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# GLOBAL AND LOCALISED A POSTERIORI ERROR ANALYSIS IN THE MAXIMUM NORM FOR FINITE ELEMENT APPROXIMATIONS OF A CONVECTION-DIFFUSION PROBLEM

MATS BOMAN

ABSTRACT. We analyse finite element approximations of a stationary convection-diffusion problem. We prove global and localised a posteriori error estimates in the maximum norm. For the discretisation we use the Streamline Diffusion method.

## 1. INTRODUCTION

In this note we study finite element solution of the problem

$$\begin{aligned} \alpha u + \beta \cdot \nabla u - \epsilon \Delta u = f \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\alpha$, $\beta$, and $f$ are given functions and $\epsilon$ is a positive number. We assume that $\alpha - \nabla\cdot\beta \geq c > 0$, where $c \approx 1$, and $\|\beta\|_{L_\infty(\Omega)} \leq C$. Further, $\Omega \subset \mathbf{R}^d$, $d = 1, 2, 3$, is a bounded, convex and polyhedral domain. We prove a posteriori error estimates in the maximum norm for the error $u - U$, where $U$ is an approximation obtained by a finite element method. When $\epsilon \approx 1$ we use a standard Galerkin method for the discretisation and our a posteriori error estimate takes the form:

$$\|u - U\|_{L_\infty} \leq C(1 + |\log h_{\min}|)^2 \|h^2 r(U)\|_{L_\infty},$$

where $h = h(x)$ is the mesh function, and $r(U)$ is a computable realisation of the residual

$$R(U) = f - \alpha U - \beta \cdot \nabla U + \epsilon \Delta U \in H^{-1}.$$

In the elliptic case, $\epsilon \approx 1$, our result is essentially included in [3], [15] and [2]. These works also consider the non convex case.

However our analysis also gives results for small $\epsilon$. In this case we use a Streamline Diffusion method, see [4], and our result is of the form:

$$\|u - U\|_{L_\infty} \leq C(1 + |\log h_{\min}|)^{\frac{3}{2}} \left\| \left(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\right) r(U) \right\|_{L_\infty}.$$

This result is probably not optimal in the $\epsilon$ dependence. We believe that the factor $h^2\epsilon^{-3/2}$ should be replaced by $h^2\epsilon^{-1}$, which is the result that we obtain in one dimension, $d = 1$. In one dimension, $d = 1$, we prove an estimate with this $\epsilon$ dependence.

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It is known that if $\epsilon$ is small, then the solution $u$ is nonsmooth in certain regions, typically near characteristic layers and boundary layers. When used in an adaptive algorithm, a global error bound (as the above) would always require heavy refinement near such singular layers, even if we are only interested in the behavior in regions where the solution is smooth.

In order to avoid this we prove, in a special case, that the above result may be localised to special regions $\Omega_0$, oriented along the streamlines and avoiding singular layers.

More precisely we consider the following equation with $\Omega \subset \mathbf{R}^2$:

$$\begin{aligned} u + u_x - \epsilon \Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

Our localised result, essentially, takes the form

$$\|\varphi(u-U)\|_{L_\infty} \leq C(1 + |\log h_{\min}|)^{3/2} \Big\|\varphi \min\Big(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) r(U)\Big\|_{L_\infty},$$

where the cut-off function $\varphi$ is such that $\varphi \approx 1$ in $\Omega_0$ and $\varphi$ decays exponentially with $s/\sqrt{\epsilon}$ where $s$ is the distance to $\Omega_0$.

For this result to be useful it is important that the residual $r(U)$ can be made small in $\Omega_0$ without resolving the the singular layers. The standard Galerkin finite element method does not have this property.

However, a priori error analysis indicates that this is possible for the Streamline Diffusion method. In [13] it is shown that in regions $\Omega_0$, of the same type as above, and where the solution $u$ of (1.2) is assumed to be smooth, we have $\|u - U\|_{L_\infty(\Omega_0)} \leq C h_{\max}^{5/4}$. This result is improved in [14] to $O(h_{\max}^{11/8})$. Further in [18] it is proved, again with the assumption that $u$ is smooth in $\Omega_0$, that for certain triangulations $\|u - U\|_{L_\infty(\Omega_0)} \leq C h_{\max}^2$. Numerical experiments indicate that this is valid also for more general meshes. However there are also numerical examples, using certain meshes, in which one only gets the convergence rate $O(h_{\max}^{3/2})$. This has been further discussed in [17]. An important feature of these works is that they do not require that the singular layers are resolved.

In the case when $\epsilon$ is small our work is related to [4], where an a posteriori error estimate in the $L_2$-norm is proved. The localised result is inspired by [13]. We also recall that there is a localised a priori error estimate in $L_2$ for the Streamline Diffusion method, see [12].

As in [4], [7], our a posteriori error estimates are proved by using a duality argument involving a continuous linear adjoint problem. The analysis relies on the regularity of the solution $G$ of the adjoint problem. Since we consider estimates in the maximum norm we use an $L_1$-$L_\infty$ duality argument, where $G$ acts as a regularised Green function, a technique introduced in [15].

This paper is organised as follows. In Section 2 we introduce the space discretisation and formulate our finite element method. In Section 3 we state our global error estimates. The localised result is stated and proved in Section 10. In sections 4–9 we prove the results of Section 3.

In Section 4 we prove a lemma which splits the estimate of the error into a small a priori part and a part which can be estimated by a posteriori quantities. This lemma makes it possible to replace the exact Green function by a regularised Green function. In Section

5 we introduce the adjoint problem and perform the duality argument, expressing the a posteriori part of the error in terms of the residual $R(U)$ and the solution $G$ of the adjoint problem. In Section 6 we estimate the residual in terms of computable quantities and derivatives of $G$. In Section 7 we analyse the regularity of $G$. In Section 8 we prove a sharper regularity estimate for $G$ in the case $d = 1$. In Section 9 we conclude the proofs of the theorems stated in Section 3.

Throughout this paper we use the standard Lebesgue spaces $L_p(\omega)$ for $\omega \subset \Omega$ with the convention that $L_p = L_p(\Omega)$, and the corresponding Sobolev spaces $W_p^k(\omega)$, $W_p^k = W_p^k(\Omega)$, $H^k = W_2^k$ and $H_0^1 = \{u \in H^1 : u|_{\partial\Omega} = 0\}$. Moreover $(u, v)_\omega = \int_\omega uv\,dx$ and $(u, v) = (u, v)_\Omega$. We use the notation $\overline{f}_\omega = \sup_{x\in\omega} f(x)$ and $\underline{f}_\omega = \inf_{x\in\omega} f(x)$. Further, we write $D^k v(x) = \sqrt{\sum_{|\alpha|=k} |D^\alpha v(x)|^2}$, so that the $W_p^k$ seminorm may be conveniently written $\|D^k v\|_{L_p}$.

## 2. The discretisation

In this section we formulate a discretisation of (1.1) using the Streamline Diffusion method, see [4]. For the discretisation let $\mathcal{F} = \{\mathcal{T}\}$ be a family of triangulations, where a triangulation $\mathcal{T} = \{K\}$ is a partition of $\Omega$ into open simplices $K$ which are face to face so that $\overline{\Omega} = \cup_{K\in\mathcal{T}} \overline{K}$. Let $h_K = \mathrm{diam}(K)$ and let $\rho_K$ denote the radius of the largest closed ball contained in $\overline{K}$. We assume that $\mathcal{F}$ is nondegenerate, i.e., we assume that there is a constant $c_0$ such that for all triangulations $\mathcal{T} \in \mathcal{F}$ we have

$$\max_{K\in\mathcal{T}} \frac{h_K}{\rho_K} \leq c_0. \tag{2.1}$$

To each triangulation $\mathcal{T} \in \mathcal{F}$ we associate a positive, piecewise constant function $h(x)$, defined on $\overline{\Omega}$ by

$$h|_K = h_K, \quad \forall K \in \mathcal{T}.$$

We also need a measure of the "regularity" of a triangulation. We therefore introduce the quantity $\xi = \xi(\mathcal{T})$ as follows. Let $\mathcal{T} \in \mathcal{F}$ be a triangulation and $K$ be a simplex in $\mathcal{T}$. We define the set

$$S_K = \{K' \in \mathcal{T} : \overline{K'} \cap \overline{K} \neq \emptyset\}, \tag{2.2}$$

and

$$\xi = \max_{K\in\mathcal{T}} \max_{K'\in S_K} |1 - h_{K'}^2/h_K^2|. \tag{2.3}$$

For each triangulation $\mathcal{T} \in \mathcal{F}$ we have an associated function space $V = V(\mathcal{T})$, consisting of all continuous functions on $\overline{\Omega}$ which are linear on each $K \in \mathcal{T}$ and vanish on $\partial\Omega$. We discretise (1.1) by the Streamline Diffusion method, see [4]: Find $U \in V$ such that

$$\begin{aligned}(\alpha U, \chi + \delta\beta\cdot\nabla\chi) + (\beta\cdot\nabla U, \chi + \delta\beta\cdot\nabla\chi) + \epsilon(\nabla U, \nabla\chi) \\ = (f, \chi + \delta\beta\cdot\nabla\chi), \qquad \forall\chi \in V.\end{aligned} \tag{2.4}$$

The streamline diffusion coefficient $\delta = \delta(x)$ is defined by

$$\delta = c_1 \max(0, h - \epsilon), \tag{2.5}$$

where $c_1 \geq 0$ is a constant. Note that $c_1 = 0$ gives the standard Galerkin method.

## 3. A posteriori error estimates

In this section we state the global a posteriori error estimates. Our error estimates take the form of a small a priori term plus an a posteriori term. The a priori term is of the form $Ch_{\min}^{\sigma}$, where $\sigma \geq 2$ can be chosen arbitrarily. However, $\sigma$ enters also in the factor

$$L = 1 + \sigma |\log h_{\min}|, \tag{3.1}$$

which multiplies the a posteriori term.

Let $u$ be the solution of (1.1) and let $U$ be the solution of (2.4) for some $c_1 \geq 0$. The term $[\partial_\nu U](x)$ denotes the jump across $\partial K$ of the exterior normal derivative $\partial_\nu U$ at $x \in \partial K$.

