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*PREPRINT 2002–18*

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NO 2002–18
ISSN 1404–4382

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Printed in Sweden
Chalmers University of Technology
Göteborg, Sweden 2002

# ADAPTIVE STRATEGIES AND ERROR CONTROL FOR COMPUTING MATERIAL FORCES IN FRACTURE MECHANICS

PER HEINTZ, FREDRIK LARSSON, PETER HANSBO, AND KENNETH RUNESSON

ABSTRACT. The concept of material forces pertains to a variation of the inverse motion map while the placement field is kept fixed. From the weak formulation of the self-equilibrating Eshelby (material) stress tensor it turns out that the classical J-integral formulations in fracture mechanics are just special cases due to the choice of particular weight functions. In this contribution, we discuss a posteriori error control of the material forces as part of an adaptive strategy to reduce the discretization error to an acceptable level. The data of the dual problem involves the quite non-conventional tangent stiffness of the (material) Eshelby stress tensor with respect to a variation of the (physical) strain field. The suggested strategy is applied to the common fracture mechanics problem of a single-edged crack, whereby different strategies for computing the J-integral are compared. We also consider the case in which the crack edges are not parallel, i.e a notch.

## 1. INTRODUCTION

In this paper we discuss a goal-oriented error control algorithm for computing material forces in fracture mechanics. For an in-depth discussion of the theoretical background and the design of the algorithm for computing the material forces from the finite element solution, we refer to Steinmann et al. [1],[2]

The considered goal-quantities are derived from an equilibrium assumption and their corresponding linearization are presented as part of the data to the dual problem. We make a comparison between the different methods and present some numerical results. We also consider the case where the crack edges are not parallel so that the contribution from the edges can not be neglected.

The outline of the paper is as follows: In section 2 we recall the mathematical framework behind the adaptive strategy. In section 3, we give a brief introduction to the concept of material forces and its relation to the classical J-integral formulations in fracture mechanics. In section 4, we discuss some aspects on the chosen goal functionals and their linearization. Finally, in section 5 we consider the model problem of a single edge crack/notch in a linear elastic plate, whereby we present some adaptive results together with some conclusions.

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*Date*: December 20, 2002.

*Key words and phrases.* Adaptivity, Error Control, Fracture mechanics, Material forces.

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## 2. Error Control based on a Dual Solution

We give a short review of the theoretical background and refer to [3], [4] and [5] for more in-depth discussions. See also [6] for work on goal-oriented a posteriori error estimates in elastic fracture mechanics.

The weak form of the primal problem is described with its semi-linear and linear forms

$$\text{Find } \boldsymbol{u} \in V: \quad a(\boldsymbol{u};\boldsymbol{v}) = F(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V. \tag{2.1}$$

The FE approximation yields $\boldsymbol{u}_h \in V_h$ and we are interested in the error in a particular quantity. We denote the exact quantity by $Q(\boldsymbol{u})$ and introduce $E(\boldsymbol{u},\boldsymbol{u}_h)$ that measures the difference between the exact and the approximated quantity

$$E(\boldsymbol{u},\boldsymbol{u}_h) = Q(\boldsymbol{u}) - Q(\boldsymbol{u}_h). \tag{2.2}$$

The directional derivative for $E(\boldsymbol{u},\boldsymbol{u}_h)$ in the direction $\boldsymbol{w}$ is defined as

$$E'(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{w}) = \frac{\partial}{\partial \epsilon} E(\boldsymbol{u}+\epsilon\boldsymbol{w},\boldsymbol{u_h})|_{\epsilon=0}, \tag{2.3}$$

and the secant form of $E(\boldsymbol{u},\boldsymbol{u}_h)$ is obtained as

$$E_S(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{w}) \ = \ \int_0^1 E'(\bar{\boldsymbol{u}}(s),\boldsymbol{u}_h;\boldsymbol{w})\mathrm{d}s, \tag{2.4}$$

where $\bar{\boldsymbol{u}}(s) = \boldsymbol{u}_h + s\boldsymbol{e}$. Choosing $\boldsymbol{w} = \boldsymbol{e} := \boldsymbol{u} - \boldsymbol{u}_h$ leads to

$$E_S(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{e}) = E(\boldsymbol{u},\boldsymbol{u}_h) - E(\boldsymbol{u}_h,\boldsymbol{u}_h) = E(\boldsymbol{u},\boldsymbol{u}_h). \tag{2.5}$$

The dual bilinear form $a_S^*(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{w},\boldsymbol{v})$ is defined as

$$a_S^*(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{w},\boldsymbol{v}) = a_S(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{v},\boldsymbol{w}) = \int_0^1 a'(\bar{\boldsymbol{u}}(s),\boldsymbol{v};\boldsymbol{w})\mathrm{d}s, \tag{2.6}$$

where $a_S(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{v},\boldsymbol{w})$ is the secant stiffness of the primal problem. The abstract variational format of the dual problem is now defined as

$$\text{Find } \boldsymbol{\varphi} \in V: \quad a_S^*(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{\varphi},\boldsymbol{v}) = E_S(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V. \tag{2.7}$$

Using the above definitions, the following exact error representation holds with $\boldsymbol{v} = \boldsymbol{e}$

$$\begin{aligned} E_S(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{e}) &= a_S^*(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{\varphi},\boldsymbol{e}) = a_S(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{e},\boldsymbol{\varphi}) \\ &= a_S(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{e},\boldsymbol{\varphi}-\pi_h\boldsymbol{\varphi}) \\ &= F(\boldsymbol{\varphi}-\pi_h\boldsymbol{\varphi}) - a(\boldsymbol{u}_h;\boldsymbol{\varphi}-\pi_h\boldsymbol{\varphi}). \end{aligned} \tag{2.8}$$

The third equality is obtained using the Galerkin orthogonality. In an adaptive scheme, the error representation formula is evaluated numerically on the element level, and elements with large contribution to the total error are chosen to be refined to the next grid-level.

