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# A POSTERIORI ERROR ANALYSIS IN THE MAXIMUM NORM FOR FINITE ELEMENT APPROXIMATIONS OF A TIME-DEPENDENT CONVECTION-DIFFUSION PROBLEM

MATS BOMAN

ABSTRACT. We analyse finite element approximations of a time dependent convection-diffusion problem. We prove an a posteriori error estimate in the maximum norm. For the discretisation we use the Streamline Diffusion method.

## 1. INTRODUCTION

We consider numerical approximations of the solution of a general linear parabolic equation. Let $\Omega$ be a convex, polyhedral, bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$. Let $u$ be the solution of

$$\begin{aligned} u_t + \alpha u + \beta \cdot \nabla u - \epsilon \Delta u &= f \quad \text{in} \quad \Omega \times (0, T), \\ u &= 0 \quad \text{in} \quad \partial\Omega \times (0, T), \\ u &= v_0 \quad \text{in} \quad \Omega \times \{0\}, \end{aligned} \tag{1.1}$$

where $\alpha, \beta$, and $f$ are functions of $x$ and $t$. The initial value $v_0$ is a function of $x$ and $\epsilon$ is a positive number. We assume that $\alpha \geq 0$, $\alpha - \nabla \cdot \beta \geq 0$, and $\|\beta\|_{L_\infty(0,T;L_\infty(\Omega))} \leq C$. In order to discretise this equation we use a Discontinuous Galerkin method, when $\epsilon$ is small we combine this with a Streamline Diffusion method, see [6].

Our main goal is an a posteriori error estimate, in the maximum norm, for the parabolic case, $\epsilon \approx 1$. However our technique yields a result also for small $\epsilon$. Let $U$ be the discrete solution. The main result of this paper takes the following form:

$$\begin{aligned} \max_{1 \leq n \leq N} \|u(t_n) - U_n^-\|_{L_\infty(\Omega)} \leq CL\Big( &\Big\| \frac{k}{\epsilon^{1/2}} R^t \Big\|_{L_\infty(I_n;L_\infty(\Omega))} \\ &+ \Big\| \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R^x \Big\|_{L_\infty(I_n;\infty(\Omega))} \Big), \end{aligned} \tag{1.2}$$

where $L$ is a logarithmic factor. $R^t$, $R^x$ are residuals arising from the time and space discretisation, respectively.

The proof of (1.2) is based on a duality argument, see [3] and [4]. Since we consider estimates in the maximum norm we use an $L_1$-$L_\infty$ duality argument. An important feature of our proof technique is the use of a regularised Green function $G$, see [11], [5] and [2].

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*Key words and phrases.* parabolic, time dependent, convection-diffusion, a posteriori error estimate, residual, maximum norm, pointwise, discontinuous Galerkin method, adaptive, finite element, duality argument, streamline diffusion.

In order to prove the regularity estimates for $G$ we combine the argument in [2], using maximal regularity estimates, see [7], with an energy method as in [1].

In Section 2 we formulate the discretisation method. In Section 3 we state the main result. In Section 4 we prove a lemma which allows the us to replace the true Green function by an approximate one. In Section 5 we do the duality argument. In Section 6 we estimate the residual in terms of computable quantities. In Section 7 we estimate the regularity of the approximate Green function $G$. In Section 8 we conclude the proof of the main result.

We conclude this section by introducing some notation. Let $\Omega$ be a bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$. We use the standard Lebesgue spaces $L_p(\omega)$ for $\omega \subset \Omega$, with the convention that $L_p = L_p(\Omega)$, and the corresponding Sobolev spaces $W_p^k(\omega)$, $W_p^k = W_p^k(\Omega)$, $H^k = W_2^k$ and $H_0^1 = \{u \in H^1 : u|_{\partial\Omega} = 0\}$. Moreover, we write $(u, v)_\omega = \int_\omega uv\,dx$, $(u, v) = (u, v)_\Omega$. Let $X$ be a Banach space with norm $\|\cdot\|_X$ and let $I$ be an interval. We define $\|u\|_{L_p(I;X)} = (\int_I \|u(t)\|_X^p\,dt)^{1/p}$ for $1 \le p < \infty$ and $\|u\|_{L_\infty(I;X)} = \operatorname{ess\,sup}_I \|u(t)\|_X$ for $p = \infty$. We introduce the notation $D^j v(x) = \sqrt{\sum_{|\alpha|=j} |D^\alpha v(x)|^2}$, so that the $W_p^m$ seminorm may be conveniently written $\|D^m v\|_{L_p}$.

## 2. The discretisation method

In this section we formulate a discretisation of (1.1) in space and time using the Discontinuous Galerkin method with Streamline Diffusion, see [6]. For the discretisation in space let $\mathcal{F} = \{\mathcal{T}\}$ be a family of triangulations, where a triangulation $\mathcal{T} = \{K\}$ is a partition of $\Omega$ into open simplices $K$ which are face to face. Let $h_K = \operatorname{diam}(\overline{K})$ and let $\rho_K$ denote the radius of the largest closed ball contained in $\overline{K}$. We assume that $\mathcal{F}$ is nondegenerate, i.e., we assume that there is a constant $c_0$ such that for all triangulations $\mathcal{T} \in \mathcal{F}$ we have

$$\max_{K \in \mathcal{T}} \frac{h_K}{\rho_K} \le c_0. \tag{2.1}$$

To each triangulation $\mathcal{T} \in \mathcal{F}$ we associate a positive, piecewise constant function $h(\mathcal{T})$, defined on $\overline{\Omega}$ by

$$h|_K = h_K, \quad \forall K \in \mathcal{T}. \tag{2.2}$$

To each triangulation $\mathcal{T} \in \mathcal{F}$ we have an associated function space $S = S(\mathcal{T})$, consisting of all continuous functions on $\overline{\Omega}$ which are linear in $x$ on each $K \in T$ and vanish on $\partial\Omega$.

We now consider the discretisation in time. Let $0 = t_0 < \cdots < t_N = T$ be a partition of $[0, T]$ into subintervals $I_n = (t_{n-1}, t_n)$ of lengths $k_n = t_n - t_{n-1}$, and associate with each such time interval a triangulation $\mathcal{T}_n \in \mathcal{F}$ with corresponding function mesh $h_n(\mathcal{T})$ and function space $S_n(\mathcal{T})$. Let $q \ge 0$ be an integer. We define the following function spaces:

$$V_n = V_{q,n} = \{v : v(x,t) = \sum_{j=0}^{q} t^j \varphi_j,\ \varphi_j \in S_n\}, \tag{2.3}$$

$$V = \{\varphi : \varphi|_{I_n} \in V_n,\ n = 1, ..., N\}, \tag{2.4}$$

and the mesh functions $k(t)$ and $h = h(x,t)$ by

$$k|_{I_n} = k_n, \quad h|_{I_n} = h_n. \tag{2.5}$$

We discretise (1.1) as follows: Find $U \in V$ such that, for $n = 1, 2, ..., N$,

$$\begin{aligned} \int_{I_n} \big\{(U_t + \alpha U + \beta \cdot \nabla U, w + \delta(w_t + \beta \cdot \nabla w)) + \epsilon(\nabla U, \nabla w)\big\}\, dt \\ + ([U]_{n-1}, w_{n-1}^+) \\ = \int_{I_n} (f, w + \delta(w_t + \beta \cdot \nabla w))\, dt, \quad \forall w \in V_n, \end{aligned} \tag{2.6}$$

where $U_0^- = v_0$ and

$$[\varphi]_n = \varphi_n^+ - \varphi_n^-, \quad \varphi_n^\pm = \lim_{s \to 0^+} \varphi(t_n \pm s). \tag{2.7}$$

Further, $\delta = \delta(x,t)$ is the Streamline Diffusion coefficient, which is defined by

$$\delta = c_1 \min(0, h - \epsilon), \tag{2.8}$$

where $c_1 \geq 0$. Note that $c_1 = 0$ means that Streamline Diffusion is not used.

## 3. The discretisation error

In order to state our results we need some additional notation. Let

$$\begin{aligned} h_{\min} &= \min_{1 \leq n \leq N} \inf_{x \in \Omega} h_n(x), \\ h_{\max} &= \max_{1 \leq n \leq N} \sup_{x \in \Omega} h_n(x). \end{aligned} \tag{3.1}$$

In order to state our estimate we introduce computable residuals. Let $\chi : \Omega \times [0,T] \to \mathbf{R}$ be any function such that $\chi|_{I_n}(x, \cdot) \in P_q(I_n)$. Let $[\partial_\nu U]$ denote the jump across $\partial K$ in the outward normal derivative. We define the following computable residuals $R_e^t, R_i^t, R_e^x$, and $R_i^x$ by

$$\begin{aligned} R_e^t|_{I_n} &= R_{n,e}^t = k_n^{-1}[U]_{n-1}, \\ R_i^t|_{I_n} &= R_{n,i}^t = |\alpha U + \beta \cdot \nabla U - f - \chi|, \\ R_e^x|_{K \times I_n} &= R_{n,e}^x|_K = \left\| \epsilon h_K^{-1} k_n^{-1} \int_{I_n} |[\partial_\nu U]|\, dt \right\|_{L_\infty(\partial K \setminus \partial\Omega)}, \\ R_i^x|_{I_n} &= R_{n,i}^x = k_n^{-1} \int_{I_n} |U_t + \alpha U + \beta \cdot \nabla U - f|\, dt, \end{aligned} \tag{3.2}$$

where the indices $e$ and $i$ refer to the edge and the interior parts of the residual, $x$ and $t$ refer to space and time discretisation. We now state our main result.

