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SMOOTHING PROPERTIES AND APPROXIMATION OF TIME DERIVATIVES IN MULTISTEP BACKWARD DIFFERENCE METHODS FOR LINEAR PARABOLIC EQUATIONS

YUBIN YAN

Abstract. In this paper we consider smoothing properties and time derivative approximation in multistep backward difference methods for nonhomogeneous parabolic equations. Smoothing properties and time derivative approximations in single step methods for homogeneous parabolic equations have been studied in Hansbo [5], Yan [12], [13]. We extend the similar results in Yan [12] to the multistep backward difference methods.

1. Introduction

In this paper we shall consider the smoothing properties and the approximation of time derivatives in multistep backward difference methods for the following nonhomogeneous linear parabolic equation

$$u_t + Au = f, \quad \text{for } t > 0, \quad \text{with } u(0) = v, \tag{1.1}$$

in a Hilbert space H with norm $\|\cdot\|$, where $u_t = du/dt$ and A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, where $v \in H$ and f is a function of t with values in H.

The theory of stability and error estimates for the approximation of the solution of (1.1) by a multistep method have been well developed, see Becker [1], Bramble, Pasciak, Sammon, and Thomée [2], Crouzeix [3], Hansbo [6], LeRoux [7], [8], Palencia and Garcia-Archilla [9], Savaré [10], Thomée [11], and the references there in. The smoothing properties and the approximation of time derivatives in single step methods for homogeneous parabolic problems have been studied by Hansbo [5], [6], Yan [12], [13].

This paper is related to Yan [12]. Let us first recall the main results in Yan [12]. Consider (1.1) with $f = 0$, i.e.,

$$u_t + Au = 0 \quad \text{for } t > 0, \quad \text{with } u(0) = v. \tag{1.2}$$

Let $U^n, n \geq 1$, be an approximation of the solution $u(t_n)$ of (1.2) at time $t_n = nk$, where k is the time step, defined by a single step method,

$$U^n = r(kA)U^{n-1}, \quad \text{for } n \geq 1, \quad \text{with } U^0 = v, \tag{1.3}$$

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where the rational function $r(\lambda)$ is accurate of order $p \geq 1$, i.e.,

$$r(\lambda) - e^{-\lambda} = O(\lambda^{p+1}), \quad \text{as } \lambda \to 0. \tag{1.4}$$

Let $j \geq 1$. Define the following finite difference quotient, with some nonnegative integers m_1, m_2 and real numbers c_ν,

$$Q_k^j U^n = \frac{1}{k^j} \sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu}, \quad \text{for } n \geq m_1. \tag{1.5}$$

Assume that Q_k^j is an approximation of order $p \geq 1$ to the time derivative D_t^j, that is, for any smooth real-valued function u,

$$D_t^j u(t_n) = Q_k^j u^n + O(k^p), \quad \text{as } k \to 0, \quad \text{with } u^n = u(t_n). \tag{1.6}$$

We then have the following smooth data error estimates

$$\|Q_k^j U^n - D_t^j u(t_n)\| \leq C k^p \|A^{p+j} v\|, \quad \text{for } n \geq m_1,\ v \in \mathcal{D}(A^{p+j}). \tag{1.7}$$

Further, if $|r(\infty)| < 1$, then we have the following smoothing properties

$$\|Q_k^j U^n\| \leq C t_n^{-j} \|v\|, \quad \text{for } n \geq m_1,\ t_n > 0,\ v \in H, \tag{1.8}$$

and nonsmooth data error estimates

$$\|Q_k^j U^n - D_t^j u(t_n)\| \leq C k^p t_n^{-(p+j)} \|v\|, \quad \text{for } n \geq m_1,\ t_n > 0,\ v \in H. \tag{1.9}$$

The purpose of this paper is to extend the above results for homogeneous parabolic equation, which is approximated by a single step method, to the nonhomogeneous parabolic equation, which will be approximated by a multistep backward difference method.

We introduce the backward difference operator $\bar{\partial}_p$, $p \geq 1$, by

$$\bar{\partial}_p U^n = \sum_{j=1}^{p} \frac{k^{j-1}}{j} \bar{\partial}^j U^n, \quad \text{where } \bar{\partial} U^n = (U^n - U^{n-1})/k. \tag{1.10}$$

With $U^0, \cdots, U^{p-1}$ given, we define our approximate solution U^n by

$$\bar{\partial}_p U^n + A U^n = f^n, \quad \text{for } n \geq p, \quad \text{where } f^n = f(t_n). \tag{1.11}$$

It is well known from the theory for numerical solution of ordinary differential equations, see, e.g., Hairer and Wanner [4], that this method is $A(\theta)$-stable for some $\theta = \theta_p > 0$ when $p \leq 6$. The error estimates for such method has been studied in Bramble, Pasciak, Sammon, and Thomée [2]. It is easy to see that, for any smooth real-valued function u, see Thomée [11, Chapter 10],

$$u_t(t_n) = \bar{\partial}_p u^n + O(k^p), \quad \text{as } k \to 0, \quad \text{with } u^n = u(t_n). \tag{1.12}$$

In Theorem 2.1 below, we obtain the following smoothing property: if U^n is the solution of (1.11) with $f = 0$, then we have, with $p \leq 6$,

$$\|\bar{\partial}_p U^n\| \leq C t_n^{-1} \sum_{j=0}^{p-1} \|U^j\|, \quad \text{for } n \geq 2p.$$

It is natural to approximate the time derivative $u_t(t_n)$ of the solution of (1.1) by $\bar{\partial}_p U^n$ $(n \geq 2p)$, where U^n, $n \geq p$, is computed by the multistep backward difference method (1.11). In Theorems 3.1 and 3.4, we obtain the following error estimates

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \leq C \sum_{j=0}^{p-1} \|A(U^j - u^j)\| + Ck^p \int_0^{t_n} \|Au^{(p+1)}(s)\| \, ds, \quad \text{for } n \geq 2p,$$

and, with $G(s) = |u^{(p+1)}(s)|_{-2p-1}^2 + s^{2p+2}|u^{(p+1)}(s)|_1^2 + s^2|u_t(s)|_1^2$,

$$t_n^{2p+2}\|\bar{\partial}_p U^n - u_t(t_n)\|^2 \leq C \sum_{j=p}^{2p-1} \Big(|U^j - u^j|_{-2p}^2 + k^{2p+2}\|A(U^j - u^j)\|^2\Big) + Ck^{2p}\Big(\int_0^{t_n} G(s)\, ds + t_{2p}^3|u_t(t_{2p})|_1^2\Big),$$

respectively.