In the case when $\epsilon \approx 1$ we have the following result. Here we assume that $\xi$ is sufficiently small, see (2.3), which is necissitated by the use of weighted estimates for the $L_2$-projection in the proof of the theorem (see [1]). The use of the $L_2$-projection, which is possible only for the standard Galerkin method, makes it possible to subtract an arbitrary function $\chi \in V$ in the residual. One can prove the theorem without using the $L_2$-projection, in which case the condition on $\xi$ can be removed, but then it is not possible to subtract the function $\chi$ in the residual.

**Theorem 3.1.** *Let $\epsilon \approx 1$. Let $c_1 = 0$ and let $\sigma \geq 2$. Assume that $\alpha(x) \geq 0$ and $\alpha(x) - \nabla \cdot \beta(x) \geq c > 0$ for all $x \in \Omega$, where $c \approx 1$. There exist constants $\xi^*$, $h^*$ and $C$ such that, if $\xi \leq \xi^*$ and $h_{\min} \leq h^*$, then*

$$\begin{aligned} \|u - U\|_{L_\infty(\Omega)} \leq Ch_{\min}^{\sigma} &+ CL^{3/2} \|h^2(f - \alpha U - \beta \cdot \nabla U - \chi)\|_{L_\infty(\Omega)} \\ &+ CL^{3/2} \max_{K \in \mathcal{T}} h^2 \|\epsilon h_K^{-1} [\partial_\nu U]\|_{L_\infty(\partial K)}, \end{aligned} \tag{3.2}$$

*where $\chi$ is an arbitrary function in $V$.*

In the general case, allowing $\epsilon$ to be small, we have the following estimate.

**Theorem 3.2.** *Let $\sigma \geq 2$. Assume that $\alpha(x) \geq c$, and $\alpha(x) - \nabla \cdot \beta(x) \geq c > 0$ for all $x \in \Omega$, where $c \approx 1$. Then there exist constants $h^*$ and $C$ such that if, $h_{\min} \leq h^*$, then*

$$\begin{aligned} \|u - U\|_{L_\infty(\Omega)} \leq C\frac{h_{\min}^{\sigma}}{\epsilon^2} &+ CL^{3/2} \left\| \min\left(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\right)(f - \alpha U - \beta \cdot \nabla U) \right\|_{L_\infty(\Omega)} \\ &+ CL^{3/2} \max_{K \in \mathcal{T}} \left( \min\left(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\right) \|\epsilon h_K^{-1} [\partial_\nu U]\|_{L_\infty(\partial K)} \right). \end{aligned} \tag{3.3}$$

In one dimension we are able to prove the following estimate, which is sharper with respect to the $\epsilon$-dependence.

**Theorem 3.3.** *Let $d = 1$ and let $\sigma \geq 2$. Assume that $\alpha(x) \geq 0$, $\alpha(x) - \beta_x(x) \geq c > 0$ and $|\beta(x)| \geq c$ for all $x \in \Omega$, where $c \approx 1$, and that $\beta$ does not change sign. Then there exist* *constants $h^*$ and $C$ such that if, $h_{\min} \le h^*$, then*

$$\|u-U\|_{L_\infty(\Omega)} \le C\frac{h_{\min}^\sigma}{\epsilon^2} + CL\Big\| \min\Big(1, h, \frac{h^2}{\epsilon}\Big)(f-\alpha U-\beta\cdot\nabla U)\Big\|_{L_\infty(\Omega)} + CL \max_{K\in\mathcal{T}} \Big(\min\Big(h, \frac{h^2}{\epsilon}\Big)\|\epsilon h_K^{-1}[\partial_\nu U]\|_{L_\infty(\partial K)}\Big). \tag{3.4}$$

## 4. A split of the error into an a priori and an a posteriori part

The main result of this section is a lemma which splits the estimate of the error $e = u-U$ into two parts, see [15] and [7]. The first part is then estimated through an a priori estimate. The remaining part is of the form $|(\omega e, g)|$, where $g$ is an approximate delta function and $\omega$ is a weight-function. The presence of the weight-function $\omega$ is motivated by the localisation argument in Section 10, where we need a more general version of the lemma.

We assume that the weight-function $\omega$ has the following properties:

$$\begin{aligned} &\omega \in W_1^2(\Omega),\\ &0 < \omega \le C_1,\\ &|D^j\omega| \le C_2\epsilon^{-j/2}\omega, \quad a.e. \text{ in } \Omega,\ j\le 2,\\ &\frac{\overline{\omega}_{S_K}}{\underline{\omega}_K} \le C_3, \quad \forall K\in\mathcal{T}, \end{aligned} \tag{4.1}$$

where $S_K$ is as in (2.2) and $\overline{w}_{S_K} = \sup_{S_K}\omega$, $\underline{w}_K = \inf_K \omega$. Note that the function $\omega(x)\equiv 1$ satisfies (4.1).

We now define the function $g$. Let $x_0\in\Omega$ and let $g = g_{x_0}$ be such that

$$\int_{\mathbf{R}^d} g\,dx = 1; \quad \operatorname{supp} g \subset \mathcal{B}(x_0;\rho); \quad 0 \le g \le C\rho^{-d}. \tag{4.2}$$

Here $\mathcal{B}(x_0;\rho)$ denotes the closed ball with center at $x_0$ and with small radius $\rho$ to be chosen. By direct calculation we have

$$\|g\|_{L_p} \le C\rho^{-d/p'}, \quad p' = p/(p-1). \tag{4.3}$$

We now state the main result of this section.

**Lemma 4.1.** *Let $U\in V$ and $u\in H_0^1(\Omega)\cap C^\gamma(\Omega)$ for some $0<\gamma\le\frac{1}{2}$. Let $e = u-U$ and let $x_0$ be such that $|\omega e(x_0)| = \|\omega e\|_{L_\infty(\Omega)}$. Let $g = g_{x_0}$ be given by (4.2) and let $\omega$ satisfy (4.1). Let $\sigma>\gamma$. There exist a constant $h_*>0$ such that, if $h_{\min}\le h_*$ and $\rho \le h_{\min}^{\sigma/\gamma}$, then*

$$\|\omega(u-U)\|_{L_\infty(\Omega)} \le Ch_{\min}^\sigma \|u\|_{C^\gamma(\Omega)} + 2|(\omega(u-U), g)|. \tag{4.4}$$

*Proof.* Let $\mathcal{B}$ denote the union of all elements $K\in\mathcal{T}$ that intersect $\mathcal{B}(x_0;\rho)$. Extend $e$ to be zero outside $\overline{\Omega}$. By the mean value theorem there is an $x_1\in\mathcal{B}(x_0,\rho)\cap\overline{\Omega}$ such that $(\omega e)(x_1) = (\omega e, g)$. We note that $\omega e = \omega u - \omega\Pi u + \omega\Pi e$, where $\Pi : C(\overline{\Omega})\to V$ is the Lagrange interpolation operator. Thus

$$|(\omega e)(x_0) - (\omega e)(x_1)| \le |(\omega u)(x_0) - (\omega u)(x_1)| + \rho\|D(\omega\Pi u)\|_{L_\infty(\mathcal{B})} + \rho\|D(\omega\Pi e)\|_{L_\infty(\mathcal{B})}. \tag{4.5}$$

We have

$$
\begin{aligned}
|\omega(x_0)u(x_0) - \omega(x_1)u(x_1)| &\le |\omega(x_0)u(x_0) - \omega(x_1)u(x_0)| \\
&\quad + |\omega(x_1)u(x_0) - \omega(x_1)u(x_1)| \\
&\le \rho\|D\omega\|_{L_\infty(\Omega)}\|u\|_{L_\infty(\Omega)} + \rho^\gamma\|u\|_{C^\gamma(\Omega)}\|\omega\|_{L_\infty(\Omega)} \\
&\le \rho^\gamma C_1\|u\|_{C^\gamma(\Omega)}(C_2\epsilon^{-1/2}\rho^{1-\gamma} + 1) \\
&\le 2\rho^\gamma C_1\|u\|_{C^\gamma(\Omega)},
\end{aligned}
\tag{4.6}
$$

where we used (4.1) and the estimate $\|u\|_{L_\infty(\Omega)} \le C\|u\|_{C^\gamma(\Omega)}$. The last inequality in (4.6) is valid if

$$
\rho \le \left(\frac{\epsilon^{1/2}}{C_2}\right)^{1/(1-\gamma)}. \tag{4.7}
$$

Further,

$$
\begin{aligned}
\rho\|D(\omega\Pi u)\|_{L_\infty(\mathcal{B})} &\le \rho\|(D\omega)\Pi u\|_{L_\infty(\mathcal{B})} + \rho\|\omega D\Pi u\|_{L_\infty(\mathcal{B})} \\
&\le \rho C_1C_2\epsilon^{-1/2}\|\Pi u\|_{L_\infty(\mathcal{B})} + \rho C_1\|D(\Pi u - \Pi u(x_0))\|_{L_\infty(\mathcal{B})} \\
&\le \rho C_1C_2\epsilon^{-1/2}\|u\|_{L_\infty(\mathcal{B})} \\
&\quad + \rho CC_1h_{\min}^{-1}\|\Pi u - \Pi u(x_0)\|_{L_\infty(\mathcal{B})} \\
&\le \rho CC_1C_2\epsilon^{-1/2}\|u\|_{C^\gamma(\Omega)} \\
&\quad + \rho^{1+\gamma}CC_1\|u\|_{C^\gamma(\Omega)} \\
&= \rho^\gamma CC_1\|u\|_{C^\gamma(\Omega)}\left(\rho^{1-\gamma}C_2\epsilon^{-1/2} + \rho h_{\min}{}^{-1}\right) \\
&\le 2\rho^\gamma CC_1\|u\|_{C^\gamma(\Omega)},
\end{aligned}
\tag{4.8}
$$

where we used (4.1), the stability of $\Pi$ in the $L_\infty$-norm and an inverse estimate. The last inequality in (4.8) is valid if

$$
\rho \le \left(\frac{\epsilon^{1/2}}{C_2}\right)^{1/(1-\gamma)} \text{ and } \rho \le h_{\min}. \tag{4.9}
$$

