FEniCS Course

Lecture 09: One-shot optimisation

Contributors
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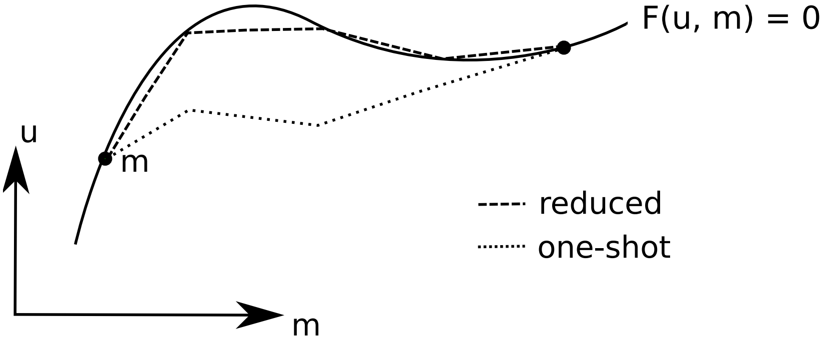
One-shot optimisation



One-shot optimisation

Consider

$$\min_{u,m} J(u, m)$$

subject to:

$$F(u, m) = 0.$$

One-shot solution strategy

1. Form Lagrangian $\mathcal{L}$
2. Set the derivative of $\mathcal{L}$ to 0 (optimality conditions)
3. Solve the resulting system

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1. Form Lagrangian $\mathcal{L} = J(u,m) + \langle \lambda, F(u,m) \rangle$ with Lagrange multipler λ.
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3. Solve the resulting system for u, m, λ simultaneously!

One-shot Hello World!

$$\min_{u,f} \frac{1}{2}\int_\Omega \|u - u_d\|^2 \,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega \|f\|^2 \,\mathrm{d}x$$

subject to:

$$-\Delta u = f \qquad \text{in } \Omega$$

1. **Lagrangian**

$$\mathcal{L} = \frac{1}{2}\int_\Omega \|u - u_d\|^2 \,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega \|f\|^2 \,\mathrm{d}x + \int_\Omega \lambda(-\Delta u - f) \,\mathrm{d}x$$

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Code

```
L = 0.5*inner(u-ud, u-ud)*dx
  + 0.5*alpha*inner(f, f)*dx
  + inner(grad(u), grad(lmbd))*dx
  - f*lmbd*dx
```

2. Optimality (KKT) conditions

$$\frac{\partial \mathcal{L}}{\partial u}\tilde{u} = 0 \quad \forall \tilde{u}$$
$$\frac{\partial \mathcal{L}}{\partial m}\tilde{m} = 0 \quad \forall \tilde{m}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda}\tilde{\lambda} = 0 \quad \forall \tilde{\lambda}$$

2. Optimality (KKT) conditions

$$\frac{\partial \mathcal{L}}{\partial u}\tilde{u} = \int_\Omega (u - u_d) \cdot \tilde{u}\,\mathrm{d}x - \int_\Omega \lambda \Delta \tilde{u}\,\mathrm{d}x = 0 \qquad \forall \tilde{u}$$

$$\frac{\partial \mathcal{L}}{\partial m}\tilde{m} = \alpha \int_\Omega m\tilde{m}\,\mathrm{d}x - \int_\Omega \lambda \tilde{m}\,\mathrm{d}x = 0 \qquad \forall \tilde{m}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}\tilde{\lambda} = \int_\Omega -\tilde{\lambda}(\Delta u - m)\,\mathrm{d}x = 0 \qquad \forall \tilde{\lambda}$$

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$$\frac{\partial \mathcal{L}}{\partial u}\tilde{u} = \int_\Omega (u - u_d) \cdot \tilde{u} \, \mathrm{d}x - \int_\Omega \lambda \Delta \tilde{u} \, \mathrm{d}x = 0 \qquad \forall \tilde{u}$$

$$\frac{\partial \mathcal{L}}{\partial m}\tilde{m} = \alpha \int_\Omega m \tilde{m} \, \mathrm{d}x - \int_\Omega \lambda \tilde{m} \, \mathrm{d}x = 0 \qquad \forall \tilde{m}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}\tilde{\lambda} = \int_\Omega -\tilde{\lambda}(\Delta u - m) \, \mathrm{d}x = 0 \qquad \forall \tilde{\lambda}$$

Code

```
# w = (u, m, lmbd)
kkt = derivative(L, w, w_test)
```

3. Solve the optimality (KKT) conditions

Easy:

```
solve(kkt == 0, w, bcs)
```