When we choose some suitable discrete starting values $U^0, U^1, \cdots, U^{p-1}$, we get the following nonsmooth data error estimates, with $f = 0$ and $p \leq 6$,

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \leq Ck^p t_n^{-p-1}\|v\|, \quad \text{for } n \geq 2p.$$

By C and c we denote large and small positive constants independent of the functions and parameters concerned, but not necessarily the same at different occurrences. When necessary for clarity we distinguish constants by subscripts.

2. Smoothing properties

In this section we will show the smoothing properties for the multistep backward difference method. Before showing this, we first discuss some properties of the backward difference operator $\bar{\partial}_p$ defined by (1.10). We first note that (1.10) can be written in another form, see, e.g., Yan [12],

(2.1)
$$\bar{\partial}_p U^n = k^{-1} \sum_{\nu=0}^{p} c_\nu U^{n-\nu},$$

where the coefficients c_ν are independent of k. Introducing $P(x) = \sum_{\nu=0}^{p} c_\nu x^\nu$, it is easy to check that (1.12) is equivalent to

(2.2)
$$P(e^{-\lambda}) - \lambda = O(\lambda^{p+1}), \quad \text{as } \lambda \to 0.$$

In fact, with $u(t) = e^t$ in (1.12), we have

$$P(e^{-k}) - k = O(k^{p+1}), \quad \text{as } k \to 0,$$

replacing k by λ, we show (2.2). On the other hand, if (2.2) holds, (1.12) follows from Taylor expansion of $\bar{\partial}_p u^n$ at t_n.

For $p = 1$, (1.11) reduces to the backward Euler method

$$(U^n - U^{n-1})/k + AU^n = f^n, \quad \text{for } n \geq 1,$$

and the starting value is $U^0 = v$.

For $p = 2$, we have

$$(\tfrac{3}{2}U^n - 2U^{n-1} + \tfrac{1}{2}U^{n-2})/k + AU^n = f^n, \quad \text{for } n \geq 2,$$

and both U^0 and U^1 are needed to start the procedure.

Bramble, Pasciak, Sammon, and Thomée [2] obtain the following stability result, i.e., with U^n the solution of (1.11),

$$\|U^n\| \leq C \sum_{j=0}^{p-1} \|U^j\| + Ck \sum_{j=p}^{n} \|f^j\|, \quad \text{for } n \geq p. \tag{2.3}$$

In this paper we first show the following smoothing property for the multistep backward difference method.

Theorem 2.1. *Let $p \leq 6$. Then there is a constant C, independent of the positive definite operator A, such that for the solution U^n of (1.11) with $f = 0$,*

$$\|\bar{\partial}_p U^n\| \leq C t_n^{-1} \sum_{j=0}^{p-1} \|U^j\|, \quad \textit{for } n \geq 2p. \tag{2.4}$$

To prove this theorem, we need the following lemma from Thomée [11, Lemma 10.3].

Lemma 2.2. *The solution of (1.11) may be written, with $g^j = kf^j = kf(t_j)$,*

$$U^n = \sum_{j=p}^{n} \beta_{n-j}(kA) g^j + \sum_{s=0}^{p-1} \beta_{ns}(kA) U^s, \quad \textit{for } n \geq p, \tag{2.5}$$

where the $\beta_j(\lambda)$ and $\beta_{ns}(\lambda)$ are defined by, with $\lambda > 0$, $P(\zeta) = \sum_{\nu=0}^{p} c_\nu \zeta^\nu$,

$$\sum_{j=0}^{\infty} \beta_j(\lambda)\zeta^j := (P(\zeta) + \lambda)^{-1}, \qquad \beta_{ns}(\lambda) = \sum_{j=p-s}^{p} \beta_{n-s-j}(\lambda) c_j. \tag{2.6}$$

If $p \leq 6$, there are positive constants c, C and λ_0 such that

$$|\beta_j(\lambda)| \leq \begin{cases} Ce^{-cj\lambda}, & \textit{for } 0 < \lambda \leq \lambda_0, \\ C\lambda^{-1} e^{-cj}, & \textit{for } \lambda \geq \lambda_0. \end{cases} \tag{2.7}$$

Proof of Theorem 2.1. By (2.5) and (2.1), we find that

$$\bar{\partial}_p U^n = k^{-1} \sum_{\nu=0}^{p} c_\nu \sum_{s=0}^{p-1} \beta_{(n-\nu)s}(kA) U^s \equiv \sum_{s=0}^{p-1} \beta'_{ns}(kA) U^s,$$

where obviously we require that $n - \nu \geq p$ $(0 \leq \nu \leq p)$ which implies $n \geq 2p$, and where $\beta'_{ns}(\lambda)$ are some functions of λ. Since $\bar{\partial}_p U^n$ is linearly dependent on U^s $(0 \leq s \leq p-1)$, it suffices to consider separately the cases when all terms but one on the right of (2.4) vanish.

We consider the case when $U^l \neq 0,\ 0 \leq l \leq p-1$ and $U^s = 0,\ 0 \leq s \leq p-1,\ s \neq l$.

