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SE–412 96 Göteborg Sweden
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Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg
Sweden
Telephone: +46 (0)31 772 1000
Fax: +46 (0)31 772 3595
www.phi.chalmers.se

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EDGE STABILIZATION FOR GALERKIN APPROXIMATIONS OF CONVECTION–DIFFUSION PROBLEMS

ERIK BURMAN AND PETER HANSBO

Abstract. In this paper we recall a stabilization technique for finite element methods for convection-diffusion-reaction equations, originally proposed by Douglas and Dupont [6]. The method uses least square stabilization of the gradient jumps across element boundaries. We prove that the method is stable in the hyperbolic limit and prove optimal a priori error estimates. We address the question of monotonicity of discrete solutions and present some numerical examples illustrating the theoretical results..

1. Introduction

The standard Galerkin for convection-diffusion-reaction problems is not stable if implemented without stabilization. Over the years many different stabilization methods have been proposed and it is by now a well established discipline with different well explored methods like the SUPG/SD-method [8], the residual free bubbles [2] and more recent contributions like subviscosity models for convection diffusion problems [7]. The relation between the different approaches is also well understood in most cases. However for complex flow problems, like the ones arising in combustion problems, most of these methods have drawbacks. The SUPG stabilization becomes non-symmetric and the formulation does not permit lumped mass; the residual free bubbles add additional degrees of freedom; the projection methods introduce the need of hierarchical meshes for the projection or the sub viscosity model. In this paper we recall a method due to Douglas and Dupont [6] which stabilizes convection-diffusion-reaction problems by adding a least-squares term based on the jump in the gradient over element boundaries. Unlike [6], we also consider the crucial case of a vanishing diffusion parameter.

The method can be seen as a higher order penalty method, or as a sub viscosity method where we have eliminated the need for patches. We also add a non-linear term adding diffusion on the element edges in the tangential direction, in order to guarantee monotonicity. We prove that the shock-capturing parameter can be chosen in such a way that a discrete maximum principle holds. The method has many of the advantages of the above methods, but no additional degrees of freedom are added, no hierarchical meshes are needed, the formulation remains symmetric, and the mass can be lumped for efficient time marching

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Erik Burman, Department of Mathematics, Ecole Polytechnique Fédérale de Lausanne, Switzerland

Peter Hansbo, Department of Applied Mechanics, Chalmers University of Technology, SE–41296 Göteborg, Sweden.

and treatment of stiff source terms. Furthermore the methods allows for the introduction of crosswind diffusion which is consistent for solutions in $H^2(\Omega)$. The price to pay is an increased number of non-zero elements in the stiffness matrix due to the fact that the gradient jump term couple neighboring elements. However for systems of PDE:s (like the ones in combustion problems) where a large number of unknowns are associated with each node, these additional blocks are diagonal, making the increased memory cost reasonable.

2. Convection–diffusion–reaction

As a first model problem, we consider, in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the problem of solving

$$\sigma\, u + \beta \cdot \nabla u - \nabla \cdot (\varepsilon \nabla u) = f \quad \text{in } \Omega \tag{2.1}$$

with, for simplicity, $u = 0$ on $\partial\Omega$. Here, f is a given source term, β is a given smooth velocity field, satisfying $\nabla \cdot \beta = 0$, and σ and ε are bounded positive functions.

The weak form of this problem is to find $u \in H_0^1(\Omega)$ such that

$$A(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$A(u, v) := \int_\Omega \big(\sigma\, u\, v + \varepsilon \nabla u \cdot \nabla v + \beta \cdot \nabla u\, v\big)\, \mathrm{d}x \quad \text{and} \quad (f, v) := \int_\Omega f\, v\, \mathrm{d}x.$$

We denote the L_2-scalar product by $(\cdot, \cdot)$ and the corresponding norm by $\|\cdot\|$. The finite element method consists of seeking a piecewise polynomial approximation $U \in V_h \subset H_0^1(\Omega)$. It is well known that the standard Galerkin approximation, in the convection dominated case, results in a wildly oscillating solution in the presence of sharp layers. To stabilize the method we propose, following [6], to add a term penalizing the gradient jumps across element boundaries of the type

$$\begin{aligned} J(U, v) &= \sum_K \frac{1}{2} \int_{\partial K} \gamma h_{\partial K}^2 [\nabla U] \cdot [\nabla v]\, \mathrm{d}s \\ &= \sum_K \frac{1}{2} \int_{\partial K} \gamma h_{\partial K}^2 [n \cdot \nabla U]\, [n \cdot \nabla v]\, \mathrm{d}s. \end{aligned} \tag{2.3}$$

Here, $h_{\partial K}$ is the size of ∂K, $[q]$ denotes the jump of q across ∂K for $\partial K \cap \partial\Omega = \emptyset$, $[q] = 0$ on $\partial K \cup \partial\Omega$, n is the outward pointing unit normal to K, and γ is a constant. We also introduce the local mesh size

$$h_K := \max_K h_{\partial K},$$

and we will assume that $h_K / h_{\partial K} < C$ where C is a fixed constant. Our finite element method then reads, find $U \in V_h$ such that

$$A(U, v) + J(U, v) = (f, v) \quad \forall v \in V_h. \tag{2.4}$$

To simplify the analysis we will assume that the exact solution belongs to $H^2(\Omega)$; it then follows that the formulation (2.4) is consistent, as put forth in the following Lemma.

Lemma 1. *For $u \in H^2(\Omega)$ there holds*

$$A(u-U,v)+J(u-U,v)=0$$

for all $v \in V^h$.

Proof This is an immediate consequence of the regularity hypothesis: if $u \in H^2(\Omega)$ then the trace of ∇u is well defined and hence $J(u,v)=0$. $\square$

Remark 1. *Another possible choice of $J(U,v)$ is*

$$\begin{aligned} J(U,v) &= \sum_K \frac{1}{2}\int_{\partial K} \gamma_\beta h_{\partial K}^2 [\beta\cdot\nabla U][\beta\cdot\nabla v]\mathrm{d}s \\ &+\sum_K \frac{1}{2}\int_{\partial K} \gamma_{\beta^\perp} h_{\partial K}^2 [\beta^\perp\cdot\nabla U][\beta^\perp\cdot\nabla v]\mathrm{d}s, \end{aligned} \tag{2.5}$$

This way the streamline and the crosswind stabilizations may be tuned independently. Note that (2.3) corresponds to the case $\gamma_\beta=\gamma_{\beta^\perp}$.

2.1. **Stability.** The main point of any stabilized method is of course that it enhances stability. The stability estimate obtained using edge stabilization is less immediate than that obtained in the case of streamline-diffusion or discontinuous galerkin. However we will show that we, thanks to the term $J(U,v)$, get the control of $\|h_K^{1/2}\beta\cdot\nabla U\|^2$ crucial for the analysis. To prove stability in a discontinuous galerkin method one exploits the fact that $h_K\beta\cdot\nabla U$ is in the finite element test space and hence can be chosen as test function. In the case of edge stabilization we proceed in a similar way. Indeed, even if $h_K\beta\cdot\nabla U$ is not in the finite element space something which is close is, and the difference is controlled by the edge stabilization term. We denote by π_h the Clément quasi interpolant [5], $\pi_h : L_2(\Omega)\to V_h$.