**Remark**: When the primal problem is linear and symmetrical we have

$$a_S^*(\boldsymbol{u},\boldsymbol{u}_h;\boldsymbol{\varphi},\boldsymbol{v}) = a(\boldsymbol{\varphi},\boldsymbol{v}). \quad \square \tag{2.9}$$

## 3. Material forces and the J-integral in fracture mechanics

3.1. **Preliminaries.** Subsequently, we consider the case of small strains and quasi-static loading conditions. In absence of physical volume forces the physical (Cauchy) stress $\boldsymbol{\sigma}$ fulfills the quasi-static equilibrium equation

$$\nabla \cdot \boldsymbol{\sigma}^{\mathrm{T}} = \mathbf{0} \qquad \text{in } \Omega. \tag{3.1}$$

Similarly, in the absence of material volume forces, the material (Newton-Eshelby) stress $\tilde{\mathbf{\Sigma}}$ is also self-equilibrating

$$\nabla \cdot \tilde{\mathbf{\Sigma}}^{\mathrm{T}} = \mathbf{0} \qquad \text{in } \Omega. \tag{3.2}$$

The Newton-Eshelby stress is defined as

$$\tilde{\mathbf{\Sigma}} \stackrel{\text{def}}{=} W\boldsymbol{I} - \boldsymbol{H}^{\mathrm{T}} \cdot \boldsymbol{\sigma}, \tag{3.3}$$

where we used the notation $\boldsymbol{H}(\boldsymbol{u}) = \boldsymbol{u} \otimes \nabla$ and $W$ is the strain energy density. The definition of $\tilde{\mathbf{\Sigma}}$ in (3.3) is commonly adopted as the generic material stress tensor in the mechanics of fracture and defects. Restricting the present discussion to linear elastic response, we have

$$W(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{\epsilon}(\boldsymbol{u}) : \mathsf{E}^{\mathrm{e}} : \boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{H}(\boldsymbol{u}) : \mathsf{E}^{\mathrm{e}} : \boldsymbol{H}(\boldsymbol{u}), \tag{3.4}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathsf{E}^{\mathrm{e}} : \boldsymbol{\epsilon}(\boldsymbol{u}) = \mathsf{E}^{\mathrm{e}} : \boldsymbol{H}(\boldsymbol{u}), \tag{3.5}$$

where $\boldsymbol{\epsilon}(\boldsymbol{u}) \stackrel{\text{def}}{=} \boldsymbol{H}(\boldsymbol{u})^{\mathrm{sym}}$ is the (small) strain operator and $\mathsf{E}^{\mathrm{e}}$ is the constant elastic stiffness tensor. It appears that $\tilde{\mathbf{\Sigma}}$ can be computed a posteriori when the solution $\boldsymbol{u}$ to the direct motion problem is available.

Consider now an arbitrary subdomain $A \subset \Omega$ with boundary $\Gamma$. From equilibrium (3.2), the resultant to the material tractions along $\Gamma$ must vanish

$$\int_{\Gamma} \tilde{\mathbf{\Sigma}} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = \mathbf{0}, \tag{3.6}$$

where $\boldsymbol{n}$ is the outward unit normal to $\Gamma$. We now consider the case when the boundary is non-smooth such that $\Gamma$ can be decomposed into a regular part $\Gamma^{\mathrm{r}}$ and a singular part $\Gamma^{\mathrm{s}}$ (comprising a notch, a crack tip, etc.) with $\Gamma = \Gamma^{\mathrm{r}} \cup \Gamma^{\mathrm{s}}$ and $\Gamma^{\mathrm{r}} \cap \Gamma^{\mathrm{s}} = \emptyset$, see Figure 1. We use (3.6) to single out the resultant vector force on the singular part $\Gamma^{\mathrm{s}}$

$$\boldsymbol{F}_{\mathrm{mat}} \stackrel{\text{def}}{=} -\int_{\Gamma^{\mathrm{s}}} \tilde{\mathbf{\Sigma}} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = \int_{\Gamma^{\mathrm{r}}} \tilde{\mathbf{\Sigma}} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma. \tag{3.7}$$

We now define the (generalized) J-integral as the projection of $\boldsymbol{F}_{\mathrm{mat}}$ in the direction of a possible unit extension $\boldsymbol{e}$ of the notch, crack, etc

$$J \stackrel{\text{def}}{=} \boldsymbol{e} \cdot \boldsymbol{F}_{\mathrm{mat}} = \int_{\Gamma^{\mathrm{r}}} \boldsymbol{e} \cdot \tilde{\mathbf{\Sigma}} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma. \tag{3.8}$$

