FEniCS Course

Lecture 4: Time-dependent PDEs

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The heat equation

We will solve the simplest extension of the Poisson problem into the time domain, the heat equation:

$$\begin{aligned}\frac{\partial u}{\partial t} - \Delta u &= f \quad \text{in } \Omega \text{ for } t > 0\\ u &= g \quad \text{on } \partial\Omega \text{ for } t > 0\\ u &= u^0 \text{ in } \Omega \text{ at } t = 0\end{aligned}$$

The solution $u = u(x, t)$, the right-hand side $f = f(x, t)$ and the boundary value $g = g(x, t)$ may vary in space ($x = (x_0, x_1, ...)$) and time (t). The initial value u^0 is a function of space only.

Time-discretization of the heat equation

We discretize in time using the implicit Euler (dG(0)) method:

$$\frac{\partial u}{\partial t}(t^n) \approx \frac{u^n - u^{n-1}}{\Delta t}, \quad u(t^n) \approx u^n, \quad f^n = f(t^n)$$

Semi-discretization of the heat equation:

$$\frac{u^n - u^{n-1}}{\Delta t} - \Delta u^n = f^n$$

Algorithm

1. Start with u^0 and choose a timestep $\Delta t > 0$.
2. For $n = 1, 2, \ldots$, solve for u^n:

$$u^n - \Delta t \Delta u^n = u^{n-1} + \Delta t f^n$$

Variational problem for the heat equation

Find $u^n \in V^n$ such that

$$a(u^n, v) = L^n(v)$$

for all $v \in \hat{V}$ where

$$a(u, v) = \int_\Omega uv + \Delta t \nabla u \cdot \nabla v \, \mathrm{d}x$$
$$L^n(v) = \int_\Omega u^{n-1} v + \Delta t f^n v \, \mathrm{d}x$$

Note that the bilinear form $a(u, v)$ is constant while the linear form L^n depends on n

Naive FEniCS code for the heat equation

```
from dolfin import *

# Mesh and function space
mesh = UnitSquare(8, 8)
V = FunctionSpace(mesh, "Lagrange", 1)

# Time variables
dt = Constant(0.3); t = float(dt); T = 1.8
g_expr = '1 + x[0]*x[0] + alpha*x[1]*x[1] + beta*t'
g = Expression(g_expr, alpha=3.0, beta=1.2, t=0,
               degree=2)

# Previous and current solution
u0 = interpolate(g, V)
u1 = Function(V)

# Variational problem at each time
u = TrialFunction(V)
v = TestFunction(V)
f = Constant(1.2 - 2. - 2*3.0)
a = u*v*dx + dt*inner(grad(u), grad(v))*dx
L = u0*v*dx + dt*f*v*dx
bc = DirichletBC(V, g, "on_boundary")

while (t <= T):
    # Solve
    g.t = t
    solve(a == L, u1, bc)

    # Update
    u0.assign(u1)
    t += float(dt)

plot(u1, title="Approximated final solution")
interactive()
```

Detailed time-stepping algorithm for the heat equation

Define the boundary condition
Compute u^0 *as the projection of the given initial value*
Define the forms a *and* L
Assemble the matrix A *from the bilinear form* a
$t \leftarrow \Delta t$
while $t \leqslant T$ **do**
  Assemble the vector b *from the linear form* L
  Apply the boundary condition
  Solve the linear system $AU = b$ *for* U *and store in* u^1
  $t \leftarrow t + \Delta t$
  $u^0 \leftarrow u^1$ (get ready for next step)
end while

Test problem

We construct a test problem for which we can easily check the answer. We first define the exact solution by

$$u = 1 + x^2 + \alpha y^2 + \beta t$$

We insert this into the heat equation:

$$f = \dot{u} - \Delta u = \beta - 2 - 2\alpha$$

The initial condition is

$$u^0 = 1 + x^2 + \alpha y^2$$

This technique is called the *method of manufactured solutions*

Handling time-dependent expressions

We need to define a time-dependent expression for the boundary value:

```
alpha = 3
beta = 1.2

g = Expression("1 + x[0]*x[0] + \
               alpha*x[1]*x[1] + beta*t",
               alpha=alpha, beta=beta, t=0)
```

Updating parameter values:

```
g.t = t
```

Projection and interpolation

We need to project the initial value into V_h:

```
u0 = project(g, V)
```

We can also interpolate the initial value into V_h:

```
u0 = interpolate(g, V)
```

A closer look at `solve`

For linear problems, this code

```
solve(a == L, u, bcs)
```

is equivalent to this

```
# Assembling a bilinear form yields a matrix
A = assemble(a)
# Assembling a linear form yields a vector
b = assemble(L)

# Applying boundary condition info to system
for bc in bcs:
    bc.apply(A, b)

# Solve Ax = b
solve(A, u.vector(), b)
```

Implementing the variational problem

```
dt = 0.3

u0 = project(g, V)
u1 = Function(V)

u = TrialFunction(V)
v = TestFunction(V)
f = Constant(beta - 2 - 2*alpha)

a = u*v*dx + dt*inner(grad(u), grad(v))*dx
L = u0*v*dx + dt*f*v*dx

bc = DirichletBC(V, g, "on_boundary")

# assemble only once, before time-stepping
A = assemble(a)
```

Implementing the time-stepping loop

```
T = 2
t = dt

while t <= T:
    b = assemble(L)
    g.t = t
    bc.apply(A, b)
    solve(A, u1.vector(), b)

    t += dt
    u0.assign(u1)
```

FEniCS programming exercise: heat equation

Consider the heat equation problem:

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega = [0,1]^2 \text{ for } t > 0$$
$$u(x,t) = g(x,t) \text{ for } x \in \partial\Omega \text{ for } t > 0$$
$$u(x,0) = g(x,0) \text{ for } x \in \Omega$$

with

$$f = \beta - 2 - 2\alpha$$
$$g(x,t) = 1 + x_0^2 + \alpha x_1^2 + \beta t \quad (x = (x_0, x_1))$$

Ex. 1 Compute an approximated solution at $T = 1.8$

Ex. 2 Compare the approximated solution to the exact solution at $T = 1.8$. How large is the error (in the eyenorm and in the $L^2(\Omega)$ norm)?

Ex. 3 Compute an approximated solution with the same set-up but on $\Omega = [0,1]^3 \subset \mathbb{R}^3$.