In the case $0 < l \leq p-1$, we need to show

$$\|\bar{\partial}_p U^n\| \leq C t_n^{-1} \|U^l\|. \tag{2.8}$$

By Lemma 2.2, we have

$$\begin{aligned}\bar{\partial}_p U^n &= k^{-1}\sum_{\nu=0}^{p} c_\nu\big(\beta_{(n-\nu)l}U^l\big) = k^{-1}\sum_{\nu=0}^{p} c_\nu\Big(\sum_{j=p-l}^{p}\beta_{n-\nu-l-j}(kA)c_j\Big)U^l \\ &= k^{-1}\sum_{j=p-l}^{p}\Big(\sum_{\nu=0}^{p} c_\nu\beta_{n-\nu-l-j}(kA)\Big)c_jU^l, \quad \text{for } 0\le l\le p-1.\end{aligned}$$

We also note that

$$\sum_{\nu=0}^{p} c_\nu\beta_{n-\nu-s}(\lambda) = -\lambda\beta_{n-s}(\lambda), \quad \text{for } p<s\le n,\ n-\nu-s\ge 0. \tag{2.9}$$

In fact, if $n-s<p$, (2.9) follows from comparing the coefficients of $\zeta^{\bar{s}}$ of (2.6) for $0\le \bar{s}\le p$. If $n-s\ge p$, by comparing the coefficients of $\zeta^{\bar{s}}$ of (2.6) for $\bar{s}\ge p$, we get

$$(c_0+\lambda)\beta_{\bar{s}}+\cdots+c_p\beta_{\bar{s}-p}=0.$$

Replacing $\bar{s}$ by $n-s$ ($n\ge 2p,\ n-s\ge p$), we get (2.9).

Thus (2.8) follows from

$$\Big|n\lambda\sum_{j=p-l}^{p}\beta_{n-l-j}(\lambda)\Big|\le C, \quad \text{for } 0<l\le p-1,$$

which follows from, for fixed l, $0<l\le p-1$,

$$\Big|n\lambda\sum_{j=p-l}^{p}\beta_{n-l-j}(\lambda)\Big|\le C\sum_{j=p-l}^{p} n\lambda e^{-c(n-l-j)\lambda}\le C, \quad \text{for } 0\le\lambda\le\lambda_0,$$

and

$$\Big|n\lambda\sum_{j=p-l}^{p}\beta_{n-l-j}(\lambda)\Big|\le C\sum_{j=p-l}^{p} ne^{-c(n-l-j)}\le C, \quad \text{for } \lambda\ge\lambda_0.$$

We now consider the case $l=0$, we have, by Lemma 2.2,

$$\bar{\partial}_p U^n = k^{-1}\sum_{\nu=0}^{p} c_\nu(\beta_{(n-\nu)0}U^0) = k^{-1}\Big(\sum_{\nu=0}^{p} c_\nu\beta_{n-\nu-p}(kA)\Big)c_pU^0.$$

We will show

$$\frac{n}{s}\Big|\sum_{\nu=0}^{p} c_\nu\beta_{n-\nu-s}(\lambda)\Big|\le C, \quad \text{for } \lambda\in\sigma(kA),\ n\ge 2p. \tag{2.10}$$

Assuming this in the moment, by spectral representation, the desired estimate $\|\bar{\partial}_pU^n\|\le Ct_n^{-1}\|U^0\|$ follows.

It remains to prove (2.10). In fact, since (2.9), it suffices to show,

$$\frac{n}{s}|\lambda\beta_{n-s}(\lambda)|\le C, \quad \text{for } \lambda\in\sigma(kA), \quad n\ge 2p, \quad p<s\le n, \tag{2.11}$$

which we will now prove. For small $\lambda < \lambda_0$, we have, by (2.7),

$$\frac{n}{s}|\lambda\beta_{n-s}(\lambda)| \le (n\lambda e^{-cn\lambda})(s^{-1}e^{cs\lambda}) \le (n\lambda e^{-cn\lambda})\max\{p^{-1}e^{cp\lambda}, n^{-1}e^{cn\lambda}\} \le C.$$

For $\lambda \ge \lambda_0$, using again (2.7), we have,

$$\frac{n}{s}|\lambda\beta_{n-s}(\lambda)| \le C(ne^{-cn})(s^{-1}e^{cs}) \le C,$$

which completes the proof of (2.11). Together these estimates complete the proof of Theorem 2.1. □

3. Error estimates

In this section, we will show the error estimates for the approximation $\bar{\partial}_p U^n$ of the time derivative $u_t(t_n)$ in both smooth and nonsmooth data cases. Recall that the error estimate for the approximation U^n of $u(t_n)$ in the smooth data case reads, see Thomée [11, Theorem 10.1],

$$\|U^n - u^n\| \le C\sum_{j=0}^{p-1}\|U^j - u^j\| + Ck^p\int_0^{t_n}\|u^{(p+1)}(s)\|\,ds, \quad \text{for } n \ge p. \tag{3.1}$$

Applying (3.1), we can easily prove the following smooth data error estimate for the time derivative approximation.

Theorem 3.1. *Let $p \le 6$. Then there is a constant C, independent of the positive definite operator A, such that*

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \le C\sum_{i=0}^{p-1}\|A(U^i - u^i)\| + Ck^p\int_0^{t_n}\|Au^{(p+1)}(s)\|\,ds, \quad \textit{for } n \ge 2p. \tag{3.2}$$

Proof. By (1.11) and (1.1), we have

$$\|\bar{\partial}_p U^n - u_t(t_n)\| = \|A(U^n - u(t_n))\|.$$

Applying (3.1) with norm $\|A\cdot\|$, we obtain (3.2). The proof is complete. □

We now turn to nonsmooth data error estimate. Below we will use the norm $|v|_s = (A^s v, v)^{1/2}$, $s \in \mathbf{R}$, defined by

$$|v|_s^2 = \sum_{l=1}^{\infty}\mu_l^s(v,\varphi_l)^2 < \infty, \quad \text{for } s \in \mathbf{R},$$

where $\{\mu_l, \varphi_l\}_{l=1}^{\infty}$ is the eigensystem of the operator A.

We first recall the following stability result, see Thomée [11, Theorem 10.4].

Lemma 3.2. *Let $p \le 6$ and $s \ge 0$, and let U^n be the solution of* (1.11)*. Then we have, with C independent of the positive definite operator A,*

$$t_n^s \|U^n\|^2 + k \sum_{j=p}^{n} t_j^s |U^j|_1^2 \le C \sum_{j=0}^{p-1} (|U^j|_{-s}^2 + k^s \|U^j\|^2)$$
$$+ Ck \sum_{j=p}^{n} (|f^j|_{-s-1}^2 + t_j^s |f^j|_{-1}^2), \quad \textit{for } n \ge p.$$

We need the following generalization of Lemma 3.2.