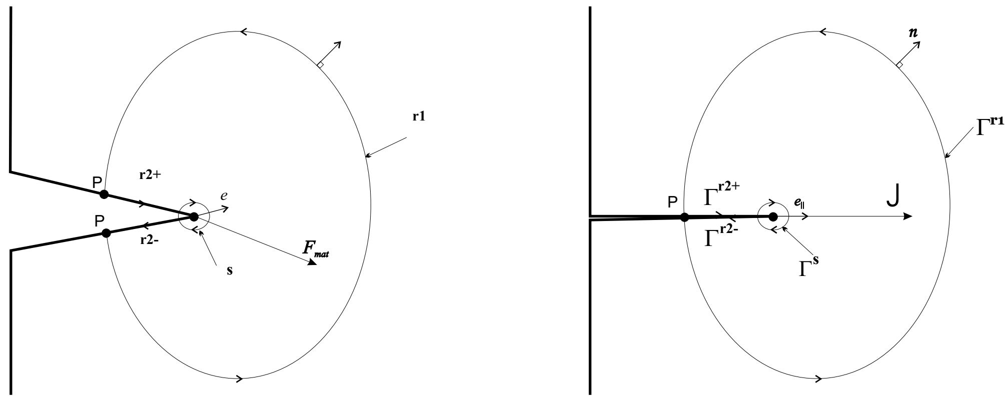


FIGURE 1. Arbitrary subdomain with regular ($\Gamma^{\mathrm{r}} = \Gamma^{\mathrm{r1}} + \Gamma^{\mathrm{r2+}} + \Gamma^{\mathrm{r2-}}$) and singular ($\Gamma^{\mathrm{s}}$) parts. a) Notch with concentrated force acting on singular point and b) straight crack and J-integral.

3.2. **Different formats of the J-integral.** Consider the weak format of (3.2), which can be rewritten as

$$F_{\mathrm{mat}}(\boldsymbol{w}) \stackrel{\mathrm{def}}{=} -\int_{\Gamma^{\mathrm{s}}} \boldsymbol{w}\cdot\tilde{\boldsymbol{\Sigma}}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma = \int_{\Gamma^{\mathrm{r}}} \boldsymbol{w}\cdot\tilde{\boldsymbol{\Sigma}}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma - \int_A \boldsymbol{H}(\boldsymbol{w}) : \tilde{\boldsymbol{\Sigma}}\,\mathrm{d}A, \tag{3.9}$$

for all virtual displacements $\boldsymbol{w}(\boldsymbol{x})$ of sufficient regularity. Next, we consider the special case of a straight, traction-free, crack, which is tested for a possible extension in the same direction, see Figure 1.

It is interesting to note that the J-integral in this case can be retrieved from (3.9) in three different ways:

- **Contour integral format**
  Setting $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{e}_{||}$ (constant) in (3.9) gives

$$J = F_{\mathrm{mat}}(\boldsymbol{e}_{||}) = \int_{\Gamma^{r_1}} \boldsymbol{e}_{||}\cdot\tilde{\boldsymbol{\Sigma}}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma, \tag{3.10}$$

  where it was used that $\boldsymbol{H}(\boldsymbol{e}_{||}) = \boldsymbol{0}$ in the subdomain $A$ and that the contribution from the crack faces vanishes (since $\boldsymbol{e}_{||}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0}$).
- **Domain integral format**
  Setting $\boldsymbol{w}(\boldsymbol{x}) = q(\boldsymbol{x})\boldsymbol{e}_{||}$ with $q = 1$ on $\Gamma^{\mathrm{s}}$ and letting $q$ decay within the chosen domain until $q = 0$ at $\Gamma^{r_1}$, we obtain from (3.9)

$$J \;=\; F_{\mathrm{mat}}(q\boldsymbol{e}_{||}) = -\int_A \boldsymbol{H}(q\boldsymbol{e}_{||}) : \tilde{\boldsymbol{\Sigma}}\,\mathrm{d}\Omega = -\int_A (\boldsymbol{e}_{||}\otimes\nabla q) : \tilde{\boldsymbol{\Sigma}}\,\mathrm{d}\Omega, \tag{3.11}$$

  where it was used that the boundary integral vanishes on $\Gamma^{r_1}$ (since $q = 0$). Introducing the material force resultant

$$\boldsymbol{F}_{\mathrm{mat}}(q) \stackrel{\mathrm{def}}{=} -\int_A \tilde{\boldsymbol{\Sigma}}\cdot\nabla q\,\mathrm{d}\Omega, \tag{3.12}$$

as a vector functional of the chosen $q(\boldsymbol{x})$ within the chosen domain $A$ yields

$$J = \boldsymbol{e}_{||} \cdot \boldsymbol{F}_{\text{mat}}(q). \tag{3.13}$$

**Remark:** In practice, $q(\boldsymbol{x}_c) = 1$ at the crack tip $\boldsymbol{x} = \boldsymbol{x}_c$. $\square$

- **Node integral format**
  As a special case of (3.13), due to the possibility to choose the subdomain arbitrary, we consider the limit situation when $A$ shrinks to nothing. Letting $A$ be parametrizised by a typical radius $r$, we formally obtain

$$J = \lim_{r\to 0} F_{\text{mat}}(q\boldsymbol{e}_{||}) = \lim_{r\to 0} \boldsymbol{e}_{||} \cdot \boldsymbol{F}_{\text{mat}}(q). \tag{3.14}$$

  **Remark**: The notation 'node integral format' alludes to the corresponding finite element approximation, cf. below. $\square$

Further background material on the contour and domain formats for computing $J$ can be found in [7], [8] and [9].