Likewise,

$$
\begin{aligned}
\rho\|D(\omega\Pi e)\|_{L_\infty(\mathcal{B})} &\le \rho C_2\epsilon^{-1/2}\|\omega\Pi e\|_{L_\infty(\mathcal{B})} + \rho\|\omega D\Pi e\|_{L_\infty(\mathcal{B})} \\
&\le \max_{K\in\mathcal{B}} \rho\overline{\omega}_K(C_2\epsilon^{-1/2}\|\Pi e\|_{L_\infty(K)} + \|D\Pi e\|_{L_\infty(K)}) \\
&\le \max_{K\in\mathcal{B}} \rho\overline{\omega}_K(C_2\epsilon^{-1/2}\|e\|_{L_\infty(K)} + Ch_{\min}^{-1}\|e\|_{L_\infty(K)}) \\
&\le \max_{K\in\mathcal{B}} \frac{\overline{\omega}_K}{\underline{\omega}_K}\left(\rho C_2\epsilon^{-1/2}\|\omega e\|_{L_\infty(K)} + Ch_{\min}^{-1}\|\omega e\|_{L_\infty(K)}\right) \\
&\le \rho C_3(C_2\epsilon^{-1/2} + Ch_{\min}^{-1})\|\omega e\|_{L_\infty(\Omega)} \\
&\le \tfrac{1}{2}\|\omega e\|_{L_\infty(\Omega)},
\end{aligned}
\tag{4.10}
$$

where we used (4.1), the stability of $\Pi$ in the $L_\infty$-norm and an inverse estimate. The last inequality in (4.10) is valid if

$$\rho \leq \tfrac{1}{4}\epsilon^{1/2}(C_3C_2)^{-1} \text{ and } \rho \leq \tfrac{1}{4}h_{\min}(C_3C)^{-1}. \tag{4.11}$$

Let now $\rho \leq h_{\min}^{\sigma/\gamma}$ for $h_{\min} \leq h_*$. We see that, since $\sigma/\gamma > 1$, for sufficiently small $h_*$ the conditions (4.7), (4.9) and (4.11) are satisfied and we get, using (4.5), (4.6), (4.8) and (4.10),

$$\begin{aligned} \|\omega e\|_{L_\infty(\Omega)} &\leq |(\omega e)(x_1)| + |(\omega e)(x_0) - (\omega e)(x_1)| \\ &\leq |(\omega e, g)| + 2CC_1\rho^\gamma\|u\|_{C^\gamma(\Omega)} + \tfrac{1}{2}\|\omega e\|_{L_\infty(\Omega)}, \end{aligned} \tag{4.12}$$

which concludes the proof. □

We have the following lemma which gives a rough estimate of the Hölder-norm of a solution $u$ of (1.1).

**Lemma 4.2.** *Let $\Omega$ be a convex polyhedral domain in $\mathbf{R}^d$, $d = 1, 2, 3$. Let $u$ be a solution of (1.1) with $\alpha - \frac{1}{2}\nabla\cdot\beta \geq 0$. For any $0 \leq \gamma < 1/2$ there is a constant $C$ such that*

$$\|u\|_{C^\gamma} \leq C\epsilon^{-2}\|f\|_{L_2}. \tag{4.13}$$

*Proof.* By a standard energy argument we get

$$\|\Delta u\|_{L_2} \leq C\epsilon^{-2}\|f\|_{L_2}. \tag{4.14}$$

Further

$$\|D^2u\|_{L_2} \leq C\|\Delta u\|_{L_2} \leq C\epsilon^{-2}\|f\|_{L_2}, \tag{4.15}$$

where we used elliptic regularity. By Sobolev's inequality we get (4.13) for $\gamma < 2 - d/2$, $d = 1, 2, 3$. □

## 5. A duality argument

In this section we express the quantity $(e, g)$ in terms of the residual $R(U)$ and the solution $G$ of an adjoint problem, using a duality argument. In order to do so we use the following adjoint problem:

$$\begin{aligned} \alpha G - \nabla\cdot(\beta G) - \epsilon\Delta G &= g \quad \text{in} \quad \Omega, \\ G &= 0 \quad \text{in} \quad \partial\Omega. \end{aligned} \tag{5.1}$$

Multiplying by $e = u - U$ gives

$$\begin{aligned} (e, g) &= (e, \alpha G - \nabla\cdot(\beta G) - \epsilon\Delta G) \\ &= (\alpha u + \beta\cdot\nabla u - \epsilon\Delta u, G) - (\alpha U + \beta\cdot\nabla U, G) - (\epsilon\nabla U, \nabla G) \\ &= (f - \alpha U - \beta\cdot\nabla U, G) - (\epsilon\nabla U, \nabla G) \\ &= \langle R(U), G\rangle, \end{aligned} \tag{5.2}$$

where we used (1.1) and where the residual $R(U) \in H^{-1}$ is defined by

$$\langle R(U), v\rangle = (f - \alpha U - \beta\cdot\nabla U, v) - (\epsilon\nabla U, \nabla v), \quad \forall v \in H_0^1. \tag{5.3}$$

We thus have

$$(e, g) = \langle R(U), G\rangle, \tag{5.4}$$

which we will use with $g$ given by (4.2).

## 6. Estimates of the residual

In this section we state and prove some estimates of the residual $R(U)$. One of the estimates involve a weight-function $\omega$ with properties as in (4.1). We note that the estimate is valid in the unweighted case, $\omega \equiv 1$. The weighted estimate will be used in the localisation argument in Section 10. The estimates are also weighted with powers of $\epsilon$, anticipating the regularity estimates of $G$ in Section 7-8.

We recall that there is an interpolant $\Pi$, see [16] and the discussion in [11], such that for $v \in W_1^2 \cap H_0^1$,

$$\begin{aligned} \|\Pi v\|_{L_1(K)} &\le C\|v\|_{L_1(S_K)}, \quad \forall K \in \mathcal{T}, \\ \|v - \Pi v\|_{L_1(K)} &\le C h_K^i \|D^i v\|_{L_1(S_K)}, \quad i = 1, 2, \quad \forall K \in \mathcal{T}, \\ \|D(v - \Pi v)\|_{L_1(K)} &\le C h_K^{i-1} \|D^i v\|_{L_1(S_K)}, \quad i = 1, 2, \quad \forall K \in \mathcal{T}. \end{aligned} \tag{6.1}$$

**Lemma 6.1.** *Let $U$ be the solution of* (2.4)*. Let $\omega$ be as in* (4.1) *and let $v \in W_1^2 \cap H_0^1$. There exist a constant $C$ such that*

$$\begin{aligned} |\langle R(U), \omega v\rangle| \le\; & C\Big\|\omega \min\Big(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)(f - \alpha U - \beta\cdot\nabla U)\Big\|_{L_\infty(\Omega)} \\ & \times \Big(\|v\|_{L_1(\Omega)} + \epsilon^{1/2}\|Dv\|_{L_1(\Omega)} + \epsilon^{3/2}\|D^2 v\|_{L_1(\Omega)}\Big) \\ & + C\max_{K\in\mathcal{T}} \Big\|\omega \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) h^{-1}\epsilon[\partial_\nu U]\Big\|_{L_\infty(\partial K)} \\ & \times \Big(\epsilon^{1/2}\|Dv\|_{L_1(\Omega)} + \epsilon^{3/2}\|D^2 v\|_{L_1(\Omega)}\Big). \end{aligned} \tag{6.2}$$

*Proof.* We first note that by expanding derivatives and using (4.1) and (6.1) we get the following inequalities

$$\begin{aligned} \underline{\omega}_K^{-1}\|\Pi(\omega v)\|_{L_1(K)} &\le C\|v\|_{L_1(S_K)}, \\ \underline{\omega}_K^{-1}\|D(\omega v)\|_{L_1(S_K)} &\le C(\epsilon^{-1/2}\|v\|_{L_1(S_K)} + \|Dv\|_{L_1(S_K)}), \\ \underline{\omega}_K^{-1}\|D^2(\omega v)\|_{L_1(S_K)} &\le C(\epsilon^{-1}\|v\|_{L_1(S_K)} + \epsilon^{-1/2}\|Dv\|_{L_1(S_K)} + \|D^2 v\|_{L_1(S_K)}). \end{aligned} \tag{6.3}$$

We also recall the inverse estimate

$$\|\nabla\chi\|_{L_1(K)} \le C h_K^{-1}\|\chi\|_{L_1(K)}, \quad \forall \chi \in V. \tag{6.4}$$

We now have

$$\begin{aligned} \langle R(U), \omega v\rangle &= (f - \alpha U - \beta\cdot\nabla U, \omega v) - \epsilon(\nabla U, \nabla(\omega v)) \\ &= (f - \alpha U - \beta\cdot\nabla U, \omega v - \Pi(\omega v) - \delta\beta\cdot\nabla\Pi(\omega v)) \\ &\quad - \epsilon(\nabla U, \nabla((\omega v) - \Pi(\omega v))) = I - II, \end{aligned} \tag{6.5}$$

where we used (2.4). Here

$$|I| \le \left\|\omega \min\left(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\right)(f - \alpha U - \beta\cdot\nabla U)\right\|_{L_\infty(\Omega)} \times \left\|\omega^{-1}\max\left(1, \frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\right)(\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla\Pi(\omega v))\right\|_{L_1(\Omega)}, \tag{6.6}$$

where

$$\begin{aligned}&\left\|\omega^{-1}\max\left(1, \frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\right)(\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla\Pi(\omega v))\right\|_{L_1(\Omega)}\\ &\qquad\le \sum_{K\in\mathcal{T}} \underline{\omega}_K^{-1}\max\left(1, \frac{\epsilon^{1/2}}{h_K}, \frac{\epsilon^{3/2}}{h_K^2}\right)\left\|\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla\Pi(\omega v)\right\|_{L_1(K)}\\ &\qquad= III.\end{aligned} \tag{6.7}$$