Lemma 3.3. *Let $p \le 6$ and $s \ge 0$, and let U^n be the solution of* (1.11)*. Assume that $m \ge p$ and $U^{m-p}, \cdots, U^{m-1}$ are given. Then we have, with C independent of the positive definite operator A,*

$$t_n^s \|U^n\|^2 + k \sum_{j=m}^{n} t_j^s |U^j|_1^2 \le C \sum_{j=m-p}^{m-1} (|U^j|_{-s}^2 + k^s \|U^j\|^2)$$
$$+ Ck \sum_{j=m}^{n} (|f^j|_{-s-1}^2 + t_j^s |f^j|_{-1}^2), \quad \textit{for } n \ge m.$$

Proof. We modify the proof of Lemma 3.2. By eigenfunction expansion, it suffices to show

$$n^s (U^n, \varphi_l)^2 + (k\mu_l) \sum_{j=m}^{n} j^s (U^j, \varphi_l)^2 \le C \sum_{j=m-p}^{m-1} \Big((k\mu_l)^{-s} + 1\Big)(U^j, \varphi_l)^2 \tag{3.3}$$
$$+ C \sum_{j=m}^{n} \Big((k\mu_l)^{-s-1} + j^s (k\mu_l)^{-1}\Big)(kf^j, \varphi_l)^2, \quad \text{for } 1 \le l < \infty.$$

By (1.11), we find that, with $1 \le l < \infty$,

$$(c_0 + k\mu_l)(U^n, \varphi_l) + c_1 (U^{n-1}, \varphi_l) + \cdots + c_p (U^{n-p}, \varphi_l) = (kf^n, \varphi_l).$$

We now instead consider the equation, with $\lambda \in \sigma(kA)$, $W^n = W^n(\lambda)$,

$$(c_0 + \lambda) W^n + c_1 W^{n-1} + \cdots + c_p W^{n-p} = F^n, \quad \text{for } n \ge m, \tag{3.4}$$

where $W^{m-p}, \cdots, W^{m-1} \in \mathbf{R}$ are given and $F^l \in \mathbf{R}, (m \le l \le n)$ are arbitrary. We shall show

$$n^s (W^n)^2 + \lambda \sum_{j=m}^{n} j^s (W^j)^2 \le C \sum_{j=m-p}^{m-1} (\lambda^{-s} + 1)(W^j)^2 \tag{3.5}$$
$$+ C \sum_{j=m}^{n} (\lambda^{-s-1} + j^s \lambda^{-1})(F^j)^2.$$

Assuming this and applying this to $W^n = (U^n, \varphi_l)$, $\lambda = k\mu_l$ and $F^n = (kf^n, \varphi_l)$, for fixed $l, 1 \le l < \infty$, we complete the proof of (3.3).

We now turn to prove (3.5). By linearity it suffices to consider separately the case when $W^{m-l}=0$, $1\le l\le p$, and then the case when $F^l=0$ for $l\ge m$.

By Lemma 2.2, we find that

$$n^s|\beta_n| + \lambda\sum_{j=0}^{\infty} j^s|\beta_j| \le C(1+\lambda^{-s}), \quad \text{for } n\ge 0. \tag{3.6}$$

In fact, by (2.7), we have, for $0\le\lambda\le\lambda_0$,

$$n^s|\beta_n| \le Cn^s e^{-cn\lambda} \le C\lambda^{-s},$$

and

$$\lambda\sum_{j=0}^{\infty} j^s|\beta_j| \le C\lambda\sum_{j=0}^{\infty} j^s e^{-c\lambda j} = C\lambda^{1-s}\sum_{j=0}^{\infty} e^{-\frac{c}{2}\lambda j} \le C\lambda^{-s}.$$

and for $\lambda\ge\lambda_0$, the left-hand side of (3.6), is less than $Cn^s e^{-cn} + C\sum_{j=0}^{\infty} j^s e^{-cj}$, which is bounded.

We also note that the solutions $W^n (n\ge m)$ of (3.4) satisfy, by (2.5),

$$W^m = \beta_0(\lambda)F^m + \sum_{s=0}^{p-1}\beta_{ps}(\lambda)W^{s+m-p},$$

$$W^{m+1} = \beta_0(\lambda)F^m + \beta_1(\lambda)F^{m+1} + \sum_{s=0}^{p-1}\beta_{(p+1)s}(\lambda)W^{s+m-p},$$

$$\vdots$$

$$W^n = \sum_{j=m}^{n}\beta_{n-j}(\lambda)F^j + \sum_{s=0}^{p-1}\beta_{(p+n-m)s}(\lambda)W^{s+m-p}, \quad n\ge m,$$

or, in general form,

$$W^n = \sum_{j=m}^{n}\beta_{n-j}F^j + \sum_{s=0}^{p-1}\beta_{(p+n-m)s}W^{s+m-p}, \quad \text{for } n\ge m. \tag{3.7}$$

After the above preparations, we now consider the proof of (3.5) in the case when $W^{m-p}=\cdots=W^{m-1}=0$. We have, by (3.7),

$$W^n = \sum_{j=m}^{n}\beta_{n-j}F^j = \sum_{l=0}^{n-m}\beta_l F^{n-l}, \quad \text{for } n\ge m,$$

so that, using the Schwarz inequality,

$$n^s(W^n)^2 = n^s\Big(\sum_{l=0}^{n-m}\beta_l F^{n-l}\Big)^2 \le n^s\Big(\sum_{l=0}^{n-m}|\beta_l|\Big)\sum_{l=0}^{n-m}|\beta_l|(F_{n-l})^2.$$

Hence, by (3.6), and noting that $n^s \leq C(l^s + (n-l)^s)$ and $1 \leq (n-l)^s$, we find

$$
\begin{aligned}
n^s(W^n)^2 &\leq C\lambda^{-1} \sum_{l=0}^{n-m} \Big(l^s|\beta_l|(F^{n-l})^2 + (n-l)^s|\beta_l|(F^{n-l})^2 \Big) \\
&\leq C\lambda^{-1} \sum_{l=0}^{n-m} (\lambda^{-s} + (n-l)^s)(F^{n-l})^2,
\end{aligned} \tag{3.8}
$$

which is the desired estimate for the first term of the left hand side in (3.5). For the second term in (3.5), we have, by (3.8),

$$
\begin{aligned}
\lambda \sum_{n=m}^{N} n^s(W^n)^2 &\leq \lambda \sum_{n=m}^{N} \Big(\lambda^{-1} \sum_{j=0}^{n-m} \big(j^s|\beta_j|(F^{n-j})^2 + |\beta_j|(n-j)^s(F^{n-j})^2 \big) \Big) \\
&\leq \sum_{n=m}^{N} \sum_{j=0}^{N-m} \Big(j^s|\beta_j| + n^s|\beta_j| \Big)(F^n)^2 \\
&\leq C\lambda^{-1} \sum_{n=m}^{N} (1+\lambda^{-s})(F^n)^2 + \lambda^{-1} \sum_{n=m}^{N} n^s(F^n)^2 \\
&\leq C\lambda^{-1} \sum_{n=m}^{N} (\lambda^{-s} + n^s)(F^n)^2,
\end{aligned}
$$

which completes the proof in the present case.