**Theorem 3.1.** *Let $u$ and $U$ be the solutions of* (1.1) *and* (2.6). *Let $\sigma \geq 1$ be arbitrary. We have*

$$\begin{aligned}\max_{1\leq n\leq N} \|u(t_n) - U_n^-\|_{L_\infty(\Omega)} &\leq C_1 h_{\min}^\sigma \\ &+ CC_r L_N \max_{1\leq n\leq N} \Big( \Big\| \frac{k}{\epsilon^{1/2}} R_e^t \Big\|_{L_\infty(I_n;L_\infty)} + \Big\| \frac{R_i^t}{\epsilon^{1/2}} \Big\|_{L_1(I_n;L_\infty)} \\ &+ \Big\| \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R_e^x \Big\|_{L_\infty(I_n;L_\infty)} \\ &+ \Big\| \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R_i^x \Big\|_{L_\infty(I_n;L_\infty)} \Big),\end{aligned} \tag{3.3}$$

*where*

$$C_r = 1 + \min\Big( \min\Big(\frac{1}{\epsilon^{1/2}}, \frac{1}{\alpha_{\min}}\Big)(1+\log t_N), \sqrt{t_N}\Big), \tag{3.4}$$

*and*

$$L_N = \Big(1 + \log\frac{t_N}{k_N}\Big)\Big(\log\frac{t_N}{k_N} + \sigma|\log h_{\min}|\Big)^2. \tag{3.5}$$

**Remark.** $C_1$ depends on $t$ and $\epsilon$. We have that $C_1 \leq C(t)(\epsilon^{-3/2} + \epsilon^{-1}k_1^{-1/2})$, where $C(t)$ depends on the time behaviour of $\alpha$, $\beta$ and $f$, see proof of Lemma 4.2.

**Remark.** It is known that the solution $u$ of (1.1) can be nonsmooth in certain regions such as boundary and characteristic layers when $\epsilon$ is small. In order to make a global error estimate, such as (3.3), small we expect that we have to refine the mesh heavily both in $k$ and $h$. However, it seems possible that a localised version of (3.3) would give a small estimate of the error in certain regions where the solution is smooth even if we do not resolve the singular layers. In order to motivate this we argue as follows.

In the stationary case we have the following result. Let $u$ be the solution of

$$\begin{aligned} u + u_x - \epsilon\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \partial\Omega, \end{aligned} \tag{3.6}$$

and let $U$ be the approximate solution obtained by the Streamline Diffusion method. It was shown in [9] that for special regions $\Omega_0$, oriented along the streamlines and where $\|u\|_{C^2(\Omega_0)} \leq C$, we have $\|u-U\|_{L_\infty(\Omega_0)} \leq Ch_{\max}^{5/4}$ even if the singular layers are not resolved. This result was improved in [10] to $O(h_{\max}^{11/8})$ and in [15] to $O(h_{\max}^2)$ for special triangulations.

In [1] it was shown, essentially, that

$$\|\varphi(u-U)\|_{L_\infty(\Omega)} \leq C \Big\| \varphi \min\Big(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R^x \Big\|_{L_\infty(\Omega)}, \tag{3.7}$$

where $\varphi \approx 1$ in $\Omega_0$ and $\varphi$ decays exponentially with $s/\sqrt{\epsilon}$ where $s$ is the distance to $\Omega_0$. This shows that the a posteriori error estimate can be localised in the same sense as in [9]. However, to make this result useful it is important that it is possible to compute $U$ without resolving the layers and still have a small residual $R^x$ in $\Omega_0$. The a priori estimates indicate that this is possible.

It seems to be possible to repeat this argument in the nonstationary case. In [14] it is shown that in certain regions $Q_0 \subset \Omega \times (0,T)$, oriented along the streamlines and where $u \in C^2(Q_0)$, we have $\|u - U\|_{L_\infty(Q_0)} \leq C h_{\max}^{(12-d)/8}$, where $d$ is the dimension of $\Omega$.

Since the proof of (3.3) is similar to the stationary case, we believe that it is possible to localise (3.3) in a similar way. This conjecture is also supported by the localised a priori analysis in [14] and [13]. We have not carried out the details.

## 4. A split of the error into an a priori and an a posteriori part

The main result of this section is a lemma which will be used to split the estimate of the error $e = u(t_N) - U_N^-$ into two parts, see [11] and [5]. The first part is then estimated through an a priori estimate. The remaining part is of the form $|(u(t_N) - U_N^-, g)|$ where $g$ is an approximate delta function. We now define the function $g$. Let $x_0 \in \Omega$ and let $g = g_{x_0}$ be such that

$$\int_{\mathbf{R}^d} g\,dx = 1; \quad \operatorname{supp} g \subset \mathcal{B}(x_0;\rho); \quad 0 \leq g \leq C\rho^{-d}. \tag{4.1}$$

Here $\mathcal{B}(x_0;\rho)$ denotes the closed ball with center at $x_0$ and with small radius $\rho$ to be chosen. By direct calculation we have

$$\|g\|_{L_p} \leq C\rho^{-d/p'}, \quad p' = p/(p-1). \tag{4.2}$$

We now state the main result of this section.

**Lemma 4.1.** *Let $w \in H_0^1(\Omega) \cap C^\gamma(\Omega)$ for some $\gamma \in (0,1)$. Let $W \in S$ and let $\sigma \geq 1$. Let $x_0$ be such that $\|w - W\|_{L_\infty(\Omega)} = |w(x_0) - W(x_0)|$ and let $g = g_{x_0}$ be given by (4.1). There exist constants $C$ and $h_* > 0$ such that, if $h_{\min} \leq h_*$ and $\rho \leq h_{\min}^{2\sigma/\gamma}$, then*

$$\|w - W\|_{L_\infty(\Omega)} \leq C h_{\min}^\sigma \|w\|_{C^\gamma(\Omega)} + 2|(w - W, g)|. \tag{4.3}$$

*Proof.* Let $e = w - W$ and recall that $|e(x_0)| = \|e\|_{L_\infty(\Omega)}$. Let $\mathcal{B}$ denote the union of all elements $K \in \mathcal{T}$ that intersect $\mathcal{B}(x_0,\rho)$. Extend $e$ to be zero outside $\overline{\Omega}$. By the mean value theorem there is an $x_1 \in \mathcal{B}(x_0,\rho) \cap \overline{\Omega}$ such that $e(x_1) = (e,g)$. We note that $e = w - \Pi w + \Pi e$, where $\Pi : C(\overline{\Omega}) \to S$ is the Lagrange interpolation operator. Thus

$$\begin{aligned} |e(x_0) - e(x_1)| \leq |w(x_0) - w(x_1)| + \rho\|D(\Pi w)\|_{L_\infty(\mathcal{B})} \\ + \rho\|D(\Pi e)\|_{L_\infty(\mathcal{B})}. \end{aligned} \tag{4.4}$$

We have by assumption $\rho \leq h_{\min}^{2\sigma/\gamma}$, so that

$$|w(x_0) - w(x_1)| \leq \rho^\gamma \|w\|_{C^\gamma(\Omega)} \leq h_{\min}^{2\sigma}\|w\|_{C^\gamma(\Omega)}. \tag{4.5}$$

Since $h_{\max}^\gamma \leq C$ and $2\sigma/\gamma - 1 \geq 1$, we have

$$\begin{aligned} \rho\|D(\Pi w)\|_{L_\infty(\mathcal{B})} = \rho\|D(\Pi w - \Pi w(x_0))\|_{L_\infty(\mathcal{B})} \leq C\rho h_{\min}^{-1}\|\Pi w - \Pi w(x_0)\|_{L_\infty(\mathcal{B})} \\ \leq C h_{\max}^\gamma h_{\min}^{2\sigma/\gamma-1}\|w\|_{C^\gamma(\Omega)} \leq C h_{\min}^\sigma \|w\|_{C^\gamma(\Omega)}, \end{aligned} \tag{4.6}$$

where we also used the stability of $\Pi$ in the $L_\infty$-norm and an inverse estimate. Likewise,

$$\rho\|D(\Pi e)\|_{L_\infty(\mathcal{B})} \le C\rho h_{\min}^{-1}\|\Pi e\|_{L_\infty(\Omega)} \le C_1\rho h_{\min}^{-1}\|e\|_{L_\infty(\Omega)} \le C_1 h_{\min}^{2\sigma/\gamma-1}\|e\|_{L_\infty(\Omega)}. \tag{4.7}$$

Since $2\sigma/\gamma \ge 2$, we have

$$C_1 h_{\min}^{2\sigma/\gamma-1} \le \frac{1}{2}, \tag{4.8}$$

for $h_{\min} \le h_*$ sufficiently small so that

$$\|e\|_{L_\infty} \le |e(x_1)| + |e(x_0) - e(x_1)| \le |(e,g)| + Ch_{\min}^{\sigma}\|w\|_{C^\gamma(\Omega)} + \tfrac{1}{2}\|e\|_{L_\infty(\Omega)}, \tag{4.9}$$

which concludes the proof. □

We will apply the previous lemma with $w = u(t_N)$, where $u$ is the solution of (1.1). We have the following lemma which gives a rough estimate of the Hölder-norm of a solution $u$ of (1.1).