3.3. **Finite element approximation.** In a finite element setting we choose $\boldsymbol{w}(\boldsymbol{x}) \approx \boldsymbol{w}_h(\boldsymbol{x}) = \sum_n N^n(\boldsymbol{x})\boldsymbol{W}^n$, where $N^n$ is the basis function associated with the node at $\boldsymbol{x} = \boldsymbol{x}_n$ and $\boldsymbol{W}^n$ is the nodal displacement vector. Hence, we obtain

$$F_{\text{mat}}(\boldsymbol{w}_h) \;=\; \sum_n \boldsymbol{W}^n \cdot \boldsymbol{F}^n_{\text{mat}}, \tag{3.15}$$

where we introduced

$$\boldsymbol{F}^n_{\text{mat}} \;\stackrel{\text{def}}{=}\; \int_{\Gamma^{\text{r}}_n} N^n \tilde{\boldsymbol{\Sigma}} \cdot \boldsymbol{n}\, \text{d}\Gamma - \int_{A_n} \tilde{\boldsymbol{\Sigma}} \cdot \nabla N^n \,\text{d}\Omega, \tag{3.16}$$

i.e. $\boldsymbol{F}^n_{\text{mat}}$ is the material nodal force (vector) associated with the basis function $N^n$ for $n = 1, 2, 3, .., N$, and where $A_n \subset A$ is the part of $A$ where $N^n$ has support. Moreover, $\Gamma_n$ is the boundary of $A_n$ and in order to account for singularities we decompose $\Gamma_n$ as $\Gamma_n = \Gamma^{\text{r1}}_n \cup \Gamma^{\text{r2+}}_n \cup \Gamma^{\text{r2-}}_n \cup \Gamma^{\text{s}}_n$. Expanding $q(\boldsymbol{x}) = \sum_n N^n(\boldsymbol{x})Q_n$, we may thus summarize:

- **Domain integral format**

$$J_h(\boldsymbol{u}) = -\sum_n Q_n \int_{A_n} (\boldsymbol{e}_{||} \otimes \boldsymbol{\nabla} N^n) : \tilde{\boldsymbol{\Sigma}}(\boldsymbol{u}) \,\text{d}\Omega. \tag{3.17}$$

- **Node integral format**

$$J_h(\boldsymbol{u}) = -\int_{A_{n_c}} (\boldsymbol{e}_{||} \otimes \boldsymbol{\nabla} N^{n_c}) : \tilde{\boldsymbol{\Sigma}}(\boldsymbol{u}) \,\text{d}\Omega. \tag{3.18}$$

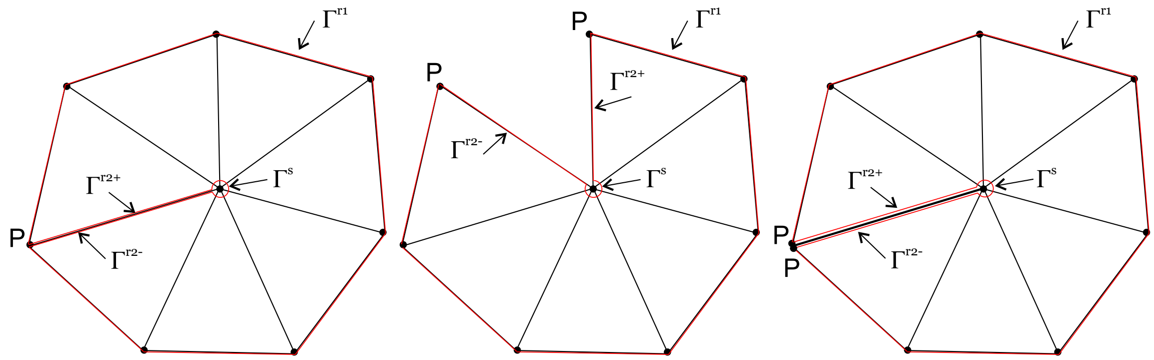


FIGURE 2. Three different situations: a) internal node, b) external node and c) crack tip node.

3.4. **The internal, external(notch) and crack tip node.** Consider the three different situations schematically described in Figure 2.

- **Internal node**
  In this case $N^n = 0$ on $\Gamma_n^{\mathrm{r1}}$. Moreover, the contributions from $\Gamma_n^{\mathrm{r2+}}$ and $\Gamma_n^{\mathrm{r2-}}$ are (in the continuous formulation) self-equilibrating. This yields

$$\boldsymbol{F}_{\mathrm{mat}}^n = -\int_{A_n} \tilde{\boldsymbol{\Sigma}} \cdot \nabla N^n \, d\Omega. \tag{3.19}$$

