Adaptive Finite Element Methods
for the
Unsteady Maxwell's Equations

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Abstract

In this paper we describe the ideas behind adaptive finite element methods. We also apply these methods to the time-dependent Maxwell system of electromagnetics.
We use the weak formulation of Lee-Madsen and Monk, for which there is an *a priori* convergence theorem derived by Monk. We then discretise the problem using a standard Galerkin method, and we show that this method is stable.
We derive the Galerkin orthogonality properties, which together with some interpolation properties for the finite element solution, and the strong stability estimates for the adjoint problem (which we also derive) enable us to prove an *a posteriori* error estimate in the H^{-1}-norm that forms the basis for the adaptive algorithm we develop.
We also give an example of a grid refinement strategy for the special case of a 2-dimensional mesh consisting of triangles. We conclude by writing out explicitly the algebraic system of equations to be solved in the 2-dimensional case.

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Contents

Chapter 1

Introduction

1.1 Overview

Here we will give some background information on Electromagnetics, and on the adaptive finite element methods used in this paper to obtain solutions to Maxwell's equations.

1.1.1 Electromagnetics

Electromagnetics is the study of the effects of electric charges at rest and in motion. From elementary physics we know that there are two kinds of charges: positive and negative. Both positive and negative charges are sources of an electric field. Moving charges produce a current, which gives rise to a magnetic field. A time-varying electric field is accompanied by a magnetic field, and vice versa. In other words, time-varying elecric and magnetic fields are coupled, resulting in an electromagnetic field.
Electromagnetic theory is indispensable in understanding the principles behind atom smashers, cathode-ray oscilloscopes, radar, satellite communication, television reception, remote sensing, radio astronomy, microwave devices, optical fiber communication, transients in transmission lines, electromagnetic compatibility problems, instrument-landing systems, electromechanical energy conversion, brain scanners and so on.
The governing equations in electromagnetics are *Maxwell's equations*, which are usually expressed as a hyperbolic system of two coupled, first-order differential or integral equations.

The equations are named after James Clerk Maxwell (1831-1879). One of his major contributions was to generalise *Ampére's circuital law*, which is one of the Maxwell's equations, by introducing the *displacement current density* term in the equation to make it consistent with the *charge conservation law.* The other equation in the Maxwell system is *Faraday's law of electromagnetic induction.* It is named after Michael Faraday, who, in 1831, discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. It is the quantitative relationship between the induced emf and the rate of change of flux linkage, based on experimental observation, that is known as Faraday's law. *Lenz's law* is the assertion that the induced emf will cause a current to flow in the closed loop in such a direction as to oppose the change in the linking magnetic flux.
Two other equations are often also included in the Maxwell system; *Gauss's (electrical) law*, and an equation stating that there are no such things as isolated magnetic charges (sometimes called *Gauss's magnetical law*). These two equations can, as we will show, be derived from the first two, by using the charge conservation law. Maxwell's equations can, together with the charge conservation law and *Lorentz's force equation*, be used to explain *all* macroscopic electromagnetic phenomena.

Analytical solutions to Maxwell's equations do exist, but techniques for obtaining them - most notably separation of variables and Fourier and Laplace transform methods - limit the solutions to those based on simple coordinate systems with fairly regular or infinite boundaries. If, for example, we require the solution on a domain with irregular finite boundaries, or if we have variable constitutive relations, then we are forced to find the solution to such problems numerically.
The earliest numerical schemes involved a staggered mesh finite difference method as developed by Yee[23] in 1966, and more recently finite element methods have been used with considerable success, particularly when the boundaries of the problem domain are irregular.

1.1.2 Adaptive Finite Element Methods

The basic ideas behind the *adaptive finite element methods* we are going to use in this paper are described, for example, in Johnson[11]. Given a norm $\|\cdot\|$, a tolerance $TOL > 0$, and a piecewise polynomial finite element discretisation of a certain degree, we want to design an algorithm for constructing a mesh $\mathcal{T}$ such that

$$\| \mathbf{u} - \mathbf{u}^h \| \leq TOL, \tag{1.1}$$

where $\mathbf{u}$ is the exact solution and $\mathbf{u}^h$ is the finite element solution on the mesh $\mathcal{T}$. There are two important factors to be considered here. We want our algoritm to be *reliable*, so that the error $\mathbf{u} - \mathbf{u}^h$ satisfies (1.1) for any specified tolerance, and we also require it to be *efficient*, so that we do not unnecessarily refine the mesh $\mathcal{T}$. We therefore want to minimise the degrees of freedom, i.e. nodes in the mesh, at every stage, whilst ensuring that (1.1) still holds. Adaptive algorithms such as those described by Johnson are based on *a posteriori* error estimates of the form

$$\| \mathbf{u} - \mathbf{u}^h \| \leq \mathcal{E}(\mathbf{u}^h, h, data), \tag{1.2}$$

and it is this procedure that is followed in this paper. This provides us with the following adaptive strategy for error control in the norm $\| \cdot \|$ to the tolerance TOL; we want to find a mesh $\mathcal{T}$, with mesh function h and corresponding approximate solution $\mathbf{u}^h$, such that

$$\mathcal{E}(\mathbf{u}^h, h, data) \leq TOL, \tag{1.3}$$

with a minimal number of degrees of freedom. This last criterion means that we want to satisfy (1.3) with as near equality as possible.
Error estimates of the form (1.2) rely on the representation of the error in terms of the solution of an *adjoint* or *dual* problem. Such estimates are usually obtained by making use of certain properties of the finite element solution $\mathbf{u}^h$, such as Galerkin orthogonality and interpolation estimates, along with strong stability estimates derived from the related adjoint problem. This error representation is fundamental in this approach to adaptivity, as from it we gain invaluable information about the structure of the global error which then forms the basis of our adaptive algorithm.

In the next section we state the definitions and notation that will be used throughout the paper.

1.2 Definitions and Notations

In this section we will give definitions of the spaces and norms to be used in the analysis, and introduce the notation by which they will be identified. A general reference for this section is Adams[1].

1.2.1 L^p-Spaces and Sobolev Spaces

Let Ω be a bounded open subset of $\mathbf{R}^n$, for n a positive integer. Then, for $1 \leq p \leq \infty$, $L^p(\Omega)$ will denote the usual Lebesgue space of real-valued functions with norm $\| \cdot \|_{L^p(\Omega)}$. For $p = 2$, we will omit the subscript, writing $\| \cdot \|$ for $\| \cdot \|_{L^2(\Omega)}$, and we define the $L^2(\Omega)$ inner product $(\cdot, \cdot)$ by

$$(u, v) = \int_\Omega u(\mathbf{x})v(\mathbf{x})d\mathbf{x},$$

for $u, v \in L^2(\Omega)$. If ω is a measurable subset of $\overline{\Omega}$, we denote by $(\cdot, \cdot)_{L^2(\omega)}$ the L^2 inner product on ω. The space-time L^2 norm is defined as

$$\| f \|_{L^2(0,T;L^2(\Omega))} = \left(\int_0^T \| f \|^2 \, dt \right)^{1/2}.$$

The space of m-times continuously differentiable functions from $[0, T]$ into the Hilbert space X is denoted $C^m(0, T; X)$. We also introduce the ρ-weighted inner product $(\cdot, \cdot)_\rho$, defined as

$$(u, v)_\rho = \int_\Omega \rho(\mathbf{x})u(\mathbf{x})v(\mathbf{x})d\mathbf{x},$$

where $\rho : \Omega \to \mathbb{R}_+$, and ρ is locally integrable on Ω. We then define $L^2_\rho(\Omega)$ to be the Hilbert space where the norm

$$\| u \|_\rho = \sqrt{(u, u)_\rho}$$

is finite.

Further, for k a non-negative integer, let $W^{k,p}(\Omega)$ denote the classical Sobolev space equipped with the norm $\| \cdot \|_{W^{k,p}(\Omega)}$ and the semi-norm $| \cdot |_{W^{k,p}(\Omega)}$. For $p = 2$ we write $H^k(\Omega)$ for $W^{k,2}(\Omega)$. Also, let $H^k_0(\Omega)$ denote the closure of the space of infinitely smooth functions with compact support in Ω in the norm of $H^k(\Omega)$. The dual space of $H_0^k(\Omega)$ will be denoted by $H^{-k}(\Omega)$, with its corresponding norm given by

$$\| L \|_{H^{-k}} = \sup_{0 \neq v \in H_0^k(\Omega)} \frac{L(v)}{\| v \|_{H_0^k(\Omega)}}$$

where L is a continuous functional on $H_0^k(\Omega)$.

1.2.2 The Space $H(div;\Omega)$

The space of functions with square integrable divergence is denoted by

$$H(div;\Omega) = \{\mathbf{u} \in L^2(\Omega)^n | \nabla \cdot \mathbf{u} \in L^2(\Omega)\},$$

and the associated (graph) norm on $H(div;\Omega)$ is

$$\| \mathbf{u} \|_{H^D} = (\| \mathbf{u} \|^2 + \| \nabla \cdot \mathbf{u} \|^2)^{1/2}.$$

With the inner product

$$(\mathbf{u}, \mathbf{v})_{H^D} = (\mathbf{u}, \mathbf{v}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}),$$

$H(div;\Omega)$ becomes a Hilbert space.

We also state here the following Green's theorem:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain in $\mathbb{R}^N$. Then the mapping $\gamma_n : \mathbf{v} \to \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega}$ defined on $(\mathcal{D}(\overline{\Omega}))^N$ can be extended by continuity to a linear continuous map γ_n from $H(div;\Omega)$ onto $H^{-1/2}(\partial\Omega)$. Furthermore the following Green's theorem holds for functions $\mathbf{v} \in H(div;\Omega)$ and $\phi \in H^1(\Omega)$*

$$(\mathbf{v}, \nabla\phi) + (\nabla \cdot \mathbf{v}, \phi) = (\phi, \mathbf{v} \cdot \mathbf{n})_{L^2(\partial\Omega)}. \tag{1.4}$$

Proof See Monk[16]. □

For a discussion on fractional order spaces, see Adams[1].

1.2.3 The Space $H(curl;\Omega)$

The curl operator is defined on a three-dimensional vector function $\mathbf{v}$ (for which the derivatives make sense) by

$$\nabla \times \mathbf{v} = (\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}). \tag{1.5}$$

In $\mathbb{R}^2$ there are two curl operators, one scalar and one vector. If $\mathbf{v}$ is a 2-component vector function, then its scalar curl is given by

$$\nabla \times \mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2},$$

which is just the third component of (1.5). For a scalar function ϕ, the corresponding vector curl is given by

$$\vec{\nabla} \times \phi = (\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}),$$

which is just the first two components of (1.5), but with $v_1 = v_2 = 0$ and $v_3 = \phi$. Corresponding to the space $H(div;\Omega)$ we define the space of three dimensional vector functions with square integrable curl by

$$H(curl;\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla \times \mathbf{v} \in L^2(\Omega)^3\},$$

with the corresponding (graph) norm

$$\| \mathbf{v} \|_{H^C} = (\| \mathbf{v} \|^2 + \| \nabla \times \mathbf{v} \|^2)^{1/2}.$$

In $\mathbb{R}^2$ there are two possible spaces corresponding to the vector and the scalar curl operators. The simplest is the space of scalar functions with square integrable vector curl given by

$$H(c\vec{u}rl;\Omega) = \{u \in L^2(\Omega) : \vec{\nabla} \times u \in L^2(\Omega)^2\},$$

with the associated (graph) norm

$$\| u \|_{H(c\vec{u}rl;\Omega)} = (\| u \|^2 + \| \vec{\nabla} \times u \|^2)^{1/2}.$$

We have that $u \in H(c\vec{url};\Omega)$ if and only if $u \in H^1(\Omega)$. Indeed the $\| \cdot \|_{H(c\vec{url},\Omega)}$ norm and the $\| \cdot \|_{H^1(\Omega)}$ norm are exactly the same so that

$$H(c\vec{url};\Omega) \cong H^1(\Omega).$$

The other case is the space of vector functions with square integrable scalar curl which is defined as

$$H(curl;\Omega) = \{\mathbf{u} \in L^2(\Omega)^2 : \nabla \times \mathbf{u} \in L^2(\Omega)\},$$

with the associated (graph) norm

$$\| \mathbf{u} \|_{H(curl;\Omega)} = (\| \mathbf{u} \|^2 + \| \nabla \times \mathbf{u} \|^2)^{1/2}.$$

We also have a Green's theorem:

Theorem 1.2 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n, n = 2, 3$, with the unit normal* $\mathbf{n}$ to $\partial\Omega$. *Then*

(i) If $n = 3$, the trace map $\gamma_t : \mathbf{v} \to \mathbf{v} \times \mathbf{n}|_{\partial\Omega}$ which is defined classically on $(\mathcal{D}(\bar{\Omega}))^3$ can be extended by continuity to a continuous linear map from $H(curl;\Omega)$ onto $H^{-1/2}(\partial\Omega)$. Furthermore the following Green's theorem holds for any $\mathbf{v} \in H(curl;\Omega)$ and $\phi \in H^1(\Omega)^3$

$$(\nabla \times \mathbf{v}, \phi) - (\mathbf{v}, \nabla \times \phi) = (\mathbf{v} \times \mathbf{n}, \phi)_{L^2(\partial\Omega)} \qquad (1.6)$$

(ii) If $n = 2$ and the unit normal $\mathbf{n} = (n_1, n_2)$, *we define* $\mathbf{v} \times \mathbf{n} = v_2 n_1 - v_1 n_2$. *Then the trace map $\gamma_t : \mathbf{v} \to \mathbf{v} \times \mathbf{n}|_{\partial}\Omega$ which is defined classically on $(\mathcal{D}(\bar{\Omega}))^2$ can be extended by continuity to a continuous linear map, still called γ_t, from $H(curl;\Omega)$ onto $H^{-1/2}(\partial\Omega)$. Furthermore the following Green's theorem holds for any $\mathbf{v} \in H(curl;\Omega)$ and $\phi \in H^1(\Omega)$*

$$(\nabla \times \mathbf{v}, \phi) - (\mathbf{v}, \vec{\nabla} \times \phi) = (\mathbf{v} \times \mathbf{n}, \phi)_{L^2(\partial\Omega)} \qquad (1.7)$$

Proof See Monk[16]. □

In the next section we are going to state Maxwell's equations, and formulate the problem which we are going to analyse in the rest of this paper.

1.3 Problem Formulation

Let Ω be a smooth, bounded, simply connected domain in $\mathbb{R}^3$ with connected boundary Γ and unit outward normal $\mathbf{n}$. Let $\epsilon(\mathbf{x})$ and $\mu(\mathbf{x})$ denote, respectively, the dielectric constant and magnetic permeability of the medium occupying Ω. Let $\sigma(\mathbf{x})$ denote the conductivity of the medium. Also let the constitutive relations $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ hold (where $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic flux densities respectively). Then, if $\mathbf{E} = \mathbf{E}(x,t)$ and $\mathbf{H} = \mathbf{H}(x,t)$ denote, respectively, the electric and magnetic field intensities, *Maxwell's equations* state that

$$\epsilon\frac{\partial \mathbf{E}}{\partial t} + \sigma\mathbf{E} - \nabla \times \mathbf{H} = \mathbf{J} \qquad \text{in } \Omega \times (0,T) \tag{1.8}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0 \qquad \text{in } \Omega \times (0,T) \tag{1.9}$$

$$\nabla \cdot (\epsilon\mathbf{E}) = \rho \qquad \text{in } \Omega \times (0,T) \tag{1.10}$$

$$\nabla \cdot (\mu\mathbf{H}) = 0 \qquad \text{in } \Omega \times (0,T), \tag{1.11}$$

where $\mathbf{J} = \mathbf{J}(x,t)$ is a known function specifying the applied current, and ρ denotes the charge density. (1.8) is called *Ampere's circuital law*, (1.9) *Faraday's law*, and (1.10) is called *Gauss's law*. (1.11) expresses that there are no such things as isolated magnetic charges. In this paper we shall assume a perfect conducting boundary condition on Ω, so that

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma \times (0,T) \tag{1.12}$$

$$\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times (0,T). \tag{1.13}$$

In addition, initial conditions must be specified so that

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \tag{1.14}$$

$$\mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \tag{1.15}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are given functions and $\mathbf{H}_0$ satisfies

$$\nabla \cdot (\mu \mathbf{H}_0) = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{H}_0 \cdot \mathbf{n} = 0 \text{ on } \Gamma. \tag{1.16}$$

The coefficients ϵ, μ, and σ are $L^\infty(\Omega)$ functions for which there exist constants ϵ_{min}, ϵ_{max}, μ_{min}, μ_{max}, and σ_{max} such that

$$\left.\begin{array}{lllll} 0 < \epsilon_{min} & \leq \epsilon(\mathbf{x}) & \leq \epsilon_{max} & < \infty \\ 0 < \mu_{min} & \leq \mu(\mathbf{x}) & \leq \mu_{max} & < \infty \\ 0 & \leq \sigma(\mathbf{x}) & \leq \sigma_{max} & < \infty \end{array}\right\} \text{a.e. in } \Omega.$$

Actually, by taking the divergence of (1.9), and using the divergence-free condition in (1.16), we can write

$$\nabla \cdot (\mu \mathbf{H}_t + \nabla \times \mathbf{E}) = \frac{\partial}{\partial t}(\nabla \cdot (\mu \mathbf{H})) = 0,$$

so that $\nabla \cdot (\mu \mathbf{H})$ is constant in time. But we have that $\nabla \cdot (\mu \mathbf{H_0}) = 0$, so (1.11) follows ($\nabla \cdot \nabla \times \mathbf{E} = 0$ by well known rules of vector calculus). In a similar way by taking the divergence of (1.8), and using the *charge conservation law*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\sigma \mathbf{E} - \mathbf{J}) = 0 \quad \text{in } \Omega, \tag{1.17}$$

we get (1.10). In addition, the boundary condition in (1.16), together with (1.9) and (1.12) implies

$$\mathbf{H} \cdot \mathbf{n} = 0 \text{ on } \Gamma \times (0, T),$$

which is our boundary condition (1.13).

