CHALMERS
FINITE ELEMENT CENTER

PREPRINT 2002–15

A unified stabilized method for Stokes' and Darcy's equations

Erik Burman and Peter Hansbo

Chalmers Finite Element Center
CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg Sweden 2002

CHALMERS FINITE ELEMENT CENTER

Preprint 2002–15

A unified stabilized method for Stokes' and Darcy's equations

Erik Burman and Peter Hansbo

CHALMERS

Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg Sweden
Göteborg, November 2002

A unified stabilized method for Stokes' and Darcy's equations
Erik Burman and Peter Hansbo
NO 2002–15
ISSN 1404–4382

Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg
Sweden
Telephone: +46 (0)31 772 1000
Fax: +46 (0)31 772 3595
www.phi.chalmers.se

Printed in Sweden
Chalmers University of Technology
Göteborg, Sweden 2002

A UNIFIED STABILIZED METHOD FOR STOKES' AND DARCY'S EQUATIONS

ERIK BURMAN AND PETER HANSBO

Abstract. We use the lowest possible approximation order, piecewise linear, continuous velocities and piecewise constant pressures to compute solutions to Stokes equation and Darcy's equation, applying an edge stabilization term to avoid locking. We prove that the formulation satisfies the discrete *inf–sup* condition, we prove optimal a priori and a posteriori error estimates for both problems, the formulation is then extended to the coupled case using a Nitsche–type weak formulation allowing for different meshes in the two subdomains. Finally we present some numerical examples verifying the theoretical predictions and showing the flexibility of the coupled approach.

1. Introduction

In this paper we will consider equations of the following form

$$
\begin{aligned}
A(u) + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega
\end{aligned}
\tag{1.1}
$$

where Ω is an open subset of $\mathbb{R}^d$, A is some selfadjoint positive definite operator, u denotes the velocity vector, p the pressure and $f \in [L^2(\Omega)]^d$. For the choice of A we focus on two cases of importance in fluid dynamics

- $A(u) := Iu$ corresponding to Darcy's equation
- $A(u) := -2\mu \nabla \cdot \boldsymbol{\varepsilon}(u)$, where $\boldsymbol{\varepsilon}(u)$ is the symmetric part of the velocity gradient, corresponding to Stokes equation.

For simplicity we assume Dirichlet conditions on the boundary, that is, $u = 0$ on $\partial\Omega$ for Stokes and $u \cdot n = 0$ on $\partial\Omega$ for Darcy. Moreover our results immediately carry over to the Brinkman model, where $A(u) := \frac{\mu}{\kappa} u - 2\mu \nabla \cdot \boldsymbol{\varepsilon}(u)$.

It is well known that the computation of solutions to such systems require that some care is taken in the choice of approximating spaces in order to make the discrete problem well posed. In particular the naive choice of piecewise linear finite elements for both the velocities and the pressure or piecewise linear finite elements for the velocities and piecewise constants for the pressure results in ill posed discretizations. The solution is either to enrich

Date: 1st November 2002.

Erik Burman, Department of Mathematics, Ecole Polytechnique Federale de Lausanne, Switzerland, *email*: erik.burman@epfl.ch

Peter Hansbo, Department of Applied Mechanics, Chalmers University of Technology, S–412 96 Göteborg, Sweden, *email*: hansbo@math.chalmers.se.

the velocity space, using higher order interpolation or local so called, bubble functions, or to stabilize the method using a Galerkin/least-squares formulation. A vast number of discretizations and stabilizations for the Stokes equation are proposed in the literature, see, e.g., [4, 9, 10, 8, 1, 6, 5]. For finite element methods treating the case of Darcy flow we refer to [13] and references therein. Our aim in this paper is to present a unified treatment of Stokes' equation and Darcy's equation. In infiltration problems, like the ones encountered in groundwater flow or bioflows, one is interested in solving a problem where the flow in one part of the domain is governed by Stokes' equation and in the other by Darcy's. It is then convenient to work with a method that may treat both equations in the same manner and also yield the same convergence orders in both cases.

To remain competitive with the approach where Darcy's equation is treated as an elliptic Poisson's equation we wish to keep down the number of degrees of freedom as much as possible. The method for the Stokes system which is in some sense minimal would be to use piecewise constant (discontinuous) approximation for the pressures and piecewise linear (continuous) approximation for the velocities. This however results in a much too rich pressure space and the only velocity that can satisfy the incompressibility constraint is $u \equiv 0$. Indeed the discrete divergence operator becomes injective instead of surjective, a phenomenon known as "locking". The key to "unlock" the problem is to add a consistent stabilizing term to the formulation. We propose to add a symmetric stabilization term penalizing the jumps over the element edges of the piecewise constant pressures. This stabilization was first introduced in the context of Stokes equation in [10] in a global form and then considered in a local form in [12]. Comparisons with other stabilized methods for the Stokes equations were carried out in [15]. The main difference between Stokes and Darcy's equations, from the point of view of analysis, is that in Stokes the velocities are $[H^1(\Omega)]^d$ whereas in the case of Darcy they are only in $H_{\mathrm{div}}(\Omega)$. This loss of regularity must be accounted for in the analysis, and this is the main reason why the stabilized mixed P_1/P_0 is an ideal candidate for the problem: since the incompressibility condition is tested with constants we obtain $H_{\mathrm{div}}(\Omega)$ stability without additional least-squares stabilization.

In this paper we apply this mixed stabilized method to Stokes' equations and Darcy's equations in a unified manner and prove optimal a priori estimates applying to both systems. The addition of the incompressibility constraint in the mesh-dependent norm allows us to prove optimal L_2 convergence for the velocities. We also propose a Nitsche type weak coupling for Stokes and Darcy which can handle non-matching meshes on the separating interface. Finally, we give basic a posteriori error estimates and show some numerical examples. Only the case of global stabilization is accounted for, but our results generalize to the local form analyzed in [12] and extend it to include Darcy flow. For some recent results on the theoretical and numerical aspects on the coupling of the Stokes and the Darcy equation we refer to [7].

2. Finite element formulation

In order to formulate our finite element method we first introduce the weak formulation of problem (1.1). We introduce the Hilbert spaces

$$W^D = \{v \in H_{\mathrm{div}}(\Omega) : v \cdot n = 0 \text{ on } \partial\Omega\},$$
$$W^S = \{v \in [H^1_0(\Omega)]^d\},$$

and

$$L^2_0 = \{q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0\},$$

with Ω some open subset of R^d. We denote the product space $W^X \times L^2_0$ by $\mathcal{W}^X$ where X is chosen to D or S depending on the choice of equation and define the following norm on $\mathcal{W}^X$,

$$\|(u,p)\|^2_{\mathcal{W}^X} = \|u\|^2_{l,\Omega} + \|\nabla \cdot u\|^2_{0,\Omega} + \|p\|^2_{0,\Omega}$$

with $l = 0$ for Darcy and $l = 1$ for Stokes. Let $a(u,v)$ be the bilinear form corresponding to the weak formulation of $A(u)$ and consider the bilinear form

$$B[(u,p),(v,q)] = a(u,v) - (p, \nabla \cdot v)_{0,\Omega} + (q, \nabla \cdot u)_{0,\Omega}. \tag{2.1}$$

The weak formulation of (1.1) now takes the form, find $(u,p) \in \mathcal{W}^X$ such that

$$B[(u,p),(v,q)] = (f,v)_{0,\Omega} \quad \forall (v,q) \in \mathcal{W}^X \tag{2.2}$$