We shall frequently use the following inequalities, which we collect in a Lemma.

Lemma 2. *For the Clément operator there holds*

$$\|\pi_h u\|_{H^s(\Omega)} \le C\|u\|_{H^s(\Omega)}, \quad \forall u\in H^s(\Omega), \tag{2.6}$$

for $s=0,1$. Further,

$$\|\pi_h h_K\beta\cdot\nabla U\| \le C\|U\|, \quad \forall U\in V_h. \tag{2.7}$$

Finally, we have the trace inequality

$$\|v\|_{L_2(\partial K)}^2 \le C\Big(h_K^{-1}\|v\|_{L_2(K)}^2 + h_K\|v\|_{H^1(K)}^2\Big), \quad \forall v\in H^1(K), \tag{2.8}$$

Here, C is a generic constant independent of h_K.

Proof Inequality (2.6) follows from the interpolation estimate

$$\|u-\pi_h u\|_{H^s(\Omega)} \le C\|u\|_{H^s(\Omega)}, \quad s=0,1,$$

cf. [5], and (2.7) follows from (2.6) and the well known inverse inequality

$$\|v\|_{H^1(K)} \le Ch_K^{-1}\|v\|_{L_2(K)}, \quad \forall v\in V_h. \tag{2.9}$$

Finally, a proof of (2.8) is given in [9]. □

As a model example we choose $\epsilon = 0$ and we assume that $h_K < \sigma^{-1/2}$ corresponding to a convection–reaction problem. Furthermore let us first assume that h_K is constant throughout the domain. The problem takes the form: find $U \in V_h$ such that

$$(\beta \cdot \nabla U, v) + (\sigma U, v) + J(U, v) = (f, v), \quad \forall v \in V_h. \tag{2.10}$$

Taking $v = U$ we obtain the basic stability estimate

$$J(U, U) + \|\sigma^{1/2} U\|^2 = (f, U) \tag{2.11}$$

Clearly we may use the fact that $\pi_h h_K \beta \cdot \nabla U \in V_h$ to write

$$\begin{aligned} \|h_K^{1/2} \beta \cdot \nabla U\|^2 &+ (\beta \cdot \nabla U, \pi_h h_K \beta \cdot \nabla U - h_K \beta \cdot \nabla U) \\ &= -J(U, \pi_h h_K \beta \cdot \nabla U) + (-\sigma U + f, \pi_h h_K \beta \cdot \nabla U). \end{aligned} \tag{2.12}$$

We use Cauchy-Schwartz inequality followed by the arithmetic-geometric inequality for the left-hand side to obtain

$$\begin{aligned} \tfrac{3}{4}\|h_K^{1/2} \beta \cdot \nabla U\|^2 &- \|h_K^{1/2}(\pi_h \beta \cdot \nabla U - \beta \cdot \nabla U)\|^2 \\ &\leq |J(U, \pi_h h_K \beta \cdot \nabla U)| + C\|\sigma^{1/2} U\|^2 + C\|f\|^2. \end{aligned} \tag{2.13}$$

Comparing the two expressions (2.11) and (2.13) we find that we need the following two results.

(1) Proof that there exists some $\zeta \geq \zeta_0 > 0$ such that

$$\|h_K^{1/2}(\pi_h \beta \cdot \nabla U - \beta \cdot \nabla U)\|^2 \leq \zeta J(U, U).$$

(2) The inverse estimate

$$J(\pi_h h_K \beta \cdot \nabla U, \pi_h h_K \beta \cdot \nabla U) \leq C_i \|h_K^{1/2} \beta \cdot \nabla U\|^2. \tag{2.14}$$

The inverse estimate is immediately proven by noting that

$$\begin{aligned} J(\pi_h h_K \beta \cdot \nabla U, \pi_h h_K \beta \cdot \nabla U) &= \sum_K \int_{\partial K} h_K^3 [\nabla \pi_h \beta \cdot \nabla U]^2 \mathrm{d}s \\ &\leq \tilde{C} \|h_K^{3/2} \nabla \pi_h \beta \cdot \nabla U\|^2 \leq C \|h_K^{1/2} \beta \cdot \nabla U\|^2 \end{aligned}$$

by virtue of (2.9) and (2.6).

2.1.1. *Bounding the projection error by the stabilization term.* The stability of the method is obtained by the fact that the edge operator controls the projection error of $h_K \beta \cdot \nabla U$ in the case of convection–diffusion. By $\{\varphi_i\}$ we denote the set of finite element basis functions spanning the space V_h. Let $\mathcal{N}_i$ be the set of all triangles K^i containing node i and assume that the cardinality of $\mathcal{N}_i$ is bounded uniformly in i. Let $\mathcal{F}_K$ be the set of all test functions φ_i such that supp $\varphi_i \cap K \neq \emptyset$ and $\Omega_i = \bigcup_{\mathcal{N}_i} K^i$. We will consider a function $p \in [P_0(K)]^2$, and its representation in the finite element basis $\tilde{p}$ defined by

$$\tilde{p}|_K = p|_K \sum_{i \in \mathcal{F}_K} \varphi_i. \tag{2.15}$$

It follows that $\tilde{p} = p$ everywhere except on elements adjacent to Dirichlet boundaries where the boundary nodes are not included in the finite element space. We note that, with $p := h_K^{1/2}\beta \cdot \nabla U$, we have on the left-hand side of (2.12) the expression $\|p\|^2 + (p, \pi_h p - p)$, and we wish to bound the second term using the first term and the jumps. This cannot be done exactly since $\pi_h p$ must obey the boundary conditions, unlike p. However, the left hand side of (2.12) can equally well be written $(p, \tilde{p}) + (p, \pi_h p - \tilde{p})$, and if we can show that $c\|p\|^2 \leq (p, \tilde{p})$ we have

$$c\|p\|^2 + (p, \pi_h p - \tilde{p}) \leq (p, \tilde{p}) + (p, \pi_h p - \tilde{p}),$$

and we can proceed to bound the second term on the left hand side in terms of the first together with the jumps. Thus, we need:

Lemma 3. *Suppose that K is an element with at least one node on a Dirichlet boundary then*

$$\|p\|_K^2 = \frac{d+1}{n_i}(p, \tilde{p}), \tag{2.16}$$

where n_i denotes the number of interior nodes of the element.