So we have that the problem (1.8)-(1.9), (1.12), and (1.14)-(1.16) is well-posed in itself, as the other equations can be derived from them by assuming that ρ and $\mathbf{J}$ are coupled through (1.17).
This is why the Maxwell system is not overdetermined, even though it may appear so. (1.8)-(1.11) gives 6 unknowns and 8 equations. But, as we have seen, (1.10) and (1.11) can be derived from (1.8) and (1.9), by using the charge conservation law.

So we have, by assuming that the charge conservation law holds, that the well-posed problem we are going to analyse is:

$$\epsilon\frac{\partial \mathbf{E}}{\partial t} + \sigma\mathbf{E} - \nabla \times \mathbf{H} = \mathbf{J} \qquad in\ \Omega \times (0,T) \tag{1.18}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0 \qquad in\ \Omega \times (0,T) \tag{1.19}$$

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma \times (0,T) \tag{1.20}$$

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \tag{1.21}$$

$$\mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \tag{1.22}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are given functions and $\mathbf{H}_0$ satisfies

$$\nabla \cdot (\mu\mathbf{H}_0) = 0 \ \text{ in } \Omega \quad \text{ and } \quad \mathbf{H}_0 \cdot \mathbf{n} = 0 \ \text{ on } \Gamma. \tag{1.23}$$

We shall assume the existence of a solution $(\mathbf{E},\mathbf{H})$ to (1.18)-(1.23) such that $\mathbf{E},\mathbf{H} \in C^1(0,T;L^2(\Omega)^3) \cap C^0(0,T;H(curl;\Omega))$. Clearly the above regularity assumption requires that $\mathbf{J} \in C^0(0,T;L^2(\Omega)^3)$.

Chapter 2

Discretisation in Space and Time

2.1 A Weak Formulation

Assuming the existence of a solution to (1.18)-(1.23), we obtain a weak formulation as follows. We multiply equation (1.18) by a test function $\phi \in L^2(\Omega)^3$ and integrate over Ω. Similarly, multiplying (1.19) by $\psi \in H(curl;\Omega)$, integrating over Ω, and integrating the curl-term by parts using the Green's theorem (1.6) and the boundary condition (1.20), we obtain a weak form for (1.19). If we let $\mathbf{E}(t) = \mathbf{E}(\cdot, t)$ and $\mathbf{H}(t) = \mathbf{H}(\cdot, t)$, we find that the solution $(\mathbf{E}, \mathbf{H}) \in [C^1(0,T; L^2(\Omega)^3) \cap C^0(0,T; H(curl;\Omega))]^2$ of (1.18)-(1.23) satisfies

$$\begin{aligned}
(\epsilon \mathbf{E}_t, \phi) + (\sigma \mathbf{E}, \phi) - (\nabla \times \mathbf{H}, \phi) &= (\mathbf{J}, \phi) \quad \forall \phi \in L^2(\Omega)^3 &(2.1)\\
(\mu \mathbf{H}_t, \psi) + (\mathbf{E}, \nabla \times \psi) &= 0 \qquad\quad \forall \psi \in H(curl;\Omega) &(2.2)
\end{aligned}$$

for $0 < t \leq T$, with the initial conditions

$$\mathbf{E}(0) = \mathbf{E}_0 \quad \text{and} \quad \mathbf{H}(0) = \mathbf{H}_0, \tag{2.3}$$

where $\mathbf{H}_0$ satisfies (1.23).

Of course for the above variational problem to make sense we need only require that $\mathbf{E} \in C^1(0,T; L^2(\Omega)^3) \cap C^0(0,T; L^2(\Omega)^3)$, so the variational problem

might be used to prove existence of a weak solution to Maxwell's equations. Notice that the boundary condition (1.20) is now imposed weakly via (2.2). This is one advantage of the weak form given in (2.1)-(2.3) since the boundary condition does not have to be imposed on trial and test spaces. The more general condition $\mathbf{n} \times \mathbf{E} = \gamma$, where γ is a tangential surface field, could also be handled easily by this formulation.
This weak formulation is called the *Lee-Madsen formulation.* It forms the basis of the finite element schemes of Monk and Lee-Madsen, see [18] and [12]. This is also the weak formulation that we are going to use in this paper. Another possibility is to apply the same Green's theorem to the curl-term in (2.1) instead. We then get the so called *Nèdèlec's formulation.*

2.2 Spatial Discretisation

Let $U_h \subset L^2(\Omega)^3$ and $V_h \subset H(curl;\Omega)$ be finite-dimensional subspaces of the given spaces (we shall define U_h and V_h in Section 3.3). Then the semidiscrete Maxwell system we will analyse in this paper is to find $(\mathbf{E}^h, \mathbf{H}^h) \in C^1(0,T;U_h) \times C^1(0,T;V_h)$ such that

$$\begin{aligned} (\epsilon \mathbf{E}_t^h, \phi^h) + (\sigma \mathbf{E}^h, \phi^h) - (\nabla \times \mathbf{H}^h, \phi^h) &= (\mathbf{J}, \phi^h) \quad \forall \phi^h \in U_h \qquad (2.4)\\ (\mu \mathbf{H}_t^h, \psi^h) + (\mathbf{E}^h, \nabla \times \psi^h) &= 0 \qquad\qquad \forall \psi^h \in V_h \qquad (2.5) \end{aligned}$$

for $0 < t \leq T$, subject to the initial conditions

$$\mathbf{E}^h(0) = \Pi_\epsilon^h \, \mathbf{E}_0 \quad \text{and} \quad \mathbf{H}^h(0) = \Pi_\mu^h \, \mathbf{H}_0, \qquad (2.6)$$

where $\Pi_\epsilon^h : L^2(\Omega)^3 \to U_h$ and $\Pi_\mu^h : H(curl;\Omega) \to V_h$ are the weighted (by ϵ and μ respectively) L^2*-projections* (see Eriksson *et al.*[5], pp. 338-339) of the initial data onto the spaces U_h, V_h respectively. The equations (2.4)-(2.6) are a system of linear ordinary differential equations, and thus existence and uniqueness of a solution are well known. An *a priori* analysis of this problem can be found in Monk[17], with a general convergence theorem on pp. 1614-1615.

2.3 Stability Analysis

Here we are going to prove that the method described in the previous section is stable in the sense that the solution at time t depends continuously on the initial data. But we start by presenting some inequalities that will be used frequently in the following analysis.

2.3.1 Useful Inequalities

If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, we denote the standard inner product between $\mathbf{a}$ and $\mathbf{b}$ by

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{N} a_i b_i,$$

and define the Euclidean length of $\mathbf{a}$ by $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.
Then we have the following useful inequalities. The first is the *Cauchy-Schwarz inequality*

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|,$$

and the second is the *arithmetic geometric mean inequality*

$$|\mathbf{a} \cdot \mathbf{b}| \leq \frac{\delta}{2}|\mathbf{a}|^2 + \frac{1}{2\delta}|\mathbf{b}|^2$$

for any $\delta > 0$.

These inequalities can also be extended to norms of functions. That is, if $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$, then the *Cauchy-Schwarz inequality* states that

$$|(f, g)| \;\leq\; \| f \| \| g \| \,.$$

The *arithmetic geometric mean inequality* states that

$$|(f,g)| \leq \frac{1}{2\delta} \parallel f \parallel^2 + \frac{\delta}{2} \parallel g \parallel^2$$

for any $\delta > 0$.

Grönwall's inequality states that if

$$y(t) \leq \ C + \int_0^t z(s)y(s) \ ds,$$

where $y(t) \geq 0$, then

$$y(t) \leq \ C \ e^{\int_0^t z(s)ds}.$$

We are also going to use the simple algebraic inequalities

$$(a+b)^2 \leq 2a^2 + 2b^2$$
$$2ab \leq a^2 + b^2$$
$$(a+b)^{1/2} \leq a^{1/2} + b^{1/2},$$

where $a, b \in \mathbb{R}$ (in the last one $a, b \geq 0$), and the more general

$$(a_1 + ... + a_k)^2 \leq k(a_1^2 + ... + a_k^2),$$

where $a_1, ..., a_k \in \mathbb{R}$. This inequality is easily proved by:

$$(a_1 + ... + a_k)^2 = (1 \cdot a_1 + ... + 1 \cdot a_k)^2 = ((1, ..., 1) \cdot (a_1, ..., a_k))^2$$
$$\leq \ (|(1, ..., 1)||(a_1, ..., a_k)|)^2 = k(a_1^2 + ... + a_k^2). \qquad \square$$

2.3.2 Stability Analysis

In this section we are going to prove the following stability theorem:

Theorem 2.1 *Let* $(\mathbf{E}^h, \mathbf{H}^h) \in C^1(0,T;U_h) \times C^1(0,T;V_h)$ *solve (2.4)-(2.6). Then the we have the following stability estimate:*

$$\begin{aligned} & \| \mathbf{E}^h(t) \|_\epsilon^2 + \| \mathbf{H}^h(t) \|_\mu^2 \\ \leq \; & e^t \left(\| \mathbf{E}^h(0) \|_\epsilon^2 + \| \mathbf{H}^h(0) \|_\mu^2 + \int_0^T \| \mathbf{J}(s) \|_{\frac{1}{\epsilon}}^2 \, ds \right). \end{aligned} \tag{2.7}$$

Proof We start by letting $\phi = \mathbf{E}^h$ and $\psi = \mathbf{H}^h$ in (2.4)-(2.5), and then we add the two equations together:

$$(\epsilon \mathbf{E}_t^h, \mathbf{E}^h) + (\sigma \mathbf{E}^h, \mathbf{E}^h) - (\nabla \times \mathbf{H}^h, \mathbf{E}^h) + (\mu \mathbf{H}_t^h, \mathbf{H}^h) + (\mathbf{E}^h, \nabla \times \mathbf{H}^h) = (\mathbf{J}, \mathbf{E}^h).$$

That is

$$(\epsilon, \frac{1}{2}\frac{d}{dt}|\mathbf{E}_t^h|^2) + (\sigma \mathbf{E}^h, \mathbf{E}^h) + (\mu, \frac{1}{2}\frac{d}{dt}|\mathbf{H}_t^h|^2) = (\mathbf{J}, \mathbf{E}^h).$$

But $\sigma > 0$ which gives

$$\frac{1}{2}\frac{d}{dt}(\| \sqrt{\epsilon}\, \mathbf{E}^h \|^2 + \| \sqrt{\mu}\, \mathbf{H}^h \|^2) \leq (\mathbf{J}, \mathbf{E}^h).$$

By integrating both sides in time from 0 to t, and using Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \| \sqrt{\epsilon}\, \mathbf{E}^h(t) \|^2 + \| \sqrt{\mu}\, \mathbf{H}^h(t) \|^2 \\ \leq \; & \| \sqrt{\epsilon}\, \mathbf{E}^h(0) \|^2 + \| \sqrt{\mu}\, \mathbf{H}^h(0) \|^2 + 2\int_0^t \| \frac{1}{\sqrt{\epsilon}} \mathbf{J}(s) \| \; \| \sqrt{\epsilon}\, \mathbf{E}^h(s) \| \, ds \\ \leq \; & \| \sqrt{\epsilon}\, \mathbf{E}^h(0) \|^2 + \| \sqrt{\mu}\, \mathbf{H}^h(0) \|^2 + \int_0^T \| \frac{1}{\sqrt{\epsilon}} \mathbf{J}(s) \|^2 \, ds + \int_0^t \| \sqrt{\epsilon}\, \mathbf{E}^h(t) \|^2 \, ds \\ \leq \; & C_0 + \int_0^t (\| \sqrt{\epsilon}\, \mathbf{E}^h(t) \|^2 + \| \sqrt{\mu}\, \mathbf{H}^h(t) \|^2) \, ds, \end{aligned}$$

where $C_0 \;=\; \parallel \sqrt{\epsilon}\, \mathbf{E}^h(0) \parallel^2 + \parallel \sqrt{\mu}\, \mathbf{H}^h(0) \parallel^2 + \int_0^T \parallel \frac{1}{\sqrt{\epsilon}}\, \mathbf{J}(s) \parallel^2 ds.$

Now we can use Grönwall's inequality, with $C = C_0$ and $z(t) = 1$;

$$\parallel \sqrt{\epsilon}\, \mathbf{E}^h(t) \parallel^2 + \parallel \sqrt{\mu}\, \mathbf{H}^h(t) \parallel^2 \leq \; C_0\, e^{\int_0^t 1\, ds}.$$

□

Remark 2.2 *This result also implies uniqueness of the solution.*

In the next section we shall consider a discrete version of Theorem 2.1; that is, the stability analysis of the time-discretised scheme.

2.4 Time Discretisation & Stability Analysis

In this section we are going to compare the stability properties and the energy conservation properties of the *Crank-Nicolson method* and the *Implicit Euler method*, that is the θ-method with $\theta = 1/2$ and $\theta = 1$ respectively.

2.4.1 The Crank-Nicolson Method

In the following stability analysis we are going to study our system (2.4)-(2.6) discretised in time using the Crank-Nicolson method. We are then going to prove that this method is stable in the sense that the solution at time level k depends continuously on the initial data. We are also going to show that this method is energy conservative for $\sigma = \mathbf{J} = 0$.

We start by adding the two space-time discretised variational equations with $\phi = (\mathbf{E}^h_{m+1} + \mathbf{E}^h_m)/2$ and $\psi = (\mathbf{H}^h_{m+1} + \mathbf{H}^h_m)/2$ together:

$$
\begin{aligned}
&(\ \epsilon \ \frac{\mathbf{E}^h_{m+1} - \mathbf{E}^h_m}{\Delta t}, \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} \ ) + (\ \sigma \ \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2}, \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} \ ) \\
&-(\nabla \times (\ \frac{\mathbf{H}^h_{m+1} + \mathbf{H}^h_m}{2}), \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} \ ) + (\ \mu \ \frac{\mathbf{H}^h_{m+1} - \mathbf{H}^h_m}{\Delta t}, \frac{\mathbf{H}^h_{m+1} + \mathbf{H}^h_m}{2} \ ) \\
&+(\ \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2}, \nabla \times (\ \frac{\mathbf{H}^h_{m+1} + \mathbf{H}^h_m}{2} \ )) = (\ \frac{\mathbf{J}_{m+1} + \mathbf{J}_m}{2}, \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} \ ).
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\| \mathbf{E}^h_{m+1} \|^2_\epsilon + \| \mathbf{H}^h_{m+1} \|^2_\mu) + \| \ \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} \ \|^2_\sigma \\
&= \frac{1}{2\Delta t}(\| \mathbf{E}^h_m \|^2_\epsilon + \| \mathbf{H}^h_m \|^2_\mu) + (\ \frac{\mathbf{J}_{m+1} + \mathbf{J}_m}{2}, \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} \ ).
\end{aligned}
$$

Now we consider two different cases.