Let $\mathcal{T}_h$ be a conforming, shape regular triangulation of Ω. We introduce the two classical finite element spaces of piecewise linears and piecewise constants

$$V^0_h = \{v : v|_K \in P_1(K); v \in C^0(\Omega); v|_{\partial\Omega} \equiv 0\},$$
$$V_h = \{v : v|_K \in P_1(K); v \in C^0(\Omega)\},$$
$$Q_h = \{q : q|_K \in P_0(K); \int_\Omega q \, \mathrm{d}x = 0\}.$$

The velocity field will be sought in $W^S_h = [V^0_h]^d$ for Stokes and in $W^D_h = \{v \in [V_h]^d : v \cdot n = 0 \text{ on } \partial\Omega\}$ for Darcy's equation and the pressure field in Q_h. In analogy with the notation above we denote the discrete counterpart of $\mathcal{W}^X$, $W^X_h \times Q_h$, by $\mathcal{W}^X_h$. We introduce the following bilinear form on which we will base our finite element method

$$B_h[(u,p),(v,q)] = a(u,v) - (p, \nabla \cdot v)_{0,\Omega} + (q, \nabla \cdot u)_{0,\Omega} + J(p,q) \tag{2.3}$$

where

$$J(p,q) = \delta \sum_K \int_{\partial K \setminus \partial\Omega} h_{\partial K} [p][q] \mathrm{d}s,$$

with $[\cdot]$ denoting the jump over the element edge (taken on interior edges only). We propose the following finite element formulation: find $(u_h, p_h) \in \mathcal{W}^X_h$ such that

$$B_h[(u_h,p_h),(v_h,q_h)] = (f, v_h)_{0,\Omega}, \quad \forall (v_h, q_h) \in \mathcal{W}^X_h. \tag{2.4}$$

This finite element formulation is simply the standard Galerkin formulation with the penalizing term $J(p,q)$ added. In the following we will assume that the pressure is in $H^1(\Omega)$: then the penalizing term is consistent and we have the following

Lemma 1. *If (u,p) is a weak solution to (1.1) with $(u,p) \in W^X \times H^1(\Omega) \cap L^2_0$ then*

$$B_h[(u-u_h, p-p_h), (v_h, q_h)] = 0 \quad \forall (v_h, q_h) \in \mathcal{W}_h^X.$$

Proof. Immediate by noting that if $p \in H^1(\Omega)$ then the trace of p is well defined and hence $J(p, q_h) = 0$ for all $q_h \in Q_h$. □

3. Stability

Since it is a well known fact that the above choice of finite element spaces results in an ill posed discrete problem if used in a standard Galerkin method, the crucial point is to show that our stabilization operator $J(p,q)$ introduces sufficient coupling between the degrees of freedom in the pressure field such that an *inf–sup* condition is satisfied. In the analysis, we will use the following norm:

$$|||(u,p)|||^2 := \|(u,p)\|^2_{\mathcal{W}^X} + J(p,p).$$

Note that the triple norm contains the L^2–norm of $\nabla \cdot u$; this term is superfluous for Stokes since we already control the H^1–norm of the velocities, but of vital importance for Darcy. In fact, the control of the divergence is what allows us to prove optimal error estimates for sufficiently regular solutions. The main result of this section is the following theorem, assuring the wellposedness of our discretization.

Theorem 1. *The finite element formulation* (2.4) *satisfies the following* inf–sup *condition*

$$\gamma |||(u_h, p_h)||| \le \sup_{(v,q) \in \mathcal{W}_h^X} \frac{B_h[(u_h,p_h),(v_h,q_h)]}{|||v_h, q_h|||}, \quad \forall (u_h, p_h) \in \mathcal{W}_h^X$$

Proof. Taking first $(v_h, q_h) = (u_h, p_h)$ we obtain

$$B_h[(u_h,p_h),(u_h,p_h)] \ge C_a \|u_h\|^2_{l,\Omega} + J(p_h,p_h), \tag{3.1}$$

where, using Korn's inequality,

$$2\mu \|\boldsymbol{\varepsilon}(v)\|^2_{L_2(\Omega)} \ge C_{\mathrm{K}} \|v\|^2_{1,\Omega} \quad \forall v \in [H^1_0]^d,$$

we have set

$$C_a = \begin{cases} 1 & \text{for} \quad l = 0, \\ C_{\mathrm{K}} & \text{for} \quad l = 1. \end{cases}$$

As a consequence of the surjectivity of the divergence operator there exists a function $v_p \in [H^1_0(\Omega)]^d$ such that $\nabla \cdot v_p = p_h$ and

$$\|v_p\|_{1,\Omega} \le c \|p_h\|_{0,\Omega} \tag{3.2}$$

Let $\pi_h v_p$ denote the L_2–projection of v_p onto $[V_h^0]^d$. By the stability of the projection we have $\|\pi_h v_p\|_{1,\Omega} \le \tilde{c} \|p_h\|_{0,\Omega}$. We now take $(v_h, q_h) = (\pi_h v_p, 0)$ and add $0 = \|p_h\|^2 - (p_h, \nabla \cdot v_p)_{0,\Omega}$ to obtain

$$B_h[(u_h,p_h),(\pi_h v_p, 0)] = a(u_h, \pi_h v_p) + \|p_h\|^2 + (p_h, \nabla \cdot (\pi_h v_p - v_p))_{0,\Omega}.$$

We integrate the third term by parts on each triangle K

$$B_h[(u_h,p_h),(\pi_h v_p,0)] = a(u_h,\pi_h v_p) + \|p_h\|^2_{0,\Omega} + \sum_K \frac{1}{2}\int_{\partial K} [p_h](\pi_h v_p - v_p)\cdot n \, \mathrm{d}s.$$

Applying now Cauchy-Schwarz inequality followed by the arithmetic–geometric inequality in the first and last term and using the stability estimate on $\pi_h v_p$ we obtain

$$B_h[(u_h,p_h),(\pi_h v_p,0)] \geq -\frac{C_b}{\alpha}\|u_h\|^2_{l,\Omega} - (1-\tilde{c}\alpha)\|p_h\|^2_{0,\Omega} - \frac{1}{\alpha}J(p_h,p_h) -\alpha\sum_K \int_{\partial K} h^{-1}\|(\pi_h v_p - v_p)\cdot n\|^2_{0,\partial K},$$

where

$$C_b = \begin{cases} 1 & \text{for} \quad l=0, \\ 2\mu & \text{for} \quad l=1. \end{cases}$$

To conclude we need the following trace inequality, cf. [16],

$$\|u\cdot n\|^2_{0,\partial K} \leq C\|u\|_{0,K}(h^{-1}\|u\|_{0,K} + \|u\|_{1,K}), \quad \forall u \in [H^1(K)]^d \tag{3.3}$$

from which we deduce

$$\|(\pi_h v_p - v_p)\cdot n\|^2_{0,\partial K} \leq Ch\|v_p\|^2_{1,K}$$

Taking into account (3.2) we may write

$$\sum_K \int_{\partial K} h^{-1}\|\pi_h v_p - v_p\|^2_{0,K} \leq c_{\mathrm{t}}\|p\|^2_{0,\Omega}$$

which leads to

$$B_h[(u_h,p_h),(\pi_h v_p,0)] \geq -\frac{C_b}{\alpha}\|u_h\|^2_{l,\Omega} + (1-(\tilde{c}+c_{\mathrm{t}})\alpha)\|p_h\|^2_{0,\Omega} -\frac{1}{\alpha}J(p_h,p_h). \tag{3.4}$$