We next consider the case when $F^j = 0$, $m \leq j \leq n$ and $W^{m-l} \neq 0$, $1 \leq l \leq p$, $W^{m-\bar{l}} = 0$, $1 \leq \bar{l} \leq p$, $\bar{l} \neq l$. We begin with the special case $l = p$. By (3.7) with $s = 0$, we have

$$
W^n = \beta_{(p+n-m)0}(\lambda)W^{m-p} = \beta_{n-m}(\lambda)c_p W^{m-p},
$$

so that, using (2.7) and $n^s \leq C\big((n-m)^s + m^s\big)$,

$$
\begin{aligned}
n^s(W^n)^2 &\leq Cn^s\beta_{n-m}(\lambda)(W^{m-p})^2 \leq C\big(1+(n-m)^s\big)\beta_{n-m}(\lambda)(W^{m-p})^2 \\
&\leq C(1+\lambda^{-s})(W^{m-p})^2.
\end{aligned}
$$

From this we also obtain

$$
\begin{aligned}
\lambda \sum_{n=m}^{N} n^s(W^n)^2 &\leq C\lambda \sum_{n=m}^{N} \Big(1+(n-m)^s\Big)\beta_{n-m}^2(\lambda)(W^{m-p})^2 \\
&\leq C(1+\lambda^{-s})(W^{m-p})^2.
\end{aligned}
$$

For the general case $l \neq p$, we have, by (3.7) with $s = p - l$

$$
W^n = \beta_{(p+n-m)(p-l)}(\lambda)W^{m-l} = \sum_{j=l}^{p} \beta_{n-m+l-j}(\lambda)c_j W^{m-l},
$$

so that, using (2.7) and $n^s \le C\big((n-m+l-j)^s + (m-l+j)^s\big)$,

$$\begin{aligned} n^s (W^n)^2 &\le C n^s \sum_{j=l}^{p} \beta_{n-m+l-j}(\lambda)(W^{m-l})^2 \\ &\le C \sum_{j=l}^{p} \big(1 + (n-m+l-j)^s\big)\beta_{n-m+l-j}(\lambda)(W^{m-l})^2 \\ &\le C \sum_{j=l}^{p} (1+\lambda^{-s})(W^{m-l})^2. \end{aligned}$$

From this we also obtain

$$\begin{aligned} \lambda \sum_{n=m}^{N} n^s (W^n)^2 &\le C\lambda \sum_{n=m}^{N} \sum_{j=l}^{p} \big(1 + (n-m+l-j)^s\big)\beta^2_{n-m+l-j}(\lambda)(W^{m-l})^2 \\ &\le C(1+\lambda^{-s})(W^{m-l})^2. \end{aligned}$$

Together these estimates complete the proof. □

Now we are the position to state our error estimate.

Theorem 3.4. *Let $p \le 6$ and let U^n and u be the solutions of* (1.11) *and* (1.1)*, respectively. Then, with* $G(s) = |u^{(p+1)}(s)|^2_{-2p-1} + s^{2p+2}|u^{(p+1)}(s)|^2_1 + s^2|u_t(s)|^2_1$,

$$\begin{aligned} t_n^{2p+2}\|\bar\partial_p U^n - u_t(t_n)\|^2 &\le C \sum_{j=p}^{2p-1} \big(|U^j - u^j|^2_{-2p} + k^{2p+2}\|A(U^j - u^j)\|^2\big) \\ &\quad + Ck^{2p}\Big(\int_0^{t_n} G(s)\,ds + t_{2p}^3 |u_t(t_{2p})|_1^2\Big), \end{aligned}$$

Proof. The error $\varepsilon^n = \bar\partial_p U^n - u_t(t_n)$ $(n \ge p)$ satisfies

$$\bar\partial_p \varepsilon^n + A\varepsilon^n = -\tau^n, \quad \text{where } \tau^n = A\big(\bar\partial_p u(t_n) - u_t(t_n)\big), \quad \text{for } n \ge 2p.$$

Applying Lemma 3.3 with $s = 2p+2, m = 2p$, we have, for $n \ge 2p$,

$$\begin{aligned} t_n^{2p+2}\|\varepsilon^n\|^2 &\le C \sum_{j=p}^{2p-1} \big(|\varepsilon^j|^2_{-2p-2} + k^{2p+2}\|\varepsilon^j\|^2\big) \\ &\quad + Ck \sum_{j=2p}^{n} \big(|\tau^j|^2_{-2p-3} + t_j^{2p+2}|\tau^j|^2_{-1}\big). \end{aligned}$$

We now estimate the term $k\sum_{j=2p}^n |\tau^j|^2_{-2p-3}$. We will show that, with any norm $\|\cdot\|$ in H,

$$\|\bar\partial_p u(t_j) - u_t(t_j)\| \le Ck^{p-1} \int_{t_{j-p}}^{t_j} \|u^{(p+1)}(s)\|\,ds, \quad \text{for } j \ge 2p. \tag{3.9}$$

Assuming this we have

$$|\tau^j|_{-2p-3}^2 \le Ck^{2p-1} \int_{t_{j-p}}^{t_j} |u^{(p+1)}(s)|_{-2p-1}^2 \, ds, \quad \text{for } j \ge 2p.$$

