FEniCS Course

Lecture 10: Discontinuous Galerkin methods for elliptic equations

Contributors
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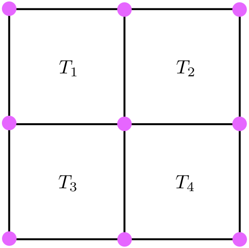
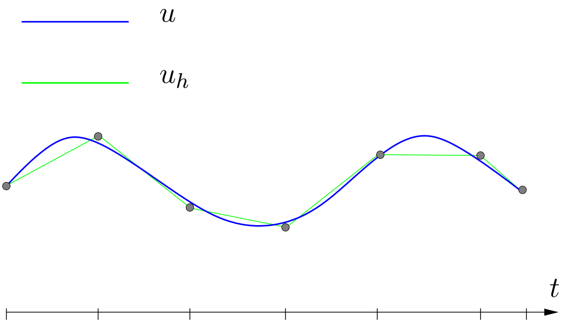
FEniCS PROJECT

The discontinuous Galerkin (DG) method uses discontinuous basis functions





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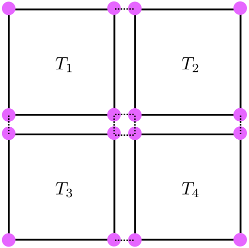

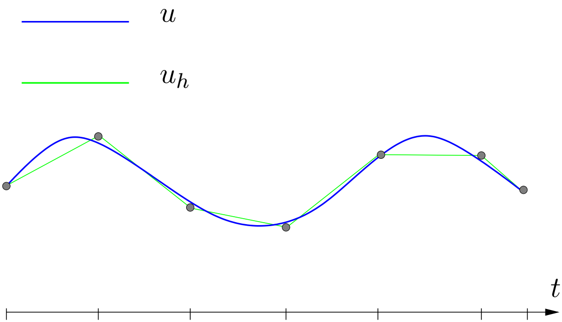

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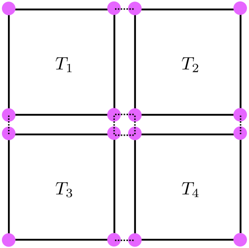

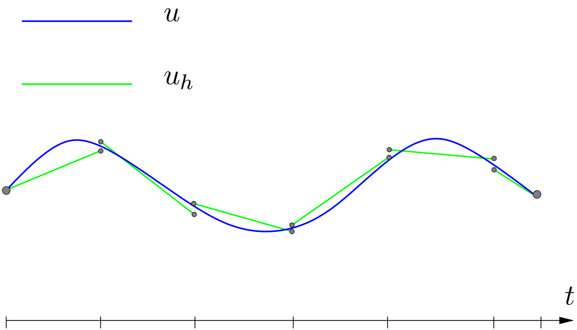

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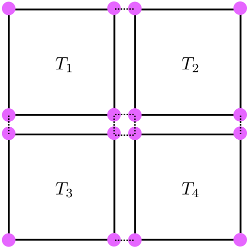
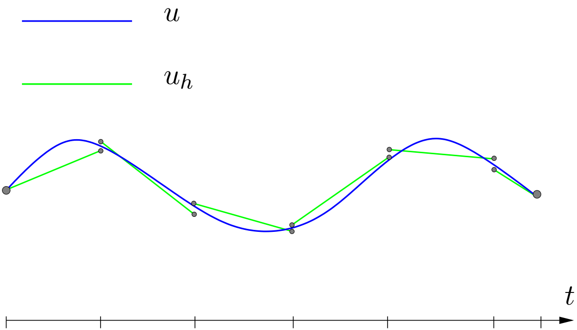
The DG method eases mesh adaptivity





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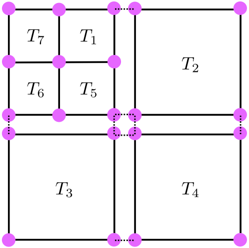