Proof The proof is immediate noting that

$$(p, \tilde{p}) = p_K^2 \int_K \sum_{i \in \mathcal{F}_K} \varphi_i \mathrm{d}x = \frac{n_i}{d+1} p_K^2 m(K).$$

□ We will now proceed to prove that

$$\|h^{s/2}(\tilde{p} - \pi_h p)\|^2 \leq C \tilde{J}_s(p, p)$$

with

$$\tilde{J}_s(p, p) = \sum_K \int_{\partial K} h^{s+1} [p]^2 \mathrm{d}s.$$

The operator $\pi_h : [P_0(K)]^2 \to [V_h]^2$, which denotes the lowest order Clément operator is constructed as follows.

$$\pi_h p = \sum_i p_i \varphi_i \tag{2.17}$$

with

$$p_i = \frac{1}{m(\Omega_i)} \sum_{\mathcal{N}_i} p|_{K^i} m(K^i). \tag{2.18}$$

In the following we will also write $p|_{K^i} - p|_K = \sum_{K_i}^{K} [p]$, with $[p]$ denoting the jump across element boundaries and the sum is taken over the shortest "path" from element K^i to element K.

It is now straightforward to show that the projection error is controlled by the operator $\tilde{J}_s(p,p)$

$$\begin{aligned}\|h_K^{s/2}(\pi_h p-\tilde{p})\|^2 &= \sum_K \int_K h_K^s\Big(\sum_{i\in\mathcal{F}_K}(\frac{1}{m(\Omega_i)}\sum_{K^i\in\mathcal{N}_i} p|_{K^i}\; m(K^i))\varphi_i-\tilde{p}\Big)^2\mathrm{d}x\\ &= \sum_K \int_K h_K^s\Big(\sum_{i\in\mathcal{F}_K}\frac{1}{m(\Omega_i)}\sum_{K^i\in\mathcal{N}_i}(p|_{K^i}-p|_K)m(K^i)\varphi_i\Big)^2\mathrm{d}x\\ &\leq C \sum_K \int_K h_K^s\Big(\sum_{i\in\mathcal{F}_K}\frac{1}{m(\Omega_i)}\sum_{K^i\in\mathcal{N}_i}\Big\{\sum_{K_i}^{K}[p]\Big\}^2 m(K^i)\mathrm{d}x\\ &\leq C \sum_K\int_{\partial K} h_K^{s+1}[p]^2\mathrm{d}s \leq C\tilde{J}_s(p,p).\end{aligned}$$

Where we used the upper bound on the number of triangles neighboring to a node and a scaling argument. We have proved the following:

Lemma 4. *If p is some piecewise constant function, $\tilde{p}$ is defined by* (2.15) *and π_h is the Clément interpolant on V_h, then the edge stabilization term satisfies*

$$\|h_K^{s/2}(\pi_h p-\tilde{p})\|^2 \leq \zeta\tilde{J}_s(p,p) \tag{2.19}$$

for some $\zeta \geq \zeta_0 > 0$

From this the stability of our method now follows noting that by Lemma 3 we have $c\|p\| \leq (p,\tilde{p})$.

Remark 2. *Note that by the construction of $\tilde{p}$ we get less stabilization in elements adjacent to Dirichlet boundaries than in the interior of the domain, hence we expect to get poorer stabilizing properties close to sharp out flow layers (when diffusion is present), something which is confirmed by the numerical experiments.*

Remark 3. *When the mesh parameter h and or the velocity β varies in the domain we get using Lemma 4, and assuming for simplicity that β is constant on each element.*

$$\|h_K^{-1/2}(\pi_h h\beta\cdot\nabla U - h\beta\cdot\nabla U)\|^2 \leq \sum_K\int_{\partial K}[h|\beta|\nabla U]^2\mathrm{d}s.$$

Noting that

$$\begin{aligned}[h|\beta|\nabla U] &= h_{K'}|\beta_{K'}|\nabla U|_{K'} - h_K|\beta_K|\nabla U|_K\\ &= \{h|\beta|\}[\nabla U] + [h|\beta|]\{\nabla U\}\end{aligned}$$

, where $\{x\} = (x_{K'}+x_K)/2$ we see that the right hand side may be rewritten as

$$\begin{aligned}\sum_K\int_{\partial K}[h|\beta|\cdot\nabla U]^2\mathrm{d}s \leq C_0 &\sum_K\int_{\partial K}\{h^2|\beta|\}[\nabla U]^2\mathrm{d}s\\ &+C_1\sum_K\int_{\partial K}[h|\beta|]^2\{\nabla U\}^2\mathrm{d}s.\end{aligned}$$

From this we may conclude, assuming a condition on the variation of the mesh and the velocities of the following type $[h|\beta|] \leq c \max(h_K|\beta_K|, h_{K'}|\beta_{K'}|)$, *with* $c << 1$ *and applying a scaling argument in the second term on the right hand side*

$$(\beta \cdot \nabla U, \pi_h h_K \beta \cdot \nabla U - h_K \beta \cdot \nabla U) \leq \alpha\zeta C_0 J(U,U)$$
$$+(\frac{1}{\alpha} + \alpha\zeta c C)\|h^{1/2}\beta \cdot \nabla U\|^2,$$

where the constant C *depends essentially on the shape regularity of the mesh. This shows that stability does not deteriorate with variations in the meshsize.*

2.1.2. *The* inf–sup *condition.* We may now combine the above results to prove a discrete inf–sup condition for our method. We consider the following mesh dependent norm

$$|||u|||^2 = \|h_K^{1/2}\beta \cdot \nabla u\|^2 + \|\varepsilon^{1/2}\nabla u\|^2 + \|\sigma^{1/2}u\|^2 + J(u,u).$$

Theorem 1. *With the triple norm defined above we have for some* α

$$\alpha |||U||| \leq \sup_{w_h \in V_h} \frac{A(U,w_h) + J(U,w_h)}{|||w_h|||}, \quad \forall U \in V_h.$$

Proof The proof is straightforward using the inverse inequalities and Lemma 4 of the previous section. We start by proving that

$$\alpha |||U|||^2 \leq A(U, U + C\pi_h h_K \beta \cdot \nabla U) + J(U, U + C\pi_h h_K \beta \cdot \nabla U).$$

Writing out the right hand side with the term $\|C^{1/2}h_K^{1/2}\beta \cdot \nabla U\|^2$ added and subtracted and using that $(\beta \cdot \nabla U, U) = 0$ leads to.

$$\begin{aligned}
&A(U, U + C\pi_h h_K \beta \cdot \nabla U) + J(U, U + C\pi_h h_K \beta \cdot \nabla U)\\
&\geq \frac{3}{4}\Big(\|C^{1/2}h_K^{1/2}\beta \cdot \nabla U\|^2 + \|\varepsilon^{1/2}\nabla U\|^2 + \|\sigma^{1/2}U\|^2 + J(U,U)\Big)\\
&\quad -\|C^{1/2}\pi_h h_K^{1/2}\beta \cdot \nabla U - C^{1/2}h_K^{1/2}\beta \cdot \nabla U\|^2 - \|\varepsilon^{1/2}\nabla C\pi_h h_K \beta \cdot \nabla U\|^2\\
&\quad -\|\sigma^{1/2}C\pi_h h_K \beta \cdot \nabla U\|^2 - J(C\pi_h h_K \beta \cdot \nabla U, C\pi_h h_K \beta \cdot \nabla U).
\end{aligned}$$