Further, since $\delta$ is constant on $K$,

$$\begin{aligned}&\underline{\omega}_K^{-1}\|\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla\Pi(\omega v)\|_{L_1(K)}\\ &\qquad\le \underline{\omega}_K^{-1}(\|\omega v\|_{L_1(K)} + \|\Pi(\omega v)\|_{L_1(K)} + \delta\|\beta\|_{L_\infty}\|D\Pi(\omega v)\|_{L_1(K)})\\ &\qquad\le \underline{\omega}_K^{-1}(\|\omega v\|_{L_1(K)} + C\|\omega v\|_{L_1(S_K)} + C\delta h_K^{-1}\|\omega v\|_{L_1(S_K)})\\ &\qquad\le C\underline{\omega}_K^{-1}\|\omega v\|_{L_1(S_K)} \le C\|v\|_{L_1(S_K)},\end{aligned} \tag{6.8}$$

where we used (4.1), (6.1) and (6.4) combined with the fact that $\delta \le Ch$, see (2.5). We also have the estimate

$$\begin{aligned}&\underline{\omega}_K^{-1}\|\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla\Pi(\omega v)\|_{L_1(K)}\\ &\qquad\le \underline{\omega}_K^{-1}(\|\omega v - \Pi(\omega v)\|_{L_1(K)} + \delta\|\beta\|_{L_\infty}\|D(\omega v - \Pi(\omega v))\|_{L_1(K)}\\ &\qquad\quad + \delta\|\beta\|_{L_\infty}\|D(\omega v)\|_{L_1(K)})\\ &\qquad\le Ch_K\underline{\omega}_K^{-1}\|D(\omega v)\|_{L_1(S_K)}\\ &\qquad\le Ch_K(\epsilon^{-1/2}\|v\|_{L_1(S_K)} + \|Dv\|_{L_1(S_K)}),\end{aligned} \tag{6.9}$$

where we used (4.1), (6.1), (6.3) combined with $\delta \le Ch$. In a similar way

$$\begin{aligned}&\underline{\omega}_K^{-1}\|\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla\Pi(\omega v)\|_{L_1(K)}\\ &\qquad\le C\underline{\omega}_K^{-1}h_K^2(\|D^2(\omega v)\|_{L_1(S_K)} + \epsilon^{-1}\|D(\omega v)\|_{L_1(S_K)})\\ &\qquad\le Ch_K^2(\epsilon^{-3/2}\|v\|_{L_1(S_K)} + \epsilon^{-1}\|Dv\|_{L_1(S_K)} + \|D^2 v\|_{L_1(S_K)}),\end{aligned} \tag{6.10}$$

where we used (4.1), (6.1), (6.3) and the inequality $\delta \le C\frac{h^2}{\epsilon}$ which follows from (2.5). Let $A = \{K \in \mathcal{T} : \min(1, \frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}) = 1\}$, $B = \{K \in \mathcal{T} : \max(1, \frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}) = \frac{\epsilon^{1/2}}{h})\}$ and let

$C = \{K \in \mathcal{T} : \max(1, \frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}) = \frac{\epsilon^{3/2}}{h^2})\}$. By combining (6.8), (6.9), and (6.10) we get

$$
\begin{aligned}
III &= \sum_{K\in A} \underline{\omega}_K^{-1} \|\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla(\omega v)\|_{L_1(K)} \\
&\quad + \sum_{K\in B} \underline{\omega}_K^{-1} \frac{\epsilon^{1/2}}{h} \|\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla(\omega v)\|_{L_1(K)} \\
&\quad + \sum_{K\in C} \underline{\omega}_K^{-1} \frac{\epsilon^{3/2}}{h^2} \|\omega v - \Pi(\omega v) - \delta\beta\cdot\nabla(\omega v)\|_{L_1(K)} \\
&\le C \sum_{K\in A} \|v\|_{L_1(S_K)} + C \sum_{K\in B} (\|v\|_{L_1(S_K)} + \epsilon^{1/2}\|Dv\|_{L_1(S_K)}) \\
&\quad + C \sum_{K\in B} (\|v\|_{L_1(S_K)} + \epsilon^{1/2}\|Dv\|_{L_1(S_K)} + \epsilon^{3/2}\|D^2 v\|_{L_1(S_K)}) \\
&\le C(\|v\|_{L_1(\Omega)} + \epsilon^{1/2}\|Dv\|_{L_1(\Omega)} + \epsilon^{3/2}\|D^2 v\|_{L_1(\Omega)}).
\end{aligned} \tag{6.11}
$$

Therefore

$$
\begin{aligned}
|I| &\le C\Big\|\omega \min\Big(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)(f - \alpha U - \beta\cdot\nabla U)\Big\|_{L_\infty(\Omega)} \\
&\quad \times (\|v\|_{L_1(\Omega)} + \epsilon^{1/2}\|Dv\|_{L_1(\Omega)} + \epsilon^{3/2}\|D^2 v\|_{L_1(\Omega)}).
\end{aligned} \tag{6.12}
$$

We now estimate II. We first note that

$$
\begin{aligned}
(\epsilon\nabla U, \nabla(\omega v - \Pi(\omega v))) &= \sum_{K\in\mathcal{T}} (\epsilon\nabla U, \nabla(\omega v - \Pi(\omega v)))_K \\
&= \sum_{K\in\mathcal{T}} (\epsilon\partial_\nu U, \omega v - \Pi(\omega v))_{\partial K} \\
&= -\frac{1}{2}\sum_{K\in\mathcal{T}} (\epsilon[\partial_\nu U], \omega v - \Pi(\omega v))_{\partial K},
\end{aligned} \tag{6.13}
$$

where

$$
\begin{aligned}
|(\epsilon[\partial_\nu U], \omega v - \Pi(\omega v))_{\partial K}| &\le \Big\|\omega \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) h^{-1}\epsilon[\partial_\nu U]\Big\|_{L_\infty(\partial K)} \\
&\quad \times \Big\|\omega^{-1} \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big) h(\omega v - \Pi(\omega v))\Big\|_{L_1(\partial K)}.
\end{aligned} \tag{6.14}
$$

Further,

$$
\begin{aligned}
&\Big\|\omega^{-1} \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big) h(\omega v - \Pi(\omega v))\Big\|_{L_1(\partial K)} \\
&\quad \le C\underline{\omega}_K^{-1} \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big)\Big(\|\omega v - \Pi(\omega v)\|_{L_1(K)} \\
&\qquad\qquad + h_K\|D(\omega v - \Pi(\omega v))\|_{L_1(K)}\Big),
\end{aligned} \tag{6.15}
$$

where we used a trace inequality. We here have that

$$
\begin{aligned}
\underline{\omega}_K^{-1}(\|\omega v-\Pi(\omega v)\|_{L_1(K)} + h_K\|D(\omega v-\Pi(\omega v))\|_{L_1(K)}) \\
\le C h_K \underline{\omega}_K^{-1}\|D(\omega v)\|_{L_1(S_K)} \\
\le C h_K\Big(\epsilon^{-1/2}\|v\|_{L_1(S_K)} + \|Dv\|_{L_1(S_K)}\Big),
\end{aligned} \tag{6.16}
$$

where we used (6.1) and (6.3). But we also have the estimate

$$
\begin{aligned}
&\underline{\omega}_K^{-1}(\|\omega v-\Pi(\omega v)\|_{L_1(K)} + h_K\|D(\omega v-\Pi(\omega v))\|_{L_1(K)}) \\
&\le C h_K^2 \underline{\omega}_K^{-1}\|D^2(\omega v)\|_{L_1(S_K)} \\
&\le C h_K^2\Big(\epsilon^{-1}\|v\|_{L_1(S_K)} + \epsilon^{-1/2}\|Dv\|_{L_1(S_K)} + \|D^2 v\|_{L_1(S_K)}\Big),
\end{aligned} \tag{6.17}
$$

where we again used (6.1) and (6.3). By combining (6.14), (6.16), and (6.17) we get the following estimate

$$
\begin{aligned}
|(\epsilon\nabla U, \nabla(\omega v-\Pi(\omega v)))| &\le C \max_{K\in\mathcal{T}}\Big\|\omega \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) h^{-1}\epsilon[\partial_\nu U]\Big\|_{L_\infty(\partial K)} \\
&\times \Big(\sum_{K\in B}\frac{\epsilon^{1/2}}{h}\underline{\omega}_K^{-1}(\|\omega v-\Pi(\omega v)\|_{L_1(K)} + h_K\|D(\omega v-\Pi(\omega v))\|_{L_1(K)}) \\
&+ \sum_{K\in C}\frac{\epsilon^{3/2}}{h^2}\underline{\omega}_K^{-1}(\|\omega v-\Pi(\omega v)\|_{L_1(K)} + h_K\|D(\omega v-\Pi(\omega v))\|_{L_1(K)})\Big) \\
&\le C \max_{K\in\mathcal{T}}\Big\|\omega \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) h^{-1}\epsilon[\partial_\nu U]\Big\|_{L_\infty(\partial K)} \\
&\times \Big(\|v\|_{L_1(\Omega)} + \epsilon^{1/2}\|Dv\|_{L_1(\Omega)} + \epsilon^{3/2}\|D^2 v\|_{L_1(\Omega)}\Big),
\end{aligned} \tag{6.18}
$$

which proves the lemma. □

In the proof of Theorem 3.1, where $\epsilon \approx 1$, we will use the following version of Lemma 6.1. The proof of this lemma is similar to the proof of Lemma 6.1, but uses the $L_2$-projection instead of the interpolant $\Pi$, which is possible since we use the standard Galerkin method in this case. In order to use the $L_2$-projection one needs a regularity assumption on the triangulation. This is expressed through the parameter $\xi$ in (2.3), see [5], [1].

**Lemma 6.2.** *Let $U$ be the solution of (2.4) with $c_1 = 0$. Let $v \in W_1^2 \cap H_0^1$ and let the residual $R(U)$ be defined by (5.3). For sufficiently small $\xi$ there exist a constant $C$ such that*

$$
\begin{aligned}
|\langle R(U), v\rangle| \le C\Big(\Big\|\frac{h^2}{\epsilon^2}(f-\alpha U-\beta\cdot\nabla U-\chi)\Big\|_{L_\infty(\Omega)} \\
+ \max_{K\in\mathcal{T}}\frac{h^2}{\epsilon^2}\|h_K^{-1}\epsilon[\partial_\nu U]\|_{L_\infty(\partial K)}\Big)\epsilon^2\|D^2 v\|_{L_1(\Omega)},
\end{aligned} \tag{6.19}
$$

*where $\chi$ is an arbitrary function in $V$.*

The following lemma will be used in the proof of Theorem 3.3 where $d = 1$. The proof of this lemma is an obvious modification of the proof of Lemma 6.1.