  In the discrete setting there are jumps in the tractions along $\Gamma_n^{\mathrm{r2}}$, hence, the contribution is in general not self-equilibrating. For a sufficient resolution, however, the contribution is small and could be neglected without loss of accuracy.
- **External (Notch) node**
  In this case $N^n = 0$ on $\Gamma_n^{\mathrm{r1}}$, whereas $N^n \neq 0$ on $\Gamma_n^{\mathrm{r2+}} \cup \Gamma_n^{\mathrm{r2-}}$. Hence, the contribution from the surface integral does not vanish and we have

$$\boldsymbol{F}_{\mathrm{mat}}^n = \int_{\Gamma_n^{\mathrm{r2}}} N^n \tilde{\boldsymbol{\Sigma}} \cdot \boldsymbol{n} \, d\Gamma - \int_{A_n} \tilde{\boldsymbol{\Sigma}} \cdot \nabla N^n \, d\Omega \tag{3.20}$$

  In case there are no physical tractions along $\Gamma_n^{\mathrm{r2+}} \cup \Gamma_n^{\mathrm{r2-}}$ we have $\tilde{\Sigma} \cdot \boldsymbol{n} = W\boldsymbol{n}$ and the material tractions will not vanish.
- **Crack tip node**
  A special case, the notch degenerates into a crack with parallel surfaces. Further, in the case that we want to compute the J-integral of a straight, traction free crack for a possible straight extension, then $\boldsymbol{e}_{||} \cdot \boldsymbol{n} = 0$ and the contribution from $\Gamma_n^{\mathrm{r2}}$ vanishes. Hence, (3.20) becomes

$$\boldsymbol{F}_{\mathrm{mat}}^n = -\int_{A_n} \tilde{\boldsymbol{\Sigma}} \cdot \nabla N^n \, d\Omega \tag{3.21}$$

  For $n = n_c$, defining the crack tip node, we may compute the FE-discretized $J \approx J_h$ from the domain or node integral format by setting $n = n_c$ in (3.21).

## 4. Goal quantities and their linearization

**4.1. Contour integral format.** Choosing the contour integral format for computing $J$ leads to

$$Q(\boldsymbol{u}) = \int_\Gamma \boldsymbol{e}_{||} \cdot \tilde{\boldsymbol{\Sigma}}(\boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma, \tag{4.1}$$

with the associated linearization

$$E'(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{w}) = \int_\Gamma \left(\boldsymbol{e}_{||} \otimes \boldsymbol{n}\right) : \tilde{\mathbf{C}} : \boldsymbol{H}(\boldsymbol{w}) \, \mathrm{d}\Gamma, \tag{4.2}$$

where $\tilde{\mathbf{C}}$ is the tangent stiffness tensor associated with the Newton-Eshelby material stress $\tilde{\boldsymbol{\Sigma}}$ with respect to a variation of the (physical) strain field:

$$\tilde{\mathbf{C}} = \boldsymbol{I} \otimes \boldsymbol{\sigma} - \boldsymbol{I} \underline{\otimes} \boldsymbol{\sigma} - \boldsymbol{H}^T \cdot \mathbf{E}^{\mathrm{e}}. \tag{4.3}$$

In practice, we linearize $\tilde{\mathbf{C}}$ using a p-refined approximation for $\boldsymbol{u}$.

The contour $\Gamma$ can, without restriction on the theoretical applicability, be chosen as a path of element edges (defining the boundary of $A$) in the finite element subdivision of $\Omega$.

**4.2. Domain integral format.** Choosing the domain integral format for computing $J$ leads to

$$Q(\boldsymbol{u}) = -\int_A \left(\boldsymbol{e}_{||} \otimes \boldsymbol{\nabla} q\right) : \tilde{\boldsymbol{\Sigma}}(\boldsymbol{u}) \, \mathrm{d}\Omega, \tag{4.4}$$

with the associated linearization

$$E'(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{w}) = -\int_A \left(\boldsymbol{e}_{||} \otimes \boldsymbol{\nabla} q\right) : \tilde{\mathbf{C}}(\bar{\boldsymbol{u}}) : \boldsymbol{H}(\boldsymbol{w}) \, \mathrm{d}A. \tag{4.5}$$

In the finite element approximation of $Q(\boldsymbol{u}) = J_h(\boldsymbol{u}) \approx J_h(\boldsymbol{u}_h)$, the expression (3.17) is used.

**4.3. Node integral format.** Choosing the node integral format for computing $J$ leads to (in practice)

$$Q(\boldsymbol{u}) = -\lim_{r \to 0} \int_{A_{n_c}} \left(\boldsymbol{e}_{||} \otimes \boldsymbol{\nabla} N^{n_c}\right) : \tilde{\boldsymbol{\Sigma}}(\boldsymbol{u}) \, \mathrm{d}\Omega, \tag{4.6}$$

where $r$ is a typical diameter of $A_{n_c}$. The associated linearization obviously becomes

$$E'(\boldsymbol{u}, \boldsymbol{u}_h; \boldsymbol{w}) = -\lim_{r \to 0} \int_{A_{n_c}} \left(\boldsymbol{e}_{||} \otimes \boldsymbol{\nabla} N^{n_c}\right) : \tilde{\mathbf{C}}(\bar{\boldsymbol{u}}) : \boldsymbol{H}(\boldsymbol{w}) \, \mathrm{d}A. \tag{4.7}$$

In the finite element approximation $Q(\boldsymbol{u}) = J_h(\boldsymbol{u}) \approx J_h(\boldsymbol{u}_h)$, the expression in (3.18) is used.