The claim now follows by applying (2.7) in the two last terms, (2.14) and Lemma 4 for the two other non-positive terms and finally choosing C sufficiently small. To conclude we need to show that $\exists c$ such that $|||U + C\pi_h h_K \beta \cdot \nabla U||| \leq c|||U|||$, but this is immediate by the inverse inequalities (2.7) and (2.14). □

2.2. ***A priori* error estimates.** We now proceed to prove a priori error estimates for the discrete solution using the triple norm and the inf–sup condition defined above. For the a priori analysis we need the following approximation result

Lemma 5. *The following interpolation estimate holds:*

$$|||u - \pi_h u||| \leq C(\varepsilon^{1/2}h + h^{3/2} + \sigma^{1/2}h^2)\|u\|_{H^2(\Omega)}.$$

Proof The estimates
$$\|\varepsilon^{1/2}\nabla(u-\pi_h u)\|_{L_2(\Omega)} \le Ch\varepsilon^{1/2}\|u\|_{H^2(\Omega)}$$
and
$$\|\sigma^{1/2}(u-\pi_h u)\|_{L_2(\Omega)} \le Ch^2\sigma^{1/2}\|u\|_{H^2(\Omega)}$$
follow from standard interpolation theory. Further, we have, using (2.8),
$$\begin{aligned}\|\nabla(u-\pi_h u)\|^2_{L_2(\partial K)} &\le C\Big(h_K^{-1}\|\nabla(u-\pi_h u)\|^2_{L_2(K)} + h_K\|u\|^2_{H^2(K)}\Big)\\ &\le Ch_K\|u\|^2_{H^2(K)},\end{aligned}$$
and it follows by summation that $J(u-\pi_h u, u-\pi_h u)^{1/2} \le Ch^{3/2}\|u\|_{H^2(\Omega)}$. □ Using this

interpolation estimate and the consistency we prove the following a priori estimate in the convection dominated case when $\varepsilon < h$.

Theorem 2. *Let $u \in H^2(\Omega)$ be the solution of* (2.2) *and $U \in V_h$ the finite element solution of* (2.4)*; then*
$$|||u-U||| \le C(\varepsilon^{1/2}h + h^{3/2} + \sigma^{1/2}h^2)\|u\|_{H^2(\Omega)}.$$

Proof We decompose the error into
$$|||u-U||| \le |||u-\pi_h u||| + |||U-\pi_h u|||$$
the first part is bounded by Lemma 5 and for the second part we use the inf–sup condition of Theorem 1 and the consistency to obtain, using the notation $\tilde e = U - \pi_h u$
$$\begin{aligned}\alpha|||\tilde e||| &\le sup_{w_h\in V_h}\frac{A(\tilde e, w_h) + J(\tilde e, w_h)}{|||w_h|||}\\ &= sup_{w_h\in V_h}\frac{A(u-\pi_h u, w_h) + J(u-\pi_h u, w_h)}{|||w_h|||},\end{aligned}$$
where nominator may be written
$$\begin{aligned}A(u-\pi_h u, w_h) + J(u-\pi_h u, w_h) &= (\varepsilon\nabla(u-\pi_h u), \nabla w_h)\\ &+(\sigma(u-\pi_h u), w_h) + (\beta\cdot\nabla(u-\pi_h u), w_h)\\ &+J(u-\pi_h u, w_h) = i + ii + iii + iv.\end{aligned}$$
We now bound the four contributions. The first and second terms are handled by applying Cauchy-Schwartz inequality followed by the inverse inequality (2.7)
$$i \le \tilde c\|\varepsilon^{1/2}\nabla(u-\pi_h u)\|\|\varepsilon^{1/2}\nabla w_h\| \le \tilde c\|\varepsilon^{1/2}\nabla(u-\pi_h u)\|\;|||w_h|||$$
$$ii \le \|\sigma^{1/2}(u-\pi_h u)\|\|\sigma^{1/2}w_h\| \le \tilde c\|\sigma^{1/2}(u-\pi_h u)\|\;|||w_h|||.$$
In the third term we integrate by parts in the $\tilde e$ part and use (2.6) in the second part to obtain
$$\begin{aligned}iii \le (u-\pi_h u, \beta\cdot\nabla\tilde w_h) &\le \tilde c\|h_K^{-1/2}(u-\pi_h u)\|\|h_K^{1/2}\beta\cdot\nabla w_h\|\\ &\le \tilde c\|h_K^{-1/2}(u-\pi_h u)\||||w_h|||.\end{aligned}$$

For the edge penalty term finally we simply apply Cauchy-Schwartz inequality

$$\begin{aligned} iv &\leq \tilde{c} J(u-\pi_h u, u-\pi_h u)^{1/2} J(w_h, w_h)^{1/2} \\ &\leq c J(u-\pi_h u, u-\pi_h u)^{1/2} |||w_h||| \end{aligned}$$

The claim now follows by applying Lemma 5. □

We proceed to prove an a priori L_2-estimate for the diffusion dominated case using a duality argument. Consider the following adjoint problem: find $\varphi \in H_0^1(\Omega)$ such that

$$A(v,\varphi) = (\psi, v), \quad \forall v \in H_0^1(\Omega) \tag{2.20}$$

this problem is well-posed and if $\psi = u - U$ then $\|\varepsilon\varphi\|_{H^2(\Omega)} \leq C\|u-U\|$. Using the Aubin-Nitsche duality argument we obtain

Theorem 3. *Assume that ε is bounded away from zero and let $u \in H^2(\Omega)$ be the solution of* (2.2) *and $U \in V_h$ the finite element solution of* (2.4). *Then*

$$\|u-U\| \leq Ch^2 \|u\|_{H^2(\Omega)}$$

Proof Choosing $\psi = v = u - U$ in equation (2.20) we obtain, since $\varphi \in H^2(\Omega)$

$$\begin{aligned} \|u-U\|^2 &= A(u-U,\varphi) = A(u-U,\varphi-\pi_h\varphi) + J(u-U,\pi_h\varphi) \\ &= A(u-U,\varphi-\pi_h\varphi) + J(u-U,\varphi-\pi_h\varphi) \end{aligned}$$

Writing out the different contributions and applying Cauchy-Schwarz in the last term we have

$$\begin{aligned} \|u-U\|^2 &\leq (\beta\cdot\nabla(u-U),\varphi-\pi_h\varphi) + (\sigma(u-U),\varphi-\pi_h\varphi) \\ &\quad +(\varepsilon\nabla(u-U),\nabla(\varphi-\pi_h\varphi)) \\ &\quad +J(u-U,u-U)^{1/2} J(\varphi-\pi_h\varphi,\varphi-\pi_h\varphi)^{1/2} \\ &\leq C|||u-U|||\Big(\|h_K^{-1/2}(\varphi-\pi_h\varphi)\| \\ &\quad +\|\varepsilon^{1/2}(\varphi-\pi_h\varphi)\|_{H^1(\Omega)} + J(\varphi-\pi_h\varphi,\varphi-\pi_h\varphi)^{1/2}\Big). \end{aligned}$$