Thus

$$\begin{aligned} k \sum_{j=2p}^{n} |\tau^j|_{-2p-3}^2 &\le Ck^{2p} \sum_{j=2p}^{n} \int_{t_{j-p}}^{t_j} |u^{(p+1)}(s)|_{-2p-1}^2 \, ds \\ &\le Ck^{2p} \int_0^{t_n} |u^{(p+1)}(s)|_{-2p-1}^2 \, ds. \end{aligned}$$

It remains to estimate $k \sum_{j=2p}^{n} t_j^{2p+2} |\tau^j|_{-1}^2$. If $j \ne 2p$, we have, by (3.9) with norm $\|A^{1/2} \cdot \|$,

$$k \sum_{j=2p+1}^{n} t_j^{2p+2} |\tau^j|_{-1}^2 \le Ck^{2p} \sum_{j=2p+1}^{n} t_j^{2p+2} \int_{t_{j-p}}^{t_j} |u^{(p+1)}(s)|_1^2 \, ds.$$

Here we have $t_j \le cs$ for $s \in [t_{j-p}, t_j], \ j \ge 2p+1$ which follows from

$$t_j \le s \frac{t_j}{t_{j-p}} \le s \frac{t_{2p+1}}{t_{p+1}} \le cs, \quad \text{for } j \ge 2p+1.$$

Hence

$$k \sum_{j=2p+1}^{n} t_j^{2p+2} |\tau^j|_{-1}^2 \le Ck^{2p} \sum_{j=2p+1}^{n} \int_{t_{j-p}}^{t_j} s^{2p+2} |u^{(p+1)}(s)|_1^2 \, ds.$$

For $j = 2p$, we write, since $\sum_{\nu=0}^{p} c_\nu = 0$,

$$\begin{aligned} \tau^{2p} &= k^{-1} A \Big(\sum_{\nu=0}^{p} c_\nu u(t_{2p-\nu}) - u_t(t_{2p}) \Big) \\ &= k^{-1} A \Big(\sum_{\nu=0}^{p} c_\nu \int_{t_p}^{t_{2p-\nu}} u_t(s) \, ds - u_t(t_{2p}) \Big), \end{aligned}$$

and we obtain

$$k |\tau^{2p}|_{-1}^2 \le C \int_{t_p}^{t_{2p}} |u_t(s)|_1^2 \, ds + k |u_t(t_{2p})|_1^2,$$

which follows from

$$\begin{aligned} |\tau^{2p}|_{-1}^2 &\le C \Big(k^{-2} \sum_{\nu=0}^{p} \Big| \int_{t_p}^{t_{2p-\nu}} u_t(s) \, ds \Big|_1^2 + |u_t(t_{2p})|_1^2 \Big) \\ &\le Ck^{-2} \sum_{\nu=0}^{p} (pk) \int_{t_p}^{t_{2p-\nu}} |u_t(s)|_1^2 \, ds + |u_t(t_{2p})|_1^2 \\ &\le Ck^{-1} \int_{t_p}^{t_{2p}} |u_t(s)|_1^2 \, ds + |u_t(t_{2p})|_1^2. \end{aligned}$$

Thus, we get

$$\begin{aligned} kt_{2p}^{2p+2}|\tau^{2p}|_{-1}^2 &\le Ck^{2p+2}\Big(\int_{t_p}^{t_{2p}} |u_t(s)|_1^2\,ds + k|u_t(t_{2p})|_1^2\Big) \\ &\le Ck^{2p}\Big(\int_{t_p}^{t_{2p}} s^2|u_t(s)|_1^2\,ds + t_{2p}^3|u_t(t_{2p})|_1^2\Big). \end{aligned}$$

It remains to estimate (3.9). We write, by Taylor expansion around t_{j-p},

$$\begin{aligned} u(t) &= \sum_{l=0}^{p} \frac{u^{(l)}(t_{j-p})}{l!}(t-t_{j-p})^l + \frac{1}{p!}\int_{t_{j-p}}^{t} (t-s)^p u^{(p+1)}(s)\,ds \\ &\equiv Q(t) + R(t). \end{aligned}$$

By (1.12) and since $Q(t)$ is a polynomial of degree p, we have $\bar{\partial}_p Q(t) - Q_t(t) = 0$. Thus, by (2.1),

$$\bar{\partial}_p u(t_j) - u_t(t_j) = \bar{\partial}_p R(t_j) - R_t(t_j) = k^{-1}\sum_{\nu=0}^{p} c_\nu R(t_{j-\nu}) - R_t(t_j).$$

Noting that

$$\|R(t_{j-\nu})\| \le Ck^p \int_{t_{j-p}}^{t_j} \|u^{(p+1)}(s)\|\,ds, \quad \text{for } 0 \le \nu \le p, \quad j \ge 2p,$$

and

$$\begin{aligned} \|R_t(t_j)\| &= \frac{1}{(p-1)!}\Big\| \int_{t_{j-p}}^{t_j} (t_j-s)^{p-1}u^{(p+1)}(s)\,ds\Big\| \\ &\le Ck^{p-1}\int_{t_{j-p}}^{t_j} \|u^{(p+1)}(s)\|\,ds, \end{aligned}$$

we complete the proof of (3.9).

Together these estimates complete the proof. □

In the homogeneous case, i.e., $f = 0$, we have the following nonsmooth data error estimates.

Theorem 3.5. *Let $p \le 6$ and let U^n and u be the solutions of* (1.11) *and* (1.1)*, respectively. Assume that $f = 0$ and the discrete initial values satisfy*

$$|U^j - u^j|_{-2p} + k^{p+1}\|A(U^j - u^j)\| \le Ck^p\|v\|, \quad \textit{for } p \le j \le 2p-1. \tag{3.10}$$

Then, with C independent of the positive definite operator A,

$$\|\bar{\partial}_p U^n - u_t(t_n)\| \le Ck^p t_n^{-p-1}\|v\|, \quad \textit{for } n \ge 2p.$$

Proof. For the solution u of homogeneous parabolic equation, it is easy to show that

$$\int_0^{t_n} |u^{(p+1)}(s)|_{-2p-1}^2 ds \le C\|v\|^2, \quad \int_0^{t_n} s^{2p+2}|u^{(p+1)}(s)|_1^2 ds \le C\|v\|^2,$$

and $t_{2p}^3|u_t(t_{2p}|_1^2 \le C\|v\|^2$. Applying for Theorem 3.4, we complete the proof. □

4. Error estimates for the starting values

In Theorems 3.4 and 3.5, we see that it is necessary to define starting approximations $\{U^j\}_{j=0}^{p-1}$ such that

$$|U^j - u^j|_{-2p} + k^{p+1}\|A(U^j - u^j)\| = O(k^p), \quad \text{for } p \le j \le 2p-1.$$

In this section we will investigate two simplest cases $p = 1, 2$. The approach can be extended to the general case for $p > 2$, but the proof is more complicated.