First we consider the case when $\sigma = \mathbf{J} = 0$. We then have that the Crank-Nicholson scheme is *energy conservative* in the sense that if we define the energy function $\mathcal{E}$ at time level m as

$$
\mathcal{E}_m = \ \| \mathbf{E}^h_m \|^2_\epsilon + \| \mathbf{H}^h_m \|^2_\mu,
$$

we have that $\mathcal{E}_{m+1} = \mathcal{E}_m = ... = \mathcal{E}_0$.

In particular we have that for any k

$$
\| \mathbf{E}^h_k \|^2_\epsilon + \| \mathbf{H}^h_k \|^2_\mu \ = \ \| \mathbf{E}^h_0 \|^2_\epsilon + \| \mathbf{H}^h_0 \|^2_\mu \, . \tag{2.8}
$$

The second case we consider is when $\sigma \neq 0$ and $\mathbf{J} \neq 0$. We then have, by using the arithmetic mean inequality with $\delta = 2$,

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\| \mathbf{E}^h_{m+1} \|^2_\epsilon + \| \mathbf{H}^h_{m+1} \|^2_\mu) + \| \ \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} \ \|^2_\sigma \\
&\leq \frac{1}{2\Delta t}(\| \mathbf{E}^h_m \|^2_\epsilon + \| \mathbf{H}^h_m \|^2_\mu) + \frac{1}{4} \| \ \frac{\mathbf{J}_{m+1} + \mathbf{J}_m}{2} \ \|^2_{\frac{1}{\sigma}} + \| \ \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} \ \|^2_\sigma \, .
\end{aligned}
$$

This gives

$$\mathcal{E}_{m+1} \ \leq \ \mathcal{E}_m + \frac{\Delta t}{2} \parallel \frac{\mathbf{J}_{m+1} + \mathbf{J}_m}{2} \parallel^2_{\frac{1}{\sigma}} .$$

By summing from $m = 0$ to $m = k-1$ we obtain the following stability theorem:

Theorem 2.3 *Let* $(\mathbf{E}^h, \mathbf{H}^h) \in U_h \times V_h$ *solve (2.4)-(2.6), discretised in time by the Crank-Nicolson method. If* $(\mathbf{E}^h_k, \mathbf{H}^h_k)$ *denotes* $(\mathbf{E}^h(t_k), \mathbf{H}^h(t_k))$*, with* $t_k = k \cdot \Delta t$*, then the following stability estimate holds:*

$$\parallel \mathbf{E}^h_k \parallel^2_\epsilon + \parallel \mathbf{H}^h_k \parallel^2_\mu \ \leq \ \parallel \mathbf{E}^h(0) \parallel^2_\epsilon + \parallel \mathbf{H}^h(0) \parallel^2_\mu + \frac{\Delta t}{2} \sum_{m=0}^{k-1} \parallel \frac{\mathbf{J}_{m+1} + \mathbf{J}_m}{2} \parallel^2_{\frac{1}{\sigma}} . \tag{2.9}$$

Also, if $\sigma = \mathbf{J} = 0$*, we have that our Crank-Nicolson scheme is energy conservative, in the sense of (2.8).*

2.4.2 The Implicit Euler Method

In this section we are going to derive similar stability estimates as we did in the previous section for the case of the Crank-Nicolson scheme. We are also going to show that the Implicit Euler method is not energy conservative for $\sigma = \mathbf{J} = 0$, in contrast with the Crank-Nicolson scheme.

We start, as in the previous section, by adding the two space-time discretised (now by using the Implicit Euler method) variational equations with $\phi = \mathbf{E}^h_{m+1}$ and $\psi = \mathbf{H}^h_{m+1}$ together:

$$\begin{aligned} &(\ \epsilon \ \frac{\mathbf{E}^h_{m+1} - \mathbf{E}^h_m}{\Delta t}, \mathbf{E}^h_{m+1}) + (\sigma \mathbf{E}^h_{m+1}, \mathbf{E}^h_{m+1}) - (\nabla \times \mathbf{H}^h_{m+1}, \mathbf{E}^h_{m+1}) \\ &+(\ \mu \ \frac{\mathbf{H}^h_{m+1} - \mathbf{H}^h_m}{\Delta t}, \mathbf{H}^h_{m+1}) + (\mathbf{E}^h_{m+1}, \nabla \times \mathbf{H}^h_{m+1}) = (\mathbf{J}_{m+1}, \mathbf{E}^h_{m+1}). \end{aligned}$$

Now consider the first term

$$
\begin{aligned}
&(\ \epsilon \ \frac{\mathbf{E}^h_{m+1} - \mathbf{E}^h_m}{\Delta t}, \mathbf{E}^h_{m+1}) = (\ \epsilon \ \frac{\mathbf{E}^h_{m+1} - \mathbf{E}^h_m}{\Delta t}, \frac{\mathbf{E}^h_{m+1} + \mathbf{E}^h_m}{2} + \frac{\mathbf{E}^h_{m+1} - \mathbf{E}^h_m}{2} \ ) \\
&= \frac{1}{2\Delta t}(\| \mathbf{E}^h_{m+1} \|^2_\epsilon - \| \mathbf{E}^h_m \|^2_\epsilon) + \frac{1}{2\Delta t} \| \mathbf{E}^h_{m+1} - \mathbf{E}^h_m \|^2_\epsilon .
\end{aligned}
$$

Here the second term is non-negative. Even if we have that $\sigma = \mathbf{J} = 0$, we still have this term. Therefore the Implicit Euler scheme is *energy-dissipative*, not energy-conservative as was the case with the Crank-Nicolson scheme. By using the fact that the second term is non-negative and also using the arithmetic mean inequality with $\delta = 2$ we have that

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\| \mathbf{E}^h_{m+1} \|^2_\epsilon - \| \mathbf{E}^h_m \|^2_\epsilon) + \frac{1}{2\Delta t}(\| \mathbf{H}^h_{m+1} \|^2_\mu - \| \mathbf{H}^h_m \|^2_\mu) + \| \mathbf{E}^h_{m+1} \|^2_\sigma \\
&\leq \ (\mathbf{J}_{m+1}, \mathbf{E}^h_{m+1}) \ \leq \ \frac{1}{4} \| \mathbf{J}_{m+1} \|^2_{\frac{1}{\sigma}} + \| \mathbf{E}^h_{m+1} \|^2_\sigma .
\end{aligned}
$$

Finally by summing from $m = 0$ to $m = k - 1$, we get the following stability estimate:

Theorem 2.4 *Let* $(\mathbf{E}^h, \mathbf{H}^h) \in U_h \times V_h$ *solve (2.4)-(2.6), discretised in time by the Implicit Euler method. If* $(\mathbf{E}^h_k, \mathbf{H}^h_k)$ *denotes* $(\mathbf{E}^h(t_k), \mathbf{H}^h(t_k))$*, with* $t_k = k \cdot \Delta t$*, then the following stability estimate holds:*

$$
\| \mathbf{E}^h_k \|^2_\epsilon + \| \mathbf{H}^h_k \|^2_\mu \ \leq \ \| \mathbf{E}^h(0) \|^2_\epsilon + \| \mathbf{H}^h(0) \|^2_\mu + \frac{\Delta t}{2} \sum_{m=1}^{k} \| \mathbf{J}_m \|^2_{\frac{1}{\sigma}} . \quad (2.10)
$$

2.5 Galerkin Orthogonality Properties

Finally, in this chapter we are going to derive the *Galerkin orthogonality properties*. These play a key role in the *a posteriori* error analysis in Chapter 3. We get them by considering the weak formulation (2.1)-(2.2) and the semidiscrete system (2.4)-(2.5). Since $U_h \subset L^2(\Omega)^3$ and $V_h \subset H(curl;\Omega)$, the following is true:

$$\begin{aligned} (\epsilon \mathbf{E}_t, \phi^h) + (\sigma \mathbf{E}, \phi^h) - (\nabla \times \mathbf{H}, \phi^h) &= (\mathbf{J}, \phi^h) \quad \forall \phi^h \in U_h & (2.11) \\ (\mu \mathbf{H}_t, \psi^h) + (\mathbf{E}, \nabla \times \psi^h) &= 0 \qquad \forall \psi^h \in V_h, & (2.12) \end{aligned}$$

for $0 < t \leq T$. Therefore by subtracting (2.4) from (2.11), and (2.5) from (2.12), and denoting $\mathbf{E} - \mathbf{E}^h$ by $\mathbf{e}$, and $\mathbf{H} - \mathbf{H}^h$ by $\mathbf{h}$, we get

$$\begin{aligned} (\epsilon \mathbf{e}_t + \sigma \mathbf{e} - \nabla \times \mathbf{h}, \phi^h) &= 0 \quad \forall \phi^h \in U_h & (2.13) \\ (\mu \mathbf{h}_t, \psi^h) + (\mathbf{e}, \nabla \times \psi^h) &= 0 \quad \forall \psi^h \in V_h, & (2.14) \end{aligned}$$

for $0 < t \leq T$.

These are the very important Galerkin orthogonality properties. We are now prepared to start the *a posteriori* analysis in the next chapter.

Chapter 3

A Posteriori Error Analysis

In this chapter we will derive an *a posteriori* error bound for the Maxwell system (1.18)-(1.23). It is this bound that is used for designing an error indicator when adapting the mesh.

3.1 Preparation

In this section we are going to present some results that are necessary for the subsequent error analysis. We shall state some Interpolation Theorems, a Trace Theorem and, for us the very important, *Friedrich's div-curl inequality.*

3.1.1 Interpolation Theorems

We start by defining exactly what we mean by a finite element. We do this following the definitions of Brenner and Scott[3], pp. 67. We also introduce the idea of the local interpolant.

Definition 3.1 *Let*

(i) $K \subseteq \mathbb{R}^n$ *be an open, bounded, polyhedral domain (the* **element domain***),*

(ii) $\mathcal{P}$ *be a finite-dimensional space of functions on* K *(the* **shape functions**) *and*

(iii) $\mathcal{N} = \{N_1, N_2, \ldots, N_k\}$ *be a basis for* $\mathcal{P}'$ *(the* **nodal variables***).*

Then $(K, \mathcal{P}, \mathcal{N})$ *is called a* **finite element***.*

Definition 3.2 *Given a finite element* $(K, \mathcal{P}, \mathcal{N})$*, let the set* $\{\psi_i : 1 \leq i \leq d\} \subseteq \mathcal{P}$ *be the basis dual to* $\mathcal{N}$*. If* v *is a function for which all* $N_i \in \mathcal{N}, i = 1, \ldots, d$ *, are defined, then we define the* **local interpolant** *by*

$$\mathcal{I}_K v := \sum_{i=1}^{d} N_i(v)\psi_i. \tag{3.1}$$

Various properties of the (local) interpolant are discussed in Brenner and Scott[3], pp. 75-79.

Definition 3.3 *Let* Ω *be a given domain and let* $\{\mathcal{T}^h\}, 0 < h \leq 1$*, be a family of subdivisions such that*

$$max\{h_T : T \in \mathcal{T}^h\} \leq h \; diam \; \Omega,$$

where $h_T =$ *diam* T*. Then the family is said to be* **nondegenerate** *if there exists* $\rho > 0$ *such that, for all* $T \in \mathcal{T}^h$ *and for all* $h \in (0, 1]$*,*

$$diam \; B_T \geq \rho \; h_T$$

where B_T *is the largest ball contained in* T*.*

With these definitions in mind we give the following Interpolation Theorems.

Theorem 3.4 *Let* $(K, \mathcal{P}, \mathcal{N})$ *be a finite element, satisfying*

(i) K is star-shaped with respect to some ball,

(ii) $\mathcal{P}_{m-1} \subseteq \mathcal{P} \subseteq W^{m,\infty}(K)$ *, where* $\mathcal{P}_k$ *is the set of polynomials in n variables of degree less than or equal to k,*

(iii) $\mathcal{N} \subseteq (C^l(\bar{K}))'$ *(so that the nodal variables in* $\mathcal{N}$ *involve derivatives up to order l) and*

(iv) $1 \leq p \leq \infty$ *and either* $m - l - n/p > 0$ *when* $p > 1$ *or* $m - l - n \geq 0$ *when* $p = 1$.

Then for $0 \leq s \leq m$, *and* $v \in W^{m,p}(K)$ *we have*

$$|v - \mathcal{I}^h v|_{W^{s,p}(K)}) \leq C(diamK)^{m-s}|v|_{W^{m,p}(K)} \tag{3.2}$$

where C depends on m,n, and K.

Proof See Brenner and Scott[3], pp. 104-105. □

Theorem 3.5 *Let* $\{\mathcal{T}^h\}, 0 < h \leq 1$ *, be a nondegenerate family of subdivisions of a polyhedral domain* Ω *in* $\mathbb{R}^n$. *Let* $(K, \mathcal{P}, \mathcal{N})$ *be a reference element, satisfying the same conditions (i)-(iv) for some l,m, and p as in Theorem 3.4.*
Then for all $T \in \mathcal{T}^h$, $0 < h \leq 1$, *let* $(K, \mathcal{P}_T, \mathcal{N}_T)$ *be the affine equivalent element. Then there exists a positive constant* C_I *depending on the reference element, n,m and p such that, for* $0 \leq s \leq m$,

$$(\sum_{T \in \mathcal{T}^h} \| h_T^{s-m}(v - \mathcal{I}^h v) \|_{W^{s,p}(T)}^p)^{\frac{1}{p}} \leq C_I |v|_{W^{m,p}(\Omega)} \tag{3.3}$$

for all $v \in W^{m,p}(\Omega)$, *where the left-hand side should be interpreted, in the case* $p = \infty$ *as* $\max_{T \in \mathcal{T}^h} \| h_T^{s-m}(v - \mathcal{I}^h v) \|_{W^{s,\infty}(T)}$.

Proof See Brenner and Scott[3], pp. 104-109. □

In the subsequent analysis, we need to have a bound of the form

$$(\sum_{T \in \mathcal{T}^h} \| h_T^{-1}(v - \mathcal{I}^h v) \|_{L^2(T)}^2)^{\frac{1}{2}} \leq C_I |v|_{H^1(\Omega)}.$$

However suitable values of l, m and p cannot be found for $l = 0$ and $n = 2$ or 3, which will satisfy the conditions of Theorem 3.4. An alternative to this is outlined in in Brenner and Scott[3], pp. 118-120, where the notion of the *quasi-interpolant* is introduced. This allows us to modify Theorem 3.4 and Theorem 3.5 in the following way so that we get the results we want.

Theorem 3.6 *For* $v \in W^{k,p}(K), 0 \leq k \leq m$ *and* $1 \leq p \leq \infty$,

$$\| v - \tilde{\mathcal{I}}^h v \|_{W^{s,p}(K)} \leq C h_K^{k-s} |v|_{W^{k,p}(K)} \tag{3.4}$$

for $0 \leq s \leq k \leq m$, *where* $h_K = diam\ K$, *and* $\tilde{\mathcal{I}}^h$ *is the quasi-interpolant defined by relaxing the amount of smoothness required by the function being approximated through the use of local projections (see Brenner and Scott[3]).*

Proof See Scott and Zhang[19]. □

Theorem 3.7 *If all elements' sets of shape functions contain all polynomials of degree less than* m *and* $\mathcal{T}^h$ *is nondegenerate then, for* $v \in W^{k,p}(\Omega), 0 \leq k \leq m$ *and* $1 \leq p \leq \infty$,

$$\Big(\sum_{T \in \mathcal{T}^h} \| h_T^{s-k}(v - \tilde{\mathcal{I}}^h v) \|_{W^{s,p}(T)}^p \Big)^{\frac{1}{p}} \leq C_I |v|_{W^{k,p}(\Omega)} \tag{3.5}$$

for $0 \leq s \leq k \leq m$, *where* $\tilde{\mathcal{I}}^h$ *is the quasi-interpolant defined by relaxing the amount of smoothness required by the function being approximated through the use of local projections (see Brenner and Scott[3]).*

Proof See Scott and Zhang[19]. □

Letting $s = k$ and applying the *triangle inequality*, the following corollary is derived:

Corollary 3.8 *Under the conditions of Theorem 3.7*

$$\Big(\sum_{T \in \mathcal{T}^h} \| \tilde{\mathcal{I}}^h v \|_{W^{k,p}(T)}^p \Big)^{\frac{1}{p}} \leq C_I |v|_{W^{k,p}(\Omega)}. \tag{3.6}$$

A more detailed discussion of the quasi-interpolant can be found in Süli and Houston[21].

