -\operatorname{div} P &= B && \text{in } \Omega \\ u &= g && \text{on } \Gamma_{\mathrm{D}} \\ P \cdot n &= T && \text{on } \Gamma_{\mathrm{N}} \end{aligned}$$

- u is the displacement
- $P = P(u)$ is the first Piola–Kirchoff stress tensor
- B is a given body force per unit volume
- g is a given boundary displacement
- T is a given boundary traction

Variational problem

Multiply by a test function $v \in \hat{V}$ and integrate by parts:

$$-\int_\Omega \operatorname{div} P \cdot v \,\mathrm{d}x = \int_\Omega P : \operatorname{grad} v \,\mathrm{d}x - \int_{\partial\Omega} (P \cdot n) \cdot v \,\mathrm{d}s$$

Note that $v = 0$ on Γ_D and $P \cdot n = T$ on Γ_N

Find $u \in V$ such that

$$\int_\Omega P : \operatorname{grad} v \,\mathrm{d}x = \int_\Omega B \cdot v \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} T \cdot v \,\mathrm{d}s$$

for all $v \in \hat{V}$

Stress–strain relations

- $F = I + \operatorname{grad} u$ is the deformation gradient
- $C = F^\top F$ is the right Cauchy–Green tensor
- $E = \frac{1}{2}(C - I)$ is the Green–Lagrange strain tensor
- $W = W(E)$ is the strain energy density
- $S_{ij} = \frac{\partial W}{\partial E_{ij}}$ is the second Piola–Kirchoff stress tensor
- $P = FS$ is the first Piola–Kirchoff stress tensor

St. Venant–Kirchoff strain energy function:

$$W(E) = \frac{\lambda}{2}(\operatorname{tr}(E))^2 + \mu \operatorname{tr}(E^2)$$

Useful FEniCS tools (I)

Defining subdomains/boundaries:

```
class MyBoundary(SubDomain):
    def inside(self, x, on_boundary):
        return on_boundary and \
               x[0] > 1.0 - DOLFIN_EPS
```

Marking boundaries:

```
my_boundary_1 = MyBoundary1()
my_boundary_2 = MyBoundary2()
boundaries = FacetFunction("uint", mesh)
boundaries.set_all(0)
my_boundary_1.mark(boundaries, 1)
my_boundary_2.mark(boundaries, 2)
ds = ds[boundaries]
a = ...*ds(0) + ...*ds(1) + ...*ds(2)
```

Useful FEniCS tools (II)

Computing derivatives of expressions:

```
E = variable(...)
W = ...
S = diff(W, E)
```

Computing derivatives of forms (linearization):

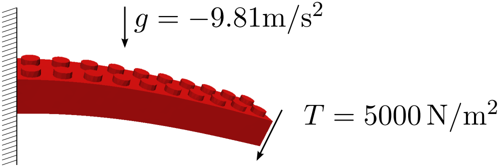
```
u  = Function(V)
du = TrialFunction(V)
F  = ...u...*dx
J  = derivative(F, u, du)
```

Solving nonlinear variational problems:

```
solve(F == 0, u, bcs)
solve(F == 0, u, bcs, J=J)
```

The FEniCS challenge!

Compute the deflection of a regular 10×2 LEGO brick. Use the St. Venant–Kirchhoff model and assume that the LEGO brick is made of PVC plastic. The LEGO brick is subject to gravity of size $g = -9.81$ m/s^2 and a downward traction of size 5000 N/m^2 at its end point.



To check your solution, compute the average value of the displacement in the z-direction.
The student(s) who first produce the right anwswer will be rewarded with an exclusive FEniCS coffee mug!