The control of $\|\nabla\cdot u_h\|^2_{0,\Omega}$ is obtained by choosing $(v_h,q_h) = (0,\nabla\cdot u_h)$.

$$\begin{aligned} B_h[(u_h,p_h),(0,\nabla\cdot u_h)] &= \|\nabla\cdot u_h\|^2_{0,\Omega} + J(p_h,\nabla\cdot u_h) \\ &\geq (1-C\alpha)\|\nabla\cdot u_h\|^2_{0,\Omega} - \tfrac{1}{\alpha}J(p_h,p_h). \end{aligned} \tag{3.5}$$

Where we used that $\|h^{1/2}\nabla\cdot u_h\|^2_{\partial K} \leq C\|\nabla\cdot u_h\|^2_K$ by a scaling argument if $\nabla\cdot u_h$ is elementwise constant. Finally we take $(v_h,q_h) = (\beta u_h + \pi_h v_p, \beta p_h + \nabla\cdot u_h)$, with

$$\beta \geq (1-(\tilde{c}+c_{\mathrm{t}})\alpha) + \alpha^{-1}\left(\frac{C_b}{C_a}+2\right),$$

which yields by (3.1), (3.4), (3.5)

$$B_h[(u_h,p_h),(v_h,q_h)] \geq (1-(\tilde{c}+c_{\mathrm{t}})\alpha)|||(u_h,p_h)|||^2.$$

The claim now follows by taking α sufficiently small and noting that $\exists C$ such that $|||(u_h,p_h)||| \geq C|||(v_h,q_h)|||$. $\square$

4. Error analysis

4.1. **A priori estimates.** First of all we note that applying the trace inequality (3.3) we easily derive the following approximation property for couples of functions $(u,p) \in [H^2(\Omega)]^d \times H^1(\Omega)$,

$$|||(u-\pi_h u, p-\pi_h p)||| \leq ch(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}) \tag{4.1}$$

Proposition 1. *Assume that the solution* (u,p) *to problem* (1.1) *resides in* $[H^2(\Omega)]^d \times H^1(\Omega) \cap L^2_0(\Omega)$*; then the finite element solution* (2.4) *satisfies the error estimate*

$$|||(u-u_h, p-p_h)||| \leq ch(\|u\|_{2,\Omega} + \|p\|_{1,\Omega})$$

Proof. In view of (4.1) we only need to show the inequality for $|||(u_h-\pi_h u, p_h-\pi_h p)|||$. By Theorem 1 and using Galerkin orthogonality we obtain, with the notation $\eta_h = u_h - \pi_h u$ and $\zeta_h = p_h - \pi_h p$,

$$\begin{aligned} |||(\eta_h,\zeta_h)||| &\leq \frac{1}{\gamma} \sup_{(v_h,q_h)\in\mathcal{W}_h^X} \frac{B_h[(\eta_h,\zeta_h),(v_h,q_h)]}{|||(v_h,q_h)|||} \\ &\leq \frac{1}{\gamma} \sup_{(v_h,q_h)\in\mathcal{W}_h^X} \frac{B_h[(u-\pi_h u, p-\pi_h p),(v_h,q_h)]}{|||(v_h,q_h)|||}. \end{aligned}$$

It remains to use interpolation estimates to bound the terms on the right hand side. The result follows from standard interpolation theory and (3.3). We have

$$\begin{aligned} &a(u-\pi_h u, v_h) \leq ch\|u\|_{2,\Omega}|||(v_h,q_h)|||, \\ &-(p-\pi_h p, \nabla\cdot v_h)_{0,K} = 0, \\ &(q_h, \nabla\cdot(u-\pi_h u))_{0,K} \leq ch\|u\|_{2,K}|||(v_h,q_h)|||, \\ &J(p-\pi_h p, q_h) \leq ch\|p\|_{1,\Omega}|||(v_h,q_h)|||. \end{aligned}$$

□

Using the Aubin-Nitsche duality argument we prove the following $L_2(\Omega)$–estimate for the velocities

Proposition 2.

$$\|u-u_h\|_{0,\Omega} \leq ch^2(\|u\|_{2,\Omega} + \|p\|_{1,\Omega})$$

Proof. Let $(\varphi, r) \in \mathcal{W}^X$ be the solution of the dual equation

$$B[(v,q),(\varphi,r)] = (\psi, v)_{0,\Omega} \quad \forall (v,q) \in \mathcal{W}^X, \tag{4.2}$$

and we assume that this dual solution enjoys the additional regularity

$$\|\varphi\|_{2,\Omega} + \|r\|_{1,\Omega} \leq c\|\psi\|_{0,\Omega}. \tag{4.3}$$

Choosing $v = u - u_h$, $q = 0$ and $\psi = u - u_h$, we may write

$$\|u-u_h\|^2_{0,\Omega} = a(u-u_h,\varphi) + (\nabla\cdot(u-u_h), r)_{0,\Omega}$$

and proceed using Galerkin orthogonality to obtain

$$\begin{aligned}\|u-u_h\|_{0,\Omega}^2 &= a(u-u_h,\varphi-\pi_h\varphi)+(\nabla\cdot(u-u_h),(r-\pi_h r))_{0,\Omega}\\ &\quad +(\nabla\cdot\pi_h\varphi,p-p_h)_{0,\Omega}+J(p-p_h,\pi_h r)\\ &\le \|u-u_h\|_{l,\Omega}\|\varphi-\pi_h\varphi\|_{l,\Omega}+\|\nabla\cdot(u-u_h)\|_{0,\Omega}\|r-\pi_h r\|_{0,\Omega}\\ &\quad +|\pi_h\varphi-\varphi|_{1,\Omega}\|p-p_h\|_{0,\Omega}\\ &\quad +J(p-p_h,p-p_h)^{1/2}J(r-\pi_h r,r-\pi_h r)^{1/2}.\end{aligned}$$

As a consequence of proposition 1 and the regularity hypothesis (4.3) we may conclude, keeping in mind that

$$\|\nabla\cdot(u-u_h)\|_{0,\Omega}\le |||(u-u_h,0)|||$$

and using the interpolation result

$$J(r-\pi_h r,r-\pi_h r)^{1/2}\le h\|r\|_{1,\Omega},$$

that

$$\begin{aligned}\|u-u_h\|_{0,\Omega}^2 &\le ch^2\|\varphi\|_{2,\Omega}+ch^2\|r\|_{1,\Omega}+ch^2\|r\|_{1,\Omega}\\ &\le ch^2\|u-u_h\|_{0,\Omega}.\end{aligned}$$

□

4.2. **A posteriori estimates.** In this section we derive the a posteriori equivalents of proposition 1 and 2. The approaches are standard and therefore we do not detail the proofs, for details on a posteriori error estimation in this context we refer to [11] and [3]. Below we let E_K denotes the set of edges $\{E\}$ on element K and $\mathcal{E}_h$ the set of edges in the mesh $\mathcal{T}_h$.