3.1.2 A Trace Theorem

To be able to bound the error on the boundary, we need the following Trace Theorem (For a discussion of this and similar results see Brenner and Scott[3], p. 37):

Theorem 3.9 *Suppose that $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary, and that p is a real number in the range $1 \leq p \leq \infty$. Then there exists a constant, C_{Tr}, such that*

$$\| v \|_{L^p(\partial\Omega)} \leq C_{Tr} \| v \|_{L^p(\Omega)}^{1-1/p} \| v \|_{W^{1,p}(\Omega)}^{1/p} \qquad \forall v \in W^{1,p}(\Omega). \tag{3.7}$$

Proof See Brenner and Scott[3], p. 37.

From this theorem we get an important corollary by following Süli and Wilkins[20], p. 11. That is, we bound $\| v \|_{L^p(\partial T)}$ by transforming to the canonical triangle and applying Theorem 3.7. On transforming back again, we obtain the desired result.

Corollary 3.10 *Suppose that $\{\mathcal{T}^h\}$, $0 < h \leq 1$, is a nondegenerate family of subdivisions of a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and that p is a real number in the range $1 \leq p \leq \infty$. Then there exists a constant, C_{Tr}, such that*

$$\| v \|_{L^p(\partial T)} \leq C_{Tr} \| v \|_{L^p(T)}^{1-1/p} (h_T^{-1} \| v \|_{L^p(T)} + \| \nabla v \|_{L^p(T)})^{1/p}, \tag{3.8}$$

$\forall v \in W^{1,p}(\Omega)$, $\forall T \in \{\mathcal{T}^h\}$, and $\forall h \in (0,1]$.

3.1.3 Friedrichs' div-curl Inequality

This theorem implies that the div-curl-norm appearing in the right-hand side of (3.9) is equivalent to the H^1-norm. This theorem plays a key role in the following *a posteriori* analysis. For the proof we refer to Girault and Raviart[6], Krizek and Neittaanmaki[8] and Jiang *et al.*[10].

Theorem 3.11 *(Friedrichs' div-curl inequality). Let Ω be a bounded, simply connected, convex, and open domain with piecewise smooth boundary $\Gamma = \Gamma_1 \bigcup \Gamma_2$. Either Γ_1 or Γ_2 may be empty, but not both. Also Γ_1 and Γ_2 must have at least one common point. Then every function $\mathbf{u}$ of $H^1(\Omega)^3$ with $\mathbf{n} \cdot \mathbf{u} = 0$ on Γ_1 and $\mathbf{n} \times \mathbf{u} = \mathbf{0}$ on Γ_2 satisfies*

$$\| \mathbf{u} \|_1^2 \leq C_F(\| \nabla \cdot \mathbf{u} \|_0^2 + \| \nabla \times \mathbf{u} \|_0^2), \tag{3.9}$$

where the constant $C_F > 0$ depends only on Ω.

3.2 Adjoint Problem

In this section we are going to introduce an adjoint problem related to (1.18)-(1.23), and then we will derive strong stability estimates for this problem. The introduction of this adjoint, or dual, problem enables us to find the error bounds in the norm $\| \cdot \|_{H^{-1}(\Omega)}$. Given $\xi, \eta \in H^1(\Omega)^3$, consider the following adjoint problem on $\Omega \times [0, T]$:

$$\begin{aligned} -(\epsilon\xi)_t + \sigma\xi + \nabla \times \eta &= 0 \quad \text{in } \Omega \times (0,T) & (3.10) \\ -(\mu\eta)_t - \nabla \times \xi &= 0 \quad \text{in } \Omega \times (0,T), & (3.11) \end{aligned}$$

subject to the final conditions

$$\xi(\cdot, T) = \mathbf{\Psi} \quad \text{and} \quad \eta(\cdot, T) = \mathbf{\Upsilon} \tag{3.12}$$

with $\Psi, \Upsilon \in H_0^1(\Omega)$, and boundary conditions

$$\begin{aligned} \mathbf{n} \times \xi &= 0 \quad \text{on } \Gamma \times (0,T) && (3.13) \\ \eta \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma \times (0,T) && (3.14) \end{aligned}$$

3.2.1 Constant Coefficients

Now for simplicity consider the Adjoint Problem (3.10)-(3.14) when the coefficients ϵ, μ, σ are (positive) constants. Our aim is to use the *Friedrichs' div-curl inequality*, so we want to bound the curls and divs of ξ and η.

Theorem 3.12 *The solution (ξ, η) of the adjoint problem (3.10)-(3.14) satisfies the following strong stability estimate $\forall t \in [0,T]$:*

$$\begin{aligned} &\mu\epsilon \parallel \xi_t \parallel^2 + 2\mu\sigma \int_t^T \parallel \xi_t \parallel^2 d\tau + \parallel \nabla \times \xi \parallel^2 + \parallel \nabla \cdot \xi \parallel^2 + \parallel \nabla \times \eta \parallel^2 + \parallel \nabla \cdot \eta \parallel^2 \\ &\leq \frac{\mu}{\epsilon} \parallel \sigma \mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \parallel^2 + \parallel \nabla \times \mathbf{\Psi} \parallel^2 + e^{-\frac{2\sigma}{\epsilon}(T-t)} \parallel \nabla \cdot \mathbf{\Psi} \parallel^2 + \parallel \nabla \cdot \mathbf{\Upsilon} \parallel^2 \\ &+ \left((\parallel \sigma \mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \parallel^2 + \frac{\epsilon}{\mu} \parallel \nabla \times \mathbf{\Psi} \parallel^2)^{\frac{1}{2}} + \sigma (\parallel \mathbf{\Psi} \parallel^2 + \frac{\mu}{\epsilon} \parallel \mathbf{\Upsilon} \parallel^2)^{\frac{1}{2}} \right)^2, \qquad (3.15) \end{aligned}$$

where ϵ,μ and σ are (positive) constants.

Proof To prove (3.15) we start by differentiating the first equation with respect to time to give

$$-\epsilon\xi_{tt} + \sigma\xi_t + \nabla \times \eta_t = -\epsilon\xi_{tt} + \sigma\xi_t - \tfrac{1}{\mu}\nabla \times \nabla \times \xi = 0,$$

and we obtain the reduced problem for ξ

$$\begin{aligned} &\epsilon\xi_{tt} - \sigma\xi_t + \frac{1}{\mu}\nabla \times \nabla \times \xi = 0 && (3.16) \\ &\mathbf{n} \times \xi = 0 \qquad \text{on} \quad \Gamma \times (0,T) && (3.17) \\ &\xi(\cdot, T) = \mathbf{\Psi} \qquad \text{on} \quad \Omega \times (0,T). && (3.18) \end{aligned}$$

Now multiply (3.16) by ξ_t and integrate over Ω:

$$
\begin{aligned}
0 &= (\epsilon\xi_{tt} - \sigma\xi_t + \frac{1}{\mu}\nabla \times \nabla \times \xi, \xi_t) \\
&= \epsilon(\xi_{tt}, \xi_t) - \sigma(\xi_t, \xi_t) + \frac{1}{\mu}(\nabla \times \xi, \nabla \times \xi_t) + \frac{1}{\mu}((\nabla \times \xi) \times \mathbf{n}, \xi_t)_{L^2(\Gamma)}.
\end{aligned}
$$

But $((\nabla \times \xi) \times \mathbf{n}, \xi_t)_{L^2(\Gamma)} = 0$ due to the boundary condition (3.17), so

$$
\frac{\epsilon}{2}\frac{d}{dt} \| \xi_t \|^2 - \sigma \| \xi_t \|^2 + \frac{1}{2\mu}\frac{d}{dt} \| \nabla \times \xi \|^2 = 0.
$$

We then integrate in time from t to T, to give

$$
\begin{aligned}
&\mu\epsilon \| \xi_t \|^2 + 2\mu\sigma \int_t^T \| \xi_t \|^2 \, d\tau + \| \nabla \times \xi \|^2 \\
&= \mu\epsilon \| \xi_t(\cdot, T) \|^2 + \| \nabla \times \xi(\cdot, T) \|^2 \\
&= \mu\epsilon \| \frac{1}{\epsilon}(\sigma\xi(\cdot, T) + \nabla \times \eta(\cdot, T)) \|^2 + \| \nabla \times \xi(\cdot, T) \|^2 \\
&= \frac{\mu}{\epsilon} \| \sigma\mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|^2 + \| \nabla \times \mathbf{\Psi} \|^2 .
\end{aligned}
$$

So we get a bound for $\| \nabla \times \xi \|^2$ from

$$
\begin{aligned}
&\mu\epsilon \| \xi_t \|^2 + 2\mu\sigma \int_t^T \| \xi_t \|^2 \, d\tau + \| \nabla \times \xi \|^2 \\
&= \frac{\mu}{\epsilon} \| \sigma\mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|^2 + \| \nabla \times \mathbf{\Psi} \|^2 . \qquad (3.19)
\end{aligned}
$$

Now we want to derive a bound on $\| \nabla \cdot \xi \|^2$; we obtain this by taking the divergence of the first adjoint equation (3.10):

$$\nabla \cdot (-\epsilon \xi_t + \sigma \xi + \nabla \times \eta) = -\epsilon(\nabla \cdot \xi_t) + \sigma(\nabla \cdot \xi) = 0.$$

(Note that $\nabla \cdot \nabla \times \eta = 0$ due to well known rules of vector calculus.)

$$\Rightarrow \;\; (\nabla \cdot \xi)_t - \frac{\sigma}{\epsilon}(\nabla \cdot \xi) = 0.$$

This is a linear ordinary differential equation with the solution

$$(\nabla \cdot \xi)(\cdot, t) = Ce^{\frac{\sigma}{\epsilon}t}.$$

We get the constant C, from the final conditions, as follows:

$$(\nabla \cdot \xi)(\cdot, T) = Ce^{\frac{\sigma}{\epsilon}T} = \nabla \cdot \mathbf{\Psi}$$

$$\Rightarrow \;\; \nabla \cdot \xi = e^{-\frac{\sigma}{\epsilon}(T-t)}(\nabla \cdot \mathbf{\Psi}).$$

The bound we are interested in is therefore

$$\| \nabla \cdot \xi \|^2 = e^{-\frac{2\sigma}{\epsilon}(T-t)} \| \nabla \cdot \mathbf{\Psi} \|^2 . \tag{3.20}$$

To obtain a bound on $\| \nabla \cdot \eta \|^2$, we take the divergence of the second adjoint equation (3.11):

$$0 = \nabla \cdot (\mu \eta_t + \nabla \times \xi) = \mu(\nabla \cdot \eta_t) + \nabla \cdot (\nabla \times \xi) = \mu(\nabla \cdot \eta_t).$$

That is,

$$(\nabla \cdot \eta)_t = 0 \;\; \Rightarrow \;\; \nabla \cdot \eta = \nabla \cdot \mathbf{\Upsilon}$$

$$\Rightarrow \;\; \| \nabla \cdot \eta \|^2 \;\; = \;\; \| \nabla \cdot \mathbf{\Upsilon} \|^2 . \tag{3.21}$$

To get a bound on $\| \nabla \times \eta \|^2$, we need, as we shall see later, a bound on $\| \xi \|^2$. To get this; multiply the first adjoint equation (3.10) by ξ, the second adjoint equation (3.11) by η, integrate over Ω and add them together. This gives:

$$
\begin{aligned}
0 &= (-\epsilon\xi_t + \sigma\xi + \nabla \times \eta, \xi) + (-\mu\eta_t - \nabla \times \xi, \eta) \\
&= -\epsilon(\xi_t, \xi) + \sigma(\xi, \xi) + (\nabla \times \eta, \xi) - \mu(\eta_t, \eta) - (\nabla \times \xi, \eta) \\
&= -\frac{\epsilon}{2}\frac{d}{dt} \| \xi \|^2 + \sigma \| \xi \|^2 + (\eta \times \mathbf{n}, \xi)_{L^2(\Gamma)} - \frac{\mu}{2}\frac{d}{dt} \| \eta \|^2 \\
&= -\frac{\epsilon}{2}\frac{d}{dt} \| \xi \|^2 + \sigma \| \xi \|^2 - \frac{\mu}{2}\frac{d}{dt} \| \eta \|^2
\end{aligned}
$$

$$
\Rightarrow \; \epsilon \| \xi \|^2 + 2\sigma \int_t^T \| \xi \|^2 \, d\tau + \mu \| \eta \|^2 = \epsilon \| \mathbf{\Psi} \|^2 + \mu \| \mathbf{\Upsilon} \|^2 .
$$

With this bound on $\| \xi \|^2$, we can now deal with $\| \nabla \times \eta \|^2$:

$$
\begin{aligned}
\| \nabla \times \eta \|^2 &= \| \epsilon\xi_t - \sigma\xi \|^2 \le (\| \epsilon\xi_t \| + \| \sigma\xi \|)^2 = \left((\| \epsilon\xi_t \|^2)^{1/2} + (\| \sigma\xi \|^2)^{1/2} \right)^2 \\
&\le \left((\| \sigma\mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|^2 + \frac{\epsilon}{\mu} \| \nabla \times \mathbf{\Psi} \|^2)^{\frac{1}{2}} + \sigma(\| \mathbf{\Psi} \|^2 + \frac{\mu}{\epsilon} \| \mathbf{\Upsilon} \|^2)^{\frac{1}{2}} \right)^2 .
\end{aligned}
\tag{3.22}
$$

So finally we get, by (3.19),(3.20),(3.21) and (3.22);

$$
\begin{aligned}
&\mu\epsilon \| \xi_t \|^2 + 2\mu\sigma \int_t^T \| \xi_t \|^2 \, d\tau + \| \nabla \times \xi \|^2 + \| \nabla \cdot \xi \|^2 + \| \nabla \times \eta \|^2 + \| \nabla \cdot \eta \|^2 \\
&\le \frac{\mu}{\epsilon} \| \sigma\mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|^2 + \| \nabla \times \mathbf{\Psi} \|^2 + e^{-\frac{2\sigma}{\epsilon}(T-t)} \| \nabla \cdot \mathbf{\Psi} \|^2 + \| \nabla \cdot \mathbf{\Upsilon} \|^2 \\
&+ \left((\| \sigma\mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|^2 + \frac{\epsilon}{\mu} \| \nabla \times \mathbf{\Psi} \|^2)^{\frac{1}{2}} + \sigma(\| \mathbf{\Psi} \|^2 + \frac{\mu}{\epsilon} \| \mathbf{\Upsilon} \|^2)^{\frac{1}{2}} \right)^2 .
\end{aligned}
\tag{3.23}
$$

This completes the proof of (3.15). □

3.2.2 Variable Coefficients

Now we turn our attention to the case of variable coefficients, that is when $\epsilon = \epsilon(\mathbf{x})$, $\mu = \mu(\mathbf{x})$, and $\sigma = \sigma(\mathbf{x})$. The different steps in the proof of the following theorem are analogous to the case of constant coefficients, though the analysis is slightly more technical.