From Theorem 2 we now that $|||u-U||| \leq C(\varepsilon^{1/2}h + h^{3/2})$ and applying standard interpolation estimates we estimate the norms of the dual solution by

$$\begin{aligned} \|h_K^{-1/2}(\varphi-\pi_h\varphi)\| + \|\varepsilon^{1/2}(\varphi-\pi_h\varphi)\|_{H^1(\Omega)} + J(\varphi-\pi_h\varphi,\varphi-\pi_h\varphi)^{1/2} \\ \leq C\Big(\frac{h^{3/2}}{\varepsilon} + \frac{h}{\varepsilon^{1/2}}\Big)\|\varepsilon\varphi\|_{H^2(\Omega)}. \end{aligned}$$

and consequently $\|u-U\| \leq Ch^2\Big(\frac{h}{\varepsilon} + \frac{h^{1/2}}{\varepsilon^{1/2}} + 1\Big)\|u\|_{H^2(\Omega)}$. □

2.3. ***A posteriori* error estimates.** We consider estimates of general linear functionals of the error, following Becker and Rannacher [1].

Theorem 4. *Let u be a solution to (2.4), let ψ and φ be the data and solution to (2.20), and define $I_\psi(u-U)=(u-U,\psi)$. Then*

$$|I_\psi(u-U)| \leq \sum_K (\rho_K\omega_K + \tilde\rho_K\tilde\omega_K)$$

where

$$\rho_K = \|\sigma U + \beta\cdot\nabla U - f\|_K + h_K^{-1/2}\|[\varepsilon n\cdot\nabla U]\|_{\partial K},\ \tilde\rho_K = \gamma h_K^{1/2}\|[n\cdot\nabla U]\|_{\partial K}$$

and

$$\omega_K = \max\{\|\varphi-\pi_h\varphi\|_K, h_K^{1/2}\|\varphi-\pi_h\varphi\|_{\partial K}\},\ \tilde\omega_K = h_K^{3/2}\|[n\cdot\nabla\pi_h\varphi]\|_{\partial K}.$$

Proof We have, using Lemma 1, that

$$\begin{aligned} I_\psi(u-U) &= A(u-U,\varphi) = A(u-U,\varphi-\pi_h\varphi) - J(U,\pi_h\varphi)\\ &= A(u-U,\varphi-\pi_h\varphi) - J(U,\pi_h\varphi)\\ &= (f,\varphi-\pi_h\varphi) - (\beta\cdot\nabla U+\sigma U,\varphi-\pi_h\varphi) -\\ &\quad (\varepsilon\nabla U,\nabla(\varphi-\pi_h\varphi)) - J(U,\pi_h\varphi). \end{aligned}$$

We note that the stabilizing term is bounded by

$$J(U,\pi_h\varphi) \leq \sum_K \frac{1}{2}\gamma h_K^2\|[n\cdot\nabla U]\|_{\partial K}\,\|[n\cdot\nabla\pi_h\varphi]\|_{\partial K}$$

The desired estimate is then obtained by an integration by parts in the third term together with an element wise application of the Cauchy-Schwartz inequality. □

Remark 4. *The sum over $\tilde\rho_K\tilde\omega_K$ may be replaced by $|J(U,\pi_h\varphi)|$.*

We now prove that for sufficiently regular solution and adjoint solution the stabilizing term contribution is of the right order.

Corollary 1. *If $\varphi\in H^2(\Omega)$ and $u\in H^2(\Omega)$ then*

$$|J(U,\pi_h\varphi)| \leq Ch^2$$

Proof The result is an immediate consequence of the consistency and the interpolation.

$$\begin{aligned} |J(U,\pi_h\varphi)| &= |J(U-u,\pi_h\varphi-\varphi)|\\ &\leq |J(U-u,U-u)|^{1/2}|J(\pi_h\varphi-\varphi,\pi_h\varphi-\varphi)|^{1/2}\\ &\leq C(\varepsilon^{1/2}h + h^{3/2} + \sigma^{1/2}h^2)^2 \end{aligned}$$

where the last inequality follows from Lemma 5 and Theorem 2. □

Remark 5. *We note that the stabilization term $J(U,U)$ will not make convergence deteriorrate when $\varepsilon > h$; hence there is no need to tune the stabilization parameter in such a way that it tends to zero when the fine scales of the flow are resolved to preserve order. This is another advantage of our method compared with the SUPG method.*

2.4. **Monotonicity.** In a recent paper [3] the authors constructed shock-capturing terms, for which they rigorously proved a discrete maximum principle (DMP). This was in the case of the streamline diffusion method and only for strictly acute meshes. These monotonicity results were then developed further in [4]. Here we follow their example and construct an edge based shock-capturing term. Moreover we prove that our method can be tuned with respect to the mesh to satisfy a DMP. We wish to point out that this result differs from the results in [3] in several ways, first of all, the shock-capturing we propose is not residual based, but staying true to our concept of edge stabilization, we use the jumps in the gradient over element edges, this time giving diffusion in the edge tangent direction. This latter concept also permits us to lift the hypothesis of strictly acute meshes, instead the size of the shock-capturing term will depend on the smallest angle of the mesh. Moreover, the right hand side does not play any role in the stabilization, making it possible to use nodal quadrature for source terms and making the shock capturing term independent of data. We proceed by presenting some elementary Lemmas for local minima of piecewise affine functions, for the proofs of which we refer to [4]. We recall the notation of section 2.1, consider some node S_i, let $\mathcal{N}_i$ be the set of all triangles K containing node i, $\Omega_i = \cup_{K\in\mathcal{N}_i}\mathrm{supp}(K)$, $\mathcal{S}_i$ the set of all edges connected to S_i and $\tilde{\mathcal{S}}_i$ the set of all edges in $\bar{\Omega}_i$. Furthermore we denote by v_i the function in V_h, such that $v_i = \delta_{ij}$ in node S_j and by $[x]_e$ we denote the jump of the quantity x across the edge e.

Lemma 6. *Let τ denote a unit vector tangent to the edge e. If $U \in V_h$ and U has a local minimum in the node S_i then*

$$\mathrm{sign}(\tau \cdot \nabla U)\ \tau \cdot \nabla v_i|_e \leq 0 \quad \forall e \in \mathcal{S}_i.$$

Lemma 7. *If $U \in V_h$ and U has a local minimum in the node S_i then*

$$\|U - \bar{U}\|_{L_1(\Omega_i)} \leq C_0 \|h_K \nabla U\|_{L_1(\Omega_i)} \leq C_1 \|h^2_{\partial K}[\nabla U]\|_{L_1(\mathcal{S}_i)} \tag{2.21}$$

where $\bar{U}$ is a constant such that

$$\int_{S_i} (U - \bar{U})\ dx = 0.$$

Lemma 8. *There exists a constant C, depending only on the mesh geometry, such that*

$$\|\nabla v_i\|_{L_\infty(K)} \leq C \min_{\partial K \in \mathcal{S}_i} |\tau \cdot \nabla v_i|$$

and

$$\|\nabla v_i\|_{L_\infty(S_i)} \leq C \min_{e \in \mathcal{S}_i} |\tau \cdot \nabla v_i|$$

Theorem 5. *If $U \in V_h$ is a function such that*

$$A(U, v) + J(U, v) + J_{sc}(U, v) = (f, v) \quad \forall v \in V_h \tag{2.22}$$

with $\sigma = 0$ in $A(U, v)$, $f \geq 0$ and

$$J_{sc}(U, v) = \sum_K \int_{\partial K} \Psi(U)\ \mathrm{sign}(\tau \cdot \nabla U)\, \tau \cdot \nabla v\ \mathrm{d}s$$

where $\Psi(U)|_K = h_K(C_\varepsilon \varepsilon + C_{\beta,\gamma} h_K) \max_{e\in K} |[n \cdot \nabla U]_e|$, then $U \geq 0$.