In the case of $p = 1$, the approximate solution is defined by the backward Euler method

$$\bar\partial_1 U^n + AU^n = f^n, \quad \text{for } n \ge 1, \quad \text{with } U^0 = v, \tag{4.1}$$

or, with $r(\lambda) = 1/(1+\lambda)$,

$$U^n = r(kA)U^{n-1} + kr(kA)f^n, \quad \text{for } n \ge 1, \quad \text{with } U^0 = v.$$

We then have the following lemma.

Lemma 4.1. *Let U^1 and u be the solutions of* (4.1) *and* (1.1), *respectively. Then we have*

$$\begin{aligned}|U^1 - u^1|_{-2} &+ k^2\|A(U^1 - u^1)\| \\ &\le Ck\|v - A^{-1}f(0)\| + Ck\int_0^k \|A^{-1}f'(\tau)\|\,d\tau + Ck^2\int_0^k \|f'(\tau)\|\,d\tau.\end{aligned} \tag{4.2}$$

In particular, if $f = 0$, then

$$|U^1 - u^1|_{-2} + k^2\|A(U^1 - u^1)\| \le Ck\|v\|. \tag{4.3}$$

Proof. Noting that $u^1 = e^{-kA}v + \int_0^k e^{-(k-s)A}f(s)\,ds$ and using Taylor's formula, we have

$$\begin{aligned}U^1 - u^1 &= \big(r(kA) - e^{-kA}\big)v + kr(kA)f^1 - \int_0^k e^{-(k-s)A}f(s)\,ds \\ &= \big(r(kA) - e^{-kA}\big)v + kr(kA)\Big(f(0) + \int_0^k f'(\tau)\,d\tau\Big) \\ &\quad - k\int_0^1 e^{-(1-s)kA}\Big(f(0) + \int_0^{ks} f'(\tau)\,d\tau\Big)\,ds \\ &= \big(r(kA) - e^{-kA}\big)v + kb_0(kA)f(0) + kR(f),\end{aligned}$$

where

$$b_0(\lambda) = r(\lambda) - \int_0^1 e^{-(1-s)\lambda}\,ds,$$

and

$$R(f) = r(kA)\int_0^k f'(\tau)\,d\tau - \int_0^1 e^{-(1-s)kA}\int_0^{ks} f'(\tau)\,d\tau\,ds. \tag{4.4}$$

Thus we have, noting that $\lambda b_0(\lambda) = -(r(\lambda) - e^{-\lambda})$,

$$A(U^1 - u^1) = \big(r(kA) - e^{-kA}\big)(Av - f(0)) + kAR(f) \tag{4.5}$$
$$= -\big(r(kA) - e^{-kA}\big)u_t(0) + kAR(f).$$

We first show that

$$k^2\|A(U^1 - u^1)\| \le Ck\|A^{-1}u_t(0)\| + Ck^2\int_0^k \|f'(\tau)\|\,d\tau. \tag{4.6}$$

In fact, by (4.4) and (4.5),

$$\begin{aligned} k^2\|A(U^1 - u^1)\| &\le k\|kA(r(kA) - e^{-kA})A^{-1}u_t(0)\| \\ &\quad + k^2\Big\|kAr(kA)\int_0^k f'(\tau)\,d\tau\Big\| \\ &\quad + k^2\Big\|\int_0^1 kAe^{-(1-s)kA}\int_0^{ks} f'(\tau)\,d\tau\,ds\Big\| \\ &= I + II + III. \end{aligned}$$

For I, we have, since $|\lambda(r(\lambda) - e^{-\lambda})| \le C$ for $0 < \lambda < \infty$,

$$I \le Ck\|A^{-1}u_t(0)\|.$$

For II, we have, since $|\lambda r(\lambda)| \le C$ for $0 < \lambda < \infty$,

$$II \le Ck^2\int_0^k \|f'(\tau)\|\,d\tau.$$

For III, we have, since $\Big|\int_\epsilon^1 \lambda e^{-(1-s)\lambda}\,ds\Big| \le C$ for $0 \le \epsilon \le 1$,

$$\begin{aligned} III &= k^2\Big\|\int_0^k\int_{\tau/k}^1 kAe^{-(1-s)kA}f'(\tau)\,ds\,d\tau\Big\| \\ &\le Ck^2\Big(\int_0^k \|f'(\tau)\|\,d\tau\Big)\Big\|\int_{\tau/k}^1 kAe^{-(1-s)kA}\,ds\Big\| \\ &\le Ck^2\int_0^k \|f'(\tau)\|\,d\tau. \end{aligned}$$

Thus we obtain (4.6).

We next show that

$$|A(U^1 - u^1)|_{-4} \le Ck\|A^{-1}u_t(0)\| + Ck\int_0^k \|A^{-1}f'(\tau)\|\,d\tau. \tag{4.7}$$

In fact, by (4.4) and (4.5),

$$\begin{aligned}|A(U^1-u^1)|_{-4} &\le k\|(kA)^{-1}(r(kA)-e^{-kA})A^{-1}u_t(0)\| \\ &\quad + k\Big\|r(kA)\int_0^k A^{-1}f'(\tau)\,d\tau\Big\| \\ &\quad + k\Big\|\int_0^1 e^{-(1-s)kA}\int_0^{ks} A^{-1}f'(\tau)\,d\tau\,ds\Big\| \\ &= I + II + III.\end{aligned}$$