**Remark**: At any 'regular' node, the associated material force $\boldsymbol{F}^n_{\mathrm{mat}}$ should vanish at sufficient mesh refinement, i.e.,

$$\lim_{h \to 0} \boldsymbol{e}_{||} \cdot \boldsymbol{F}^n_{\mathrm{mat}} = \begin{pmatrix} J & \text{when} & n = n_c \\ 0 & \text{when} & n \neq n_c \end{pmatrix}. \tag{4.8}$$

Hence, it would be possible to define a global measure $Q$ that accounts for this fact. For example, choosing $A = \Omega$ and $\boldsymbol{w} = \boldsymbol{u}$ (the exact solution), we expect to obtain $F_{\text{mat}}(\boldsymbol{u}) = \boldsymbol{u}(\boldsymbol{x_c}) \cdot \boldsymbol{F}_{\text{mat}}$. At least, the values $\boldsymbol{F}^n_{\text{mat}}$ should give information that can be used to move the nodes and improve the quality of the mesh. Indeed, such a strategy was recently suggested by Mueller et al. [10]. □

## 5. A comparison between the integral formats for computing J

Consider the pre-cracked plate in Figure 3 with the dimension w = 0.5m, h = 1.0m, and crack length a = 0.1m. The plate is loaded along its upper boundary by the traction $\boldsymbol{t} = [0, 1\text{e}{+}06]^T$N. The elasticity parameters are E = 210e+09Pa and $\nu = 0.3$.

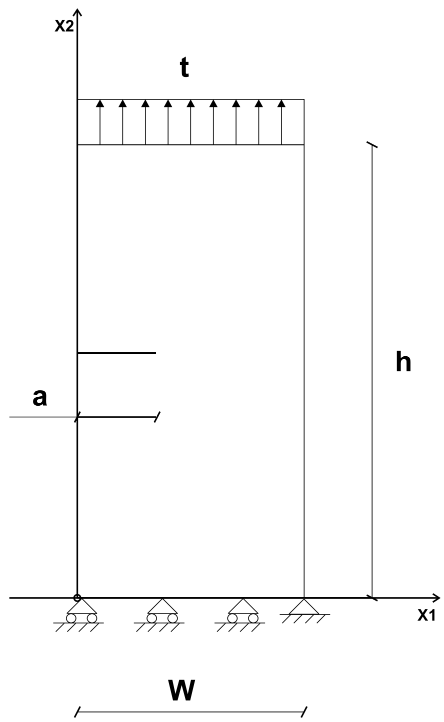


Figure 3. Plate with a single-edged crack of length a.

Figure 4 shows how the different methods used for computing $J$ result in different adapted grids to meet the required tolerance $TOL = 0.5\%$. Figures 5-6 show the convergence rate and the effectivity index $\eta$, respectively. In these figures we use the notation

$$\bar{h} = \frac{\text{m}(\Omega)}{\sqrt{N^{el}}}, \qquad \eta = \frac{F(\tilde{\boldsymbol{\varphi}} - \boldsymbol{\pi}_h\tilde{\boldsymbol{\varphi}}) - a(\boldsymbol{u}_h; \tilde{\boldsymbol{\varphi}} - \boldsymbol{\pi}_h\tilde{\boldsymbol{\varphi}})}{Q(\boldsymbol{u}) - Q(\boldsymbol{u}_h)}, \tag{5.1}$$

where $N^{el}$ is the number of elements in the mesh. The improved dual solution $\tilde{\boldsymbol{\varphi}}$ is computed with quadratic approximation. In Figure 5 we can see that the convergence rate is much higher for the contour and domain integral methods. The reason for this is that linear elements do not capture the $\frac{1}{\sqrt{r}}$ singularity very well. The plot in Figure 6 shows that the error representation formula for the node integral method gives an error estimate approximately 40% below the actual error, which should be compared with $10-20\%$ for the other considered methods.

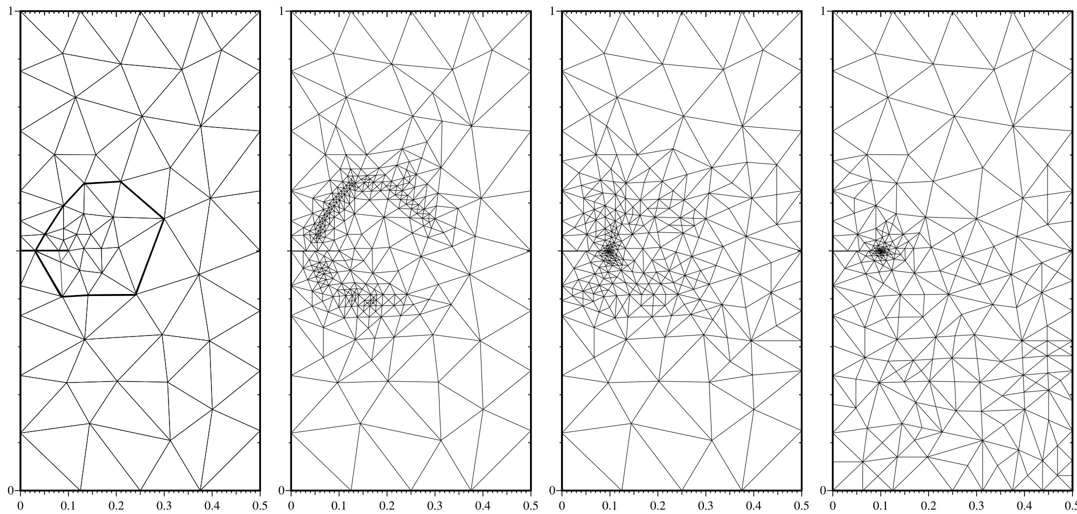

FIGURE 4. Adaptive meshes for the different methods used to compute $J$. From left: initial mesh and contour $\Gamma$, contour integral method, domain integral method and, finally, node integral method.