Proof First some remarks are in order, we notice that the shock capturing term is divided into two parts one of order $h_K\varepsilon$ and the other of order h_K^2. The first contribution is needed to control violations of the DMP due to the Laplace operator discretized on non strictly acute meshes, the other term controls violations of the DMP provoked by the convective term and the stabilization. We will assume that there is a local minimum in the node S_i and test (2.22) with the corresponiding testfunction v_i. First, we integrate by parts to obtain

$$A(U,v_i) = \int_{\mathcal{S}_i} [\varepsilon n\cdot\nabla U] v_i \,\mathrm{d}s + \int_{S_i} (\bar{U}-U)\,\beta\cdot\nabla v_i \,\mathrm{d}x.$$

Next, we apply Lemma 6 to bound the first term

$$\begin{aligned}&\int_{\mathcal{S}_i} [\varepsilon\, n\cdot\nabla U] v_i \,\mathrm{d}s\\ &\qquad + C_\varepsilon \varepsilon \int_{\tilde{\mathcal{S}}_i} h_{\partial K} \max_{e\in K} |[n\cdot\nabla U]_e| \mathrm{sign}(\tau\cdot\nabla U)\,\tau\cdot\nabla v_i \,\mathrm{d}s\\ &\leq \varepsilon\frac{1}{2}\|[n\cdot\nabla U]\|_{L_1(\mathcal{S}_i)} - C_\varepsilon\varepsilon \sum_{K\in\Omega_i} \|\max_{e\in K}|[n\cdot\nabla U]_e|\|_{L_1(\partial K)} \leq 0\end{aligned}$$

with $C_\varepsilon \geq \frac{1}{2}$. In the same manner we write for the second term

$$\begin{aligned}&\int_{\Omega_i} (\bar{U}-U)\,\beta\cdot\nabla v_i \,\mathrm{d}x\\ &\qquad + C_{\gamma,\beta} \sum_{K\in\Omega_i} \int_{\partial K} h^2_{\partial K} \max_{e\in K} |[n\cdot\nabla U]_e|\, \mathrm{sign}(\tau\cdot\nabla U)\,\tau\cdot\nabla v_i \,\mathrm{d}s\\ &\leq \|\bar{U}-U\|_{\Omega_i)} \|\beta\cdot\nabla v_i\|_{L_\infty(\Omega_i)}\\ &\qquad - C_{\gamma,\beta} h_K^2 \sum_{K\in\Omega_i} \|\max_{e\in K}|[n\cdot\nabla U]_e|\|_{L_1(\partial K)} \min_{\mathcal{S}_i} |\tau\cdot\nabla v_i| \leq 0\end{aligned}$$

where the last inequality is a consequence of Lemma 7 and Lemma 8. Consider finally the least-squares stabilization term $J(U,v_i)$:

$$\begin{aligned}&\sum_K \frac{1}{2}\int_{\partial K} \gamma h^2_{\partial K} [n\cdot\nabla U][n\cdot\nabla v_i] \mathrm{d}s\\ &\qquad + C_{\gamma,\beta} \int_{\mathcal{S}_i} h^2_{\partial K} \max_{e\in K} |[n\cdot\nabla U]_e|\, \mathrm{sign}(\tau\cdot\nabla U)\,\tau\cdot\nabla v_i \,\mathrm{d}s\\ &\leq h_K^2 \frac{\gamma}{2} \|[n\cdot\nabla U]\|_{L_1(\tilde{\mathcal{S}}_i)} \|\nabla v_i\|_{L_\infty(S_i)}\\ &\qquad - C_{\gamma,\beta} h_K^2 \|\max_{e\in K}|[n\cdot\nabla U]_e|\|_{L_1(\tilde{\mathcal{S}}_i)} \min_{\mathcal{S}_i} |\tau\cdot\nabla v_i| \leq 0.\end{aligned}$$

It follows that all three contributions are negative which leads to a contradiction, since the right hand side is positive. Hence a function $U\in V_h$ presenting a local minimum in node S_i can not be solution to the discrete problem. The same argument may be repeated if there are several connected nodes which are taking the same minimal value by choosing a test function v which takes the value 1 in all these nodes. Since there can be no local minimum in the interior of the domain and $U = 0$ on the boundary we conclude that $U \geq 0$. $\square$

Remark 6. *We note that this holds true also for elliptic problems, allowing for the discrete maximum principle to hold in this case on meshes that are not strictly acute. However when* $\varepsilon > h$ *we do not expect the shock capturing term to have the right order. The* $\sigma > 0$ *case may be included in the above framework, either by using nodal quadrature (lumped mass) for the source term, or by adding a shock-capturing term tailored to control the source term. For further detail on these issues we refer to* [4].

Remark 7. *The above form of* $\Psi(U)$ *has been chosen in order to enhance clarity of the argument, however it is not the minimal coefficient assuring a DMP. Indeed a more detailed study allows for a minimal shock capturing term where each of the terms is accounted for separately.*

3. Numerical examples

In this section we will illustrate the theoretical results obtained above with some computational experiments.

3.1. **Convection-diffusion-reaction.** The model problem (2.1) is considered, choosing $\sigma = 1$, $\beta = (1, 0)$ and $\varepsilon = 10^{-5}$, corresponding to the convection dominated case. We let $\Omega = [0, 1] \times [0, 1]$ and use two different source terms f in order to get the following exact solutions, see figure 1

- Test case 1: $u = \exp(-\frac{(x-0.5)^2}{a_w} - \frac{3(y-0.5)^2}{a_w})$,
- Test case 2: $u = \frac{1}{2}(1 - \tanh(\frac{x-0.5}{a_w}))$.