**Lemma 4.2.** *Let $\Omega$ be a convex polyhedral domain in $\mathbf{R}^d$, $d = 1, 2, 3$. Let $u$ be a solution of (1.1) with $\alpha - \frac{1}{2}\nabla\cdot\beta \ge 0$. Assume that the initial data $v_0 \in H_0^1$. For any $0 \le \gamma < \frac{1}{2}$ we have*

$$\|u(t)\|_{C^\gamma(\Omega)} \le C(t)(\epsilon^{-3/2} + \epsilon^{-1}t^{-1/2}). \tag{4.10}$$

*Proof.* Multiply (1.1) by $u_t$ and integrate in space to get

$$\|u_t\|_{L_2}^2 + \frac{1}{2}\frac{d}{dt}(\epsilon\|\nabla u\|_{L_2}^2) = (f, u_t) \le \frac{1}{2}\|f\|_{L_2}^2 + \frac{1}{2}\|u_t\|_{L_2}^2, \tag{4.11}$$

integration in time gives

$$\int_0^T \|u_t\|_{L_2}^2\,dt + \epsilon\|\nabla u(T)\|_{L_2}^2 \le \int_0^T \|f\|_{L_2}^2\,dt + \epsilon\|\nabla v_0\|_{L_2}^2. \tag{4.12}$$

We now differentiate (1.1) with respect to $t$ and multiply with $tu_t$, after an integration in space we get

$$\frac{d}{dt}(t\|u_t\|_{L_2}^2) - \|u_t\|_{L_2}^2 + (\alpha_t u, tu_t) + (\alpha u_t, tu_t) + (\beta_t\cdot\nabla u, tu_t) + (\beta\cdot\nabla u_t, tu_t) + \epsilon t\|\nabla u_t\|_{L_2}^2 = (f_t, tu_t). \tag{4.13}$$

Rearranging and using the fact that $(\alpha - \frac{1}{2}\nabla\cdot\beta, tu_t^2) \ge 0$ gives

$$T\|u_t\|_{L_2}^2 + \int_0^T \{(\alpha_t u, tu_t) + (\beta_t\cdot\nabla u, tu_t)\}\,dt + \int_0^T \epsilon t\|\nabla u_t\|_{L_2}^2\,dt = \int_0^T \{(f_t, tu_t) + \|u_t\|_{L_2}^2\}\,dt. \tag{4.14}$$

Further

$$
(4.15)\qquad
\begin{aligned}
\int_0^T |(\alpha_t u, tu_t)|\,dt &\le \|t^{1/2}u_t\|_{L_\infty(0,T;L_2)} T^{1/2} \int_0^T \|\alpha_t u\|_{L_2}\,dt \\
&\le \frac{1}{8}\max_{0\le t\le T} t\|u_t(t)\|_{L_2}^2 + CT\Big(\int_0^T \|\alpha_t u\|_{L_2}\,dt\Big)^2 \\
&\le \frac{1}{8}\max_{0\le t\le T} t\|u_t(t)\|_{L_2}^2 + CT\|\alpha_t\|_{L_1(0,T;L_\infty)}^2 \|u\|_{L_\infty(0,T;L_2)}^2 \\
&\le \frac{1}{8}\max_{0\le t\le T} t\|u_t(t)\|_{L_2}^2 + CT\|\alpha_t\|_{L_1(0,T;L_\infty)}^2 \|f\|_{L_1(0,T;L_2)}^2.
\end{aligned}
$$

Similarly

$$
(4.16)\qquad
\begin{aligned}
\int_0^T |(\beta_t\cdot\nabla u, tu_t)|\,dt &\le \|t^{1/2}u_t\|_{L_\infty(0,T;L_2)} T^{1/2} \int_0^T \|\beta_t\cdot\nabla u\|_{L_2}\,dt \\
&\le \frac{1}{8}\max_{0\le t\le T} t\|u_t(t)\|_{L_2}^2 + CT\Big(\int_0^T \|\beta_t\cdot\nabla u\|_{L_2}\,dt\Big)^2 \\
&\le \frac{1}{8}\max_{0\le t\le T} t\|u_t(t)\|_{L_2}^2 + CT\|\beta_t\|_{L_2(0,T;L_\infty)}^2 \|\nabla u\|_{L_2(0,T;L_2)}^2 \\
&\le \frac{1}{8}\max_{0\le t\le T} t\|u_t(t)\|_{L_2}^2 \\
&\quad + CT\|\beta_t\|_{L_2(0,T;L_\infty)}^2 \epsilon^{-1}(\|f\|_{L_2(0,T;L_2)}^2 + \|\nabla v_0\|_{L_2}^2),
\end{aligned}
$$

where we used (4.12) in the last inequality. Further

$$
(4.17)\qquad \int_0^T |(f_t, tu_t)| \le \frac{1}{8}\max_{0\le t\le T} t\|u_t(t)\|^2 + CT\Big(\int_0^T \|f_t\|_{L_2}\,dt\Big)^2.
$$

By combining (4.14), (4.15), (4.16), (4.17) and (4.12) we arrive at the estimate

$$
(4.18)\qquad t\|u_t(t)\|^2 \le C_1(t)\frac{t}{\sqrt{\epsilon}} + C_2(t),
$$

where $C_1(t)$ and $C_2(t)$ depends on the time behaviour of $\alpha, \beta$ and $f$. We thus have

$$
(4.19)\qquad
\begin{aligned}
\epsilon\|\Delta u(t)\|_{L_2} &\le \|\alpha\|_{L_\infty}\|u(t)\|_{L_2} + \|\beta\|_{L_\infty}\|\nabla u(t)\|_{L_2} + \|u_t(t)\|_{L_2} \\
&\le C(t)(\epsilon^{-1/2} + t^{-1/2}),
\end{aligned}
$$

where we used (4.12) and (4.18). Further

$$
(4.20)\qquad \|D^2 u(t)\|_{L_2} \le C\|\Delta u(t)\|_{L_2} \le C(t)(\epsilon^{-3/2} + \epsilon^{-1}t^{-1/2}),
$$

where we used elliptic regularity. By Sobolev's inequality we get (4.10) for $\gamma < 2 - d/2$, $d = 1, 2, 3$. □

## 5. A duality argument

The equation (2.6) defining the finite element solution $U$ can be written in compact form as

$$\begin{aligned}\sum_{n=1}^{N}\int_{I_n}\big\{(U_t+\alpha U+\beta\cdot\nabla U, w+\delta(w_t+\beta\cdot\nabla w))+\epsilon(\nabla U,\nabla w)\big\}\,dt \\ +\sum_{n=1}^{N-1}([U]_n, w_n^+)+(U_0^+, w_0^+) \\ =(v_0, w_0^+)+\int_0^{t_N}(f, w+\delta(w_t+\beta\cdot\nabla w))\,dt,\quad \forall w\in V,\end{aligned}\tag{5.1}$$

where $U_0^-=v_0$. Let

$$\begin{aligned}\mathcal{V}=\{w : w|_{I_n}\text{ is smooth in time, } w(t)\in H_0^1(\Omega) \\ \text{and } w_n^\pm\in H_0^1(\Omega)\text{ exist}\}.\end{aligned}\tag{5.2}$$

We note that the solution $u$ of (1.1) satisfies

$$\begin{aligned}\sum_{n=1}^{N}\int_{I_n}\{(u_t+\alpha u+\beta\cdot\nabla u, w)+\epsilon(\nabla u,\nabla w)\}\,dt \\ +\sum_{n=1}^{N-1}([u]_n, w_n^+)+(u_0^+, w_0^+) \\ =(v_0, w_0^+)+\int_0^{t_N}(f,w)\,dt,\quad\forall w\in\mathcal{V}.\end{aligned}\tag{5.3}$$

We define the residual $r(U)$ as a linear functional on $\mathcal{V}$,

$$\langle r(U), w\rangle=\sum_{n=1}^{N}\langle r(U), w\rangle_n,\quad\forall w\in\mathcal{V},\tag{5.4}$$

where

$$\begin{aligned}\langle r(U), w\rangle_n=\int_{I_n}\{(U_t+\alpha U+\beta\cdot\nabla U-f, w)+\epsilon(\nabla U,\nabla w)\}\,dt \\ +([U]_{n-1}, w_{n-1}^+).\end{aligned}\tag{5.5}$$

We note that by (5.1)

$$\begin{aligned}\langle r(U), w\rangle_n+\int_{I_n}(U_t+\alpha U+\beta\cdot\nabla U-f,\delta(w_t+\beta\cdot\nabla w))\,dt \\ =0,\quad\forall w\in V_n.\end{aligned}\tag{5.6}$$

We define the bilinear form $B(\cdot,\cdot)$ by

$$B(\xi,\eta) = \sum_{n=1}^{N} \int_{I_n} \{(\xi_t + \alpha\xi + \beta\cdot\nabla\xi, \eta) + \epsilon(\nabla\xi,\nabla\eta)\}\, dt \\ + \sum_{n=1}^{N-1} ([\xi]_n, \eta_n^+) + (\xi_0^+, \eta_0^+), \quad \forall \xi,\eta \in \mathcal{V}. \tag{5.7}$$

Let the error $e$ be defined by $e = u - U$. Combining (5.3), (5.4), (5.5) and (5.7) gives

$$e \in \mathcal{V}; \quad B(e,w) = -\langle r(U), w\rangle, \quad \forall w \in \mathcal{V}. \tag{5.8}$$

Let us consider the following adjoint problem: Given $g$, find $G = G(x,t)$ such that

$$\begin{aligned} -G_t + \alpha G - \nabla\cdot(\beta G) - \epsilon\Delta G = 0, &\quad \text{in } \Omega\times[0,t_N], \\ G = 0, &\quad \text{in } \partial\Omega\times[0,t_N], \\ G = g, &\quad \text{in } \Omega\times\{t_N\}. \end{aligned} \tag{5.9}$$

We multiply (5.9) by $w \in \mathcal{V}$ and integrate to find

$$\begin{aligned} 0 &= \sum_{n=1}^{N} \int_{I_n} \{(w, -G_t + \alpha G - \nabla\cdot(\beta G) - \epsilon\Delta G)\}\, dt \\ &= \sum_{n=1}^{N} \int_{I_n} \{(w_t + \alpha w + \beta\cdot\nabla w, G) + (\epsilon\nabla w, \nabla G)\}\, dt \\ &\quad + \sum_{n=1}^{N} ([w]_{n-1}, G_{n-1}^+) + (w_0^-, G_0^+) - (w_N^-, G_N^+), \end{aligned} \tag{5.10}$$

where the second equality follows from integration by parts in space and time and the continuity of $G$. We thus have, using also $G_N^+ = g$,

$$(w_N^-, g) - (w_0^-, G_0) = B(w,G), \quad \forall v \in \mathcal{V}. \tag{5.11}$$

Since $U_{\epsilon,0}^- = v_0$, we have $e_0^- = 0$, so by choosing $w = e$ in (5.11) we get

$$(e_N^-, g) = B(e,G). \tag{5.12}$$

By (5.8), with $w = G$, we conclude

$$(e_N^-, g) = -\langle r(U), G\rangle. \tag{5.13}$$

## 6. An estimate of the residual

In this section we state and prove an estimate of the residual $\langle r(U), v\rangle$ in terms of the computable residuals defined in (3.2) and derivatives of $v$, the proof is inspired by [8]. The estimate is weighted with powers of $\epsilon$ anticipating the $\epsilon$-dependence in our regularity estimates for $G$.