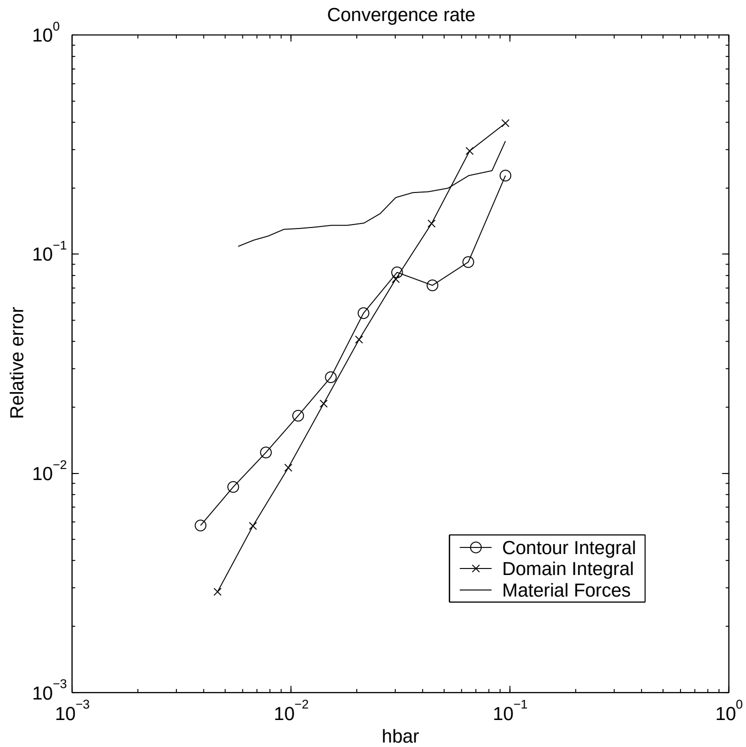


FIGURE 5. Convergence rate for different methods to compute $J$.

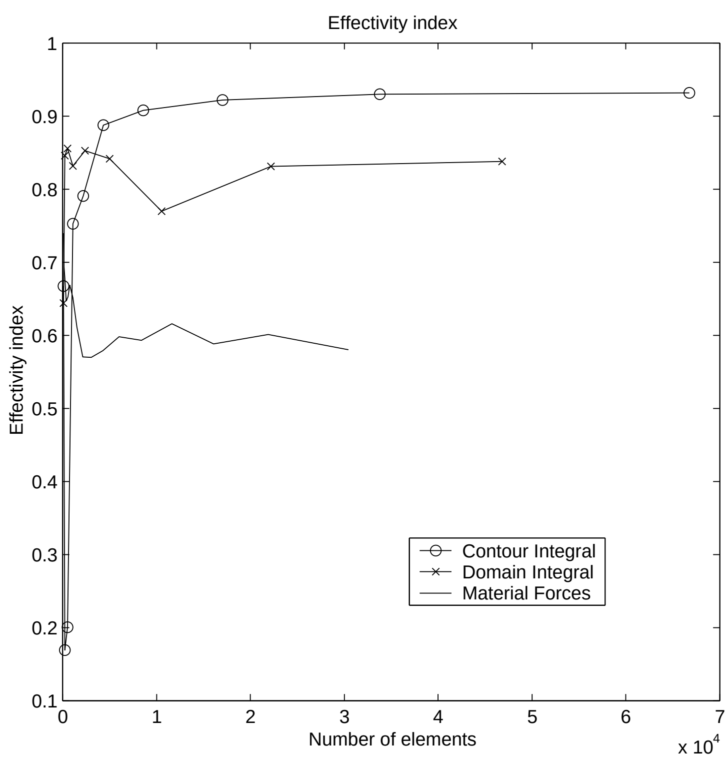


FIGURE 6. Effectivity index for different methods to compute $J$.

5.1. **Edge notch.** We now concider the case when the crack edges are not parallel such that the contribution from $\Gamma^{r2}$ is not zero.