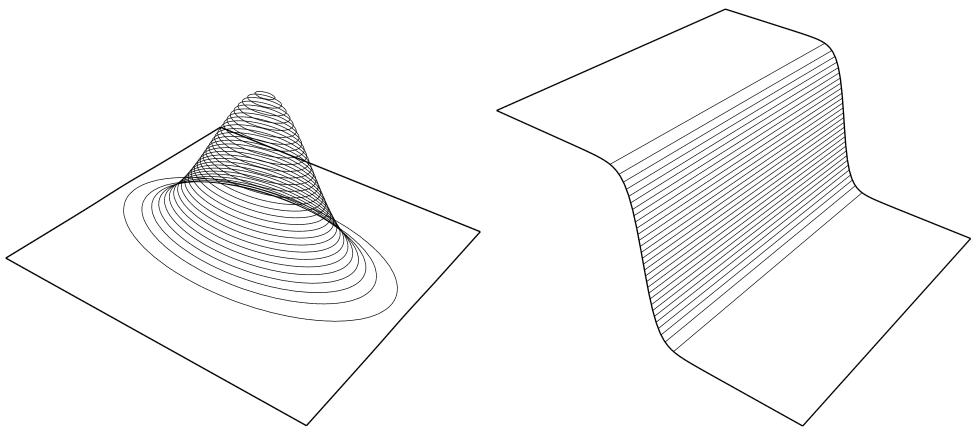

Figure 1. The two exact solutions, left: the Gaussian, right: the hyperbolic tangent.

For the Gaussian the parameter controlling the slope was chosen to $a_w = 0.2$ and for the hyperbolic tangent the parameter was chosen to $a_w = 0.05$. Two different types of meshes

	SD			ES			EC		
N	L_2	H^1	L_∞	L_2	H^1	L_∞	L_2	H^1	L_∞
20	0.0014	0.17	0.0060	0.0014	0.17	0.0060	0.0020	0.14	0.0040
40	3.3 E-4	0.080	0.0014	3.1 E-4	0.080	0.0014	3.7 E-4	0.070	0.0010
80	7.9 E-5	0.040	3.5 E-4	7.7 E-5	0.040	3.5 E-4	8.5 E-5	0.034	2.9 E-4

TABLE 1. Convergence results for test case 1 on mesh 1

	SD			ES			EC		
N	L_2	H^1	L_∞	L_2	H^1	L_∞	L_2	H^1	L_∞
20	0.0023	0.20	0.0070	0.0025	0.20	0.0070	0.0050	0.20	0.010
40	5.4 E-4	0.10	0.0016	5.8 E-4	0.097	0.0017	8.1 E-4	0.097	0.0013
80	1.4 E-4	0.050	3.5 E-4	1.5 E-4	0.048	3.9 E-4	1.7 E-4	0.048	3.3 E-4

TABLE 2. Convergence results for test case 1 on mesh 2

have been used, illustrated in figure 2, both are based on square elements, in the first case (denoted *mesh 1*) they are cut into four triangles and in the other (denoted *mesh 2*) the square elements are cut into two triangles, with the diagonal chosen randomly. We have

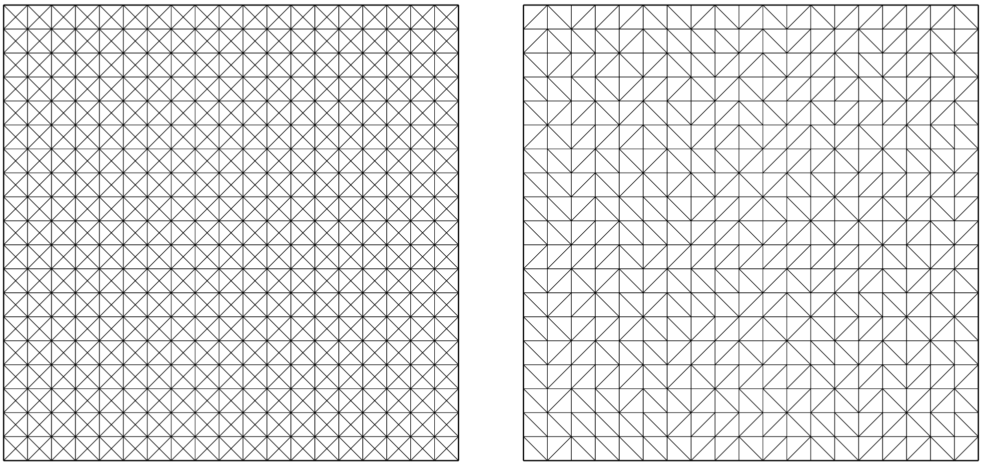

FIGURE 2. The two different meshes used, left mesh 1, right mesh 2

computed the solution using the streamline diffusion method, edge stabilization, with the term given by (2.3) (abbreviated EC) and the one given by (2.5) with $\gamma_{\beta^\perp} = 0$ (abbreviated ES). The stabilization parameter for the edge stabilization was choosen to $\gamma = 0.025$ and no shock capturing was used. The solutions were computed on three consecutive meshes having $N = 20$, $N = 40$ and $N = 80$ elements on each side respectively. We present the errors in the L_2 norm, the H^1 norm and the L_∞ norm for the three methods applied to the two test cases in tables 1 to 4.

For the first test case we note the following approximate convergence orders

	SD			ES			EC		
N	L_2	H^1	L_∞	L_2	H^1	L_∞	L_2	H^1	L_∞
20	0.0051	0.63	0.019	0.0068	0.76	0.039	0.0084	0.6	0.037
40	0.0014	0.34	0.0052	0.0015	0.37	0.0067	0.0015	0.29	0.013
80	3.4 E-4	0.17	0.0013	3.5 E-4	0.18	0.0017	3.3 E-4	0.14	0.0045

TABLE 3. Convergence results for test case 2 on mesh 1

	SD			ES			EC		
N	L_2	H^1	L_∞	L_2	H^1	L_∞	L_2	H^1	L_∞
20	0.015	0.90	0.067	0.017	0.97	0.073	0.013	0.75	0.06
40	0.0060	0.65	0.032	0.0060	0.061	0.03	0.0029	0.35	0.014
80	0.0020	0.45	0.014	0.0020	0.45	0.015	6.6 E-4	0.17	0.0044

TABLE 4. Convergence results for test case 2 on mesh 2

- $\|u - u_h\|_{0,\Omega} \approx O(h^2)$
- $\|\nabla(u - u_h)\|_{0,\Omega} \approx O(h)$
- $\|u - u_h\|_{0,\infty} \approx O(h^2)$

on both meshes. We note that the edge stabilization method ES, using the jumps only in the streamline derivative gives results very similar to that of the streamline–diffusion method, whereas the method EC, where the jump of the whole gradient is used for stabilization gives slightly larger errors on the coarsest mesh. On finer meshes the errors of all three methods are comparable.

For the second test case the results differ dramatically for the two meshes. On mesh 1 the behavior of the three methods compare to that of the previous testcase. One can note a slight degradation in the L_∞ convergence for for the method EC compared to the other methods.