**Lemma 6.1.** *Let $1 \leq n \leq N$ and let $r(U)$ be defined as in (5.4) and $R_e^t$, $R_i^t$, $R_e^x$, $R_i^x$ be defined as in (3.2). Let $v : \Omega \times I_n \to \mathbf{R}$ be a smooth function such that $v|_{\partial\Omega} = 0$. We have*

(6.1)
$$\begin{aligned}
|\langle r(U), v\rangle_n| \leq C\Big(&\epsilon^{-1/2}\|kR_e^t\|_{L_\infty(I_n;L_\infty(\Omega))} + \epsilon^{-1/2}\|R_i^t\|_{L_1(I_n;L_\infty(\Omega))} \\
&+ c_1\Big\|\min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)R_i^x\Big\|_{L_\infty(I_n;L_\infty(\Omega))}\Big) \\
&\times \epsilon^{1/2}\min\Big(\|v\|_{L_\infty(I_n;L_1(\Omega))}, \|v_t\|_{L_1(I_n;L_1(\Omega))}\Big) \\
&+ C\Big(\Big\|\min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)R_e^x\Big\|_{L_\infty(I_n;L_\infty(\Omega))} + \Big\|\min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)R_i^x\Big\|_{L_\infty(I_n;L_\infty(\Omega))} \\
&+ \Big\|\min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)R_e^t\Big\|_{L_\infty(I_n;L_\infty(\Omega))}\Big) \\
&\times \Big(\epsilon^{1/2}\int_{I_n}\|Dv\|_{L_1(\Omega)}\,dt + \epsilon^{3/2}\sup_{s\in I_n}\Big\|D^2\int_s^{t_n} v\,dt\Big\|_{L_1(\Omega)}\Big).
\end{aligned}$$

We recall that there is an interpolant $\Pi : H_0^1 \to V_n$, such that for $v \in W_1^2$, see [12] and the discussion in [8],

$$\begin{aligned}
\|D^i\Pi v\|_{L_1(K)} &\leq C\|D^i v\|_{L_1(S_K)}, \quad i = 0, 1, \\
\|v - \Pi v\|_{L_1(K)} &\leq Ch_K^i\|D^i v\|_{L_1(S_K)}, \quad i = 1, 2, \qquad (6.2)\\
\|D(v - \Pi v)\|_{L_1(K)} &\leq Ch_K^{i-1}\|D^i v\|_{L_1(S_K)}, \quad i = 1, 2,
\end{aligned}$$

where $S_K = \{K' \in \mathcal{T} : \overline{K'} \cap \overline{K} \neq \emptyset\}$.

Let $\omega \subset \Omega$. For functions $v = v(x,t)$, $x \in \omega$, $t \in I_n$ and $1 \leq p \leq \infty$, we denote

$$\big\|\|v\|_{L_\infty(I_n)}\big\|_{L_p(\omega)} = \Big(\int_\omega\Big(\sup_{t\in I_n}|v(x,t)|\Big)^p dx\Big)^{1/p}. \qquad (6.3)$$

Let $J : L_2(I_n) \to P_q(I_n)$ denote the orthogonal projection. It can be shown that $J$ has the following properties:

$$\begin{aligned}
\big\|\|Jv\|_{L_\infty(I_n)}\big\|_{L_p(\omega)} &\leq \frac{C}{k_n}\sup_{s\in I_n}\Big\|\int_s^{t_n} v\,dt\Big\|_{L_p(\omega)}, \\
\|v - Jv\|_{L_\infty(I_n;L_p(\omega))} &\leq C\min(\|v\|_{L_\infty(I_n;L_p(\omega))}, \|v_t\|_{L_1(I_n;L_p(\omega))}), \qquad (6.4)\\
\|(Jv)_t\|_{L_\infty(I_n;L_p(\omega))} &\leq \frac{C}{k_n}\min(\|v\|_{L_\infty(I_n;L_p(\omega))}, \|v_t\|_{L_1(I_n;L_p(\omega))}), \\
\big\|\|(Jv)_t\|_{L_\infty(I_n)}\big\|_{L_p(\omega)} &\leq \frac{C}{k_n}\min(\|v\|_{L_\infty(I_n;L_p(\omega))}, \|v_t\|_{L_1(I_n;L_p(\omega))}).
\end{aligned}$$

*Proof.* By the orthogonality property (5.6) of $r(U)$ it follows that

$$\begin{aligned}\langle r(U), v\rangle_n &= \langle r(U), v - J\Pi v\rangle_n \\ &\quad - \int_{I_n} \Big(U_t + \alpha U + \beta\cdot\nabla U - f, \delta\big((J\Pi v)_t + \beta\cdot\nabla J\Pi v\big)\Big)\,dt \\ &= \langle r(U), v - Jv\rangle_n + \langle r(U), Jv - J\Pi v\rangle_n \\ &\quad - \int_{I_n} \Big(U_t + \alpha U + \beta\cdot\nabla U - f, \delta\big((J\Pi v)_t + \beta\cdot\nabla J\Pi v\big)\Big)\,dt.\end{aligned} \tag{6.5}$$

We first estimate $\langle r(U), v - Jv\rangle_n$. Since $U_t(x,\cdot)\in P_q(I_n)$ we have

$$\int_{I_n} (U_t, v - Jv)\,dt = 0, \tag{6.6}$$

and similarly

$$\int_{I_n} \epsilon(\nabla U, \nabla(v - Jv))\,dt = 0. \tag{6.7}$$

By Hölder's inequality we get

$$\begin{aligned}|([U]_{n-1}, (v-Jv)^+_{n-1})| &\le \|[U]_{n-1}\|_{L_\infty(\Omega)}\|v - Jv\|_{L_\infty(I_n,L_1(\Omega))} \\ &= \|kR^t_e\|_{L_\infty(I_n;L_\infty(\Omega))}\|v - Jv\|_{L_\infty(I_n,L_1(\Omega))}.\end{aligned} \tag{6.8}$$

Let $\chi : \Omega\times[0,T]\to\mathbf{R}$ be any function such that $\chi|_{I_n}(x,\cdot)\in P_q(I_n)$. Then

$$\begin{aligned}&\Big|\int_{I_n} (\alpha U + \beta\cdot\nabla U - f, v - Jv)\,dt\Big| \\ &\quad\le \|\alpha U + \beta\cdot\nabla U - f - \chi\|_{L_1(I_n;L_\infty(\Omega))}\|v - Jv\|_{L_\infty(I_n;L_1(\Omega))} \\ &\quad= \|R^t_i\|_{L_1(I_n;L_\infty(\Omega))}\|v - Jv\|_{L_\infty(I_n;L_1(\Omega))}.\end{aligned} \tag{6.9}$$

But by (6.4) we have

$$\|v - Jv\|_{L_\infty(I_n;L_1(\Omega))} \le C\min(\|v\|_{L_\infty(I_n;L_1(\Omega))}, \|v_t\|_{L_1(I_n;L_1(\Omega))}), \tag{6.10}$$

so that

$$\begin{aligned}|\langle r(U), v - Jv\rangle_n| &\le C(\epsilon^{-1/2}\|kR^t_e\|_{L_\infty(I_n;L_\infty(\Omega))} + \epsilon^{-1/2}\|R^t_i\|_{L_1(I_n,L_\infty(\Omega))}) \\ &\quad\times \epsilon^{1/2}\min(\|v\|_{L_\infty(I_n;L_1(\Omega))}, \|v_t\|_{L_1(I_n;L_1(\Omega))}).\end{aligned} \tag{6.11}$$

Further,

$$
\begin{aligned}
&\Big|\int_{I_n} (U_t+\alpha U + \beta\cdot\nabla U - f, \delta(J\Pi v)_t)\,dt\Big| \\
&\qquad \le \Big\|\epsilon^{-1/2}\delta k_n^{-1}\int_{I_n} |U_t + \alpha U + \beta\cdot\nabla U - f|\,dt\Big\|_{L_\infty(\Omega)} \\
(6.12)\qquad &\qquad\quad \times \epsilon^{1/2} k_n \big\|\|(J\Pi v)_t\|_{L_\infty(I_n)}\big\|_{L_1(\Omega)} \\
&\qquad \le Cc_1 \Big\|\min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R_i^x\Big\|_{L_\infty(I_n;L_\infty(\Omega))} \\
&\qquad\quad \times \epsilon^{1/2}\min(\|v\|_{L_\infty(I_n,L_1(\Omega))}, \|v_t\|_{L_1(I_n,L_1(\Omega))}),
\end{aligned}
$$

where we used that, by the definition (2.8) of $\delta$, $\delta \le c_1\min(h, \epsilon^{-1}h^2)$, the definition of $R_i^x$, (6.4) and (6.2).

