FEniCS Course

Lecture 6: Computing sensitivities

Contributors
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But often we are also interested the sensitivity with respect to certain parameters, for example

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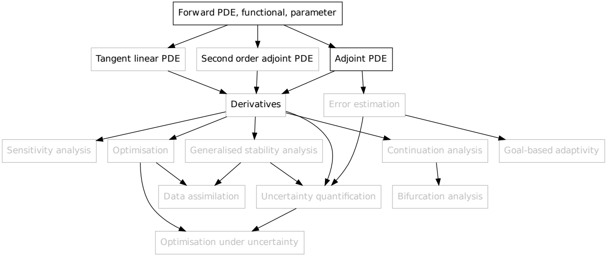

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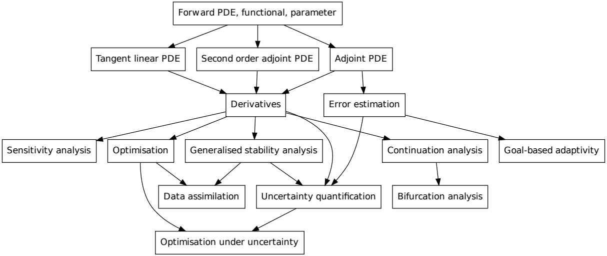

Example

Consider the Poisson's equation

$$\begin{aligned} -\Delta u &= m \quad \text{in } \Omega, \\ u &= u_0 \quad \text{on } \partial\Omega, \end{aligned}$$

together with the *objective functional*

$$J(u) = \frac{1}{2} \int_\Omega \|u - u_d\|^2 \, \mathrm{d}x,$$

where u_d is a known function.

Goal

Compute the sensitivity of J with respect to the *parameter* m: $\mathrm{d}J/\mathrm{d}m$.

Comput. deriv. (i) General formulation

Given

- Parameter m,
- PDE $F(u, m) = 0$ with solution u.
- Objective functional $J(u, m) \to \mathbb{R}$,

Goal

Compute $\mathrm{d}J/\mathrm{d}m$.

Reduced functional

Consider u as an implicit function of m by solving the PDE.
With that we define the *reduced functional* $\tilde{J}$:

$$\tilde{J}(m) = J(u(m), m)$$

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$$\tilde{J}(m) \equiv J(u(m), m).$$

Taking the derivative of with respect to m yields:

$$\frac{\mathrm{d}\tilde{J}}{\mathrm{d}m} = \frac{\mathrm{d}J}{\mathrm{d}m} = \frac{\partial J}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}m} + \frac{\partial J}{\partial m}.$$

Computing $\frac{\partial J}{\partial u}$ and $\frac{\partial J}{\partial m}$ is straight-forward, but how handle $\frac{\mathrm{d}u}{\mathrm{d}m}$?

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Comput. deriv. (iii) Computing $\frac{\mathrm{d}u}{\mathrm{d}m}$

Taking the derivative of $F(u, m) = 0$ with respect to m yields:

$$\frac{\mathrm{d}F}{\mathrm{d}m} = \frac{\partial F}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}m} + \frac{\partial F}{\partial m} = 0$$

Hence:

$$\frac{\mathrm{d}u}{\mathrm{d}m} = -\left(\frac{\partial F}{\partial u}\right)^{-1}\frac{\partial F}{\partial m}$$

Final formula for functional derivative

$$\frac{\mathrm{d}J}{\mathrm{d}m} = -\overbrace{\frac{\partial J}{\partial u}\left(\frac{\partial F}{\partial u}\right)^{-1}}^{\text{adjoint PDE}}\underbrace{\left(\frac{\partial F}{\partial u}\right)^{-1}\frac{\partial F}{\partial m}}_{\text{tangent linear PDE}} + \frac{\partial J}{\partial m},$$

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Dimensions of a finite dimensional example

$$\frac{\mathrm{d}J}{\mathrm{d}m} = \overbrace{-\frac{\partial J}{\partial u} \times \mathrlap{\underbrace{\phantom{\left(\frac{\partial F}{\partial u}\right)^{-1} \times \frac{\partial F}{\partial m}}}_{\text{discretised tangent linear PDE}}}\left(\frac{\partial F}{\partial u}\right)^{-1}}^{\text{discretised adjoint PDE}} \times \frac{\partial F}{\partial m} + \frac{\partial J}{\partial m}$$

The tangent linear solution is a matrix of dimension $|u| \times |m|$ and requires the solution of m linear systems. The adjoint solution is a vector of dimension $|u|$ and requires the solution of one linear systems.

Adjoint approach

1. Solve the adjoint equation for λ

$$\frac{\partial F}{\partial u}^* \lambda = -\frac{\partial J^*}{\partial u}.$$

2. Compute

$$\frac{\mathrm{d}J}{\mathrm{d}m} = \lambda^* \frac{\partial F}{\partial m} + \frac{\partial J}{\partial m}.$$

The computational expensive part is (1). It requires solving the (linear) adjoint PDE, and its cost is independent of the choice of parameter m.

Static example

Poisson problem

Consider

$$J(u) = \frac{1}{2}\int_\Omega \|u - u_d\|^2 \,\mathrm{d}x$$

and

$$F(u, m) = -\Delta u - m = 0.$$

```
bcs = DirichletBC(V, 0.0, "on_boundary")
a = inner(grad(u), grad(v))*dx
L = m*v*dx
solve(a == L, s, bcs)
print "J=", assemble(0.5*inner(u-ud, u-ud)*dx)
```

Static example

Adjoint system

$$\frac{\partial F^*}{\partial u}\lambda = -\frac{\partial J^*}{\partial u}$$

$$\Rightarrow \quad -\Delta\lambda = -(u - u_d) \qquad \text{(adjoint PDE)}$$

```
a = inner(grad(u), grad(v))*dx
L = -(s-ud)*v*dx
solve(a == L, lmbd, bcs)
```

Static example

Derivative computation

$$\frac{\mathrm{d}J}{\mathrm{d}m} = \lambda^* \frac{\partial F}{\partial m} + \frac{\partial J}{\partial m}$$
$$= -\lambda^*$$

```
dJdm = -lmbd
plot(dJdm, interactive=True)
```

The FEniCS challenge!

Solve the partial differential equation

$$-\Delta u = m$$

with homogeneous Dirichlet boundary conditions on the unit square for $m(x, y) = 1$.
Then solve the adjoint system for the functional

$$J(u) = \int_\Omega \|u - u_d\|^2 \, \mathrm{d}x,$$

with $u_d(x, y) = \sin(\pi x)$. Finally use the adjoint solution to compute the derivative of J with respect to m.
Can you interpret the result?