Theorem 3.13 *The solution (ξ, η) of the adjoint problem (3.10)-(3.14) satisfies the following strong stability estimate $\forall t \in [0, T]$:*

$$\begin{aligned}
&\| \nabla \times \xi \|^2 + \| \nabla \times \eta \|^2 + \| \nabla \cdot \xi \|^2 + \| \nabla \cdot \eta \|^2 \\
&+(\max_{\mathbf{x}} \mu) \| \xi_t \|_\epsilon^2 + 2(\max_{\mathbf{x}} \mu) \int_t^T \| \xi_t \|_\sigma \, d\tau \\
\leq\ & (\max_{\mathbf{x}} \mu)(\| \sigma \boldsymbol{\Psi} + \nabla \times \boldsymbol{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla \times \boldsymbol{\Upsilon} \|_{\mu^{-1}}^2) \\
&+ \frac{1}{(\min_{\mathbf{x}} \epsilon)^2} \Big(\| e^{-\sigma\epsilon^{-1}(T-t)} (\nabla \cdot (\epsilon \boldsymbol{\Psi})) \| \\
&+ (\max_{\mathbf{x}} \epsilon) \| e^{\sigma\epsilon^{-1} t} (\nabla(\sigma \epsilon^{-1})) \| \int_t^T \| e^{\sigma\epsilon^{-1} t} \| (\| \boldsymbol{\Psi} \|_\epsilon^2 + \| \boldsymbol{\Upsilon} \|_\mu^2)^{1/2} d\tau \\
&+ (\max_{\mathbf{x}} \frac{1}{\sqrt{\epsilon}} |\nabla \epsilon|)(\| \boldsymbol{\Psi} \|_\epsilon^2 + \| \boldsymbol{\Upsilon} \|_\mu^2)^{1/2} \Big)^2 \\
&+ (\max_{\mathbf{x}} \epsilon)(\| \sigma \boldsymbol{\Psi} + \nabla \times \boldsymbol{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla \times \boldsymbol{\Upsilon} \|_{\mu^{-1}}^2) \\
&+ (\max_{\mathbf{x}} \sigma)(\| \boldsymbol{\Psi} \|_\epsilon^2 + \| \boldsymbol{\Upsilon} \|_\mu^2) + 2(\max_{\mathbf{x}} \epsilon)^{1/2} (\max_{\mathbf{x}} \sigma)^{1/2} \\
&\Big((\| \sigma \boldsymbol{\Psi} + \nabla \times \boldsymbol{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla \times \boldsymbol{\Upsilon} \|_{\mu^{-1}}^2)(\| \boldsymbol{\Psi} \|_\epsilon^2 + \| \boldsymbol{\Upsilon} \|_\mu^2) \Big)^{1/2} \\
&+ \frac{1}{(\min_{\mathbf{x}} \mu)^2} \Big(\| \nabla \cdot (\mu \boldsymbol{\Upsilon}) \| + (\max_{\mathbf{x}} \frac{1}{\sqrt{\mu}} |\nabla \mu|)(\| \boldsymbol{\Psi} \|_\epsilon^2 + \| \boldsymbol{\Upsilon} \|_\mu^2)^{\frac{1}{2}} \Big)^2,
\end{aligned} \tag{3.24}$$

where $\epsilon = \epsilon(\mathbf{x})$, $\mu = \mu(\mathbf{x})$, and $\sigma = \sigma(\mathbf{x})$.

Proof To prove (3.24) we start by differentiating the first adjoint equation (3.10) with respect to time:

$$\frac{d}{dt}(-\epsilon\xi_t + \sigma\xi + \nabla\times\eta) = -\epsilon\xi_{tt} + \sigma\xi_t + \nabla\times\eta_t$$
$$= -\epsilon\xi_{tt} + \sigma\xi_t - \nabla\times(\mu^{-1}(\nabla\times\xi)) = 0,$$

and we get the following reduced problem for ξ:

$$-\epsilon\xi_{tt} + \sigma\xi_t - \nabla\times(\mu^{-1}(\nabla\times\xi)) \tag{3.25}$$
$$\mathbf{n}\times\xi = 0 \quad \text{on } \Gamma \tag{3.26}$$
$$\xi(\cdot,T) = \mathbf{\Psi}. \tag{3.27}$$

Now we multiply (3.25) by ξ_t and integrate over Ω to get

$$0 = -(\epsilon\xi_{tt},\xi_t) + (\sigma\xi_t,\xi_t) - (\nabla\times(\mu^{-1}(\nabla\times\xi)),\xi_t)$$
$$= -(\epsilon\xi_{tt},\xi_t) + (\sigma\xi_t,\xi_t) - (\mu^{-1}(\nabla\times\xi),\nabla\times\xi_t) - ((\mu^{-1}(\nabla\times\xi)\times\mathbf{n},\xi_t)_{L^2(\Gamma)}.$$

But $((\mu^{-1}(\nabla\times\xi)\times\mathbf{n},\xi_t)_{L^2(\Gamma)} = 0$ by the boundary conditions (3.26), so by integrating in time from t to T we get

$$\Rightarrow \quad \| \xi_t \|_\epsilon^2 + 2\int_t^T \| \xi_t \|_\sigma^2 \, d\tau + \| \nabla\times\xi \|_{\mu^{-1}}^2 = \| \xi_t(\cdot,T) \|_\epsilon^2 + \| \nabla\times\xi(\cdot,T) \|_{\mu^{-1}}^2$$
$$= \| \epsilon^{-1}(\sigma\xi(\cdot,T) + \nabla\times\eta(\cdot,T)) \|_\epsilon^2 + \| \nabla\times\xi(\cdot,T) \|_{\mu^{-1}}^2$$
$$= \| \sigma\mathbf{\Psi} + \nabla\times\mathbf{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla\times\mathbf{\Upsilon} \|_{\mu^{-1}}^2 .$$

Hence we obtain the equality

$$\| \xi_t \|_\epsilon^2 + 2\int_t^T \| \xi_t \|_\sigma^2 \, d\tau + \| \nabla\times\xi \|_{\mu^{-1}}^2$$
$$= \| \sigma\mathbf{\Psi} + \nabla\times\mathbf{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla\times\mathbf{\Upsilon} \|_{\mu^{-1}}^2, \tag{3.28}$$

which gives

$$
\begin{aligned}
&(\max_{\mathbf{x}} \mu) \parallel \xi_t \parallel_\epsilon^2 + 2(\max_{\mathbf{x}} \mu) \int_t^T \parallel \xi_t \parallel_\sigma d\tau + \parallel \nabla \times \xi \parallel^2 \\
&\leq \ (\max_{\mathbf{x}} \mu)(\parallel \sigma \boldsymbol{\Psi} + \nabla \times \boldsymbol{\Upsilon} \parallel_{\epsilon^{-1}}^2 + \parallel \nabla \times \boldsymbol{\Upsilon} \parallel_{\mu^{-1}}^2).
\end{aligned} \tag{3.29}
$$

Here, as in the case of constant coefficients, we also need a bound on $\parallel \xi \parallel_\epsilon^2$. We get it by multiplying the first adjoint equation (3.10) by ξ and integrating over Ω, and multiplying the second adjoint equation (3.11) by η and integrating over Ω. Adding them together we get

$$
\begin{aligned}
0 &= (-\epsilon\xi_t + \sigma\xi + \nabla \times \eta, \xi) + (-\mu\eta_t - \nabla \times \xi, \eta) \\
&= -\frac{1}{2}\frac{d}{dt} \parallel \xi \parallel_\epsilon^2 + \parallel \xi \parallel_\sigma^2 + (\nabla \times \eta, \xi) - \frac{1}{2}\frac{d}{dt} \parallel \eta \parallel_\mu^2 - (\nabla \times \xi, \eta) \\
&= -\frac{1}{2}\frac{d}{dt} \parallel \xi \parallel_\epsilon^2 + \parallel \xi \parallel_\sigma^2 + (\eta \times \mathbf{n}, \xi)_{L^2(\Gamma)} - \frac{1}{2}\frac{d}{dt} \parallel \eta \parallel_\mu^2 .
\end{aligned}
$$

But $(\eta \times \mathbf{n}, \xi)_{L^2(\Gamma)} = 0$, so by integrating in time from t to T we have

$$
\parallel \xi \parallel_\epsilon^2 + 2\int_t^T \parallel \xi \parallel_\sigma^2 d\tau + \parallel \eta \parallel_\mu^2 = \parallel \boldsymbol{\Psi} \parallel_\epsilon^2 + \parallel \boldsymbol{\Upsilon} \parallel_\mu^2 . \tag{3.30}
$$

We also need a bound on $\parallel \xi \parallel_\sigma^2$, we deduce it by the *Fundamental Theorem of Calculus*

$$
\xi(t) = \xi(T) - \int_t^T \xi_t(\tau) d\tau.
$$

Taking the absolute value of both sides gives

$$
|\xi(t)| = |\xi(T) - \int_t^T \xi_t(\tau) d\tau| \ \leq \ |\xi(T)| + \int_t^T |\xi_t(\tau)| d\tau.
$$

Then, by applying the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we get

$$|\xi(t)|^2 \ \le \ 2|\xi(T)|^2 + 2(\int_t^T |\xi_t(\tau)|d\tau)^2.$$

Now multiplying both sides by σ,

$$\sigma|\xi(t)|^2 \ \le \ 2\sigma|\xi(T)|^2 + 2\sigma(\int_t^T |\xi_t(\tau)|d\tau)^2 = 2\sigma|\xi(T)|^2 + 2\sigma(\int_t^T 1\cdot|\xi_t(\tau)|d\tau)^2$$
$$\le \ 2\sigma|\xi(T)|^2 + 2(\int_t^T 1^2 d\tau)(\int_t^T \sigma|\xi_t(\tau)|^2 d\tau). \tag{3.31}$$

Which gives (according to *Fubini's theorem*)

$$\int_\Omega \sigma|\xi(t)|^2 dx \ \le \ 2\int_\Omega \sigma|\xi(T)|^2 dx + 2(T-t)\int_t^T \left(\int_\Omega \sigma|\xi_t(\tau)|^2 dx\right) d\tau;$$

that is

$$\| \xi \|_\sigma^2 \ \ \le \ 2 \| \mathbf{\Psi} \|_\sigma^2 + (T-t)2\int_t^T \| \xi_t \|_\sigma^2 \, d\tau.$$

But then (3.28) yields

$$\| \xi \|_\sigma^2 \ \ \le \ 2 \| \mathbf{\Psi} \|_\sigma^2 + (T-t)(\| \sigma\mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla \times \mathbf{\Upsilon} \|_{\mu^{-1}}^2). \tag{3.32}$$

Now we can get a bound on $\| \nabla \cdot \eta \|^2$ by taking the divergence of the second adjoint equation (3.11). Namely,

$$\nabla \cdot (-(\mu\eta_t) - \nabla \times \xi) = -\nabla \cdot (\mu\eta_t) - \nabla \cdot \nabla \times \xi = 0$$
$$\Rightarrow \ (\nabla \cdot (\mu\eta))_t = 0 \quad \Rightarrow \ \nabla \cdot (\mu\eta) = \nabla \cdot (\mu\mathbf{\Upsilon});$$

that is

$$\| \nabla \cdot (\mu\eta) \| \;=\; \| \nabla \cdot (\mu \boldsymbol{\Upsilon}) \| \,. \tag{3.33}$$

We also have that

$$\begin{aligned} \| \mu \nabla \cdot \eta \| \;&=\; \| \nabla \cdot (\mu\eta) - \eta \cdot \nabla\mu \| \;\leq\; \| \nabla \cdot (\mu\eta) \| + \| \eta \cdot \nabla\mu \| \\ \Rightarrow\; \| \nabla \cdot \eta \| \;&\leq\; \frac{1}{(\min_{\mathbf{x}} \mu)} (\| \nabla \cdot (\mu \boldsymbol{\Upsilon}) \| + \| \eta \cdot \nabla\mu \|) \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \| \eta \cdot \nabla\mu \| \;&=\; \left(\int_\Omega |\eta \cdot \nabla\mu|^2 d\mathbf{x} \right)^{1/2} \;\leq\; \left(\int_\Omega |\eta|^2 |\nabla\mu|^2 d\mathbf{x} \right)^{1/2} \\ &=\; \left(\int_\Omega \mu|\eta|^2 \cdot \frac{1}{\mu} |\nabla\mu|^2 d\mathbf{x} \right)^{1/2} \;\leq\; (\max_{\mathbf{x}} \frac{1}{\sqrt{\mu}} |\nabla\mu|) \| \eta \|_\mu \,. \end{aligned} \tag{3.35}$$

So we get by (3.33),(3.34) and (3.35) that

$$\| \nabla \cdot \eta \|^2 \;\leq\; \frac{1}{(\min_{\mathbf{x}} \mu)^2} \left(\| \nabla \cdot (\mu \boldsymbol{\Upsilon}) \| + (\max_{\mathbf{x}} \frac{1}{\sqrt{\mu}} |\nabla\mu|)(\| \boldsymbol{\Psi} \|_\epsilon^2 + \| \boldsymbol{\Upsilon} \|_\mu^2)^{1/2} \right)^2 . \tag{3.36}$$

We can also derive a bound on $\| \nabla \times \eta \|^2$:

$$\begin{aligned} \| \nabla \times \eta \|^2 \;&=\; \| \epsilon\xi_t - \sigma\xi \|^2 \;\leq\; \| \epsilon\xi_t \|^2 + \| \sigma\xi \|^2 + 2(\| \epsilon\xi_t \|^2 \| \sigma\xi \|^2)^{1/2} \\ &\leq\; (\max_{\mathbf{x}} \epsilon) \| \xi_t \|_\epsilon^2 + (\max_{\mathbf{x}} \sigma) \| \xi \|_\sigma^2 + 2(\max_{\mathbf{x}} \epsilon)^{1/2} (\max_{\mathbf{x}} \sigma)^{1/2} (\| \xi_t \|_\epsilon^2 \| \xi \|_\sigma^2)^{1/2}. \end{aligned}$$

Using (3.28) and (3.32), this gives

$$\| \nabla \times \eta \|^2 \ \leq \ (\max_{\mathbf{x}} \epsilon)(\| \sigma \mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|^2_{\epsilon^{-1}} + \| \nabla \times \mathbf{\Upsilon} \|^2_{\mu^{-1}})$$
$$+(\max_{\mathbf{x}} \sigma)(\| \mathbf{\Psi} \|^2_{\epsilon} + \| \mathbf{\Upsilon} \|^2_{\mu}) + 2(\max_{\mathbf{x}} \epsilon)^{1/2}(\max_{\mathbf{x}} \sigma)^{1/2}$$
$$\Big((\| \sigma \mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|^2_{\epsilon^{-1}} + \| \nabla \times \mathbf{\Upsilon} \|^2_{\mu^{-1}})(\| \mathbf{\Psi} \|^2_{\epsilon} + \| \mathbf{\Upsilon} \|^2_{\mu})\Big)^{1/2} \quad (3.37)$$

To get a bound on the divergence of ξ, we take the divergence of the first dual equation (3.10):

$$\nabla \cdot (-\epsilon \xi_t + \sigma \xi + \nabla \times \eta) = -\nabla \cdot (\epsilon \xi_t) + \nabla \cdot (\sigma \xi) + \nabla \cdot \nabla \times \eta$$
$$= -(\nabla \cdot (\epsilon \xi))_t + \nabla \cdot (\sigma \xi) = -(\nabla \cdot (\epsilon \xi))_t + (\nabla \cdot (\sigma \epsilon^{-1}(\epsilon \xi)))$$
$$= -(\nabla \cdot (\epsilon \xi))_t + (\sigma \epsilon^{-1}) \nabla \cdot (\epsilon \xi) + \nabla(\sigma \epsilon^{-1}) \cdot (\epsilon \xi) = 0$$

$$\Rightarrow \ (\nabla \cdot (\epsilon \xi))_t - (\sigma \epsilon^{-1}) \nabla \cdot (\epsilon \xi) = \nabla(\sigma \epsilon^{-1}) \cdot (\epsilon \xi).$$

This is a linear ordinary differential equation in $\nabla \cdot (\epsilon \xi)$, with the well known solution

$$\nabla \cdot (\epsilon \xi) = e^{\sigma \epsilon^{-1} t}(C - \int_t^T e^{-\sigma \epsilon^{-1} t}(\nabla(\sigma \epsilon^{-1}) \cdot (\epsilon \xi)) d\tau).$$

We get the constant C from the final conditions (3.12) as follows:

$$\nabla \cdot (\epsilon \xi)(\cdot, T) = C e^{\sigma \epsilon^{-1} T} = \nabla \cdot (\epsilon \mathbf{\Psi})$$
$$\Rightarrow \ C = e^{-\sigma \epsilon^{-1} T}(\nabla \cdot (\epsilon \mathbf{\Psi})).$$

So the complete solution subject to the final conditions is

$$\nabla \cdot (\epsilon \xi) = e^{\sigma \epsilon^{-1} t}(e^{-\sigma \epsilon^{-1} T}(\nabla \cdot (\epsilon \mathbf{\Psi})) - \int_t^T e^{-\sigma \epsilon^{-1} t}(\nabla(\sigma \epsilon^{-1}) \cdot (\epsilon \xi)) d\tau).$$