Proposition 3. *Suppose that $(u,p)\in\mathcal{W}^X$ is the solution of (2.2) and $(u_h,p_h)\in\mathcal{W}_h^X$ is the solution of (2.4) then the following a posteriori error estimate holds*

$$\|(u-u_h,p-p_h)\|_{\mathcal{W}^X}\le c\sum_K e_K(u_h,p_h,f)$$

where the error indicator $e_K(u_h,p_h,f)$ is given by

$$\begin{aligned} e_K(u_h,p_h,f) &= h_K\|f-A(u_h)-\nabla p_h\|_{0,K}+\|\nabla\cdot u_h\|_{0,K}\\ &+\frac{1}{2}\sum_{E\in E_K}h_E^{1/2}\Big(\|[l2\mu\boldsymbol{\varepsilon}(u)\cdot n+p_h n]\|_{0,E}+\|[p_h]\|_{0,E}\Big).\end{aligned}$$

Proof. Using the fact that the bilinear form $B[(u_h,p_h),(v_h,q_h)]$ corresponding to the continuous problem satisfies the *inf–sup* condition with respect to $\|\cdot\|_{\mathcal{W}}^X$ we write, using the notation $\eta=u-u_h$, $\zeta=p-p_h$

$$\gamma\|(\eta,\zeta)\|_{\mathcal{W}^X}\le\sup_{(v,q)\in\mathcal{W}^X}\frac{B[(\eta,\zeta),(v,q)]}{\|(v,q)\|_{\mathcal{W}^X}}.$$

As a consequence of this relation and Galerkin orthogonality we obtain

$$\gamma\|(\eta,\zeta)\|_{\mathcal{W}^X}\le\sup_{(v,q)\in\mathcal{W}^X}\Big(\frac{B[(\eta,\zeta),(v-v_h,q-q_h)]}{\|(v,q)\|_{\mathcal{W}^X}}+\frac{J(p_h,q_h)}{\|(v,q)\|_{\mathcal{W}^X}}\Big)$$

where v_h denotes the Clément interpolant and q_h the L_2–projection on the piecewise constants. It then follows by integration by parts and the fact that $v - v_h$, or $(v - v_h)\cdot n$, is zero on $\partial\Omega$ and $\nabla\cdot u = 0$ that

$$B[(\eta,\zeta),(v - v_h, q - q_h)] = \sum_{K\in\mathcal{T}_h} (f - A(u_h) - \nabla p_h, v - v_h)_{0,K}$$
$$- \sum_{K\in\mathcal{T}_h} (q, \nabla\cdot u_h)_{0,K} - \sum_{E\in\mathcal{E}_h} \int_E [l2\mu\boldsymbol{\varepsilon}(u)\cdot n - pn]\cdot(v - v_h)\ \mathrm{d}s$$
$$+J(p_h, q_h) = i + ii + iii + iv$$

and we conclude by deriving the following upper bounds on the terms i – iv □

$$i \le c \sum_{K\in\mathcal{T}_h} h_k \|f - A(u_h) - \nabla p_h\| \|v\|_{1,\Omega}$$
$$ii \le c \sum_{K\in\mathcal{T}_h} \|\nabla\cdot u_h\|_{0,K}\|q\|_{0,K}$$
$$iii \le c \sum_{E\in\mathcal{E}_h} h_K^{1/2} \|[l2\mu\boldsymbol{\varepsilon}(u)\cdot n - pn]\|_{0,E}\|v\|_{1,\Omega}$$
$$iv \le \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E h_K[p_h][q_h]\ \mathrm{d}s \le c\sum_{E\in\mathcal{E}_h} h_K^{1/2}\|[p_h]\|_{0,E}\|q_h\|_{0,\Omega}$$

Recalling the dual problem, find $(\varphi, r)\in\mathcal{W}^X$ such that

$$B[(v,q),(\varphi,r)] = (\psi, v)_{0,\Omega} \quad \forall (v,q)\in\mathcal{W}^X, \tag{4.4}$$

and assuming the following additional regularity on the solution

$$|\varphi|_{2,\Omega} + |r|_{1,\Omega} \le c_S\|\psi\|_{0,\Omega} \quad \forall\psi\in[L^2(\Omega)]^d, \tag{4.5}$$

we now derive an a posteriori error estimate using duality

Proposition 4. *Under the above hypothesis the following a posteriori estimate holds for the solution u_h of (2.4).*

$$|\int_\Omega (u - u_h)\psi\ dx| \le \sum_{K\in\mathcal{T}_h}\rho_K\eta_K$$

with the local residuals ρ_K given by

$$\rho_K = h_K\|f - A(u_h) - \nabla p_h\|_{0,K} + \|\nabla\cdot u_h\|_{0,K}$$
$$+\frac{1}{2}h_K^{1/2}(\|[l2\mu\boldsymbol{\varepsilon}(u)\cdot n - p_h n]\|_{0,\partial K} + \|[p_h]\|_{0,\partial K}),$$

and the dual weights η_K by

$$\eta_K = c_i h_K \max(|\varphi|_{2,K}, |r|_{1,K}).$$

Moreover we have

$$\|u - u_h\|_{0,\Omega} \le c_i c_S\Big(\sum_{K\in\mathcal{T}_h} h_K^2\rho_K^2\Big)^{1/2}$$

Proof. We only give the outlines of the proof since it is a standard duality argument. Keeping the notation from the previous proof we may write (under the regularity hypothesis (4.5))

$$\begin{aligned}|\int_\Omega \eta\psi \,\mathrm{d}x| = |B[(\eta,\zeta),(\varphi,r)]| &= |B_h[(\eta,\zeta),(\varphi-\varphi_h,r-r_h)]| \\ &= |a(u-u_h,\varphi-\varphi_h) - (p-p_h,\nabla\cdot(\varphi-\varphi_h))_{0,\Omega} \\ &+(r-r_h,\nabla\cdot u_h)_{0,\Omega} + \frac{1}{2}\sum_K \int_{\partial K} h_K[p_h][r_h-r]\,\mathrm{d}s|\end{aligned}$$

and the first claim follows proceeding in the same spirit as the previous proof by integrating by parts, applying the Cauchy-Schwarz inequality and interpolation inequalities. The last term is handled using the regularity of r and the trace inequality (3.3)

$$\begin{aligned}\int_{\partial K} h_K[p_h][r_h-r]\,\mathrm{d}s &\leq 2h_K^{1/2}\|[p_h]\|_{0,\partial K} h_K^{1/2}\|r_h-r\|_{0,\partial K} \\ &\leq 2h_K^{1/2}\|[p_h]\|_{0,\partial K} c_i h_K |r|_{1,K}\end{aligned}$$

The second claim follows by an application of the Cauchy-Schwarz inequality and the regularity estimate (4.5) applied with $\psi = u - u_h$. □

5. Higher order elements

In some cases it might be desirable to be able to use higher order elements, primarily equal order interpolation for u and p or second order polynomial approximation for u and first order for p. The second case can be proved to be stable in our stabilized setting using standard techniques. However the inequality in (3.5) does no longer hold true when $\nabla\cdot u$ is not constant, we therefore propose to add another consistent, symmetric term of edge stabilization type, namely

$$\tilde{J}(\nabla\cdot u,\nabla\cdot v) = \tilde{\delta}\sum_K \int_{\partial K\setminus\partial\Omega} h_{\partial K}[\nabla\cdot u][\nabla\cdot v]\mathrm{d}s,$$

to the formulation, giving the following modifications

$$B_h[(u_h,p_h),(u_h,p_h)] \geq C_a\|u_h\|^2_{l,\Omega} + J(p_h,p_h) + \tilde{J}(\nabla\cdot u_h,\nabla\cdot u_h), \tag{5.1}$$

instead of ineqality (3.1) and

$$\begin{aligned}B_h[(u_h,p_h),(0,\nabla\cdot u_h)] &= \|\nabla\cdot u_h\|^2_{0,\Omega} + J(p_h,\nabla\cdot u_h) \\ &\geq \|\nabla\cdot u_h\|^2_{0,\Omega} - \tfrac{1}{\alpha}J(p_h,p_h) - \alpha\tilde{J}(\nabla\cdot u_h,\nabla\cdot u_h).\end{aligned} \tag{5.2}$$

instead of (3.5) leading to the desired $\nabla\cdot u$ stability (still taking the projection of v_p onto the piecewise linears). The modification of the error analysis is straightforward.