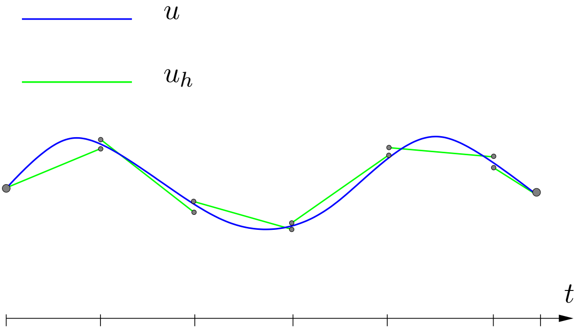

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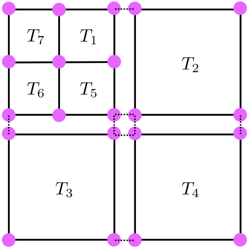

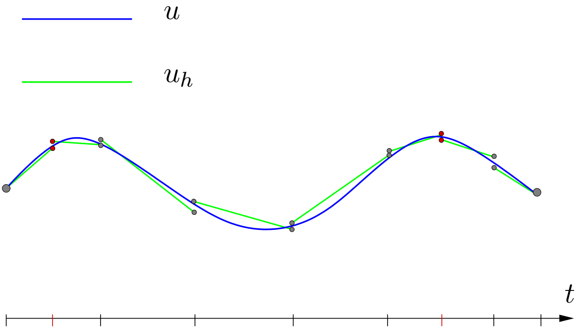

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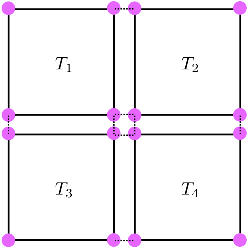

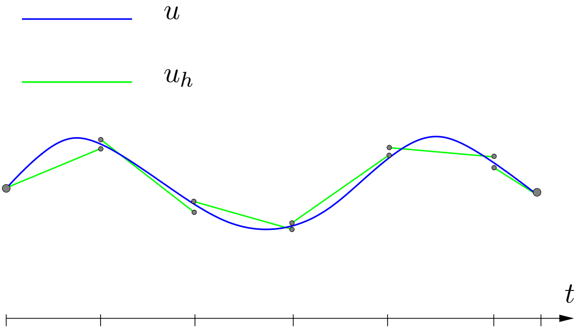

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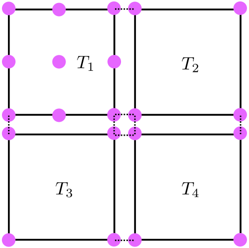

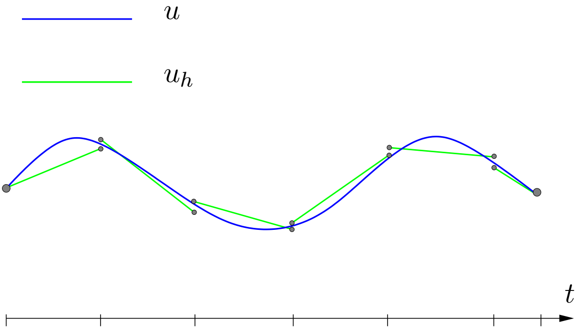

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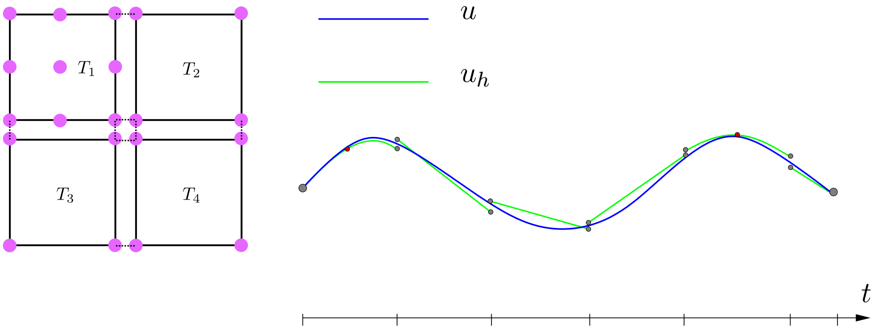
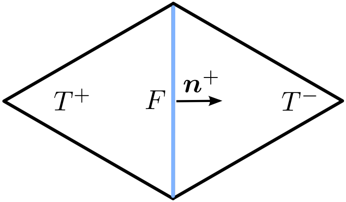