In the last case, test case 2 on mesh 2, the velocity is aligned with the mesh and orthogonal to the gradient, in this case the L_2 norm convergence of SD and ES degenerates to approximately $O(h^{3/2})$, the H^1 norm convergence to $O(h^{1/2})$ and the L_∞ norm convergence degenerates to $O(h)$. The method EC on the other hand, having some intrinsic crosswind diffusion, retains optimal convergence order in both L_2 and H^1 and shows only a minor loss of convergence in L_∞. We conclude that ES, the edge stabilization using the stabilizing term with only streamline derivative jumps (2.5) behaves essentially as the streamline diffusion method, whereas EC where the whole gradient jump is taken into account (2.3) yields a method having the same order but giving somewhat larger errors especially on coarser meshes, on the other hand, this latter method is more robust and does not seem degenerate to $O(h^{3/2})$ in the same fashion as methods giving only diffusion in the streamline direction does.

3.2. **Outflow layers and discrete maximum principle.** In this section we will show qualitatively the loss of stability in outflow layers discussed in remark 2 and how this instability can be countered using the shock capturing term proposed in section 2.4. We propose a classical testcase with a convection–diffusion problem, ($\sigma = 0$, $|\beta| = 1, \eta = 10^{-5}$). The geometry, the boundary conditions and the orientation of β are resumed in figure 3.

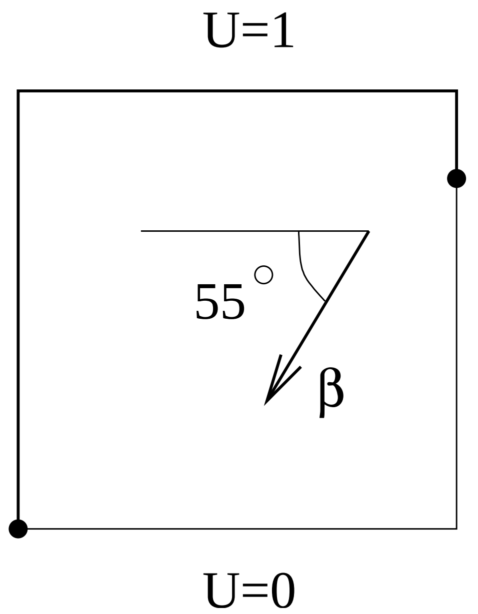

FIGURE 3. Boundary conditions and flow orientation, for outflow layer test case.

As was noted in [4] the DMP satisfying shock capturing methods result in very ill conditioned non-linear equations due to the lack of continuity of the operator. We counter this by regularizing the sign operator, replacing it by sign_ϵ defined by $\text{sign}_\epsilon(x) = \tanh(x/\epsilon)$, we choose $\epsilon = 1$ and $C_{\beta,\gamma} = 10$, a choice for which Newton's method remains reasonably well behaved and spurious oscillations are eliminated. The results of the three methods

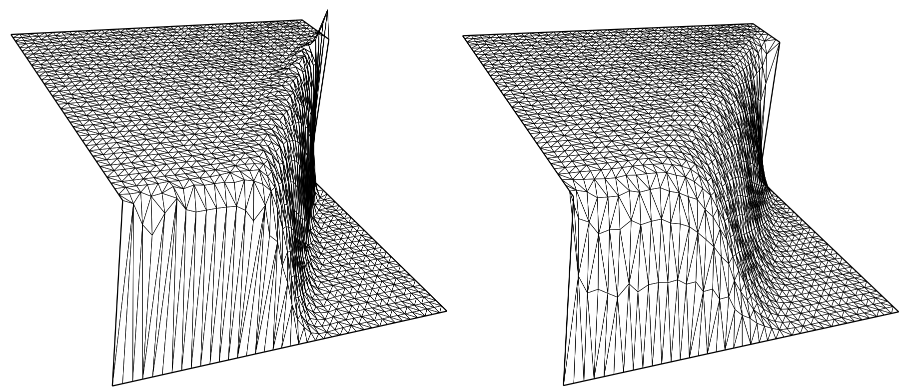

FIGURE 4. Outflow boundary layer testcase using SD, left without shock capturing, right with shock capturing

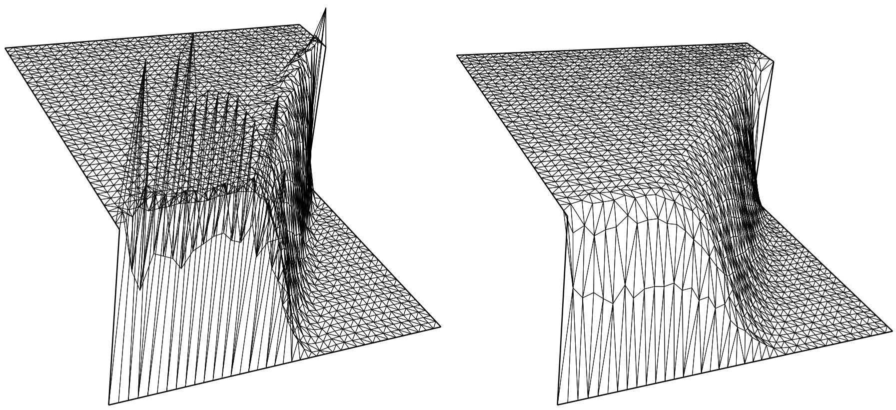

Figure 5. Outflow boundary layer testcase using ES, left without shock capturing, right with shock capturing

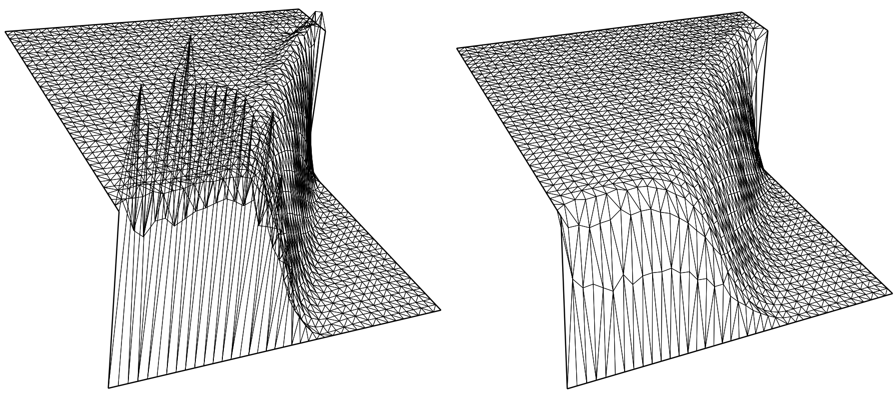

Figure 6. Outflow boundary layer testcase using EC, left without shock capturing, right with shock capturing

method	SD	ES	EC
SC	0.38	0.99	1.2
No SC	20	95	85

Table 5. Maximum violation of the DMP in % for the different methods

applied with and without shock capturing term is presented in figure 4-6. We note the large oscillations on the outflow layer for both edge stabilization approaches. In the case of the streamline diffusion method the violation of the DMP is localised essentially at the inflow in this case. The maximal overshoot for the respective cases are reported in table 5. Although the weaker ouflow stability of the edge stabilization method results in huge overshoots we see that the DMP satisfying shock capturing term wipes them out almost entirely, the remaining violation of the DMP of about one percent is due to the regularization of the sign operator, see [3].

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