This additional stabilization of the incompressibility condition can also be used in the first case where a discontinuous pressure would make the degrees of freedom skyrocket (relative to the velocity space). Instead we propose to use piecewise linear continuous approximation for both the velocities and the pressure. We stabilize the pressure as before using an edge stabilization term, this time acting on the jump of the pressure gradient, since the pressures are continuous. The resulting method takes the form, find $(u_h, p_h) \in [V_h^0]^d \times \tilde{V}_h$ such that

$$B[(u_h, p_h), (v_h, q_h)] + \tilde{J}(\nabla \cdot u_h, \nabla \cdot v_h) + J(p_h, q_h) = (f, v)_{0,\Omega} \tag{5.3}$$

for all $(v_h, q_h) \in [V_h^0]^3 \times \tilde{V}_h$, where now the stabilizing term $J(p_h, q_h)$ takes the form

$$J(p_h, q_h) = \delta \sum_K \int_{\partial K} h_{\partial K}^3 [\nabla p_h \cdot n][\nabla q_h \cdot n] \mathrm{d}s,$$

and the pressure space is given by $\tilde{V}_h = \{q : q \in V_h; \int_\Omega q \ \mathrm{d}x = 0\}$. This discretization can be shown to be stable and to have optimal convergence in L_2. For the complete analysis of this method we refer to [5].

6. The coupled problem

The aim of this paper is to propose a unified approach to Stokes and Darcy's equation in order to solve problems where the flow in one part of the domain $\Omega_1 := \Omega_{\mathrm{S}}$ is approximated by the former system of equations and in another part $\Omega_2 := \Omega_{\mathrm{D}}$ by the latter. To be able to handle completely independent triangulations of the different domains, we apply a Nitsche-type method of weak coupling between the domains. We will also split the viscous stress vector $2\mu\,\boldsymbol{\varepsilon} \cdot n$ into a scalar normal stress $\sigma_n = 2\mu\, n \cdot (\boldsymbol{\varepsilon} \cdot n)$ and a tangential stress vector $\sigma_t = 2\mu\,\boldsymbol{\varepsilon} \cdot n - \sigma_n\, n$ on the interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, where $n := n_1$ is the outer unit normal to Ω_1. We note in particular that

$$\begin{aligned} (2\mu\boldsymbol{\varepsilon} \cdot n) \cdot v &= (\sigma_t + \sigma_n n) \cdot v \\ &= \sigma_n v \cdot n + \sigma_t \cdot v. \end{aligned} \tag{6.1}$$

Denoting by $(u|_{\Omega_i}, p|_{\Omega_i}) = (u_i, p_i), \quad i = 1, 2$, we consider the following conditions on Γ:

$$\begin{aligned} \sigma_n(u_1) + p_1 &= p_2,\ \sigma_t(u_1) = 0 \quad \text{(force balance)}, \\ n \cdot u_1 &= n \cdot u_2 \quad \text{(continuity of normal velocity)}. \end{aligned} \tag{6.2}$$

We remark that the "no slip" condition in the tangential direction is not physically realistic. Robin-like conditions like

$$\sigma_t \cdot t = -k(u_1 - u_2) \cdot t,$$

with k a stiffness parameter and t a tangential vector in the direction $2\mu\,\boldsymbol{\varepsilon}(u_1) \cdot n - \sigma_n(u_1)\, n$, can easily be incorporated into the bilinear form in the standard way, but for ease of presentation we choose the simpler form in (6.2).

In the following we will write $\tilde{u} = (u_1, u_2) \in V_1 \times V_2$ with the continuous spaces

$$V_1 = \left\{ v \in [H^1(\Omega_i)]^d : \quad v|_{\partial\Omega \cap \partial\Omega_i} = 0 \right\},$$

$$V_2 = \left\{ v \in H_{\mathrm{div}}(\Omega_i) : \quad v \cdot n|_{\partial\Omega \cap \partial\Omega_i} = 0 \right\}.$$

To formulate our method, we suppose that we have regular finite element partitionings $\mathcal{T}_h^i$ of the subdomains Ω_i into shape regular simplexes. We shall consider one-sided mortaring using the trace mesh

$$\mathcal{G}_h = \{\ E \ : E = K \cap \Gamma,\ K \in \mathcal{T}_h^2\ \}. \tag{6.3}$$

We seek the approximation $\tilde{u}_h = (u_{1,h}, u_{2,h}) \in V^h = V_1^h \times V_2^h$ and $\tilde{p}_h = (p_{1,h}, p_{2,h}) \in Q^h = Q_1^h \times Q_2^h$, where

$$V_i^h = \left\{ v_i \in V_i : \ v_i|_K \text{ is linear for all } K \in \mathcal{T}_h^i \right\},$$

$$Q_i^h = \left\{ q_i \in Q_i : \ q_i|_K \text{ is constant for all } K \in \mathcal{T}_h^i \right\}.$$

On the interface we will use the notation $[\tilde{v}] = v_1 - v_2$ and we denote the diameter of $E \in \mathcal{G}_h$ by h_E. A variant of the method of Nitsche [14, 2] can now be formulated as follows: Find $\tilde{u}_h \in V^h$ such that

$$a_h(\tilde{u}_h, \tilde{v}) + b_h(\tilde{p}_h, \tilde{v}) + b_h(\tilde{q}, \tilde{u}_h) + J(\tilde{p}_h, \tilde{q}) = f_h(\tilde{v}) \tag{6.4}$$

for all $\tilde{v} \in V^h$ and $\tilde{q} \in Q^h$, with

$$\begin{aligned} a_h(\tilde{w}, \tilde{v}) \ &:= \ a(w_1, v_1) + a(w_2, v_2) \\ &\quad + \gamma_0 \sum_{E \in \mathcal{G}_h} h_E^{-1} \int_E [\tilde{w} \cdot n] \ [\tilde{v} \cdot n] \, \mathrm{d}s, \end{aligned} \tag{6.5}$$

$$b_h(\tilde{p}, \tilde{v}) := -\sum_i (p_i, \nabla \cdot v_i)_{\Omega_i} + \int_\Gamma p_2 \, [\tilde{v} \cdot n] \, \mathrm{d}s, \tag{6.6}$$

$$J(\tilde{p}, \tilde{q}) = \sum_i \delta_i \sum_{K \in \mathcal{T}_h^i} \int_{\partial K \setminus \Gamma} h_{\partial K} [p_i][q_i] \mathrm{d}s,$$

and

$$f_h(\tilde{v}) := \sum_{i=1}^{2} (f, v_i)_{\Omega_i}, \tag{6.7}$$

with γ_0 sufficiently large (see below). The method is clearly consistent in the sense that it holds for the exact solution, and we also have stability by the following proposition.

Proposition 5. *The coupled formulation (6.4) satisfies the* inf-sup *condition of theorem 1 with the triple norm given by*

$$|||(\tilde{u}, \tilde{p})|||_C^2 = \sum_{i=1}^{2} |||(u_i, p_i)|||_{\Omega_i}^2 + \sum_{E \in \mathcal{G}_h} h_E^{-1} \int_E [\tilde{u} \cdot n]^2 \ ds.$$

Proof. We will only point out how to modify theorem 1 to account for the coupled case.