We now estimate $\langle r(U), Jv - J\Pi v\rangle_n$. Let $w = J(I-\Pi)v$. Then

$$
\begin{aligned}
&\epsilon\int_{I_n} (\nabla U, \nabla w) = \epsilon \sum_{K\in\mathcal{T}_n}\int_{I_n} (\nabla U, \nabla w)_K\,dt \\
(6.13)\qquad &\qquad = \epsilon\sum_{K\in\mathcal{T}_n}\int_{I_n}\{-(\Delta U, w)_K + (\partial_\nu U, w)_{\partial K}\}\,dt \\
&\qquad = -\frac{1}{2}\epsilon\sum_{K\in\mathcal{T}_n}\int_{I_n} ([\partial_\nu U], w)_{\partial K\setminus\partial\Omega}\,dt,
\end{aligned}
$$

where $[\partial_\nu U]$ denotes the jump of the outward normal derivative and where we used that $\Delta U|_K = 0$, since $U$ is piecewise linear. Further, by Hölder's inequality in time and the definition of $R_e^x$, we get

$$
\begin{aligned}
&\epsilon\Big|\int_{I_n} ([\partial_\nu U], w)_{\partial K}\,dt\Big| \le \Big|\Big(\min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)\epsilon k_n^{-1}\int_{I_n} |h_K^{-1}[\partial_\nu U]_{n-1}|\,dt, \\
(6.14)\qquad &\qquad\qquad \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big) h_K k_n \|w\|_{L_\infty(I_n)}\Big)_{\partial K}\Big| \\
&\qquad \le \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R_e^x|_K \\
&\qquad\quad \times \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big) h_K k_n \big\|\|w\|_{L_\infty(I_n)}\big\|_{L_1(\partial K)}.
\end{aligned}
$$

By (6.4) we have

$$
\begin{aligned}
&k_n\big\|\|w\|_{L_\infty(I_n)}\big\|_{L_1(\partial K)} = k_n \big\|\|J(I-\Pi)v\|_{L_\infty(I_n)}\big\|_{L_1(\partial K)} \\
(6.15)\qquad &\qquad \le C\sup_{s\in I_n}\Big\|(I-\Pi)\int_t^{t_n} v\,dt\Big\|_{L_1(\partial K)}.
\end{aligned}
$$

By the trace inequality we have

$$\|\chi\|_{L_1(\partial K)} \le C(h_K^{-1}\|\chi\|_{L_1(K)} + \|D\chi\|_{L_1(K)}). \tag{6.16}$$

By combining (6.15), (6.16) and the approximation properties (6.2) of $\Pi_n$, we get

$$\begin{aligned} k_n\big\|h\|w\|_{L_\infty(I_n)}\big\|_{L_1(\partial K)} &\le \int_{I_n} \|hDv\|_{L_1(S_K)}\,dt, \\ k_n\big\|h\|w\|_{L_\infty(I_n)}\big\|_{L_1(\partial K)} &\le \sup_{s\in I_n}\Big\|h^2D^2\int_{I_n} v\,dt\Big\|_{L_1(S_K)}. \end{aligned} \tag{6.17}$$

Let

$$\begin{aligned} \mathcal{A} &= \{K\in\mathcal{T}_n : \min(\frac{\epsilon^{1/2}}{h},\frac{\epsilon^{3/2}}{h^2}) = \frac{\epsilon^{1/2}}{h}\}, \\ \mathcal{B} &= \{K\in\mathcal{T}_n : \min(\frac{\epsilon^{1/2}}{h},\frac{\epsilon^{3/2}}{h^2}) = \frac{\epsilon^{3/2}}{h^2}\}. \end{aligned} \tag{6.18}$$

Hence

$$\begin{aligned} \Big|\epsilon\int_{I_n}(\nabla U,\nabla(Jv-J\Pi v))\,dt\Big| &\le C\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_e^x\Big\|_{L_\infty(\Omega)} \\ &\quad\times\Big(\sum_{K\in\mathcal{A}}\epsilon^{1/2}\int_{I_n}\|Dv\|_{L_1(S_K)}\,dt + \sum_{K\in\mathcal{B}}\epsilon^{3/2}\sup_{s\in I_n}\Big\|D^2\int_s^{t_n} v\,dt\Big\|_{L_1(S_K)}\Big) \\ &\le C\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_e^x\Big\|_{L_\infty(\Omega)} \\ &\quad\times\Big(\epsilon^{1/2}\int_{I_n}\|Dv\|_{L_1(\Omega)}\,dt + \epsilon^{3/2}\sup_{s\in I_n}\Big\|D^2\int_s^{t_n} v\,dt\Big\|_{L_1(\Omega)}\Big). \end{aligned} \tag{6.19}$$

Further, by Hölder's inequality,

$$\begin{aligned} &|([U]_{n-1},(J(v-\Pi v))_{n-1}^+)| \\ &\quad\le\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)k_n^{-1}[U]_{n-1}\Big\|_{L_\infty(\Omega)} \\ &\qquad\times\sum_{K\in\mathcal{T}}\max\Big(\frac{\epsilon^{1/2}}{h},\frac{\epsilon^{3/2}}{h^2}\Big)k_n\big\|\|J(v-\Pi v)\|_{L_\infty(I_n)}\big\|_{L_1(K)} \\ &\quad=\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_e^t\Big\|_{L_\infty(I_n;L_\infty(\Omega))} \\ &\qquad\times\sum_{K\in\mathcal{T}}\max\Big(\frac{\epsilon^{1/2}}{h},\frac{\epsilon^{3/2}}{h^2}\Big)k_n\big\|\|J(v-\Pi v)\|_{L_\infty(I_n)}\big\|_{L_1(K)}, \end{aligned} \tag{6.20}$$

and similarly

$$
\begin{aligned}
&\Big|\int_{I_n} (U_t + \alpha U + \beta\cdot\nabla U - f, J(v - \Pi v))\,dt\Big| \\
&\qquad \le \Big\| \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) k_n^{-1} \int_{I_n} |U_t + \alpha U + \beta\cdot\nabla U - f|\,dt \Big\|_{L_\infty(\Omega)} \\
&\qquad\quad \times \sum_{K\in\mathcal{T}} \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big) k_n \big\| \|J(v-\Pi v)\|_{L_\infty(I_n)} \big\|_{L_1(K)} \\
&\qquad = \Big\| \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R_i^x \Big\|_{L_\infty(I_n;L_\infty(\Omega))} \\
&\qquad\quad \times \sum_{K\in\mathcal{T}} \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big) k_n \big\| \|J(v-\Pi v)\|_{L_\infty(I_n)} \big\|_{L_1(K)}.
\end{aligned}
\tag{6.21}
$$

As above

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big) k_n \big\| \|J(v-\Pi v)\|_{L_\infty(I_n)} \big\|_{L_1(K)} \\
&\quad \le C \sum_{K\in\mathcal{T}} \max\Big(\frac{\epsilon^{1/2}}{h}, \frac{\epsilon^{3/2}}{h^2}\Big) \sup_{s\in I_n} \Big\| (I-\Pi)\int_s^{t_n} v\,dt \Big\|_{L_1(K)} \\
&\quad \le C \sum_{K\in\mathcal{A}} \epsilon^{1/2} \int_{I_n} \|Dv\|_{L_1(S_K)}\,dt + C \sum_{K\in\mathcal{B}} \epsilon^{3/2} \sup_{s\in I_n} \Big\| D^2 \int_s^{t_n} v\,dt \Big\|_{L_1(S_K)} \\
&\quad \le C\Big( \epsilon^{1/2} \int_{I_n} \|Dv\|_{L_1(\Omega)}\,dt + \epsilon^{3/2} \sup_{s\in I_n} \Big\| D^2 \int_s^{t_n} v\,dt \Big\|_{L_1(\Omega)} \Big).
\end{aligned}
\tag{6.22}
$$

Finally,

$$
\begin{aligned}
&\Big|\int_{I_n} (U_t + \alpha U + \beta\cdot\nabla U - f, \delta\beta\cdot\nabla J\Pi v)\,dt\Big| \\
&\qquad \le C \Big\| \delta\epsilon^{-1/2} k_n^{-1} \int_{I_n} |U_t + \alpha U + \beta\cdot\nabla U - f|\,dt \Big\|_{L_\infty(\Omega)} \\
&\qquad\quad \times \epsilon^{1/2} k_n \big\| \|J\nabla\Pi v\|_{L_\infty(I_n)} \big\|_{L_1(\Omega)} \\
&\qquad \le C \Big\| \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R_i^x \Big\|_{L_\infty(I_n;L_\infty(\Omega))} \epsilon^{1/2} \sup_{s\in I_n} \Big\| \int_s^{t_n} \nabla\Pi v\,dt \Big\|_{L_1(\Omega)} \\
&\qquad \le C \Big\| \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R_i^x \Big\|_{L_\infty(I_n;L_\infty(\Omega))} \epsilon^{1/2} \int_{I_n} \|Dv\|_{L_1(\Omega)}\,dt,
\end{aligned}
\tag{6.23}
$$

where we again used $\delta \le c_1 \min(h, \epsilon^{-1}h^2)$ and the inequality $\|\nabla\Pi v\|_{L_1(K)} \le C\|Dv\|_{L_1(S_K)}$ from (6.2). This concludes the proof. □

## 7. Regularity estimate of the solution $G$ of the adjoint problem

In this section we prove regularity estimate for the solution $G$ of (5.9) in the case when the data $g$ is a regularised $\delta$-function. Our estimates are based on the following lemma. Let $\tilde{f} \in L_p(0,T;L_p)$ and let $g \in L_p$. Let $v$ be the solution, in the $L_p(0,T;L_p)$ sense, of

$$
\begin{aligned}
v_t - \epsilon \Delta v &= \tilde{f}, \quad \text{in } \Omega \times [0,T], \\
v &= 0, \quad \text{in } \partial\Omega \times [0,T], \\
v &= g, \quad \text{in } \Omega \times \{0\}.
\end{aligned} \tag{7.1}
$$

We now state the estimate of the solution $v$ of (7.1), the proof of this lemma can be found in [2].

**Lemma 7.1.** *Let $1 < p \le 2$, let $p' = p/(p-1)$ and let $v$ be the solution of (7.1). Then, for $0 < \tau < T$,*

$$
\int_\tau^T \|v_t\|_{L_1}\, dt \le Cp' \Big( \log \frac{T}{\tau} \|g\|_{L_p} + T^{1/p'} \|\tilde{f}\|_{L_p(0,T;L_p)} \Big), \tag{7.2}
$$

$$
\int_\tau^T \epsilon \|D^2 v\|_{L_1}\, dt \le Cp'^2 \Big( \log \frac{T}{\tau} \|g\|_{L_p} + T^{1/p'} \|\tilde{f}\|_{L_p(0,T;L_p)} \Big), \tag{7.3}
$$

$$
\sup_{s \in [0,\tau]} \epsilon \Big\| D^2 \int_s^\tau v\, dt \Big\|_{L_1} \le Cp' \Big( \|g\|_{L_p} + p' \tau^{1/p'} \|\tilde{f}\|_{L_p(0,T;L_p)} \Big). \tag{7.4}
$$

We then apply these estimates to the solution $G$ of the adjoint problem (5.9) with $\tilde{f} = -\alpha G + \nabla \cdot (\beta G)$ and with the data $g$ chosen to be a regularised $\delta$-function. A key step in this part is to establish an estimate of $\|(\alpha - \nabla \cdot \beta) G - \beta \cdot \nabla G\|_{L_p(0,t_N;L_p)}$. This is done in the following lemma.