Now take the L^2-norm of both sides:

$$\begin{aligned}
\| \nabla \cdot (\epsilon \xi) \| &= \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) - e^{\sigma \epsilon^{-1} t} \int_t^T e^{-\sigma \epsilon^{-1} \tau}(\nabla(\sigma \epsilon^{-1}) \cdot (\epsilon \xi)) d\tau \| \\
\leq & \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| + \| e^{\sigma \epsilon^{-1} t} \int_t^T e^{-\sigma \epsilon^{-1} \tau}(\nabla(\sigma \epsilon^{-1}) \cdot (\epsilon \xi)) d\tau \| \\
\leq & \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| + \| e^{\sigma \epsilon^{-1} t} \int_t^T e^{-\sigma \epsilon^{-1} \tau} |\nabla(\sigma \epsilon^{-1})| |\epsilon \xi| d\tau \| \\
= & \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| + \| e^{\sigma \epsilon^{-1} t} |\nabla(\sigma \epsilon^{-1})| \int_t^T e^{-\sigma \epsilon^{-1} \tau} |\epsilon \xi| d\tau \| \\
\leq & \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| + \| e^{\sigma \epsilon^{-1} t} |\nabla(\sigma \epsilon^{-1})| \| \| \int_t^T e^{-\sigma \epsilon^{-1} \tau} |\epsilon \xi| d\tau \| \\
\leq & \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| + \| e^{\sigma \epsilon^{-1} t}(\nabla(\sigma \epsilon^{-1})) \| \int_t^T \| e^{\sigma \epsilon^{-1} \tau}(\epsilon \xi) \| d\tau \\
\leq & \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| + \| e^{\sigma \epsilon^{-1} t}(\nabla(\sigma \epsilon^{-1})) \| \int_t^T \| e^{\sigma \epsilon^{-1} \tau} \| \| \epsilon \xi \| d\tau \\
\leq & \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| + (\max_{\mathbf{x}} \epsilon) \| e^{\sigma \epsilon^{-1} t}(\nabla(\sigma \epsilon^{-1})) \| \int_t^T \| e^{\sigma \epsilon^{-1} \tau} \| (\| \xi \|_\epsilon^2)^{1/2} d\tau
\end{aligned}$$

But by (3.30), we have a bound for $\| \xi \|_\epsilon^2$, so we have

$$\begin{aligned}
\| \nabla \cdot (\epsilon \xi) \| \leq & \| e^{-\sigma \epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| \\
+ & (\max_{\mathbf{x}} \epsilon) \| e^{\sigma \epsilon^{-1} t}(\nabla(\sigma \epsilon^{-1})) \| \int_t^T \| e^{\sigma \epsilon^{-1} t} \| (\| \mathbf{\Psi} \|_\epsilon^2 + \| \mathbf{\Upsilon} \|_\mu^2)^{1/2} d\tau.
\end{aligned} \tag{3.38}$$

Now we can write

$$\| \epsilon\nabla \cdot \xi \| \;=\; \| \nabla \cdot (\epsilon\xi) - \xi \cdot \nabla\epsilon \| \;\leq\; \| \nabla \cdot (\epsilon\xi) \| + \| \xi \cdot \nabla\epsilon \|,$$

which gives

$$\| \nabla \cdot \xi \| \;\leq\; \frac{1}{(\min\limits_{\mathbf{x}} \epsilon)}(\| \nabla \cdot (\epsilon\xi) \| + \| \xi \cdot \nabla\epsilon \|), \tag{3.39}$$

and we also have that

$$\begin{aligned} \| \xi \cdot \nabla\epsilon \| \;&=\; \left(\int_\Omega |\xi \cdot \nabla\epsilon|^2 d\mathbf{x}\right)^{1/2} \;\leq\; \left(\int_\Omega |\xi|^2|\nabla\epsilon|^2 d\mathbf{x}\right)^{1/2} \\ =\; &\left(\int_\Omega \epsilon|\xi|^2 \cdot \frac{1}{\epsilon}|\nabla\epsilon|^2 d\mathbf{x}\right)^{1/2} \;\leq\; (\max_{\mathbf{x}} \frac{1}{\sqrt{\epsilon}}|\nabla\epsilon|) \| \xi \|_\epsilon \,. \end{aligned} \tag{3.40}$$

So we get, by (3.38),(3.39),(3.40) and (3.30),

$$\begin{aligned} \| \nabla \cdot \xi \|^2 \;\leq\; &\frac{1}{(\min\limits_{\mathbf{x}} \epsilon)^2}\Big(\| e^{-\sigma\epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon\mathbf{\Psi})) \| \\ &+ (\max_{\mathbf{x}} \epsilon) \| e^{\sigma\epsilon^{-1}t}(\nabla(\sigma\epsilon^{-1})) \| \int_t^T \| e^{\sigma\epsilon^{-1}t} \| (\| \mathbf{\Psi} \|_\epsilon^2 + \| \mathbf{\Upsilon} \|_\mu^2)^{1/2} d\tau \\ &+(\max_{\mathbf{x}} \frac{1}{\sqrt{\epsilon}}|\nabla\epsilon|)(\| \mathbf{\Psi} \|_\epsilon^2 + \| \mathbf{\Upsilon} \|_\mu^2)^{1/2}\Big)^2 . \end{aligned} \tag{3.41}$$

Finally, by (3.29),(3.36),(3.37) and (3.41),

$$
\begin{aligned}
&\| \nabla \times \xi \|^2 + \| \nabla \times \eta \|^2 + \| \nabla \cdot \xi \|^2 + \| \nabla \cdot \eta \|^2 \\
&+(\max_{\mathbf{x}} \mu) \| \xi_t \|_\epsilon^2 + 2(\max_{\mathbf{x}} \mu) \int_t^T \| \xi_t \|_\sigma \, d\tau \\
&\leq (\max_{\mathbf{x}} \mu)(\| \sigma \mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla \times \mathbf{\Upsilon} \|_{\mu^{-1}}^2) \\
&+\frac{1}{(\min_{\mathbf{x}} \epsilon)^2} \Big(\| e^{-\sigma\epsilon^{-1}(T-t)}(\nabla \cdot (\epsilon \mathbf{\Psi})) \| \\
&+ (\max_{\mathbf{x}} \epsilon) \| e^{\sigma\epsilon^{-1}t}(\nabla(\sigma\epsilon^{-1})) \| \int_t^T \| e^{\sigma\epsilon^{-1}t} \| (\| \mathbf{\Psi} \|_\epsilon^2 + \| \mathbf{\Upsilon} \|_\mu^2)^{1/2} d\tau \\
&+(\max_{\mathbf{x}} \frac{1}{\sqrt{\epsilon}} |\nabla \epsilon|)(\| \mathbf{\Psi} \|_\epsilon^2 + \| \mathbf{\Upsilon} \|_\mu^2)^{1/2} \Big)^2 \\
&+(\max_{\mathbf{x}} \epsilon)(\| \sigma \mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla \times \mathbf{\Upsilon} \|_{\mu^{-1}}^2) \\
&+(\max_{\mathbf{x}} \sigma)(\| \mathbf{\Psi} \|_\epsilon^2 + \| \mathbf{\Upsilon} \|_\mu^2) + 2(\max_{\mathbf{x}} \epsilon)^{1/2}(\max_{\mathbf{x}} \sigma)^{1/2} \\
&\Big((\| \sigma \mathbf{\Psi} + \nabla \times \mathbf{\Upsilon} \|_{\epsilon^{-1}}^2 + \| \nabla \times \mathbf{\Upsilon} \|_{\mu^{-1}}^2)(\| \mathbf{\Psi} \|_\epsilon^2 + \| \mathbf{\Upsilon} \|_\mu^2)\Big)^{1/2} \\
&+\frac{1}{(\min_{\mathbf{x}} \mu)^2} \Big(\| \nabla \cdot (\mu \mathbf{\Upsilon}) \| + (\max_{\mathbf{x}} \frac{1}{\sqrt{\mu}} |\nabla \mu|)(\| \mathbf{\Psi} \|_\epsilon^2 + \| \mathbf{\Upsilon} \|_\mu^2)^{1/2} \Big)^2 .
\end{aligned}
\tag{3.42}
$$

This completes the proof of (3.24). □

3.3 A Posteriori Error Bound

The work in the last two sections now enables us to present the main results of this chapter. From now on, we shall suppose that the finite element spaces $U_h \subset L^2(\Omega)^3$ and $V_h \subset H(curl;\Omega)$ consists of piecewise polynomial functions of degree k.

Theorem 3.14 *The finite element approximation $(\mathbf{E}^h, \mathbf{H}^h)$ to the problem (1.18)-(1.23) defined by (2.4)-(2.6), satisfies the following a posteriori error bound:*

$$\| \mathbf{e}(\cdot,T) \|_{H^{-1}} + \| \mathbf{h}(\cdot,T) \|_{H^{-1}} \;\le\; C_* \left(\sum_{\kappa} \zeta(\kappa) \right)^{1/2}, \tag{3.43}$$

where C_* is a computable constant and $\zeta(\kappa)$ is the local error estimator given by:

$$\zeta(\kappa) = \| h_\kappa \mathbf{R}_1 \|^2_{L^2(0,T;L^2(\kappa))} + \| h_\kappa \mathbf{R}_2 \|^2_{L^2(0,T;L^2(\kappa))} + \| h_\kappa \mathbf{R}_3 \|^2_{L^2(0,T;L^2(\partial\kappa))} + \| h_\kappa \mu \mathbf{h}(0) \|^2_{L^2(\kappa)} + \| h_\kappa \epsilon \mathbf{e}(0) \|^2_{L^2(\kappa)}, \tag{3.44}$$

here h_κ denotes the diameter of the triangle κ. $\mathbf{R}_1$,$\mathbf{R}_2$, and $\mathbf{R}_3$ are the residuals defined by:

$$\mathbf{R}_1 = \mathbf{J} - \epsilon \mathbf{E}^h_t - \sigma \mathbf{E}^h + \nabla \times \mathbf{H}^h = \epsilon \mathbf{e}_t + \sigma \mathbf{e} - \nabla \times \mathbf{h} \tag{3.45}$$

$$\mathbf{R}_2 = -\mu \mathbf{H}^h_t - \nabla \times \mathbf{E}^h = \mu \mathbf{h}_t + \nabla \times \mathbf{e} \tag{3.46}$$

$$\mathbf{R}_3 = \mathbf{n} \times \mathbf{E}^h \tag{3.47}$$

Proof We start by considering

$$(\mathbf{e},\epsilon\xi)|_0^T + (\mathbf{h},\mu\eta)|_0^T = \int_0^T \frac{d}{dt}(\mathbf{e},\epsilon\xi)d\tau + \int_0^T \frac{d}{dt}(\mathbf{h},\mu\eta)d\tau$$
$$= \int_0^T (\mathbf{e}_t,\epsilon\xi) + (\mathbf{e},(\epsilon\xi)_t)d\tau + \int_0^T (\mathbf{h}_t,\mu\eta) + (\mathbf{h},(\mu\eta)_t)d\tau$$
$$= \int_0^T (\mathbf{e}_t,\epsilon\xi) + (\mathbf{e},\sigma\xi + \nabla\times\eta)d\tau + \int_0^T (\mathbf{h}_t,\mu\eta) + (\mathbf{h},-\nabla\times\xi)d\tau$$
$$= \int_0^T (\mathbf{e}_t,\epsilon\xi) + (\mathbf{e},\sigma\xi) + (\mathbf{e},\nabla\times\eta)d\tau$$
$$+ \int_0^T (\mathbf{h}_t,\mu\eta) - (\nabla\times\mathbf{h},\xi) + (\mathbf{h}\times\mathbf{n},\xi)_{L^2(\Gamma)}d\tau.$$

The boundary condition (3.13) implies that

$$(\mathbf{h}\times\mathbf{n},\xi)_{L^2(\Gamma)} = (\mathbf{n}\times\xi,\mathbf{h})_{L^2(\Gamma)} = 0,$$

so we have

$$(\mathbf{e},\epsilon\xi)|_0^T + (\mathbf{h},\mu\eta)|_0^T = \int_0^T (\epsilon\mathbf{e}_t + \sigma\mathbf{e} - \nabla\times\mathbf{h},\xi) + (\mu\mathbf{h}_t,\eta) + (\mathbf{e},\nabla\times\eta)d\tau.$$

Now we have, by the Galerkin orthogonality properties (2.13)-(2.14), that

$$(\mathbf{e},\epsilon\xi)|_0^T + (\mathbf{h},\mu\eta)|_0^T = \int_0^T (\epsilon\mathbf{e}_t + \sigma\mathbf{e} - \nabla\times\mathbf{h},\xi-\xi^h)d\tau$$
$$+ \int_0^T (\mu\mathbf{h}_t,\eta-\eta^h) + (\mathbf{e},\nabla\times(\eta-\eta^h))d\tau.$$

Applying (1.6) to $(\mathbf{e},\nabla\times(\eta-\eta^h))$ we get

$$(\mathbf{e},\epsilon\xi)|_0^T + (\mathbf{h},\mu\eta)|_0^T = \int_0^T (\epsilon\mathbf{e}_t + \sigma\mathbf{e} - \nabla\times\mathbf{h},\xi-\xi^h)d\tau$$

$$+\int_0^T (\mu \mathbf{h}_t, \eta - \eta^h) + (\nabla \times \mathbf{e}, \eta - \eta^h) - (\mathbf{n} \times \mathbf{e}, \eta - \eta^h)_{L^2(\Gamma)} d\tau.$$

But $(\mathbf{n} \times \mathbf{e}, \eta - \eta^h)_{L^2(\Gamma)} = -(\mathbf{n} \times \mathbf{E}^h, \eta - \eta^h)_{L^2(\Gamma)}$,

and by identifying the residuals defined in (3.45)-3.47) we get

$$
\begin{aligned}
&(\mathbf{e}, \epsilon\xi)|_0^T + (\mathbf{h}, \mu\eta)|_0^T = \int_0^T (\mathbf{R}_1, \xi - \xi^h) + (\mathbf{R}_2, \eta - \eta^h) + \langle \mathbf{R}_3, \eta - \eta^{\mathbf{h}} \rangle_\Gamma d\tau \\
&= \sum_\kappa \int_0^T \Big((h_\kappa \mathbf{R}_1, h_\kappa^{-1}(\xi - \xi^h))_\kappa + (h_\kappa \mathbf{R}_2, h_\kappa^{-1}(\eta - \eta^h))_\kappa \Big) d\tau \\
&+ \sum_{\partial\kappa \subset \Gamma} \int_0^T (h_\kappa \mathbf{R}_3, h_\kappa^{-1}(\eta - \eta^{\mathbf{h}}))_{L^2(\partial\kappa)} d\tau \\
&\leq \sum_\kappa \int_0^T \| h_\kappa \mathbf{R}_1 \|_{L^2(\kappa)} \| h_\kappa^{-1}(\xi - \xi^h) \|_{L^2(\kappa)} \, d\tau \\
&+ \sum_\kappa \int_0^T \| h_\kappa \mathbf{R}_2 \|_{L^2(\kappa)} \| h_\kappa^{-1}(\eta - \eta^h) \|_{L^2(\kappa)} \, d\tau \\
&+ \left(\int_0^T \sum_{\partial\kappa \subset \Gamma} \| h_\kappa \mathbf{R}_3 \|_{L^2(\partial\kappa)}^2 \, d\tau \right)^{1/2} \left(\int_0^T \sum_{\partial\kappa \subset \Gamma} h_\kappa^{-1} \| \eta - \eta^h \|_{L^2(\partial\kappa)}^2 \, d\tau \right)^{1/2} \\
&= \mathcal{A} + \mathcal{B} + \mathcal{C}.
\end{aligned}
$$