(1) Note that when testing with $(\tilde{u}_h, \tilde{p}_h)$ we obtain the additional stabilizing term $\gamma_0 \sum_{E \in \mathcal{G}_h} h_E^{-1} \int_E [\tilde{u}_h \cdot n]^2 \ \mathrm{d}s$.

(2) To control the pressure we choose $v_{i,p} \in H_0^1(\Omega_i)$ such that $\nabla \cdot v_{i,p} = p_{i,h}$ in the two domains separately. This way the coupling terms do not interfere, since $[\pi_h \tilde{v}_p \cdot n] = (v_{1,p} - v_{2,p}) \cdot n = 0$.

(3) When choosing $(\tilde v_h, \tilde q_h) = (0, \nabla\cdot\tilde u_h)$ the one-sided mortaring produces a term

$$\int_\Gamma \nabla\cdot u_{2,h}\,[\tilde u_h\cdot n]\,\mathrm{d}s.$$

to control this term we use Cauchy-Schwarz inequality, the arithmetic–geometric inequality followed by a scaling argument to obtain

$$\begin{aligned}\langle\nabla\cdot u_{2,h}, [\tilde u_h\cdot n]\rangle_\Gamma \;\leq\; & \frac{C\,c}{2}\|\nabla\cdot u_{2,h}\|^2_{\Omega_2^\Gamma} \\ & +\frac{1}{2c}\sum_{E\in\mathcal{G}_h} h_E^{-1}\int_E [\tilde u_h\cdot n]^2\,\mathrm{d}s,\end{aligned}$$

where Ω_2^Γ denotes the union of the triangles in Ω_2 neighbouring to the boundary Γ. The second term on the right-hand side is controlled by the additional stabilizing term from 1., choosing γ_0 sufficiently large, and the proof is complete.

□

We thus have stability and consistency, and optimal convergence follows using the same techniques of proof as previously noting that

$$\gamma_0\sum_{E\in\mathcal{G}_h} h_E^{-1}\int_E[(\tilde u-\pi_h\tilde u)\cdot n]^2\,\mathrm{d}s \leq Ch^2\sum_{i=1}^2\|u_i\|^2_{2,\Omega_i}$$

by the trace inequality (3.3).

7. Numerical results

7.1. **Convergence study for Darcy flow.** The first numerical example, taken from [13], is a study of convergence rates for Darcy flow. The domain under consideration is the unit square with a given exact pressure solution $p = \sin 2\pi\,x \sin 2\pi\,y$. The exact velocity field is then computed from Darcy's law to give boundary conditions and a source term for the divergence. In order to create a unique pressure field we also impose zero mean pressure. We set $\delta = 10$.

In Figure 1, we show the approximate velocities and pressures on the final mesh in a sequence. In Figure 2, we show the convergence of the method in the L_2-norm, which confirms second order accuracy for the velocities and first order for the pressure.

7.2. **Convergence study for Stokes flow.** Again, we consider the unit square with exact flow solution (from [15]) given by $u = (20\,x\,y^3,\ 5\,x^4-5\,y^4)$ and $p = 60\,x^2y-20\,y^3+C$. Choosing $\delta = 1/10$ and imposing zero mean pressure ($C=-5$), we obtain the optimal convergence shown in Figure 3.

7.3. **Coupling of Stokes and Darcy.** We consider an artificial example: in a domain $(0,3)\times(0,1)$ the flow is governed by Darcy on $(0,1)\times(0,1)$ and by Stokes on $(1,3)\times(0,1)$. The velocity solution for Darcy is given by the exact pressure solution

$$p = (1-x)\,y\,(1-y) - x + x^2 - x^3/3 + C_1,$$

i.e.,

$$u = (1 - 2\,x + x^2 + y - y^2, -1 + x + 2\,y - 2\,x\,y),$$

which is divergence free and has a parabolic profile at $x = 1$. We prescribe $u \cdot n$ at $y = \pm 1$ and $x = 0$. For Stokes, we prescribe $u = 0$ at $y = \pm 1$ and $u = (y\,(1-y), 0)$ at $x = 2$, corresponding to Poiseuille flow. Here we have used $A = -\mu\Delta u$ instead of $A = -2\mu\nabla\cdot\boldsymbol{\varepsilon}(u)$ in the Stokes domain to obtain the usual Poiseuille linear pressure increase also at in- and outflow. Note that this does not affect the coupling terms at $x = 1$.

In Figure 5, we show the effect of a coarse triangulation on one side; note that the solution on the interface is not parabolic due to the poor resolution on the Stokes domain. In Figure 6, we give the corresponding solution using a finer resolution for the Stokes part. Note that the meshes still do not match across the interface.

For the convergence check we use the same example and note that the pressure from the Darcy problem is constant at $x = 1$. Thus, we have $p = -2\,x + C_2$ in the Poiseuille flow and continuity of the pressure across the interface. Imposing mean pressure zero, these conditions yield $C_1 = 29/18$, $C_2 = 59/18$. The convergence of the pressure and the velocity in L_2, on a sequence of unfitted meshes (one of which is shown in in Figure 6) is given in Figure 4, showing first order and second order convergence, respectively.

8. Conclusion

We have applied the mixed P_1/P_0 stabilized finite element method allowing the use of piecewise linear approximation for the velocities and piecewise constant approximation for the pressures to Stokes and Darcy's equation. This formulation is a natural generalization of the Brezzi-Pitkäranta penalization [4], but remains consistent for sufficiently smooth exact solutions. We have proved optimal a priori estimates for both problems indicating that this method might be a suitable candidate for problems where one wishes to compute flows where (Navier-) Stokes and Darcy's equations are coupled. Moreover we discussed the possible extension to higher order finite elements and the coupling of the two systems using a Nitsche–type method. Some numerical results were reported showing good agreement with the theoretical predictions. Future extensions include different aspects of the coupling between Navier-Stokes equations and Darcy's equation from a theoretical and numerical viewpoint.

Acknowledgments
The authors wish to thank Paolo Zunino for inspiring discussions concerning the coupled problem.

References

[1] R. Becker and M. Braack, A finite element pressure gradient stabilization for the Stokes equation based on local projection, *Calcolo* **38** (2001) 173–199.

[2] R. Becker, P. Hansbo and R. Stenberg, A finite element method for domain decomposition with non-matching grids, *PREPRINT 2001-15, Chalmers Finite Element Center*, http://www.phi.chalmers.se/pub/preprints/ps/phiprint-2001-15.ps, 2001.

[3] R. Becker and R. Rannacher, A feed-back approach to error control in finite element methods: basic analysis and examples, *East-West J. Numer. Math.*, **4** (1996) 237–264.

[4] F. Brezzi and J. Pitkäranta, On the stabilization of finite element approximations of the Stokes equations, in W. Hackbusch, ed., *Efficient Solution of Elliptic Systems*, Vieweg, (1984).

[5] E. Burman and P. Hansbo, Edge stabilization for finite element methods, submitted to *Comp. Meth. Mech. Eng.* (2002).

[6] R. Codina and J. Blasco, Analysis of a pressure-stabilized finite element approximation of the stationary Navier-Stokes equations, *Numer. Math.* **87** (2000) 59–81.

[7] M. Discacciati, E. Miglio, A. Quarteroni Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.* (2002).

[8] J. Douglas and J. P. Wang, An absolutely stabilized finite-element method for the Stokes problem. *Math. Comp.* **52** (1989) 495–508.