**Lemma 7.2.** *Let $G$ be the solution of (5.9) with data $g$ as in (4.1). Assume that $\alpha \ge 0$, and $\alpha - \nabla \cdot \beta \ge 0$. Let $1 \le p \le \infty$ and let $p' = \frac{p}{p-1}$. Then*

$$
\begin{aligned}
\|G\|_{L_\infty(0,T;L_1)} &\le 1, \\
\|(\alpha - \nabla \cdot \beta) G\|_{L_p(0,T;L_p)} &\le C C_r \epsilon^{-1/2} \rho^{-d/p'}, \\
\|DG\|_{L_1(0,T;L_1)} &\le C C_r \epsilon^{-1/2}, \\
\|\beta \cdot \nabla G\|_{L_p(0,T;L_p)} &\le C C_r \epsilon^{-1/2} \rho^{-d/p'},
\end{aligned} \tag{7.5}
$$

*where*

$$
C_r = 1 + \min\Big( \min\Big( \frac{1}{\sqrt{\epsilon}}, \frac{1}{\alpha_{\min}} \Big)(1 + |\log T|), (1 + |\log \rho|)\sqrt{T} \Big). \tag{7.6}
$$

*Proof.* It is convenient to use the change of variables $t \to t_N - t$. Note that this also changes the mesh so that $I_N$ becomes $I_1$ and $k_N$ becomes $k_1$. In these variables the adjoint problem (5.9) takes the form:

$$\begin{aligned} G_t + \alpha G - \nabla \cdot (\beta G) - \epsilon \Delta G = 0, &\quad \text{in } \Omega \times [0, t_N], \\ G = 0, &\quad \text{in } \partial\Omega \times [0, t_N], \\ G = g, &\quad \text{in } \Omega \times \{0\}. \end{aligned} \tag{7.7}$$

We start by estimating $\|(\alpha - \nabla \cdot \beta)G\|_{L_2(0,T;L_2)}$ and $\|\beta \cdot \nabla G\|_{L_2(0,T;L_2)}$. Multiply (7.7) by $G$ and integrate over $\Omega$. We get

$$\tfrac{1}{2}\frac{d}{dt}\|G\|_{L_2}^2 + (\alpha G, G) - ((\nabla \cdot \beta)G, G) - (\beta \cdot \nabla G, G) + \epsilon \|\nabla G\|_{L_2}^2 = 0, \tag{7.8}$$

$$(\beta \cdot \nabla G, G) = -\frac{1}{2}((\nabla \cdot \beta)G, G). \tag{7.9}$$

Therefore we get from (7.8) using (7.9) and integrating in time

$$\frac{1}{2}\|G(T)\|_{L_2}^2 + \int_0^T \|(\alpha - \tfrac{1}{2}\nabla \cdot \beta)G^2\|_{L_1}\, dt + \int_0^T \epsilon \|\nabla G\|_{L_2}^2\, dt = \frac{1}{2}\|g\|_{L_2}^2, \tag{7.10}$$

where we used that by assumption $\alpha - \frac{1}{2}\nabla \cdot \beta = \frac{1}{2}\alpha + \frac{1}{2}(\alpha - \nabla \cdot \beta) \geq 0$. (7.10) gives an estimate for $\|\nabla G\|_{L_2(0,T;L_2)}$, and by using the inequality $\|G\|_{L_2} \leq C\|\nabla G\|_{L_2}$, we also get an estimate for $\|G\|_{L_2(0,T;L_2)}$. We thus have the following estimates

$$\begin{aligned} \|G\|_{L_2(0,T;L_2)} &\leq C\epsilon^{-1/2}\|g\|_{L_2} = C\epsilon^{-1/2}\rho^{-d/2}, \\ \|\nabla G\|_{L_2(0,T;L_2)} &\leq \epsilon^{-1/2}\|g\|_{L_2} = C\epsilon^{-1/2}\rho^{-d/2}, \end{aligned} \tag{7.11}$$

where we used (4.2). Moreover by using (7.11) we get

$$\begin{aligned} \|(\alpha - \nabla \cdot \beta)G\|_{L_2(0,T;L_2)} &\leq C\epsilon^{-1/2}\|\alpha - \nabla \cdot \beta\|_{L_\infty(0,T;L_\infty)}\|g\|_{L_2}, \\ \|\beta \cdot \nabla G\|_{L_2(0,T;L_2)} &\leq \epsilon^{-1/2}\|\beta\|_{L_\infty(0,T;L_\infty)}\|g\|_{L_2}. \end{aligned} \tag{7.12}$$

We now estimate $\|(\alpha - \nabla \cdot \beta)G\|_{L_1(0,T;L_1)}$ and $\|\beta \cdot \nabla G\|_{L_1(0,T;L_1)}$. We start by estimating $\|G\|_{L_\infty(0,T;L_1)}$ and $\|G\|_{L_1(0,T;L_1)}$. We multiply (7.7) with $\frac{G}{\sqrt{\xi + G^2}}$, $\xi > 0$, which is a regularisation of $\operatorname{sign}(G)$, and integrate in space and time.

Since $\nabla \frac{G}{\sqrt{\xi+G^2}} = \frac{\xi}{\sqrt{\xi+G^2}^3}\nabla G$, we get

$$\begin{aligned} \int_0^T \Big\{ \Big(G_t, \frac{G}{\sqrt{\xi + G^2}}\Big) + \Big(\alpha G, \frac{G}{\sqrt{\xi + G^2}}\Big) + \Big(\beta G, \frac{\xi \nabla G}{\sqrt{\xi + G^2}^3}\Big) \\ + \Big(\epsilon \nabla G, \frac{\xi \nabla G}{\sqrt{\xi + G^2}^3}\Big) \Big\}\, dt = 0. \end{aligned} \tag{7.13}$$

Further, let $E = \{x \in \Omega : G(x) > 0\}$,

$$\begin{aligned}\int_0^T \Big|\Big(\beta G, \frac{\xi \nabla G}{\sqrt{\xi + G^2}^3}\Big)\, dt\Big| &\leq \int_0^T \Big( |\beta \cdot \nabla G| \frac{\xi^{1/2}}{\sqrt{\xi + G^2}}\Big)\, dt \\ &\leq \|\beta\|_{L_\infty(0,T;L_\infty)} \|\nabla G\|_{L_2(0,T;L_2)} \Big\| \frac{\xi^{1/2}}{\sqrt{\xi + G^2}} \Big\|_{L_2(0,T;L_2(E))} \\ &\leq C \Big\| \frac{\xi^{1/2}}{\sqrt{\xi + G^2}} \Big\|_{L_2(0,T;L_2(E))},\end{aligned} \tag{7.14}$$

where we in the last inequality used (7.11). We conclude by dominated convergence that

$$\Big\| \frac{\xi^{1/2}}{\sqrt{\xi + G^2}} \Big\|_{L_2(0,T;L_2(E))} \to 0 \text{ as } \xi \to 0. \tag{7.15}$$

Hence

$$\int_0^T \frac{d}{dt} \|G\|_{L_1}\, dt + \int_0^T \|\alpha G\|_{L_1}\, dt \leq 0, \tag{7.16}$$

where we used that $(G_t, \mathrm{sign}(G)) = (\frac{d}{dt}|G|, 1) = \frac{d}{dt}\|G\|_{L_1}$ and the assumption $\alpha \geq 0$. We therefore conclude

$$\|G(T)\|_{L_1} + \int_0^T \|\alpha G\|_{L_1}\, dt \leq \|g\|_{L_1} = 1, \tag{7.17}$$

which gives the estimate for $\|G\|_{L_\infty(0,T;L_1)}$ and $\|\alpha G\|_{L_1(0,T;L_1)}$. We also note that if we don't integrate by parts in the term $\int_0^T -(\nabla \cdot (\beta G), \frac{G}{G^2 + \xi})\, dt$ we get the estimate

$$\|G(T)\|_{L_1} + \int_0^T \|(\alpha - \nabla \cdot \beta) G\|_{L_1}\, dt \leq 1 + \|\beta \cdot \nabla G\|_{L_1}. \tag{7.18}$$

We also need an estimate of $t\|G(T)\|_{L_\infty(0,T;L_1)}$. Multiply (7.7) by $\frac{tG}{\sqrt{\xi + G^2}}$. Note that $t\frac{d}{dt}\|G(t)\|_{L_1} = \frac{d}{dt}\|tG(t)\|_{L_1} - \|G(t)\|_{L_1}$. By the same argument as above we get

$$T\|G(T)\|_{L_1} + \int_0^T t\|\alpha G\|_{L_1}\, dt \leq \int_0^T \|G\|_{L_1}. \tag{7.19}$$