DG-FEM Notation

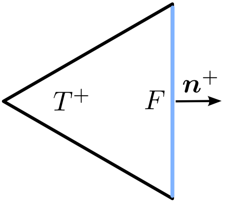
Interface facets



Average $\langle v \rangle = \frac{1}{2}(v^+ + v^-)$

Jump $[v] = (v^+ - v^-)$

Boundary facet



$\langle v \rangle = [v] = v$

Jump identity

$$[(\nabla_h v) w_h] = [\nabla_h v]\langle w_h \rangle + \langle \nabla_h v \rangle [w_h]$$

The symmetric interior penalty method (SIP)

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}} \int_T \nabla u_h \cdot \nabla v_h \,\mathrm{d}x - \underbrace{\sum_{F \in \mathcal{F}} \int_F \langle \nabla u_h \rangle \cdot \boldsymbol{n}[v_h] \,\mathrm{d}S}_{\text{Consistency}}$$

$$- \underbrace{\sum_{F \in \mathcal{F}} \int_F \langle \nabla v_h \rangle \cdot \boldsymbol{n}[u_h] \,\mathrm{d}S}_{\text{Symmetry}} + \underbrace{\sum_{F \in \mathcal{F}} \frac{\gamma}{h_F} \int_F [u_h][v_h] \,\mathrm{d}S}_{\text{Penalty}}$$

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$$l_h(v_h) = \int_\Omega f v_h \,\mathrm{d}x - \sum_{F\in\mathcal{F}^b} \int_F \langle \nabla v_h \rangle \cdot \boldsymbol{n} g \,\mathrm{d}S + \sum_{F\in\mathcal{F}^b} \frac{\gamma}{h_F} \int_F g v_h \,\mathrm{d}S$$

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Split of SIP form into interior and boundary contribution

$$a_h(u_h, v_h) = \sum_{T\in\mathcal{T}} \int_T \nabla u_h \cdot \nabla v_h \,\mathrm{d}x - \underbrace{\sum_{F\in\mathcal{F}^i} \int_F \langle \nabla u_h \rangle \cdot \boldsymbol{n}[v_h] \,\mathrm{d}S}_{\text{Consistency}}$$

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$$+ \underbrace{\sum_{F\in\mathcal{F}^b} \frac{\gamma}{h_F} \int_F u_h v_h \,\mathrm{d}s}_{\text{Penalty}}$$

Useful FEniCS tools (I)

Access facet normals and local mesh size:

```
n = FacetNormal(mesh)
h = CellSize(mesh)
```

Restriction:

```
f = Function(V)
f('+')
grad(f)('+')
```

Useful FEniCS tools (II)

Average and jump:

```
# define it yourself
h_avg = (h('+') + h('-'))/2
# or use built-in expression
avg(h)
jump(v)
jump(v, n)
```

Integration on interior facets:

```
... *dS
alpha/h_avg*dot(jump(v, n), jump(u, n))*dS
```

Exercise

Solve our favorite Poisson problem given

- Domain:

$$\Omega = [0,1] \times [0,1], \qquad \partial\Omega_D = \partial\Omega$$

- Source and boundary values:

$$f(x,y) = 200\cos(10\pi x)\cos(10\pi y)$$

$$g_D(x,y) = \cos(10\pi x)\cos(10\pi y)$$

Mission: Solve this PDE numerically by using the SIP method. Print the errornorm for both the L^2 and the H^1 norm for various mesh sizes. For a `UnitSquareMesh(128,128)` the error should be 0.0009166 and 0.1962, respectively.

Extra mission: Implement the NIP variant, solve the same problem and compare the H^1 and L^2 error for a range of meshes `UnitSquareMesh(N,N)`, $N = 2^j, j = 2, \cdots, 7$. Can you determine the order of convergence?