[9] T. J. R. Hughes, L. P. Franca, and M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška–Brezzi condition: A stable Petrov–Galerkin formulation of the Stokes problem accomodating equal-order interpolations. *Comput. Methods Appl. Mech. Eng.* **59** (1986) 85–99.

[10] T.J.R. Hughes and L.P. Franca, A new finite element formulation for CFD: VII. The Stokes problem with various well-posed boundary conditions: Symmetric formulations that converge for all velocity/pressure spaces. *Comput. Methods Appl. Mech. Eng.* **65** (1987) 85–96.

[11] D. Kay and D. Silvester, A posteriori error estimation for stabilised mixed approximation of the Stokes equations. *SIAM J. Sci. Comput.* **21** (1999) 1321–1336.

[12] N. Kechar and D. Silvester, Analysis of locally stabilised mixed finite element methods for the Stokes equations. *Math. Comp.* **58** (1992) 1–10.

[13] A. Masud and T.J.R. Hughes A stabilized mixed finite element method for Darcy flow. *Comput. Methods Appl. Mech. Eng.* **191** (2002) 4341–4370.

[14] J. Nitsche, Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abh. Math. Univ. Hamburg* **36** (1971) 9–15.

[15] S. Norburn and D. Silvester, Stabilised vs stable mixed methods for incompressible flow. *Comput. Methods Appl. Mech. Eng.* **166** (1998) 131–141.

[16] R. Verfürth, A posteriori error estimators for convection-diffusion equations. *Numer. Math.* **80** (1998) 641–663.

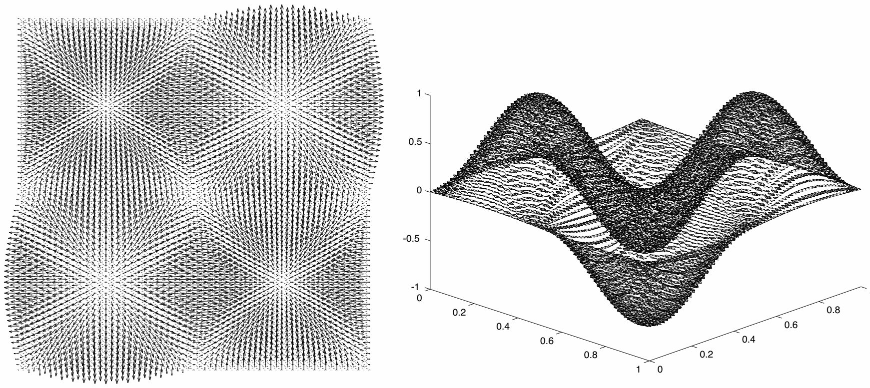


Figure 1. Approximate velocity field and elevation of the pressure on the final mesh in a sequence.

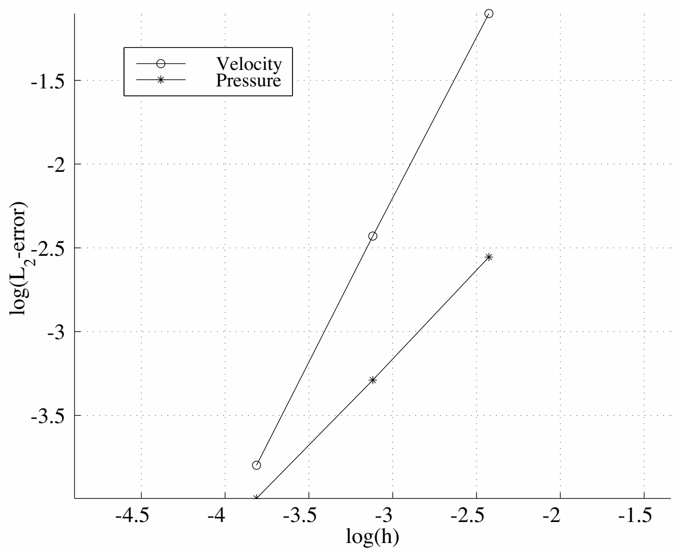


Figure 2. L_2-norm convergence of the velocity and of the pressure for Darcy.

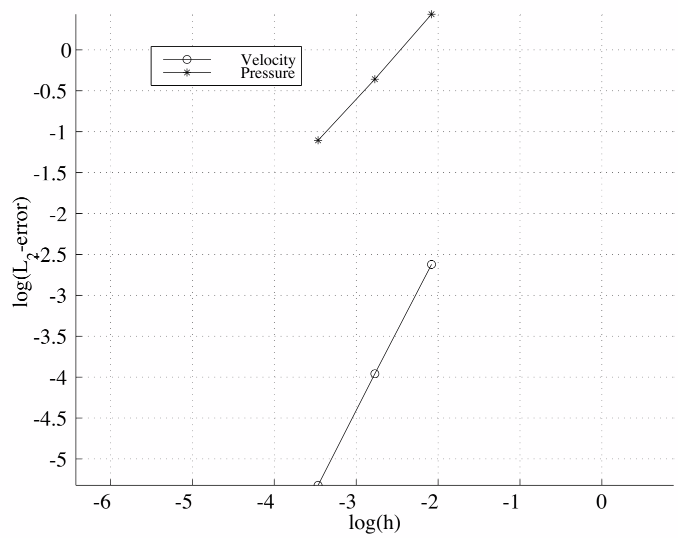


FIGURE 3. L_2-norm convergence of the velocity and of the pressure for Stokes.

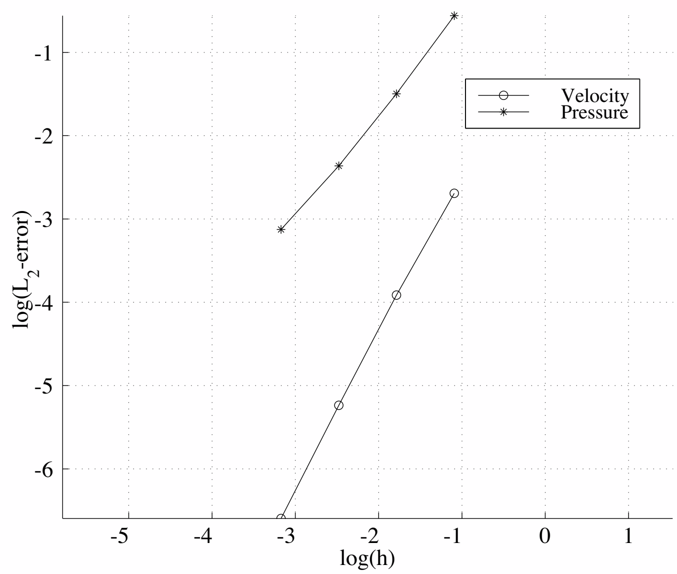


FIGURE 4. L_2-norm convergence of the velocity and of the pressure for the coupled problem.