We now estimate $\|\beta \cdot \nabla G\|_{L_1(0,T;L_1)}$. By assumption $\alpha - \nabla \cdot \beta \geq 0$, therefore $G > 0$ in $\Omega$ by the strong maximum principle, so that $\log(1 + G)$ is well defined in $\Omega$ and 0 on $\partial\Omega$. Multiply (7.7) by $\log(1 + G)$ and integrate in space and time. Since $\nabla \log(1 + G) = \frac{\nabla G}{1 + G}$, we get

$$
\begin{aligned}
\int_0^T (G_t, \log(1+G))\,dt + \int_0^T (\alpha G, \log(1+G))\,dt& \\
+ \int_0^T \Big(G\beta, \frac{\nabla G}{1+G})\Big)\,dt& \\
+ \int_0^T \Big(\epsilon\nabla G, \frac{\nabla G}{1+G}\Big)\,dt = 0,&
\end{aligned} \tag{7.20}
$$

where, by integration by parts in $t$,

(7.21)

$$
\begin{aligned}
\int_0^T (G_t, \log(1+G))\,dt &= (G(T), \log(1+G(T))) - (g, \log(1+g)) - \int_0^T \Big(G, \frac{G_t}{1+G}\Big)\,dt \\
&= (G(T), \log(1+G(T))) - (g, \log(1+g)) - \int_0^T \Big(G+1, \frac{G_t}{1+G}\Big)\,dt \\
&\quad + \int_0^T \Big(1, \frac{G_t}{1+G}\Big)\,dt \\
&= (G(T), \log(1+G(T))) - (g, \log(1+g)) \\
&\quad - \int_0^T (1, G_t)\,dt + \int_0^T (1, \log(1+G)_t)\,dt \\
&= (G(T), \log(1+G(T))) - (g, \log(1+g)) - \|G(T)\|_{L_1} + \|g\|_{L_1} \\
&\quad + \|\log(1+G(T))\|_{L_1} - \|\log(1+g)\|_{L_1}.
\end{aligned}
$$

Further

$$
\begin{aligned}
\int_\Omega G\beta\cdot\frac{\nabla G}{1+G}\,dx &= \int_\Omega (G+1)\beta\cdot\frac{\nabla G}{1+G}\,dx - \int_\Omega \beta\cdot\frac{\nabla G}{1+G}\,dx \\
&= \int_\Omega \beta\cdot\nabla G\,dx - \int_\Omega \beta\cdot\nabla\log(1+G)\,dx \\
&= \int_{\partial\Omega} G\beta\cdot n\,ds - \int_\Omega (\nabla\cdot\beta)G\,dx \\
&\quad - \int_{\partial\Omega} \log(1+G)\beta\cdot n\,ds + \int_\Omega (\nabla\cdot\beta)\log(1+G)\,dx \\
&= -\int_\Omega (\nabla\cdot\beta)G\,dx + \int_\Omega (\nabla\cdot\beta)\log(1+G)\,dx.
\end{aligned} \tag{7.22}
$$

We thus have, noting that $(G(T), \log(1+G(T))) \geq 0$ in $\Omega$,

$$
\begin{aligned}
(G(T), \log(1+G)) &+ \int_0^T (\alpha G, \log(1+G))\,dt + \int_0^T \left\| \epsilon \frac{|\nabla G|^2}{1+G} \right\|_{L_1} dt \\
&= \int_0^T ((\nabla\cdot\beta), G - \log(1+G))\,dt \\
&\quad + (g, \log(1+g)) + \|G(T)\|_{L_1} - \|g\|_{L_1} \\
&\quad - \|\log(1+G(T))\|_{L_1} + \|\log(1+g)\|_{L_1} \\
&\leq \int_0^T (\nabla\cdot\beta, G - \log(1+G))\,dt + \|g\|_{L_1}(1+\log(1+\|g\|_{L_\infty})) \\
&\leq \int_0^T (\alpha, G - \log(1+G))\,dt + C(1+|\log\rho|) \\
&\leq \int_0^T (\alpha, G)\,dt + C(1+|\log\rho|) \\
&= \int_0^T \|\alpha G\|_{L_1} + C(1+|\log\rho|)\,dt \\
&\leq C(1+|\log\rho|),
\end{aligned} \tag{7.23}
$$

where we used $\alpha \geq 0$, $G \geq \log(1+G) \geq 0$, $\alpha \geq \nabla\cdot\beta$ and (7.17). We now multiply (7.7) with $t\log(1+G)$. We first note that

$$
\begin{aligned}
\int_0^T (G_t, t\log(1+G))\,dt = (TG(T), \log(1+G(T))) &- \int_0^T (G, \log(1+G))\,dt \\
&- \int_0^T \left( tG, \frac{G_t}{1+G} \right) dt,
\end{aligned} \tag{7.24}
$$

where

$$
\begin{aligned}
\int_0^T \left( tG, \frac{G_t}{1+G} \right) &= \int_0^T \left( t(G+1), \frac{G_t}{1+G} \right) dt - \int_0^T \left( t, \frac{G_t}{1+G} \right) dt \\
&= \int_0^T (t, G_t)\,dt - \int_0^T (t, \log(1+G)_t)\,dt \\
&= (T, G(T)) - \int_0^T (1, G)\,dt - (T, \log(1+G)) \\
&\quad + \int_0^T (1, \log(1+G))\,dt.
\end{aligned} \tag{7.25}
$$

The other terms can be treated as above and we get

$$
\begin{aligned}
(TG(T), \log(1+G(T))) &+ \int_0^T t\|\alpha G\log(1+G)\|_{L_1}\,dt + \int_0^T t\Big\|\epsilon\frac{|\nabla G|^2}{1+G}\Big\|_{L_1}\,dt \\
&= \int_0^T t(\nabla\cdot\beta, G-\log(1+G))\,dt \\
&\quad + \int_0^T (G,\log(1+G))\,dt + T\|G(T)\|_{L_1} - \int_0^T \|G\|_{L_1}\,dt \\
&\quad - T\|\log(1+G)\|_{L_1} + \int_0^T \|\log(1+G)\|_{L_1}\,dt \\
&\le \int_0^T t\|\alpha G\|_{L_1}\,dt + \int_0^T \|G\|_{L_1}\,dt \\
&\le 2\int_0^T \|G\|_{L_1}\,dt,
\end{aligned}
\tag{7.26}
$$

where we used (7.19) in the last steps. We thus have the inequalities

$$
\begin{aligned}
\int_0^T \Big\|\epsilon\frac{|\nabla G|^2}{1+G}\Big\|_{L_1}\,dt &\le C(1+|\log\rho|), \\
\int_0^T t\Big\|\epsilon\frac{|\nabla G|^2}{1+G}\Big\|_{L_1}\,dt &\le 2\int_0^T \|G\|_{L_1}\,dt.
\end{aligned}
\tag{7.27}
$$

We now get

$$
\begin{aligned}
\int_0^T \|\sqrt{\epsilon}\nabla G\|_{L_1}\,dt &= \int_0^T \Big\|\frac{\sqrt{\epsilon}\nabla G}{\sqrt{1+G}}\sqrt{1+G}\Big\|_{L_1} \\
&\le \Big(\int_0^T \Big\|\frac{\epsilon|\nabla G|^2}{1+G}\Big\|_{L_1}\,dt\Big)^{1/2}\Big(\int_0^T \|1+G\|_{L_1}\,dt\Big)^{1/2} \\
&\le (C(1+|\log\rho|))^{1/2}((|\Omega|+\|g\|_{L_1})T)^{1/2},
\end{aligned}
\tag{7.28}
$$

where we used (7.27) and that by (7.17) we have $\|G\|_{L_\infty(0,T;L_1)} \le 1$. Further

$$
\begin{aligned}
\int_1^T \|\sqrt{\epsilon}\nabla G\|_{L_1}\,dt &= \int_1^T \Big\|\frac{\sqrt{\epsilon}\nabla G}{\sqrt{1+G}}\sqrt{1+G}\Big\|_{L_1} \\
&\le \int_1^T \Big\|\frac{\epsilon|\nabla G|^2}{1+G}\Big\|_{L_1}^{1/2}\|1+G\|_{L_1}^{1/2}\,dt \\
&= \int_1^T \frac{\epsilon^{1/4}}{C}t^{1/2}\Big\|\frac{\epsilon|\nabla G|^2}{(1+G)}\Big\|_{L_1}^{1/2}\frac{C}{\epsilon^{1/4}t^{1/2}}\|(1+G)\|_{L_1}^{1/2}\,dt \\
&\le \frac{1}{2}\int_1^T \Big(\frac{\epsilon^{1/2}}{C}t\Big\|\frac{\epsilon|\nabla G|^2}{1+G}\Big\|_{L_1} + \frac{C}{t\epsilon^{1/2}}\|(1+G)\|_{L_1}\Big)\,dt,
\end{aligned}
\tag{7.29}
$$

where C is a constant to be determined. We now have, by (7.26),

$$\int_1^T \frac{\sqrt{\epsilon}}{2C}\Big\|\frac{t\epsilon|\nabla G|^2}{1+G}\Big\|_{L_1} \leq \frac{\sqrt{\epsilon}}{C}\int_1^T \|G\|_{L_1}\,dt \leq \frac{\sqrt{\epsilon}}{2}\int_1^T \|\nabla G\|_{L_1}\,dt, \tag{7.30}$$

and

$$\int_1^T \frac{1}{t}\|1+G\|_{L_1(\Omega)}\,dt \leq 2\int_1^T \frac{1}{t}\,dt = 2\log T, \tag{7.31}$$

where we used that by (7.17) we have $\|G\|_{L_\infty(0,T;L_1)} \leq \|g\|_{L_1}$. We therefore have the estimate

$$\int_1^T \|\sqrt{\epsilon}\nabla G\|_{L_1}\,dt \leq \int_1^T \frac{1}{2}\|\sqrt{\epsilon}\nabla G\|_{L_1}\,dt + \frac{C}{\sqrt{\epsilon}}\log T, \tag{7.32}$$

and hence

$$\int_1^T \|\sqrt{\epsilon}\nabla G\|_{L_1}\,dt \leq \frac{C}{\sqrt{\epsilon}}\log T. \tag{7.33}$$

By a similar argument as above we get the estimate

$$\begin{aligned}\int_1^T \|\sqrt{\epsilon}\nabla G\|_{L_1}\,dt &\leq \int_1^T \|G\|_{L_1}\,dt + \int_1^T t^{-1}\|1+G\|_{L_1}\,dt \\ &\leq {\alpha_{\text{min}}}^{-1}\int_1^T \|\alpha G\|_{L_1}\,dt + C\log T(1+\|g\|_{L_1}) \\ &\leq {\alpha_{\text{min}}}^{-1} + C\log T.\end{aligned} \tag{7.34}$$