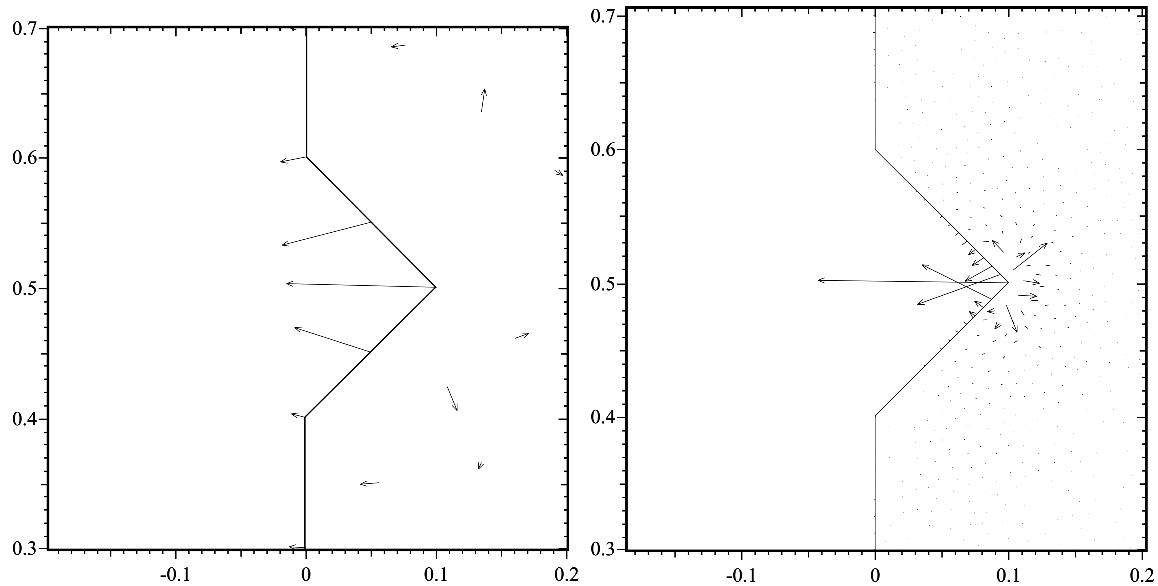


FIGURE 7. The discrete material force vectors acting on the nodes. a) Coarse mesh and large material forces along the notch boundary and b) fine mesh and more localized boundary effect.

In Figure 7 we show the resulting discrete nodal material force vectors obtained for two different meshes. As expected, the internal and external forces decrease for a refined mesh. The large forces acting around the crack tip node are due to the inability to resolve the singularity with ordinary elements. In this figure we use linear approximation, i.e. piecewise constant stresses. An asymptotic study of the contribution from the boundary term, for a sequence of refined grids is shown in Figure 8. For a typical non-singular node on the boundary the material forces vanish for a sufficiently refined mesh. However, for a singular (crack tip) node the contribution is still about 1%. This shows that the boundary term can not be neglected when computing the material force vector for a singular node.

5.2. **Summary and outlook.** We have recalled the theory for goal-oriented adaptivity together with an introduction to the 'material force' concept. An adaptive strategy was applied to the fracture mechanics problem of a straight traction free crack, whereby different methods to compute the energy release rate were compared. The results show that the contour and domain integral formulations yield the best results in terms of convergence rates and effectivity index. It was also noticed that the contribution from the crack faces cannot be neglected in case the crack edges are not perpendicular to the anticipated direction of crack extension. Future development would include the incorporation of a fictitious crack with the appropriate cohesive law. Moreover, curved cracks and closed cracks with friction, where the boundary term also includes the physical tractions, should be studied.

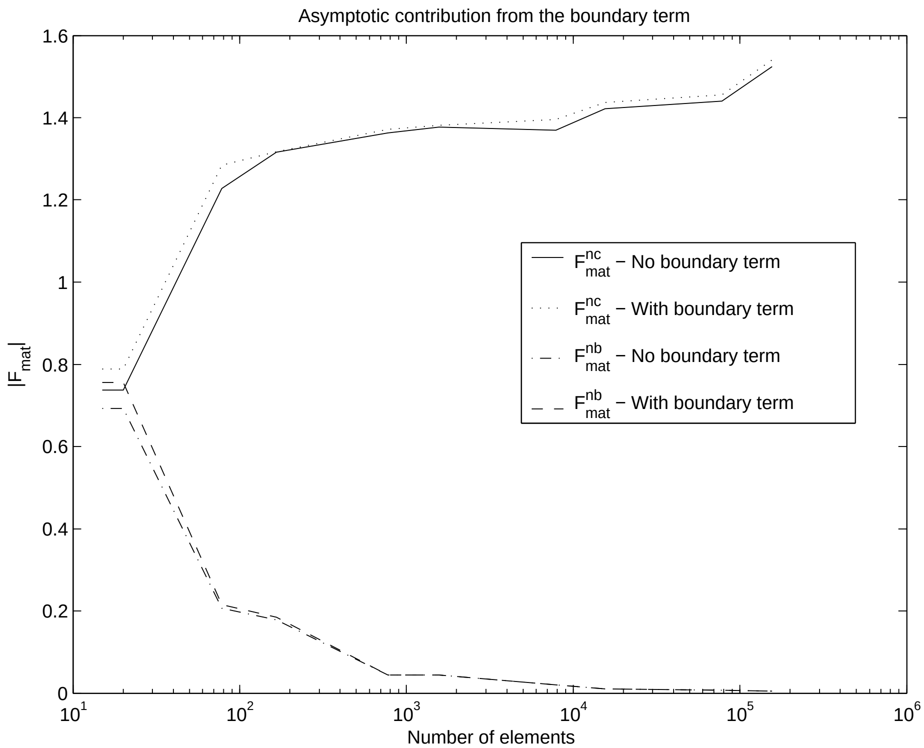


FIGURE 8. This plot shows the contribution from the boundary term. The length of the material force vector vs the of number of elements. We use the superscript *nc* for the singular (crack tip) node and *nb* for a regular (boundary) node

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