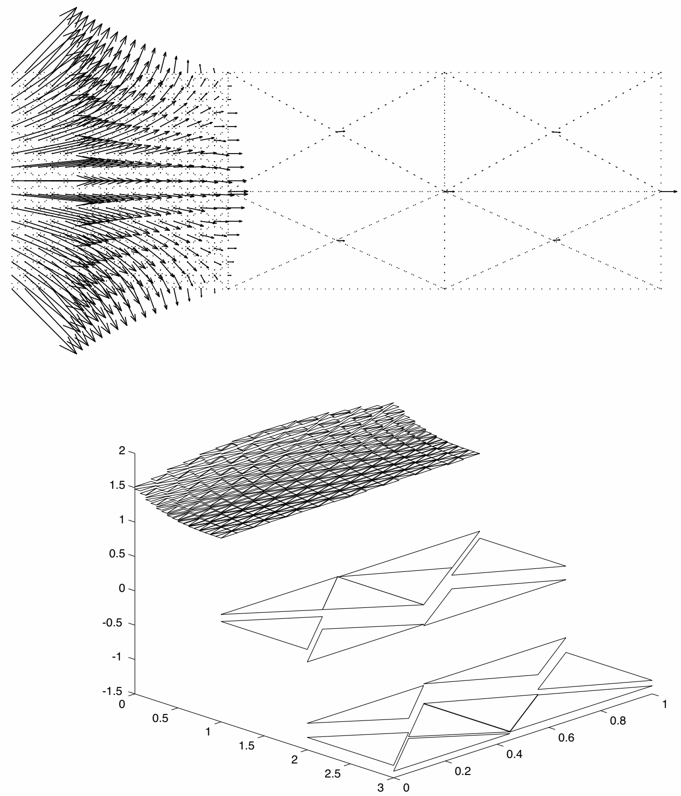
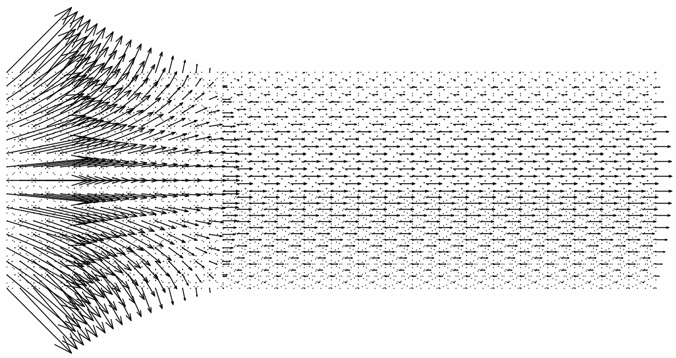


Figure 5. Velocity and pressure solutions for the coupled problem.



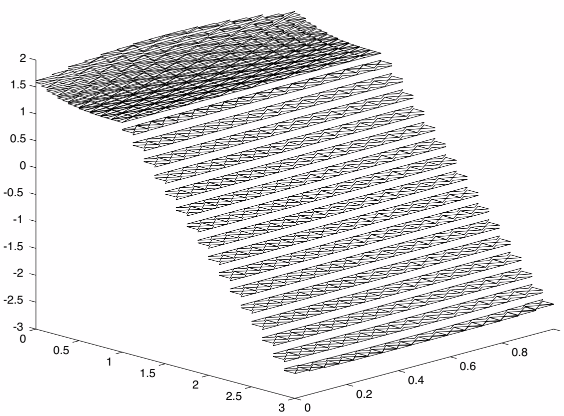


FIGURE 6. Velocity and pressure solutions for the coupled problem.

Chalmers Finite Element Center Preprints

2001–01 *A simple nonconforming bilinear element for the elasticity problem*
Peter Hansbo and Mats G. Larson

2001–02 *The $\mathcal{LL}^*$ finite element method and multigrid for the magnetostatic problem*
Rickard Bergström, Mats G. Larson, and Klas Samuelsson

2001–03 *The Fokker-Planck operator as an asymptotic limit in anisotropic media*
Mohammad Asadzadeh

2001–04 *A posteriori error estimation of functionals in elliptic problems: experiments*
Mats G. Larson and A. Jonas Niklasson

2001–05 *A note on energy conservation for Hamiltonian systems using continuous time finite elements*
Peter Hansbo

2001–06 *Stationary level set method for modelling sharp interfaces in groundwater flow*
Nahidh Sharif and Nils-Erik Wiberg

2001–07 *Integration methods for the calculation of the magnetostatic field due to coils*
Marzia Fontana

2001–08 *Adaptive finite element computation of 3D magnetostatic problems in potential formulation*
Marzia Fontana

2001–09 *Multi-adaptive galerkin methods for ODEs I: theory & algorithms*
Anders Logg

2001–10 *Multi-adaptive galerkin methods for ODEs II: applications*
Anders Logg

2001–11 *Energy norm a posteriori error estimation for discontinuous Galerkin methods*
Roland Becker, Peter Hansbo, and Mats G. Larson

2001–12 *Analysis of a family of discontinuous Galerkin methods for elliptic problems: the one dimensional case*
Mats G. Larson and A. Jonas Niklasson

2001–13 *Analysis of a nonsymmetric discontinuous Galerkin method for elliptic problems: stability and energy error estimates*
Mats G. Larson and A. Jonas Niklasson

2001–14 *A hybrid method for the wave equation*
Larisa Beilina, Klas Samuelsson and Krister Åhlander

2001–15 *A finite element method for domain decomposition with non-matching grids*
Roland Becker, Peter Hansbo and Rolf Stenberg

2001–16 *Application of stable FEM-FDTD hybrid to scattering problems*
Thomas Rylander and Anders Bondeson

2001–17 *Eddy current computations using adaptive grids and edge elements*
Y. Q. Liu, A. Bondeson, R. Bergström, C. Johnson, M. G. Larson, and K. Samuelsson

2001–18 *Adaptive finite element methods for incompressible fluid flow*
Johan Hoffman and Claes Johnson

2001–19 *Dynamic subgrid modeling for time dependent convection–diffusion–reaction equations with fractal solutions*
Johan Hoffman

2001–20 *Topics in adaptive computational methods for differential equations*
Claes Johnson, Johan Hoffman and Anders Logg

2001–21 *An unfitted finite element method for elliptic interface problems*
Anita Hansbo and Peter Hansbo

2001–22 *A P^2–continuous, P^1–discontinuous finite element method for the Mindlin-Reissner plate model*
Peter Hansbo and Mats G. Larson

2002–01 *Approximation of time derivatives for parabolic equations in Banach space: constant time steps*
Yubin Yan

2002–02 *Approximation of time derivatives for parabolic equations in Banach space: variable time steps*
Yubin Yan

2002–03 *Stability of explicit-implicit hybrid time-stepping schemes for Maxwell's equations*
Thomas Rylander and Anders Bondeson

2002–04 *A computational study of transition to turbulence in shear flow*
Johan Hoffman and Claes Johnson

2002–05 *Adaptive hybrid FEM/FDM methods for inverse scattering problems*
Larisa Beilina

2002–06 *DOLFIN - Dynamic Object oriented Library for FINite element computation*
Johan Hoffman and Anders Logg

2002–07 *Explicit time-stepping for stiff ODEs*
Kenneth Eriksson, Claes Johnson and Anders Logg

2002–08 *Adaptive finite element methods for turbulent flow*
Johan Hoffman

2002–09 *Adaptive multiscale computational modeling of complex incompressible fluid flow*
Johan Hoffman and Claes Johnson

2002–10 *Least-squares finite element method with applications in electromagnetics*
Rickard Bergström

2002–11 *Discontinuous/continuous least-squares finite element methods for Elliptic Problems*
Rickard Bergström and Mats G. Larson

2002–12 *Discontinuous least-squares finiteeElement methods for the Div-Curl problem*
Rickard Bergström and Mats G. Larson

2002–13 *Object oriented implementation of a general finite element code*
Rickard Bergström

2002-14 *On adaptive strategies and error control in fracture mechanics*
Per Heintz and Klas Samuelsson

2002-15 *A unified stabilized method for Stokes' and Darcy's equations*
Erik-Burman and Peter-Hansbo

These preprints can be obtained from

`www.phi.chalmers.se/preprints`