By combining (7.28), (7.33) and, (7.34) we get

$$\begin{aligned}\|\nabla G\|_{L_1(0,T;L_1)} &\leq \frac{C}{\sqrt{\epsilon}}\min\Big(\min\Big(\frac{1}{\sqrt{\epsilon}},\frac{1}{\alpha_{\text{min}}}\Big)(1+|\log T|),(1+|\log\rho|)\sqrt{T}\Big)) \\ &\leq CC_r\epsilon^{-1/2}.\end{aligned} \tag{7.35}$$

Further

$$\begin{aligned}\|\beta\cdot\nabla G\|_{L_1(0,T;L_1)} &\leq C\|\beta\|_{L_\infty(0,T;L_\infty)}\|\nabla G\|_{L_1(0,T;L_1)} \\ &\leq CC_r\epsilon^{-1/2},\end{aligned} \tag{7.36}$$

and by (7.18)

$$\begin{aligned}\|(\alpha-\nabla\cdot\beta)G\|_{L_1(0,T;L_1)} &\leq 1+\|\beta\cdot\nabla G\|_{L_1(0,T;L_1)} \\ &\leq CC_r\epsilon^{-1/2}.\end{aligned} \tag{7.37}$$

We now interpolate between (7.36), (7.37) and (7.11) in order to get an estimate for $\|(\alpha-\nabla\cdot\beta)G\|_{L_p(0,T;L_p)}$ and $\|\beta\cdot\nabla G\|_{L_p(0,T;L_p)}$.

$$\begin{aligned}\|(\alpha-\nabla\cdot\beta)G\|_{L_p(0,T;L_p)} &\leq \|(\alpha-\nabla\cdot\beta)G\|^{1-2/p'}_{L_1(0,T;L_1)}\|(\alpha-\nabla\cdot\beta)G\|^{2/p'}_{L_2(0,T;L_2)} \\ &\leq CC_r\epsilon^{-1/2}\rho^{-d/p'}.\end{aligned} \tag{7.38}$$

Further

$$\|\beta\cdot\nabla G\|_{L_p(0,T;L_p)} \le \|\beta\cdot\nabla G\|^{1-2/p'}_{L_1(0,T;L_1)}\|\beta\cdot\nabla G\|^{2/p'}_{L_2(0,T;L_2)} \le CC_r\epsilon^{-1/2}\rho^{-d/p'}. \tag{7.39}$$

□

We are now able to prove the main result of this section

**Lemma 7.3.** *Let $G$ be the solution of* (5.9)*, with $g$ as in* (4.1)*. Assume that $\alpha \ge 0$ and $\alpha - \nabla\cot\beta \ge 0$. There is a constant $C$ such that, if $\rho$ is sufficiently small, then*

$$\begin{aligned}
\|G\|_{L_\infty(0,t_N;L_1)} &\le 1,\\
\|G_t\|_{L_1(0,t_{N-1};L_1)} &\le CC_r\epsilon^{-1/2}\Big(1+\log\frac{t_N}{k_N}\Big)\log\frac{t_N}{k_N\rho},\\
\|DG\|_{L_1(0,t_N;L_1)} &\le CC_r\epsilon^{-1/2},\\
\|D^2G\|_{L_1(0,t_{N-1};L_1)} &\le CC_r\epsilon^{-3/2}\Big(1+\log\frac{t_N}{k_N}\Big)\Big(\log\frac{t_N}{k_N\rho}\Big)^2,\\
\sup_{s\in I_n}\Big\|D^2\int_s^{t_N} G(t)\,dt\Big\|_{L_1} &\le CC_r\epsilon^{-3/2}\Big(\log\frac{t_N}{k_N\rho}\Big)^2,
\end{aligned} \tag{7.40}$$

*where*

$$C_r = 1+\min\Big(\min\Big(\frac{1}{\sqrt{\epsilon}},\frac{1}{\alpha_{\min}}\Big)(1+|\log T|),\sqrt{T}\Big). \tag{7.41}$$

*Proof.* From Lemma 7.1 with $\tilde f = -(\alpha-\beta\cdot\nabla)G + \beta\cdot\nabla G$ we obtain,

$$\begin{aligned}
\int_{k_1}^{t_N}\|G_t\|_{L_1}\,dt &\le Cp'\Big(\log\frac{t_N}{k_1}\|g\|_{L_p} + t_N^{1/p'}\|\tilde f\|_{L_p(0,t_N;L_p)}\Big)\\
&\le CC_r\epsilon^{-1/2}p'\Big(\log\frac{t_N}{k_1}+\Big(\frac{t_N}{k_1}\Big)^{1/p'}\Big)\rho^{-d/p'}.
\end{aligned} \tag{7.42}$$

We conclude from (7.42) with $p' = \log\frac{t_N}{k_1\rho} > 1$ that

$$\begin{aligned}
\int_{k_1}^{t_N}\|G_t\|_{L_1}\,dt &\le CC_r\epsilon^{-1/2}p'\Big(\log\frac{t_N}{k_1}+\Big(\frac{t_N}{k_1}\Big)^{1/p'}\Big)\rho^{-d/p'}\\
&\le CC_r\epsilon^{-1/2}\Big(\Big(\frac{t_N}{k_1}\Big)^{-d/p'}\log\frac{t_N}{k_1}+\Big(\frac{t_N}{k_1}\Big)^{(1-d)/p'}\Big)p'\Big(\frac{t_N}{k_1\rho}\Big)^{d/p'}\\
&\le CC_r\epsilon^{-1/2}\Big(\log\frac{t_N}{k_1}+1\Big)\log\frac{t_N}{k_1\rho}.
\end{aligned} \tag{7.43}$$

Similarly we obtain

$$\int_{k_1}^{t_N}\|D^2G\|_{L_1}\,dt \le CC_r\epsilon^{-1/2}\Big(\log\frac{t_N}{k_1}+1\Big)\Big(\log\frac{t_N}{k_1\rho}\Big)^2, \tag{7.44}$$

and

$$\sup_{s\in[0,k_1]}\Big\|D^2\int_s^{k_1} G\,dt\Big\|_{L_1} \le CC_r\epsilon^{-1/2}\Big(\log\frac{t_N}{k_1\rho}\Big)^2. \tag{7.45}$$

□

## 8. Conclusion of proof of Theorem 3.1

In this section we combine the result above to get a proof of Theorem 3.1.

*Proof.* Let $n\in[1,N]$. We apply Lemma 4.1 with $w=u(t_n)$, $W=U_n^-$ , Lemma 4.2 and (5.13), to get

$$\begin{aligned}\|u(t_n)-U_n^-\|_{L_\infty(\Omega)} &\le h_{\min}^\sigma\|u(t_n)\|_{C^\gamma(\Omega)}+2|(e_n^-,g)|\\ &\le C(t_n)h_{\min}^\sigma+2|\langle r(U),G\rangle|,\end{aligned} \tag{8.1}$$

where, by combining (5.4), Lemma 6.1 and Lemma 7.3, we get

$$\begin{aligned}
|\langle r(U),G\rangle| &\le \sum_{j=1}^n|\langle r(U),G\rangle_j|\\
&\le C\sum_{j=1}^n\Big\{\Big(\Big\|\frac{k}{\epsilon^{1/2}}R_e^t\|_{L_\infty(I_j;L_\infty)}+\|\frac{R_i^t}{\epsilon^{1/2}}\Big\|_{L_1(I_j;L_\infty)}\Big)\\
&\qquad\times\min(\|G\|_{L_\infty(I_j;L_\infty)},\|G_t\|_{L_1(I_j;L_\infty)})\\
&\quad+\Big(\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_e^x\Big\|_{L_\infty(I_j;L_\infty)}+\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_i^x\Big\|_{L_\infty(I_j;L_\infty)}\\
&\qquad+\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_e^t\Big\|_{L_\infty(I_j;L_\infty)}\Big)\\
&\qquad\times\Big(\epsilon^{1/2}\int_{I_j}\|DG\|_{L_1}\,dt+\epsilon^{3/2}\sup_{s\in I_j}\|D^2\int_s^{t_j}G\,dt\|_{L_1}\Big)\Big\}\\
&\le C\max_{1\le j\le n}\Big(\Big\|\frac{k}{\epsilon^{1/2}}R_e^t\Big\|_{L_\infty(I_j;L_\infty)}+\Big\|\frac{R_i^t}{\epsilon^{1/2}}\Big\|_{L_1(I_j;L_\infty)}\\
&\quad+\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_e^x\Big\|_{L_\infty(I_j;L_\infty)}+\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_i^x\Big\|_{L_\infty(I_j;L_\infty)}\Big)\\
&\qquad\times\Big(\epsilon^{1/2}\|G_t\|_{L_1(0,t_{n-1};L_\infty)}+\|G\|_{L_\infty(I_n;L_\infty)}\\
&\quad+\epsilon^{1/2}\|Dv\|_{L_1(0,t_n;L_1)}+\epsilon^{3/2}\|D^2G\|_{L_1(0,t_{n-1};L_\infty)}+\epsilon^{3/2}\sup_{s\in I_n}\Big\|D^2\int_s^{t_n}G\,dt\Big\|_{L_1}\Big)\\
&\le CC_rL_n\max_{1\le j\le n}\Big(\Big\|\frac{k}{\epsilon^{1/2}}R_e^t\Big\|_{L_\infty(I_j;L_\infty)}+\Big\|\frac{R_i^t}{\epsilon^{1/2}}\Big\|_{L_1(I_j;L_\infty)}\\
&\quad+\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_e^x\Big\|_{L_1(I_j;L_\infty)}+\Big\|\min\Big(\frac{h}{\epsilon^{1/2}},\frac{h^2}{\epsilon^{3/2}}\Big)R_i^x\Big\|_{L_1(I_j;L_\infty)}\Big),
\end{aligned} \tag{8.2}$$

which proves Theorem 3.1. □

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