**Lemma 6.3.** *Let $U$ be the solution of* (2.4). *Let $v \in W_1^2 \cap H_0^1$ and let the residual $R(U)$ be defined by* (5.3). *Then there exist a constant $C$ such that*

$$\begin{aligned} |\langle R(U), v\rangle| \leq C\Big(\Big\|\min\Big(1,h,\frac{h^2}{\epsilon}\Big)(f-\alpha U-\beta\cdot\nabla U)\Big\|_{L_\infty(\Omega)} \\ + \max_{K\in\mathcal{T}}\Big(\min\Big(h,\frac{h^2}{\epsilon}\Big)\|h_K^{-1}\epsilon[\partial_\nu U]\|_{L_\infty(\partial K)}\Big)\Big) \\ \times\Big(\|v\|_{L_1(\Omega)}+\|Dv\|_{L_1(\Omega)}+\epsilon\|D^2v\|_{L_1(\Omega)}\Big). \end{aligned} \tag{6.20}$$

## 7. The regularity of the solution $G$ of the adjoint problem

In this section we study the regularity of the solution $G$ of the adjoint problem (5.1). We have the following lemma.

**Lemma 7.1.** *Let $G$ be the solution of* (5.1) *with $g$ given by* (4.2). *Assume that $\alpha(x) \geq c > 0$ and $\alpha(x) - \nabla\cdot\beta(x) \geq c > 0$ for all $x \in \Omega$, where $c \approx 1$. For any $\kappa > 0$ there is a constant $C$ such that, if $\rho \leq \epsilon^\kappa$, then*

$$\begin{aligned} \|G\|_{L_1} &\leq 1/c, \\ \|DG\|_{L_1} &\leq CL^{1/2}\epsilon^{-1/2}, \\ \|D^2G\|_{L_1} &\leq CL^{3/2}\epsilon^{-3/2}, \end{aligned} \tag{7.1}$$

*where $L = 1 + |\log\rho|$.*

**Remark.** Another way to get estimates for the regularity of $G$ is to use an interpolation inequality. We have the following result: Let $v \in W_2^p(\Omega) \cap H_0^1(\Omega)$, $1 \leq p \leq \infty$. Then for any $\eta > 0$, we have

$$\|Dv\|_{L_p(\Omega)} \leq \eta\|v\|_{W_2^p(\Omega)} + C\eta^{-1}\|v\|_{L_p(\Omega)}, \tag{7.2}$$

see Theorem 7.27 in [9]. An argument based on this inequality would give the estimates

$$\begin{aligned} \|DG\|_{L_1} &\leq CL\epsilon^{-1}, \\ \|D^2G\|_{L_1} &\leq CL^2\epsilon^{-2}. \end{aligned} \tag{7.3}$$

*Proof.* We first estimate $\|G\|_{L_1}$. Multiplying (5.1) by $\frac{G}{\sqrt{\eta+G^2}}$, $\eta > 0$, which is a regularisation of $\operatorname{sign}(G)$, we get, after an integration by parts,

$$\Big(\alpha G, \frac{G}{\sqrt{\eta+G^2}}\Big) + \Big(G, \frac{\eta\beta\cdot\nabla G}{(\eta+G^2)^{3/2}}\Big) + \epsilon\int_\Omega \frac{\eta\nabla G\cdot\nabla G}{(\eta+G^2)^{3/2}}\,dx = \Big(g, \frac{G}{\sqrt{\eta+G^2}}\Big). \tag{7.4}$$

We want to let $\eta \to 0^+$ here. In order to do so we note that

$$\Big\|G\frac{\eta\beta\cdot\nabla G}{(\eta+G^2)^{3/2}}\Big\|_{L_1} \leq \|\beta\|_{L_\infty}\|DG\|_{L_2}\Big\|\frac{\eta G}{(\eta+G^2)^{3/2}}\Big\|_{L_2}. \tag{7.5}$$

By a standard energy argument, using that $\alpha - \frac{1}{2}\nabla\beta \geq 0$ which follows from $\alpha \geq 0$ and $\alpha - \nabla \cdot \beta \geq 0$, we obtain

$$\|DG\|_{L_2} \leq C\epsilon^{-1}\|g\|_{L_2} \leq C\epsilon^{-1}\rho^{-d/2} < \infty. \tag{7.6}$$

Further

$$\left\|\frac{\eta G}{(\eta + G^2)^{3/2}}\right\|_{L_2}^2 = \int_\Omega \frac{\eta^2 G^2}{(\eta + G^2)^3}\,dx. \tag{7.7}$$

Here $\frac{\eta^2 G^2}{(\eta+G^2)^3} \leq 1$ and $\frac{\eta^2 G^2}{(\eta+G^2)^3} \to 0$ as $\eta \to 0^+$, so that by the dominated convergence theorem, $\int_\Omega \frac{\eta^2 G^2}{(\eta+G^2)^3} \to 0$ as $\eta \to 0^+$. By letting $\eta \to 0^+$ in (7.4) we conclude that

$$\|\alpha G\|_{L_1} \leq \|g\|_{L_1} \leq 1. \tag{7.8}$$

Since $\alpha \geq c$, where $c \approx 1$, this gives the estimate for $\|G\|_{L_1}$.

We now estimate $\|DG\|_{L_1}$. Since $\alpha - \nabla \cdot \beta \geq 0$, the maximum principle gives $G \geq 0$ in $\Omega$. Therefore $\log(1+G) \geq 0$ is well defined in $\Omega$. We note that $\log(1+G) = 0$ on $\partial\Omega$. We also note that the maximum principle combined with the assumption $\alpha - \nabla \cdot \beta \geq c$, where $c \approx 1$, and the definition of $g$ gives the estimate

$$\|G\|_{L_\infty} \leq c^{-1}\|g\|_{L_\infty} \leq C\rho^{-d}. \tag{7.9}$$

We now multiply (5.1) with $\log(1+G)$, integrate by parts using $\nabla \log(1+G) = \frac{\nabla G}{1+G}$, to get

$$(\alpha G, \log(1+G)) + \Big(\beta G, \frac{\nabla G}{1+G}\Big) + \epsilon\Big(\nabla G, \frac{\nabla G}{1+G}\Big) = (g, \log(1+G)), \tag{7.10}$$

where, by integration by parts, we have

$$\begin{aligned}\Big(\beta G, \frac{\nabla G}{1+G}\Big) &= \int_\Omega (1+G)\beta \cdot \frac{\nabla G}{1+G}\,dx - \int_\Omega \beta \cdot \nabla \log(1+G)\,dx \\ &= \int_\Omega \beta \cdot \nabla G\,dx + \int_\Omega \nabla \cdot \beta \log(1+G)\,dx \\ &= -\int_\Omega (\nabla \cdot \beta)G\,dx + \int_\Omega (\nabla \cdot \beta)\log(1+G)\,dx,\end{aligned} \tag{7.11}$$

where we used that $G$ and $\log(1+G) = 0$ vanish on $\partial\Omega$. We thus have

$$\begin{aligned}&\int_\Omega \alpha G \log(1+G)\,dx + \int_\Omega \epsilon\frac{|\nabla G|^2}{1+G}\,dx \\ &\quad= (g, \log(1+G)) + (\nabla \cdot \beta, G - \log(1+G)) \\ &\quad\leq (g, \log(1+G)) + (\alpha, G) \\ &\quad\leq \|g\|_{L_1}\log(1 + \|G\|_{L_\infty}) + \|\alpha G\|_{L_1} \leq CL,\end{aligned} \tag{7.12}$$

where we used $G \geq \log(1+G) \geq 0$, the condition $\alpha - \nabla \cdot \beta \geq 0$, $\|g\|_{L_1} = 1$, (7.9), (7.8) and where $L = 1 + |\log\rho|$. By (7.12) combined with the fact that $\alpha G \log(1+G) \geq 0$ we get

$$\int_\Omega \epsilon \frac{|\nabla G|^2}{1+G}\, dx \le CL. \tag{7.13}$$

Further, by Cauchy's inequality,

$$\begin{aligned} \|\sqrt{\epsilon}\nabla G\|_{L_1} &= \Big\| \frac{\sqrt{\epsilon}\nabla G}{\sqrt{1+G}}\sqrt{1+G}\Big\|_{L_1} \le \Big( \Big\| \frac{\epsilon|\nabla G|^2}{1+G}\Big\|_{L_1} \|1+G\|_{L_1}\Big)^{1/2} \\ &\le CL^{1/2}(1+\|G\|_{L_1})^{1/2} \le CL^{1/2}, \end{aligned} \tag{7.14}$$

where we used the estimate $\|G\|_{L_1} \le C$. Hence

$$\|DG\|_{L_1} \le CL^{1/2}\epsilon^{-1/2}. \tag{7.15}$$

The estimate of $\|D^2G\|_{L_1}$ will be obtained from an estimate of $\|\epsilon\Delta G\|_{L_p}$ for $p$ near 1. In order to estimate $\|\epsilon\Delta G\|_{L_p}$, using (5.1), we need to estimate $\|(\alpha - \nabla\cdot\beta)G\|_{L_p}$ and $\|\beta\cdot\nabla G\|_{L_p}$ for $1 \le p \le 2$. These $L_p$ estimates are obtained by interpolation between $L_1$ and $L_2$.