We shall use the Interpolation Theorem derived in Section 3.1.1. First choose $\xi^h = \tilde{\mathcal{I}}^h \xi$ and $\eta^h = \tilde{\mathcal{I}}^h \eta$; then by using (3.5) with $s = 0, k = 1$ and $p = 2$ we get for $\mathcal{A}$ and $\mathcal{B}$ the following bounds:

$$
\begin{aligned}
\mathcal{A} &\leq \sum_\kappa \int_0^T \| h_\kappa \mathbf{R}_1 \|_{L^2(\kappa)} \| h_\kappa^{-1}(\xi - \tilde{\mathcal{I}}^h \xi) \|_{L^2(\kappa)} \, d\tau \\
&\leq \sum_\kappa \int_0^T \| h_\kappa \mathbf{R}_1 \|_{L^2(\kappa)} \, (\sum_\kappa \| h_\kappa^{-1}(\xi - \tilde{\mathcal{I}}^h \xi) \|_{L^2(\kappa)}^2)^{1/2} d\tau \\
&\leq \int_0^T \sum_\kappa \| h_\kappa \mathbf{R}_1 \|_{L^2(\kappa)} \; C_I \| \nabla \xi \|_{L^2(\Omega)} \; d\tau \\
&\leq C_I \left(\int_0^T \sum_\kappa \| h_\kappa \mathbf{R}_1 \|_{L^2(\kappa)}^2 \, d\tau \right)^{1/2} \left(\int_0^T \| \nabla \xi \|_{L^2(\Omega)}^2 \, d\tau \right)^{1/2}
\end{aligned}
$$

$$\leq \ C_I \left(\sum_\kappa \| \ h_\kappa \mathbf{R}_1 \ \|^2_{L^2(0,T;L^2(\kappa))} \right)^{1/2} \sqrt{T} \ \max_t \| \ \nabla \xi \ \|_{L^2(\kappa)} \tag{3.48}$$

$$\mathcal{B} \leq \ \sum_\kappa \int_0^T \| \ h_\kappa \mathbf{R}_2 \ \|_{L^2(\kappa)} \| \ h_\kappa^{-1}(\eta - \tilde{\mathcal{I}}^h \eta) \ \|_{L^2(\kappa)} \ d\tau$$

$$\leq \ \sum_\kappa \int_0^T \| \ h_\kappa \mathbf{R}_2 \ \|_{L^2(\kappa)} \ (\sum_\kappa \| \ h_\kappa^{-1}(\eta - \tilde{\mathcal{I}}^h \eta) \ \|^2_{L^2(\kappa)})^{1/2} d\tau$$

$$\leq \ \int_0^T \sum_\kappa \| \ h_\kappa \mathbf{R}_2 \ \|_{L^2(\kappa)} \ C_I \ \| \ \nabla \eta \ \|^2_{L^2(\Omega)} \ \ d\tau$$

$$\leq \ C_I \left(\int_0^T \sum_\kappa \| \ h_\kappa \mathbf{R}_2 \ \|^2_{L^2(\kappa)} \ d\tau \right)^{1/2} \left(\int_0^T \ \| \ \nabla \eta \ \|^2_{L^2(\Omega)} \ d\tau \right)^{1/2}$$

$$\leq \ C_I \left(\sum_\kappa \| \ h_\kappa \mathbf{R}_2 \ \|^2_{L^2(0,T;L^2(\kappa))} \right)^{1/2} \sqrt{T} \ \max_t \| \ \nabla \eta \ \|_{L^2(\kappa)} \ . \tag{3.49}$$

For $\mathcal{C}$ we first use (3.8), then (3.4) to get

$$\int_0^T \sum_{\partial\kappa \subset \Gamma} h_\kappa^{-1} \ \| \ \eta - \eta^h \ \|^2_{L^2(\partial\kappa)} \ d\tau$$

$$\leq \ \int_0^T \left(\sum_\kappa h_\kappa^{-1} C_{Tr}^2 \ \| \ \eta - \tilde{\mathcal{I}}^h \eta \ \|_{L^2(\kappa)} \ (h_\kappa^{-1} \ \| \ \eta - \tilde{\mathcal{I}}^h \eta \ \|_{L^2(\kappa)} + \| \ \nabla(\eta - \tilde{\mathcal{I}}^h \eta) \ \|_{L^2(\kappa)}) \right) d\tau$$

$$\leq \ C_{Tr}^2 \int_0^T \left(\sum_\kappa \left[h_\kappa^{-2} \ \| \ \eta - \tilde{\mathcal{I}}^h \eta \ \|^2_{L^2(\kappa)} \ + C \ \| \ \nabla(\eta - \tilde{\mathcal{I}}^h \eta) \ \|^2_{L^2(\kappa)} \right] \right) d\tau.$$

Then, by using the triangle inequality, the algebraic inequality $(a+b)^2 = 2a^2 + 2b^2$, (3.5) as above, and (3.6), we get

$$\leq \ C_{Tr}^2 \int_0^T \left(\sum_\kappa \left[h_\kappa^{-2} \ \| \ \eta - \tilde{\mathcal{I}}^h \eta \ \|^2_{L^2(\kappa)} \ + C_I (|\eta|_{H^1(\kappa)} + |\tilde{\mathcal{I}}^h \eta|_{H^1(\kappa)})^2 \right] \right) d\tau$$

$$\leq \ C_{Tr}^2 \int_0^T \left(\sum_\kappa \left[h_\kappa^{-2} \ \| \ \eta - \tilde{\mathcal{I}}^h \eta \ \|^2_{L^2(\kappa)} \ + C_I (|\eta|_{H^1(\kappa)} + \| \ \tilde{\mathcal{I}}^h \eta \ \|_{H^1(\kappa)})^2 \right] \right) d\tau$$

$$\leq \ C_{Tr}^2 \int_0^T \left(\sum_\kappa \left[h_\kappa^{-2} \ \| \ \eta - \tilde{\mathcal{I}}^h \eta \ \|^2_{L^2(\kappa)} \ + C_I (2|\eta|^2_{H^1(\kappa)} + 2 \ \| \ \tilde{\mathcal{I}}^h \eta \ \|^2_{H^1(\kappa)}) \right] \right) d\tau$$

$$\leq C_{Tr}^2 C_I (3 + 2C_I) \ \| \ \nabla \eta \ \|^2_{L^2(0,T;L^2(\Omega))} \ .$$

This gives us the following bound on $\mathcal{C}$

$$\mathcal{C} \leq \; C_{Tr}\sqrt{C_I(3+2C_I)} \; \left(\sum_\kappa \| \; h_\kappa \mathbf{R}_3 \; \|^2_{L^2(0,T;L^2(\partial\kappa))}\right)^{1/2} \sqrt{T} \max_t \| \; \nabla\eta \; \|_{L^2(\kappa)} \; . \tag{3.50}$$

Now we want a bound on the magnitude of the inital error terms, i.e. on $|(\mathbf{e}(0), \epsilon\xi(0))|$, $|(\mathbf{h}(0), \mu\eta(0))|$. The approximated initial data are the ϵ- and μ-weighted L^2-projections of the exact initial data on the finite element spaces U_h, V_h respectively. Therefore if $(\xi^h, \eta^h) \in U_h \times V_h$ we have $(\epsilon\mathbf{e}(0), \xi^h(0)) = 0$. If we then choose $\xi^h(0) = \tilde{\mathcal{I}}^h\xi(0)$, and then use Cauchy-Schwarz inequality and (3.5) we get for $|(\mathbf{e}(0), \epsilon\xi(0))|$

$$\begin{aligned}
|(\mathbf{e}(0), \epsilon\xi(0))| \; &= \; |(\epsilon\mathbf{e}(0), \xi(0))| = |\sum_\kappa(\epsilon\mathbf{e}(0), \xi(0))_\kappa| \\
&\leq \; \sum_\kappa |(\epsilon\mathbf{e}(0), \xi(0) - \xi^h(0))_\kappa| \\
&\leq \; \sum_\kappa \| \; h_\kappa\epsilon\mathbf{e}(0) \; \|_{L^2(\kappa)} \| \; h_\kappa^{-1}(\xi(0) - \tilde{\mathcal{I}}^h\xi(0)) \; \|_{L^2(\kappa)} \\
&\leq \; \left(\sum_\kappa \| \; h_\kappa\epsilon\mathbf{e}(0) \; \|^2_{L^2(\kappa)}\right)^{1/2} \left(\sum_\kappa \| \; h_\kappa^{-1}(\xi(0) - \tilde{\mathcal{I}}^h\xi(0)) \; \|^2_{L^2(\kappa)}\right)^{1/2} \\
&\leq \left(\sum_\kappa \| \; h_\kappa\epsilon\mathbf{e}(0) \; \|^2_{L^2(\kappa)}\right)^{1/2} C_I \, \| \; \nabla\xi \; \|_{L^2(\kappa)} \; .
\end{aligned} \tag{3.51}$$

In the same fashion, for $|(\mathbf{h}(0), \mu\eta(0))|$,

$$|(\mathbf{h}(0), \mu\eta(0))| \leq \left(\sum_\kappa \| \; h_\kappa\mu\mathbf{h}(0) \; \|^2_{L^2(\kappa)}\right)^{1/2} C_I \, \| \; \nabla\eta \; \|_{L^2(\kappa)} \; . \tag{3.52}$$

Now we have by (3.48)-(3.52) that:

$$\begin{aligned}
|(\mathbf{e}(T), \epsilon\xi(T)) + (\mathbf{h}(T), \mu\eta(T))| \; &\leq \; C_I\sqrt{T}\left(\sum_\kappa \| \; h_\kappa\mathbf{R}_1 \; \|^2_{L^2(0,T;L^2(\kappa))}\right)^{1/2} \max_t \| \; \nabla\xi \; \|_{L^2(\kappa)} \\
+C_I\sqrt{T}\left(\sum_\kappa \| \; h_\kappa\mathbf{R}_2 \; \|^2_{L^2(0,T;L^2(\kappa))}\right)^{1/2} &\max_t \| \; \nabla\eta \; \|_{L^2(\kappa)}
\end{aligned}$$

$$+C_{Tr}\sqrt{C_I(3+2C_I)}\sqrt{T}\left(\sum_{\kappa} \| h_\kappa \mathbf{R}_3 \|^2_{L^2(0,T;L^2(\partial\kappa))}\right)^{1/2} \max_t \| \nabla\eta \|_{L^2(\kappa)}$$

$$+C_I\left(\sum_{\kappa} \| h_\kappa \epsilon \mathbf{e}(0) \|^2_{L^2(\kappa)}\right)^{1/2} \| \nabla\xi \|_{L^2(\kappa)} + C_I\left(\sum_{\kappa} \| h_\kappa \mu \mathbf{h}(0) \|^2_{L^2(\kappa)}\right)^{1/2} \| \nabla\eta \|_{L^2(\kappa)} .$$

This gives, by the inequality $(a_1+...+a_k)^2 \leq k(a_1^2+...+a_k^2)$, that

$$\begin{aligned}
&|(\mathbf{e}(T),\epsilon\xi(T)) + (\mathbf{h}(T),\mu\eta(T))|^2 \\
&\leq\ 5\left(C_I^2 T\sum_{\kappa} \| h_\kappa \mathbf{R}_1 \|^2_{L^2(0,T;L^2(\kappa))} \max_t \| \nabla\xi \|^2_{L^2(\kappa)}\right. \\
&+C_I^2 T\sum_{\kappa} \| h_\kappa \mathbf{R}_2 \|^2_{L^2(0,T;L^2(\kappa))} \max_t \| \nabla\eta \|^2_{L^2(\kappa)} \\
&+C_{Tr}^2(C+2C_I+2C_IC)T\sum_{\kappa} \| h_\kappa \mathbf{R}_3 \|^2_{L^2(0,T;L^2(\partial\kappa))} \max_t \| \nabla\eta \|^2_{L^2(\kappa)} \\
&\left.+C_I^2\sum_{\kappa} \| h_\kappa \epsilon \mathbf{e}(0) \|^2_{L^2(\kappa)} \| \nabla\xi \|^2_{L^2(\kappa)} + C_I^2\sum_{\kappa} \| h_\kappa \mu \mathbf{h}(0) \|^2_{L^2(\kappa)} \| \nabla\eta \|^2_{L^2(\kappa)}\right).
\end{aligned}$$

But now we know by *Friedrichs' div-curl Inequality* that the div-curl-norm appearing in (3.9) is equivalent to the H^1-norm. So by using *Friedrichs' div-curl Inequality*, then using the strong stability estimates derived in Section 3.2, and then bounding the $\mathbf{\Psi}, \mathbf{\Upsilon}$ terms in the H^1-norm we finally get by the inequality $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$, and by using the fact that both ϵ and μ are bounded from below by constants bigger than zero, that

$$\begin{aligned}
&|(\mathbf{e}(T),\mathbf{\Psi}) + (\mathbf{h}(T),\mathbf{\Upsilon})| \\
&\leq\ C_*\left(\sum_{\kappa}\left[\| h_\kappa \mathbf{R}_1 \|^2_{L^2(0,T;L^2(\kappa))} + \| h_\kappa \mathbf{R}_2 \|^2_{L^2(0,T;L^2(\kappa))} + \| h_\kappa \mathbf{R}_3 \|^2_{L^2(0,T;L^2(\partial\kappa))}\right.\right. \\
&\left.\left.+ \| h_\kappa \mu \mathbf{h}(0) \|^2_{L^2(\kappa)} + \| h_\kappa \epsilon \mathbf{e}(0) \|^2_{L^2(\kappa)}\right]\right)^{1/2}\left(\| \mathbf{\Psi} \|_{H^1(\Omega)} + \| \mathbf{\Upsilon} \|_{H^1(\Omega)}\right). \quad (3.53)
\end{aligned}$$

Here C_* is a computable constant. To get the bound on $\| \mathbf{e}(\cdot,T) \|_{H^{-1}} + \| \mathbf{h}(\cdot,T) \|_{H^{-1}}$, we divide through by $\| \mathbf{\Psi} \|_{H^1(\Omega)} + \| \mathbf{\Upsilon} \|_{H^1(\Omega)}$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, the inequality still holds if we take the supremum over all $\mathbf{\Psi}, \mathbf{\Upsilon} \in H_0^1(\Omega)$. This completes the proof of Theorem 3.14.

□

Chapter 4

Computational Implementation

In this section we will outline how the *a posteriori* error estimates derived in the previous section are implemented into the adaptive algorithm. We recall from Chapter 1 that we need to design an adaptive algorithm based on our *a posteriori* error estimate which is of the form

$$\| \mathbf{u} - \mathbf{u}^h \| \leq \mathcal{E}(\mathbf{u}^h, h, data),$$

and we have the stopping criterion

$$\mathcal{E}(\mathbf{u}^h, h, data) \leq TOL. \tag{4.1}$$

This guarantees *reliability*, in the sense that if the stopping criterion is satisfied, then the error is within the given tolerance. First we are going to show how to achieve reliability, and how the adaptive algorithm can be designed so that the mesh parameter h ensures that (4.1) holds. In the interest of *efficiency*, we also consider how the algorithm can allow for derefinement to ensure that (4.1) is satisfied with as near equality as possible. We then go on to show how the spatial mesh may be adapted at each time level in the special case of a 2-dimensional mesh consisting of triangles.

4.1 Adaptive Algorithm

For a given tolerance TOL, we want to find a discretisation in space at every time level such that

$$\| \mathbf{u} - \mathbf{u}^h \| \leq TOL,$$

and the mesh $\mathcal{T}$ is optimal in the sense that we minimise the number of nodes required to meet the inequality above. In the previous section we showed how to derive an *a posteriori* error bound of the form

$$\| \mathbf{e}(\cdot,T) \|_{H^{-1}} + \| \mathbf{h}(\cdot,T) \|_{H^{-1}} \;\; \leq \;\; C_* \left(\sum_{\kappa} \zeta(\kappa) \right)^{1/2},$$

where C_* is a computable constant and $\zeta(\kappa)$ a local error estimator.
We can also express the bound in the form

$$\| \mathbf{e}(\cdot,T) \|_{H^{-1}} + \| \mathbf{h}(\cdot,T) \|_{H^{-1}} \;\; \leq K_1 \mathrm{A} + K_2 \mathrm{B} + K_3 \Gamma + K_4 \Delta + K_5 \mathrm{E},$$

where

$$\mathrm{A} = \left(\sum_{\kappa} \| \, h_\kappa \mathbf{R}_1 \, \|^2_{L^2(0,T;L^2(\kappa))} \right)^{1/2}$$

$$\mathrm{B} = \left(\sum_{\kappa} \| \, h_\kappa \mathbf{R}_2 \, \|^2_{L^2(0,T;L^2(\kappa))} \right)^{1/2}$$

$$\Gamma = \left(\sum_{\kappa} \| \, h_\kappa \mathbf{R}_3 \, \|^2_{L^2(0,T;L^2(\partial\kappa))} \right)^{1/2}$$

$$\Delta = \left(\sum_{\kappa} \| \, h_\kappa \mu \mathbf{h}(0) \, \|^2_{L^2(\kappa)} \right)^{1/2}$$

$$\mathrm{E} = \left(\sum_{\kappa} \| \, h_\kappa \epsilon \mathbf{e}(0) \, \|^2_{L^2(\kappa)} \right)^{1/2},$$

and $K_1 - K_5$ are computable constants.