We first estimate $\|(\alpha-\nabla\cdot\beta)G\|_{L_1}$. As in (7.4) we multiply by a regularisation of sign$(G)$ and get

$$\begin{aligned} \Big((\alpha-\nabla\cdot\beta)G, \frac{G}{\sqrt{\eta+G^2}}\Big) - \Big(\beta\cdot\nabla G, \frac{G}{\sqrt{\eta+G^2}}\Big) + \epsilon\int_\Omega \eta\frac{\nabla G\cdot\nabla G}{(\eta+G^2)^{3/2}}\, dx \\ = \Big(g, \frac{G}{\sqrt{\eta+G^2}}\Big), \end{aligned} \tag{7.16}$$

and we conclude

$$\|(\alpha-\nabla\cdot\beta)G\|_{L_1} \le \|g\|_{L_1} + \|\beta\|_{L_\infty}\|DG\|_{L_1} \le CL^{1/2}\epsilon^{-1/2}, \tag{7.17}$$

where we used (7.15). We also need an estimate for $\|(\alpha-\nabla\cdot\beta)G\|_{L_2}$, which we get from (7.9) as follows:

$$\|(\alpha-\nabla\cdot\beta)G\|_{L_2} \le \|\alpha-\nabla\cdot\beta\|_{L_2}\|G\|_{L_\infty} \le C\rho^{-d}. \tag{7.18}$$

We now estimate $\|(\alpha-\nabla\cdot\beta)G\|_{L_p}$ and $\|\beta\cdot\nabla G\|_{L_p}$, for $1<p<2$. By Hölders inequality, combined with (7.17) and (7.18) we get

$$\|(\alpha-\nabla\cdot\beta)G\|_{L_p} \le \|(\alpha-\nabla\cdot\beta)G\|_{L_1}^{1-2/p'}\|(\alpha-\nabla\cdot\beta)G\|_{L_2}^{2/p'} \le CL^{1/2}\epsilon^{-1/2}\rho^{-2d/p'}, \tag{7.19}$$

where $p' = p/(p-1)$. Further

$$\begin{aligned} \|\beta\cdot\nabla G\|_{L_p} &\le \|\beta\|_{L_\infty}\|DG\|_{L_p} \le \|\beta\|_{L_\infty}\|DG\|_{L_1}^{1-2/p'}\|DG\|_{L_2}^{2/p'} \\ &\le CL^{1/2}\epsilon^{-1/2}\epsilon^{-2/p'}\rho^{-d/p'}, \end{aligned} \tag{7.20}$$

where we used (7.6) and (7.15). By (5.1) we get

$$
\begin{aligned}
\|\epsilon\Delta G\|_{L_p} &\le \|g\|_{L_p} + \|(\alpha - \nabla\cdot\beta)G\|_{L_p} + \|\beta\cdot\nabla G\|_{L_p} \\
&\le C\rho^{-d/p'} + CL^{1/2}\epsilon^{-1/2}\rho^{-2d/p'} + CL^{1/2}\epsilon^{-1/2}\epsilon^{-2/p'}\rho^{-d/p'} \\
&\le CL^{1/2}\epsilon^{-1/2}\epsilon^{-2/p'}\rho^{-2d/p'}.
\end{aligned} \tag{7.21}
$$

We now use the condition $\rho \le \epsilon^{\kappa}$, which gives

$$
\epsilon^{-2/p'} \le \rho^{-\frac{1}{\kappa}\frac{2}{p'}}, \tag{7.22}
$$

and therefore

$$
\|\Delta G\|_{L_p} \le CL^{1/2}\epsilon^{-3/2}\rho^{-c/p'}, \tag{7.23}
$$

where $c = 2/\kappa + 2d$. We recall the following elliptic regularity estimate: If $\Omega$ is a smooth domain or a convex domain, then

$$
\|D^2 v\|_{L_p} \le Cp'\|\Delta v\|_{L_p}, \quad \forall v \in W_p^2 \cap H_0^1, \quad 1 < p \le 2. \tag{7.24}
$$

The $p$-dependence in (7.24) is classical in in the case of a smooth domain. In the case of a convex domain we argue as in [7]. Let $v = Tf$ be the solution of the Dirichlet problem $-\Delta v = f$ in $\Omega$, $v = 0$ on $\partial\Omega$ and let $D^2_{ij}$ be a partial derivative of second order. It is well known , [10], that the operator $D^2_{ij}T$ is bounded on $L_2$, i.e, it is strong type $(2,2)$; this is the case $p = 2$ of (7.24). Moreover $D^2_{ij}T$ is weak type $(1,1)$; this is an unpublished result of Dahlberg, Verchota, and Wolff, a proof can be found in [8] and a generalisation in [6]. An application of Marcinkiewicz interpolation theorem now yields (7.24).

We now use (7.24) to get an estimate for $\|D^2G\|_{L_1}$. By Hölder's inequality and (7.24) we get

$$
\|D^2G\|_{L_1} \le C\|D^2G\|_{L_p} \le Cp'\|\Delta G\|_{L_p} \le CL^{1/2}\epsilon^{-3/2}p'\rho^{-c/p'}. \tag{7.25}
$$

The choice $p' = |\log\rho|$ gives

$$
\|D^2G\|_{L_1} \le CL^{1/2}\epsilon^{-3/2}|\log\rho| \le CL^{3/2}\epsilon^{-3/2}, \tag{7.26}
$$

which proves the lemma. □

We now prove a lemma similar to Lemma 7.1 with worse $\epsilon$-dependence, but with more general coefficients $\alpha$, $\beta$. We use this lemma in the proof of Theorem 3.1, where we assume that $\epsilon \approx 1$.

**Lemma 7.2.** *Let $G$ be the solution of* (5.1) *with $g$ given by* (4.2). *Assume that $\alpha(x) \ge 0$ and $\alpha(x) - \nabla\cdot\beta(x) \ge c > 0$ for all $x \in \Omega$, where $c \approx 1$. For any $\kappa > 0$ there is a constant $C$ such that, if $\rho \le \epsilon^{\kappa}$, then*

$$
\|D^2G\|_{L_1} \le CL^2\epsilon^{-2}, \tag{7.27}
$$

*where $L = 1 + |\log\rho|$.*

*Proof.* We first note that the proof of (7.14) is only based on the condition $\alpha \geq 0$ and $\alpha - \beta \cdot \nabla \geq c$, where $c \approx 1$. We thus have

$$\|\epsilon^{1/2} DG\|_{L_1} \leq CL^{1/2}(1 + \|G\|_{L_1})^{1/2}, \tag{7.28}$$

but we no longer get an estimate estimate for $\|G\|_{L_1}$ by (7.8). Instead we use the inequality $\|G\|_{L_1} \leq C\|DG\|_{L_1}$ in (7.28). Hence

$$\begin{aligned} \|\epsilon^{1/2} DG\|_{L_1} &\leq CL^{1/2} + CL^{1/2}\|DG\|_{L_1}^{1/2} \\ &\leq CL^{1/2} + CL\epsilon^{-1/2} + \tfrac{1}{2}\epsilon^{1/2}\|DG\|_{L_1}, \end{aligned} \tag{7.29}$$

where we used the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$. We conclude that

$$\|DG\|_{L_1} \leq CL\epsilon^{-1}, \tag{7.30}$$

and consequently, by using the inequality $\|G\|_{L_1} \leq C\|DG\|_{L_1}$ again, we get

$$\|G\|_{L_1} \leq CL\epsilon^{-1}. \tag{7.31}$$

By the same argument as in Lemma 7.1, but using (7.31) and (7.30) instead of (7.8) and (7.15), we now get (7.27). □

## 8. Regularity of the adjoint problem in 1-dimension

In one dimension, $d = 1$, our regularity estimate, for the solution $G$ of the adjoint problem (5.1), is sharper with respect to the $\epsilon$-dependence.

**Lemma 8.1.** *Assume that that $d = 1$ and $\alpha(x) \geq 0$, $\alpha(x) - \beta_x(x) \geq c > 0$, $|\beta(x)| \geq c$, where $c \approx 1$, for all $x \in \Omega$, and that $\beta$ does not change sign. Let $G$ be the solution of* (5.1). *Then*

$$\begin{aligned} \|G\|_{L_1} &\leq CL, \\ \|G_x\|_{L_1} &\leq CL, \\ \|G_{xx}\|_{L_1} &\leq CL\epsilon^{-1}, \end{aligned} \tag{8.1}$$

*where $L = 1 + |\log \rho|$.*

*Proof.* We first prove an estimate for $\|(\alpha - \beta_x)G\|_{L_1}$. Note that as in (7.8) we have $\|\alpha G\|_{L_1} \leq 1$. As in the proof of Lemma 7.1 we multiply (5.1) with $\log(1 + G)$ and similarly we get the estimate, see (7.12),

$$\begin{aligned} \int_\Omega \alpha G \log(1 + G)\, dx - \int_\Omega \beta_x G\, dx + \int_\Omega \epsilon \frac{G_x^2}{1 + G}\, dx \\ = (g - \beta_x, \log(1 + G)) \\ \leq (\|g\|_{L_1} + \|\beta_x\|_{L_1}) \log(1 + \|G\|_{L_\infty}) \leq CL, \end{aligned} \tag{8.2}$$

where we used $\|g\|_{L_1} = 1$, (7.9), and where $L = 1 + |\log \rho|$. We note that by (8.2) we get

$$\begin{aligned} &\int_{\log(1+G)\geq 1} \Big(\alpha G \log(1+G) - \beta_x G\Big)\, dx \\ &\qquad + \int_{\log(1+G)\leq 1} \Big(\alpha G \log(1+G) - \beta_x G\Big)\, dx \leq CL. \end{aligned} \tag{8.3}$$

Since $\log(1+G) \leq 1$ implies $G \leq 2$, we conclude

$$\int_{\log(1+G)\geq 1} \Big(\alpha G \log(1+G) - \beta_x G\Big)\, dx \leq CL + 2\|\beta_x\|_{L_1} \leq CL, \tag{8.4}$$

where we also used that $\alpha G \log(1+G) \geq 0$ in $\Omega$. Therefore

$$\begin{aligned} \int_\Omega (\alpha - \beta_x)\, G\, dx &= \int_{\log(1+G)\geq 1} (\alpha - \beta_x)\, G\, dx + \int_{\log(1+G)\leq 1} (\alpha - \beta_x)\, G\, dx \\ &\leq \int_{\log(1+G)\geq 1} \Big(\alpha G \log(1+G) - \beta_x G\Big)\, dx + \|\alpha G\|_{L_1} + \|\beta_x\|_{L_1} \\ &\leq CL, \end{aligned} \tag{8.5}$$

where we used (8.3), the assumption $\alpha \geq 0$ and the estimate $\|\alpha G\|_{L_1} \leq 1$. Since $\alpha - \beta_x \geq 0$ and $G \geq 0$, this proves

$$\|(\alpha - \beta_x) G\|_{L_1} \leq CL. \tag{8.6}$$

We now estimate $\|\beta G_x\|_{L_1}$. Multiply (5.1) by $-\mathrm{sign}(\beta)\frac{G_x}{\sqrt{\eta + G_x^2}}$, where $\eta > 0$. Recall that $\mathrm{sign}(\beta)$ is constant and $\frac{G_x}{\sqrt{\eta+G_x^2}}$ is a regularisation of $\mathrm{sign}(G_x)$. Let $\Omega = (a, b)$, we get, after an integration by parts,

$$\begin{aligned} &-\mathrm{sign}(\beta) \int_a^b (\alpha - \beta_x) G \frac{G_x}{\sqrt{\eta + G_x^2}}\, dx + \int_a^b |\beta| G_x \frac{G_x}{\sqrt{\eta + G_x^2}}\, dx \\ &\quad + \mathrm{sign}(\beta)\epsilon \Big( G_x(b) \frac{G_x(b)}{\sqrt{\eta + G_x(b)^2}} - G_x(a) \frac{G_x(a)}{\sqrt{\eta + G_x(a)^2}} \Big) \\ &\quad - \mathrm{sign}(\beta) \int_a^b G_x \frac{\eta G_{xx}}{(\eta + G_x^2)^{3/2}}\, dx = -\Big(g, \mathrm{sign}(\beta) \frac{G_x}{\sqrt{\eta + G_x^2}}\Big). \end{aligned} \tag{8.7}$$

Since $G \geq 0$ and $|\frac{G_x}{\sqrt{\eta+G_x^2}}| \leq 1$ this implies

$$\begin{aligned} \int_a^b |\beta| G_x \frac{G_x}{\sqrt{\eta + G_x^2}}\, dx &\leq \|(\alpha - \beta_x) G\|_{L_1} + \epsilon(|G_x(b)| + |G_x(a)|) \\ &\qquad + \Big| \int_a^b G_x \frac{\eta G_{xx}}{(\eta + G_x^2)^{3/2}}\, dx \Big| + \|g\|_{L_1}. \end{aligned} \tag{8.8}$$

We now show that

$$\Big| \int_a^b G_x \frac{\eta G_{xx}}{(\eta + G_x^2)^{3/2}}\, dx \Big| \to 0 \quad \text{as} \quad \eta \to 0^+. \tag{8.9}$$

By Hölder's inequality we have

$$\Big| \int_a^b G_x \frac{\eta G_{xx}}{(\eta + G_x^2)^{3/2}}\, dx \Big| \le \|G_{xx}\|_{L_2} \Big\| \frac{\eta G_x}{(\eta + G_x^2)^{3/2}} \Big\|_{L_2}. \tag{8.10}$$

A standard energy argument gives

$$\|G_{xx}\|_{L_2} \le C\epsilon^{-2} \|g\|_{L_2} < \infty. \tag{8.11}$$

Further,

$$\Big\| \frac{\eta G_x}{(\eta + G_x^2)^{3/2}} \Big\|_{L_2}^2 = \int_a^b \frac{\eta^2 G_x^2}{(\eta + G_x^2)^3}\, dx, \tag{8.12}$$

where $\frac{\eta^2 G_x^2}{(\eta+G_x^2)^3} \le 1$, and $\frac{\eta^2 G_x^2}{(\eta+G_x^2)^3} \to 0$ as $\eta \to 0^+$, so that by the dominated convergence theorem, we conclude that $\int_a^b \frac{\eta^2 G_x^2}{(\eta+G_x^2)^3} \to 0$ as $\eta \to 0^+$, and (8.9) follows.

In order to estimate $\epsilon(|G_x(b)| + |G_x(a)|)$ we argue as follows. Multiply (5.1) by 1 to get

$$(\alpha G, 1) - ((\beta G)_x, 1) - \epsilon(G_{xx}, 1) = (g, 1), \tag{8.13}$$

which, after an integration by parts, gives

$$\|\alpha G\|_{L_1} - \epsilon G_x(b) + \epsilon G_x(a) = 1. \tag{8.14}$$

By the maximum principle we have that $G \ge 0$ in $[a, b]$, hence $G_x(a) \ge 0$ and $G_x(b) \le 0$. Consequently, by (8.14), we get

$$\epsilon(|G_x(b)| + |G_x(a)|) \le 1. \tag{8.15}$$

Let now $\eta \to 0^+$ in (8.8). Using (8.6) and $\|g\|_{L_1(\Omega)} = 1$ we obtain

$$\|\beta G_x\|_{L_1} \le CL. \tag{8.16}$$

By assumption $|\beta| \ge c$, where $c \approx 1$, so that

$$\|G_x\|_{L_1} \le CL. \tag{8.17}$$

The inequality $\|G\|_{L_1} \le C\|G_x\|_{L_1}$ gives

$$\|G\|_{L_1} \le CL. \tag{8.18}$$

Finally, by the equation (5.1), we get

$$\|\epsilon G_{xx}\|_{L_1} \le \|(\alpha - \beta_x)G\|_{L_1} + \|\beta G_x\|_{L_1} + \|g\|_{L_1} \le CL, \tag{8.19}$$

which implies $\|G_{xx}\|_{L_1} \le CL\epsilon^{-1}$. □

## 9. Conclusion of proofs of Theorems 3.1, 3.3, and 3.2

In this section we conclude the proofs of Theorem 3.1, Theorem 3.2, and Theorem 3.3.

We recall that the parameter $\rho$ is given by the definition (4.2). In order to use Lemma 4.1 we have the conditions

$$\rho \leq h_{\min}^{\sigma/\gamma}, \tag{9.1}$$

and $h_{\min} \leq h_*$, where $h_*$ is determined by conditions in the proof of Lemma 4.1, $\gamma$ is the Hölder exponent of the solution $u$ of (1.1) and $\sigma > \gamma$ can be chosen arbitrarily. Further, in order to use Lemma 7.1 and Lemma 7.2 we have the condition

$$\rho \leq \epsilon^{\kappa}, \tag{9.2}$$

where $\kappa > 0$ can be chosen arbitrarily. The choice of the parameters $\sigma$ and $\kappa^{-1}$ affects the constant $C$ in the theorems: the larger the values of $\sigma$ and $\kappa^{-1}$, the larger the value of $C$.

Let now $h^* \leq \min(\epsilon^\kappa, h_*)$. Assume that $h_{\min} \leq 1$. Since $\sigma/\gamma \geq 1$, we have $\rho \leq h_{\min}^{\sigma/\gamma} \leq h_{\min} \leq h^* \leq \epsilon^\kappa$ and the condition (9.2) is satisfied. As a remark we note that the proof of Lemma 4.1 indicates that $\kappa = 1/2$, see (4.11), is a natural choice.

We also note that by Lemma 4.2 we have $\gamma \approx 1/2$. This implies $\rho \leq h_{\min}^{2\sigma}$. The choice $\sigma \geq 2$ is natural, since the a posteriori part is of order 2 in $h$.

In the proof of Theorem 3.1, Theorem 3.2 and Theorem 3.3, $\rho$ enters in Lemma 7.1, Lemma 7.2 and Lemma 8.1 through the logarithmic factor $L = 1 + |\log \rho|$.

We now note that the weight function $\omega \equiv 1$ satisfies (4.1). Therefore by Lemma 4.1, with the above condition on $\rho$, we have

$$\begin{aligned} \|e\|_{L_\infty} &\leq C h_{\min}^{\sigma} \|u\|_{C^\gamma} + 2|(e, g)| \\ &\leq C \frac{h_{\min}^{\sigma}}{\epsilon^2} + |\langle R(U), G \rangle|, \end{aligned} \tag{9.3}$$

where we also used Lemma 4.2 and (5.4). By combining (9.3), Lemma 6.2 and Lemma 7.2 we get Theorem 3.1. By combining (9.3), Lemma 6.1 and Lemma 7.1 we get Theorem 3.2. By combining (9.3), Lemma 6.3 and Lemma 8.1 we get Theorem 3.3.

## 10. A localisation result

In this section we state and prove a localised a posteriori error estimate in a special case. The estimate is localised in the following sense. The error and the residual are multiplied by a cut-off function $\varphi$, where $\varphi$ is such that $\varphi \approx 1$ inside $\Omega_0 \subset \Omega$ and decays exponentially with the distance from $\Omega_0$ outside $\Omega_0$. The set $\Omega_0$ can not be chosen arbitrarily. It has to be oriented along the streamlines with a cut-off in the downstream direction.

The point of this result is that it indicates that it is possible to cut-off certain regions where the residual can be very large, as in a boundary layer or a characteristic layer. We now define the problem that we will consider. Let $\Omega \subset \mathbf{R}^2$. Let $u$ be the solution of

$$\begin{aligned} u + u_x - \epsilon \Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \partial\Omega. \end{aligned} \tag{10.1}$$

In [13] it is proved, for the equation (10.1), that, for the Streamline-Diffusion method, in regions $\Omega_0$, of the type mentioned above, with the additional assumption that the the solution $u$ of (10.1) is smooth in $\Omega_0$ one have the a priori estimate $\|e\|_{L_\infty(\Omega_0)} \leq Ch_{\max}^{5/4}$, see the discussion in Section 1. It is important to note that this result show that the error in $\Omega_0$ can be made small without resolving the singular layers.

We now define the cut-off function $\varphi$. Let

$$\psi(t) = \int_t^\infty e^{-\zeta(s)}\,ds, \tag{10.2}$$

where $\zeta(s) = s$ for $t \geq 1$ and $\zeta(s) = -(s-2)$ for $t < 1$. By direct calculation we find that $\psi''$ is smooth except at $t = 1$, where $\psi''$ has a jump, and that

$$\begin{aligned} \psi(t) &> 0, \quad \forall t, \\ e^{-1} \leq \psi(t) &\leq 2e^{-1}, \quad t \leq 1, \\ \psi(t) &= e^{-t}, \quad t \geq 1, \\ \psi'(t) &< 0, \quad \forall t, \\ |\psi'(t)| &\leq \psi(t), \quad \forall t, \\ |\psi''(t)| &\leq -\psi'(t), \quad t \neq 1. \end{aligned} \tag{10.3}$$

Let $\Omega_0 = \{(x,y) \in \Omega : x \leq A,\ B_1 \leq y \leq B_2\}$ and define the cut-off function

$$\varphi(x,y) = \psi(\frac{x-A}{\sqrt{\epsilon}})\psi(\frac{y-B_1}{3\sqrt{\epsilon}})\psi(\frac{B_2-y}{3\sqrt{\epsilon}}). \tag{10.4}$$

We note that $\varphi \in C^1(\overline{\Omega})$ with a jump in the second derivatives across $\partial\Omega_0$ and

$$\frac{\overline{\varphi}_{S_K}}{\underline{\varphi}_K} \leq \begin{cases} 2, & \text{if} \quad S_K \subset \Omega_0, \\ e^{\frac{Ch}{\sqrt{\epsilon}}}, & \text{otherwise.} \end{cases} \tag{10.5}$$

We note that (10.5) guarantees that, if $Ch \leq \sqrt{\epsilon}$ in $\Omega \setminus \Omega_0$ then $\varphi$ satisfies the condition (4.1), which makes it possible to use Lemma 4.1 and Lemma 6.1 with $\omega = \varphi$.