Writing

$$K_1 \mathrm{A} + K_2 \mathrm{B} + K_3 \Gamma + K_4 \Delta + K_5 \mathrm{E} = \mathcal{E}(\mathbf{u}^h, h, data),$$

we now split $\mathcal{E}(\mathbf{u}^h, h, data)$ up into two parts to reflect the different components of $\zeta(\kappa)$; let

$$\mathcal{E}(\mathbf{u}^h, h, data) = \mathcal{E}_0(\mathbf{u}^h, h, data) + \mathcal{E}_1(\mathbf{u}^h, h, data),$$

where

$$\mathcal{E}_0(\mathbf{u}^h, h, data) = K_4\Delta + K_5\mathrm{E},$$

and

$$\mathcal{E}_1(\mathbf{u}^h, h, data) = K_1\mathrm{A} + K_2\mathrm{B} + K_3\Gamma.$$

In a similar manner we split up the tolerance TOL into two parts, an initial tolerance given by TOL_0 and a tolerance adhered to once the time stepping has started, given by TOL_1, so that

$$TOL = TOL_0 + TOL_1$$

So our desired objective of

$$\mathcal{E}(\mathbf{u}^h, h, data) \leq TOL$$

can be achieved provided that

$$\mathcal{E}_0(\mathbf{u}^h, h, data) \leq TOL_0, \tag{4.2}$$

and

$$\mathcal{E}_1(\mathbf{u}^h, h, data) \leq TOL_1. \tag{4.3}$$

Satisfying (4.2) is straightforward as this is relevant only at the start of the computation, and can be controlled by a suitable choice of background mesh. We will therefore turn our attention to (4.3), and how it is satisfied.
Now (4.3) can be written as

$$K_1\left(\sum_{\kappa} \| h_\kappa \mathbf{R}_1 \|^2_{L^2(0,T;L^2(\kappa))}\right)^{1/2} + K_2\left(\sum_{\kappa} \| h_\kappa \mathbf{R}_2 \|^2_{L^2(0,T;L^2(\kappa))}\right)^{1/2}$$
$$+K_3\left(\sum_{\kappa} \| h_\kappa \mathbf{R}_3 \|^2_{L^2(0,T;L^2(\partial\kappa))}\right)^{1/2} \leq TOL_1,$$

and, as

$$\sum_{\kappa} \| \cdot \|^2_{L^2(0,T;L^2(\mathcal{T}))} \leq N \max_{\mathbf{x}} \| \cdot \|^2_{L^2(0,T;L^2(\mathcal{T}))} \leq NT \max_{\mathbf{x},t} \| \cdot \|^2_{L^2(\mathcal{T})},$$

where N is the (predicted) number of elements in the mesh and T the final time, we see that provided we can ensure that

$$K_1\sqrt{NT} \parallel h_\kappa \mathbf{R}_1 \parallel_{L^2(\kappa)} + K_2\sqrt{NT} \parallel h_\kappa \mathbf{R}_2 \parallel^2_{L^2(\kappa)}$$

$$+ K_3\sqrt{NT} \parallel h_\kappa \mathbf{R}_3 \parallel^2_{L^2(\partial\kappa)} \leq TOL_1,$$

at every stage of the numerical calculations, (4.3) will automatically be satisfied. In practice, as we are only using the error bound as an error indicator, we flag each triangle for refinement if

$$K_1\sqrt{NT} \parallel h_\kappa \mathbf{R}_1 \parallel_{L^2(0,T;L^2(\kappa))} + K_2\sqrt{NT} \parallel h_\kappa \mathbf{R}_2 \parallel^2_{L^2(0,T;L^2(\kappa))}$$

$$+ K_3\sqrt{NT} \parallel h_\kappa \mathbf{R}_3 \parallel^2_{L^2(0,T;L^2(\partial\kappa))} \geq UPPERTOL,$$

and for derefinement if

$$K_1\sqrt{NT} \parallel h_\kappa \mathbf{R}_1 \parallel_{L^2(0,T;L^2(\kappa))} + K_2\sqrt{NT} \parallel h_\kappa \mathbf{R}_2 \parallel^2_{L^2(0,T;L^2(\kappa))}$$

$$+ K_3\sqrt{NT} \parallel h_\kappa \mathbf{R}_3 \parallel^2_{L^2(0,T;L^2(\partial\kappa))} \leq LOWERTOL,$$

where $UPPERTOL$ and $LOWERTOL$ are set to ensure that the grid modification is as effective as possible.

4.2 Grid Modification

Here we are going to describe a strategy for refining a 2D-mesh, in the case of a mesh consisting of triangles.

4.2.1 The Red-Green Isotropic Refinement Strategy

Once a triangle has been flagged for refinement or derefinement by the process described above, we need to adapt the grid accordingly so that the desired error control

$$\| \mathbf{u} - \mathbf{u}^h \| \leq TOL$$

is satisfied. This could be achieved by using the red-green isotropic refinement strategy, see Bank[2] and Hempel[7], and the references cited therein. Here, the user must first specify a (coarse) *background mesh* upon which any future refinement will be based. A red refinement corresponds to dividing a certain triangle (father) into four similar triangles (sons) by connecting the midpoints of the sides (see Figure 4.1). Green refinement is only temporary and is used to remove any hanging nodes caused by red refinement (see Figure 4.2). We note that green refinement is only used on elements which have one hanging node. For elements with two or more hanging nodes a red refinement is performed. The advantage of this refinement strategy is that the degradation of the 'quality' of the mesh is limited since red refinement is obviously harmless and green triangles can *never* be further refined, as the green refinements are always temporary and are removed at the start of the next cycle.

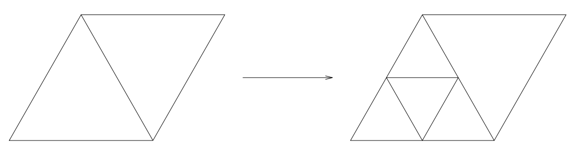

Figure 4.1: Red refinement.

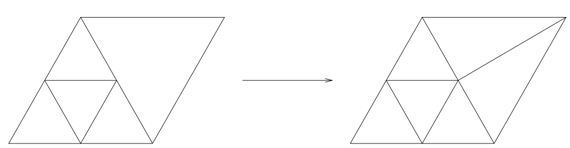

Figure 4.2: Green refinement.

Remark We note that within this mesh modification strategy, elements may also be removed from the mesh (i.e. derefined) provided that they do not lie in the original background mesh. Thus, to prevent an overly refined mesh in regions where the solution is smooth, the background mesh should be chosen to be suitably coarse.

4.3 The Restriction to 2 Dimensions

We state here some 2D results corresponding to the 3D-case investigated in Chapter 2 and 3. We also write out the system of algebraic equations resulting from the discretisation of the Maxwell system in two space dimensions.

4.3.1 Maxwell's Equations in 2D

Since we are in two space dimensions, we consider Maxwell's equations for a linear isotropic material in which the magnetic field **H** is z-polarized. That is, the third component of the electric field **E**, and the first two components of the magnetic field **H**, are zero.
Thus if the unknown electric field vector **E** is given by $\mathbf{E} = (E_x(\mathbf{x},t), E_y(\mathbf{x},t))$, and the unknown scalar magnetic field H is given by $H = H_z(\mathbf{x},t)$, then Maxwell's equations take the form of the vector equation

$$\epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} - \vec{\nabla} \times H = \mathbf{J} \qquad in\ \Omega \times (0,T) \tag{4.4}$$

and the scalar equation

$$\mu \frac{\partial H}{\partial t} + \nabla \times \mathbf{E} = 0 \qquad in\ \Omega \times (0,T), \tag{4.5}$$

with the boundary condition

$$\mathbf{n} \times \mathbf{E} = E_1 n_2 - E_2 n_1 = 0 \tag{4.6}$$

and initial data

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}) \quad \mathbf{x} \in \overline{\Omega} \tag{4.7}$$

$$H(\mathbf{x},0) = H_0(\mathbf{x}) \quad \mathbf{x} \in \overline{\Omega}, \tag{4.8}$$

where $\mathbf{E}_0$ and H_0 are given functions. Here $\mathbf{x} = (x,y)$ is a point in the plane.

4.3.2 The Space Discretisation in 2D

If we let the unknowns $(E_x, E_y, H_z) \in (V^h)^3$, where V^h is the space of continuous piecewise linear functions on the triangulation $\mathcal{T}^h$, the continuous piecewise linear 'hatfuctions' $\phi_i = \phi_i(x, y)$, defined by $\phi_i(x_j, y_j) = \delta_{ij}$, for $j = 1, ..., N$ (N is the number of nodes), form a basis for V^h. So the finite element approximation of (E_x, E_y, H_z) can be written as

$$
\begin{aligned}
E_x^h(x,y,t) &= \sum_{i=1}^{N} V_i^{E_x}(t)\phi_i(x,y) \\
E_y^h(x,y,t) &= \sum_{i=1}^{N} V_i^{E_y}(t)\phi_i(x,y) \\
H_z^h(x,y,t) &= \sum_{i=1}^{N} V_i^{H}(t)\phi_i(x,y).
\end{aligned}
$$

By choosing the hat functions $\phi_i(x, y)$ to be the test functions, we can further write the 2D-versions of the equations (2.4)-(2.5) as the matrix equations

$$
\begin{aligned}
A\dot{V}^{E_x} + BV^{E_x} + CV^{H_z} &= F^1 && (4.9) \\
A\dot{V}^{E_y} + BV^{E_y} + DV^{H_z} &= F^2 && (4.10) \\
E\dot{V}^{H_z} - DV^{E_y} - CV^{E_x} &= F^3, && (4.11)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{ij} &= (\epsilon\phi_i, \phi_j)_\Omega \\
B_{ij} &= (\sigma\phi_i, \phi_j)_\Omega \\
C_{ij} &= -(\frac{\partial\phi_i}{\partial y}, \phi_j)_\Omega \\
D_{ij} &= (\frac{\partial\phi_i}{\partial x}, \phi_j)_\Omega \\
E_{ij} &= (\mu\phi_i, \phi_j)_\Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
F^1 &= (J_x, \phi_i)_\Omega \\
F^2 &= (J_y, \phi_i)_\Omega \\
F^3 &\equiv 0.
\end{aligned}
$$

4.3.3 The Time Discretisation in 2D Using the θ-Method

We have seen in the previous section that a discretisation in space leads to an initial value problem involving a system of ordinary differential equations in time, of the form given by (4.9)-(4.11).

We then discretise (4.9)-(4.11) in time by using the θ-method, so that for some $\theta \in [0, 1]$

$$A(\frac{V_{n+1}^{E_x} - V_n^{E_x}}{\Delta t}) + \theta B V_{n+1}^{E_x} + (1-\theta) V_n^{E_x} + \theta C V_{n+1}^{H_z} + (1-\theta) C V_n^{H_z} = \theta F_{n+1}^1 + (1-\theta) F_n^1 \quad (4.12)$$

$$A(\frac{V_{n+1}^{E_y} - V_n^{E_y}}{\Delta t}) + \theta B V_{n+1}^{E_y} + (1-\theta) V_n^{E_y} + \theta D V_{n+1}^{H_z} + (1-\theta) D V_n^{H_z} = \theta F_{n+1}^2 + (1-\theta) F_n^2 \quad (4.13)$$

$$E(\frac{V_{n+1}^{H_z} - V_n^{H_z}}{\Delta t}) - \theta D V_{n+1}^{E_y} - (1-\theta) D V_n^{E_y} - \theta C V_{n+1}^{E_x} - (1-\theta) C V_n^{E_x} = F^3 \quad (4.14)$$

Here we recall that $\theta = 0$ gives the *explicit Euler's method*, $\theta = 1$ gives the *implicit Euler's method*, and $\theta = 1/2$ gives the *Crank-Nicolson method*. It can be shown that the θ-method is unconditionally stable for $\frac{1}{2} \leq \theta \leq 1$. We have shown this for the cases $\theta = 1/2$ and $\theta = 1$ in Chapter 2.3. It is also well known that the truncation error of the method is $O(\Delta t)$ for all values of $\theta \in [0, 1]$ except $\theta = 1/2$, when it is $O(\Delta t^2)$.

When we start to adapt the grid, we find that a grid at a certain time level may well be different to that at the previous time level, and we need a way of transferring information from one level to the next, so that we can approximate the numerical solution at a point on the previous grid which may not have been a node. This can be done in several ways, including straightforward interpolation. Another way is to use the idea of the L^2-*projection* (see Eriksson *et al.*[5], pp. 338-339). Denoted by $\Pi_{n+1} : \mathcal{V}_n^h(\Omega) \to \mathcal{V}_{n+1}^h(\Omega)$, this is given by

$$(\Pi_{n+1} u_n - u_{n+1}, v) \quad \forall v \in \mathcal{V}_n^h(\Omega), \quad (4.15)$$

where u_n and u_{n+1} are elements of the trial spaces at time levels n and $n+1$ respectively, and $\mathcal{V}_n^h(\Omega)$ denotes the test and trial space $\mathcal{V}^h(\Omega)$ at time $t = t_n$.

Then (4.12)-(4.14) takes the form of the following algebraic system of equations;

$$A(\frac{V_{n+1}^{E_x} - \Pi_{n+1} V_n^{E_x}}{\Delta t}) + \theta B V_{n+1}^{E_x} + (1-\theta)\Pi_{n+1} V_n^{E_x} + \theta C V_{n+1}^{H_z}$$
$$+(1-\theta) C \Pi_{n+1} V_n^{H_z} = \theta F_{n+1}^1 + (1-\theta)\Pi_{n+1} F_n^1 \quad (4.16)$$

$$A(\frac{V_{n+1}^{E_y} - \Pi_{n+1} V_n^{E_y}}{\Delta t}) + \theta B V_{n+1}^{E_y} + (1-\theta)\Pi_{n+1} V_n^{E_y} + \theta D V_{n+1}^{H_z}$$
$$+(1-\theta) D \Pi_{n+1} V_n^{H_z} = \theta F_{n+1}^2 + (1-\theta)\Pi_{n+1} F_n^2 \quad (4.17)$$

$$E(\frac{V_{n+1}^{H_z} - \Pi_{n+1} V_n^{H_z}}{\Delta t}) - \theta D V_{n+1}^{E_y} - (1-\theta) D \Pi_{n+1} V_n^{E_y} - \theta C V_{n+1}^{E_x}$$
$$-(1-\theta) C \Pi_{n+1} V_n^{E_x} = F^3, \quad (4.18)$$

these can now be solved for the unknowns $(V_{n+1}^{E_x}, V_{n+1}^{E_y}, V_{n+1}^{H_z})$.

Chapter 5

Conclusions

We have in this paper described the idea of adaptive finite element methods, following the general approach developed by C. Johnson and his co-workers (see [11] for example). We have also applied the techniques to the time-dependent Maxwell system of electromagnetics.
After a brief introduction we started by formulating the problem in Section 1.3. By using the weak formulation of Lee-Madsen[12] and Monk[18] in Section 2.1, for which we have an *a priori* convergence theorem derived by Monk[17], we applied a standard Galerkin discretisation to the problem in Section 2.2. We showed that our method is stable in Sections 2.3-2.4, and concluded Chapter 2 by deriving the Galerkin orthogonality properties in Section 2.5.
In Section 3.1 we stated some known results from approximation theory, such as some Interpolation Theorems and a Trace Theorem. We also presented the Friedrich div-curl Inequality, which states that the div-curl-norm appearing in (3.9) is equivalent to the H^1-norm. We further derived strong stability estimates for the adjoint problem in Section 3.2, and in Section 3.3 we proved *a posteriori* error estimates in the H^{-1}-norm.
In Section 4.1 we showed how this *a posteriori* error bound could be used in the adaptive algorithm to enable us to adapt the grid. We also gave an example of a grid refinement strategy for the special case of a 2-dimensional mesh consisting of triangles in Section 4.2. Finally in Section 4.3 we wrote out explicitly the algebraic system of equations (4.16)-(4.18) to be solved in the 2-dimensional case.

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