In order to localise we use the following adjoint problem:

$$\begin{aligned} \varphi G - (\varphi G)_x - \epsilon\Delta(\varphi G) &= \varphi g, \quad \text{in} \quad \Omega, \\ G &= 0, \quad \text{in} \quad \partial\Omega, \end{aligned} \tag{10.6}$$

where $g$ is given by (4.2). The duality argument reads as follows:

$$\begin{aligned} (\varphi e, g) = (e, \varphi g) &= (e, \varphi G) - (e, (\varphi G)_x) - (e, \epsilon\Delta\varphi G) \\ &= (e, \varphi G) + (e_x, \varphi G) + \epsilon(\nabla e, \nabla(\varphi G)) \\ &= (f, \varphi G) - (U, \varphi G) - (U_x, \varphi G) - \epsilon(\nabla U, \nabla(\varphi G)) \\ &= \langle R(U), \varphi G\rangle, \end{aligned} \tag{10.7}$$

where now $R(U) \in H^{-1}(\Omega)$ is takes the form

$$\langle R(U), v\rangle = (f - U - U_x, v) - (\epsilon\nabla U, \nabla v), \quad \forall v \in H_0^1. \tag{10.8}$$

In order to put (10.6) in standard form, see (5.1), we expand the derivatives, collect terms and then divide by $\varphi$ to get the equation

$$\alpha G - \nabla \cdot (\beta G) - \epsilon \Delta G = g, \tag{10.9}$$

where

$$\begin{aligned} \alpha &= \tilde{\alpha} + \nabla \beta \\ \tilde{\alpha} &= 1 - \frac{\varphi_x}{\varphi} - \epsilon \frac{\varphi_{xx}}{\varphi} - \epsilon \frac{\varphi_{yy}}{\varphi}, \\ \beta &= \Big( -1 + 2\epsilon \frac{\varphi_x}{\varphi}, 2\epsilon \frac{\varphi_y}{\varphi} \Big). \end{aligned} \tag{10.10}$$

In order to prove the localised result we need to analyse the regularity of the solution $G$ of (10.9). The regularity estimates we need follows from Lemma 7.1 if we can show that the coefficients $\alpha$ and $\beta$ satisfies the conditions of Lemma 7.1. We have the following result:

**Lemma 10.1.** *Let $\varphi$, $\alpha$, and $\beta$ be given by* (10.4) *and* (10.10). *Assume that $\epsilon \leq 1/9$. Then*

$$\begin{aligned} \alpha &\geq 1/3, \\ \alpha - \nabla \cdot \beta &\geq 5/9. \end{aligned} \tag{10.11}$$

*Proof.* We first estimate

$$\alpha - \nabla \cdot \beta = \tilde{\alpha} = 1 - \frac{\varphi_x}{\varphi} - \epsilon \frac{\varphi_{xx}}{\varphi} - \epsilon \frac{\varphi_{yy}}{\varphi}. \tag{10.12}$$

By definition and (10.3) we have that $-\frac{\varphi_x}{\varphi} \geq 0$ and $\epsilon |\frac{\varphi_{xx}}{\varphi}| \leq \sqrt{\epsilon} |\frac{\varphi_x}{\varphi}|$ and thus

$$-\frac{\varphi_x}{\varphi} - \epsilon \frac{\varphi_{xx}}{\varphi} \geq (1 - \sqrt{\epsilon}) \Big| \frac{\varphi_x}{\varphi} \Big| \geq 0. \tag{10.13}$$

It is convenient to use the notation $p(y) = \psi(\frac{y - B_1}{3\sqrt{\epsilon}})$ and $q(y) = \psi(\frac{B_2 - y}{3\sqrt{\epsilon}})$. We have

$$\begin{aligned} \Big| \epsilon \frac{\varphi_{yy}}{\varphi} \Big| &= \epsilon \Big| \frac{p_{yy} q + 2 p_y q_y + p q_{yy}}{pq} \Big| \\ &\leq \epsilon \Big( \frac{|p_{yy}|}{p} + 2 \frac{|p_y|}{p} \frac{|q_y|}{q} + \frac{|q_{yy}|}{q} \Big) \leq \frac{4}{9}, \end{aligned} \tag{10.14}$$

since by definition $|q_y| \leq \frac{q}{3\sqrt{\epsilon}}$, $|p_y| \leq \frac{p}{3\sqrt{\epsilon}}$, $|p_{yy}| \leq \frac{p}{9\epsilon}$ and, $|q_{yy}| \leq \frac{q}{9\epsilon}$. By combining (10.12), (10.13), and (10.14) we conclude that $\alpha - \nabla \cdot \beta \geq 5/9$.

We now estimate $\alpha = \tilde{\alpha} + \nabla \cdot \beta$. After differentiation and rearrangement of terms we get

$$\alpha = \tilde{\alpha} + \nabla \cdot \beta = 1 - \frac{\varphi_x}{\varphi} + \epsilon \frac{\varphi_{xx}}{\varphi} - 2\epsilon \frac{\varphi_x^2}{\varphi^2} + \epsilon \frac{\varphi_{yy}}{\varphi} - 2\epsilon \frac{\varphi_y^2}{\varphi^2}, \tag{10.15}$$

where as above

$$-\frac{\varphi_x}{\varphi} + \epsilon \frac{\varphi_{xx}}{\varphi} - 2\epsilon \frac{\varphi_x^2}{\varphi^2} \geq (1 - 3\sqrt{\epsilon}) \Big| \frac{\varphi_x}{\varphi} \Big| \geq 0, \tag{10.16}$$

since $\epsilon \le 1/9$. Further,

$$\begin{aligned}\epsilon\frac{\varphi_{yy}}{\varphi}-2\epsilon\frac{\varphi_y^2}{\varphi^2}&=\epsilon\frac{p_{yy}q+2p_yq_y+pq_{yy}}{pq}-2\epsilon\frac{(p_yq)^2+2p_yqpq_y+(pq_y)^2}{(pq)^2}\\&=\epsilon\frac{p_{yy}q+pq_{yy}}{pq}-2\epsilon\frac{(p_yq)^2+(pq_y)^2}{(pq)^2}-2\frac{p_yq_y}{pq}\\&\ge\epsilon\Big(\frac{p_{yy}}{p}+\frac{q_{yy}}{q}\Big)-2\epsilon\Big(\Big(\frac{p_y}{p}\Big)^2+\Big(\frac{q_y}{q}\Big)^2\Big),\end{aligned}\tag{10.17}$$

where we used that $-p_yq_y \ge 0$. Now, by the same argument as in (10.14) we get

$$\begin{aligned}&\Big|\epsilon\Big(\frac{p_{yy}}{p}+\frac{q_{yy}}{q}\Big)-2\epsilon\Big(\Big(\frac{p_y}{p}\Big)^2+\Big(\frac{q_y}{q}\Big)^2\Big)\Big|\\&\le\epsilon\Big(\frac{|p_{yy}|}{p}+\frac{|q_{yy}|}{q}\Big)+2\epsilon\Big(\Big(\frac{p_y}{p}\Big)^2+\Big(\frac{q_y}{q}\Big)^2\Big)\\&\le 6/9.\end{aligned}\tag{10.18}$$

By combining (10.16), (10.17), and (10.18) we conclude that

$$\alpha \ge 1+\epsilon\frac{\varphi_{yy}}{\varphi}-2\epsilon\frac{\varphi_y^2}{\varphi^2}\ge 1-6/9=1/3.\tag{10.19}$$

□

Note that the proof of Lemma 7.1 is partly based on the maximum principle. Since the classical maximum principle requires continuous coefficients and since the coefficient $\alpha$ in (10.9) has a jump discontinuity, we refer to the generalised weak maximum principle for weak solutions, see [9] Theorem 8.1.

By Lemma 10.1 and Lemma 7.1 we get

**Lemma 10.2.** *Let $G$ be the solution of (10.9) with $g$ given by (4.2). Assume $\epsilon \le 1/9$. Then there is a constant $C$ such that*

$$\begin{aligned}\|G\|_{L_1(\Omega)}&\le CL,\\\|DG\|_{L_1(\Omega)}&\le CL^{1/2}\epsilon^{-1/2},\\\|D^2G\|_{L_1(\Omega)}&\le CL^{3/2}\epsilon^{-3/2},\end{aligned}\tag{10.20}$$

*where $L = 1 + |\log\rho|$.*

We now have the following localised a posteriori error estimate:

**Theorem 10.3.** *Assume that $h\epsilon^{-1/2} \le C$ in $\Omega\setminus\Omega_0$, where $C \approx 1$. Let $\varphi$ be given by (10.4). Let $\sigma \ge 2$. Let $u$ be the solution of (10.1) and let $U$ be the corresponding solution* *of* (2.4). *There exist constants* $h^*$ *and* $C$ *such that if* $h_{\min} \le h^*$ *then*

(10.21)

$$\|\varphi(u-U)\|_{L_\infty(\Omega)} \le C\frac{h_{\min}^{\sigma}}{\epsilon^2} + CL^{3/2}\Big\|\varphi \min\Big(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)(f-U_x)\Big\|_{L_\infty(\Omega)} + CL^{3/2}\max_{K\in\mathcal{T}}\Big(\min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)\|\varphi\epsilon h_K^{-1}[\partial_\nu U]\|_{L_\infty(\partial K)}\Big),$$

*where*

(10.22)
$$L = 1 + \sigma|\log h_{\min}|.$$

*Proof.* We first note that $\varphi$ satisfies the condition (4.1) which makes it possible to use Lemma 4.1 and Lemma 6.1. With the same argument, concerning the conditions on $\rho$, as in Section 9 and the argument which gives (9.3) we get by combining Lemma 4.1, with $\omega$ replaced by $\varphi$, (10.7), Lemma 6.1, with $\omega$ replaced by $\varphi$, and, finally Lemma 10.2 the proof of Theorem 10.3. □

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