For I, we have, since $|\lambda^{-1}(r(\lambda)-e^{-\lambda})| \le C$ for $0<\lambda<\infty$,

$$I \le Ck\|A^{-1}u_t(0)\| \le Ck\|v - A^{-1}f(0)\|.$$

For II, we have, since $|r(\lambda)| \le C$ for $0<\lambda<\infty$,

$$II \le Ck\int_0^k \|A^{-1}f'(\tau)\|\,d\tau.$$

For III, we have, since $\left|\int_\epsilon^1 e^{-(1-s)\lambda}\,ds\right| \le C$ for $0\le\epsilon\le 1$,

$$\begin{aligned}III &= k\Big\|\int_0^k\int_{\tau/k}^1 kAe^{-(1-s)kA}A^{-1}f'(\tau)\,ds\,d\tau\Big\| \\ &\le Ck\Big(\int_0^k \|A^{-1}f'(\tau)\|\,d\tau\Big)\Big\|\int_{\tau/k}^1 e^{-(1-s)kA}\,ds\Big\| \\ &\le Ck\int_0^k \|A^{-1}f'(\tau)\|\,d\tau.\end{aligned}$$

Thus we obtain (4.7). □

We now turn to the case $p=2$. In this case we need two starting values U^0, U^1. We will use the backward Euler method to compute U^1, i.e., the approximation U^n of the solution $u(t_n)$ of (1.1) is defined by

(4.8) $$\bar\partial_2 U^n + AU^n = f^n \quad \text{for } n\ge 2, \qquad \bar\partial U^1 + AU^1 = f^1, \quad \text{with } U^0 = v.$$

We have the following lemma.

Lemma 4.2. *Let $U^j, j=2,3$ and u be the solutions of (4.8) and (1.1), respectively. Then we have*

(4.9) $$|U^j-u^j|_{-4} + k^3\|A(U^j-u^j)\| \le Ck^2\Big(\|v\| + \|f(0)\| + \int_0^{t_j}\|f'(\tau)\|\,d\tau\Big), \quad j=2,3.$$

In particular, if $f=0$, then

(4.10) $$|U^j-u^j|_{-4} + k^3\|A(U^j-u^j)\| \le Ck^2\|v\|, \quad j=2,3.$$

Proof. Here we only prove the case $j=2$, i.e., we will show that

$$|U^2-u^2|_{-4}+k^3\|A(U^2-u^2)\| = O(k^2). \tag{4.11}$$

The proof for the case $j=3$ is similar.

Since $\bar\partial_2 U^2 = k^{-1}\big(\frac{3}{2}U^2-2U^1+\frac{1}{2}U^0\big)$, we may write

$$U^2 = q_1(kA)U^1+q_2(kA)U^0+kP(kA)f^2,$$

where

$$q_1(\lambda)=\frac{2}{3/2+\lambda},\quad q_2(\lambda)=\frac{-1/2}{3/2+\lambda},\quad \text{and}\quad P(\lambda)=\frac{1}{3/2+\lambda}.$$

Thus, noting that $u^2=e^{-2kA}v+\int_0^{2k}e^{-(2k-s)A}f(s)\,ds$, we have

$$U^2-u^2 = q_1(kA)(U^1-u^1)+q_2(kA)(U^0-u^0)+E_2.$$

Here

$$\begin{aligned}
E_2 &= q_1(kA)u^1+q_2(kA)u^0+kP(kA)f^2-u^2\\
&= q_1(kA)\Big(e^{-kA}v+\int_0^k e^{-(k-s)A}f(s)\,ds\Big)\\
&\quad +q_2(kA)v+kP(kA)f^2\\
&\quad -\Big(e^{-2kA}v+\int_0^{2k}e^{-(2k-s)A}f(s)\,ds\Big)\\
&= \Big(q_1(kA)e^{-kA}+q_2(kA)-e^{-2k}\Big)v\\
&\quad +kq_1(kA)\int_0^1 e^{-(1-s)kA}\Big(f(0)+\int_0^{ks}f'(\tau)\,d\tau\Big)\,ds\\
&\quad +kP(kA)\Big(f(0)+\int_0^{2k}f'(\tau)\,d\tau\Big)\\
&\quad -2k\int_0^1 e^{-2(1-s)kA}\Big(f(0)+\int_0^{2ks}f'(\tau)\,d\tau\Big)\,ds\\
&= Q(kA)v+kb_0(kA)f(0)+kR(f),
\end{aligned} \tag{4.12}$$

where

$$Q(\lambda)=q_1(\lambda)e^{-\lambda}+q_2(\lambda)-e^{-2\lambda},$$

and

$$b_0(\lambda)=q_1(\lambda)\int_0^1 e^{-(1-s)\lambda}\,ds+P(\lambda)-2\int_0^1 e^{-2(1-s)\lambda}\,ds,$$

and

$$\begin{aligned} R(f) = q_1(kA) \int_0^1 e^{-(1-s)kA} \Big(\int_0^{ks} f'(\tau)\, d\tau \Big)\, ds \\ + P(kA) \int_0^{2k} f'(\tau)\, d\tau \\ - 2 \int_0^1 e^{-2(1-s)kA} \Big(\int_0^{2ks} f'(\tau)\, d\tau \Big)\, ds. \end{aligned}$$

Thus we have

$$A(U^2 - u^2) = Aq_1(kA)(U^1 - u^1) + AE_2. \tag{4.13}$$

Let us show that

$$k^3 \|A(U^2 - u^2)\| \le Ck^2 \Big(\|v\| + k\|f(0)\| + k \int_0^{2k} \|f'(\tau)\|\, d\tau \Big). \tag{4.14}$$

In fact, by (4.12) and (4.13),

$$\begin{aligned} k^3 \|A(U^2 - u^2)\| &\le k^3 \|Aq_1(kA)(U^1 - u^1)\| + k^3 \|AQ(kA)v\| \\ &\quad + k^3 \|kAb_0(kA) f(0)\| + k^3 \|kAR(f)\| \\ &= I + II + III + IV. \end{aligned}$$

We first estimate the terms II, III, and IV, then we turn to the term I.

For II, we have, since $|\lambda Q(\lambda)| < C$ for $0 < \lambda < \infty$,

$$II = k^2 \|kAQ(kA)v\| \le Ck^2 \|v\|.$$

For III, we have, since $|\lambda b_0(\lambda)| \le C$ for $0 < \lambda < \infty$,

$$III = k^3 \|kAb_0(kA) f(0)\| \le Ck^3 \|f(0)\|.$$

For IV, we have

$$\begin{aligned} IV &\le Ck^3 \Big\| kAq_1(kA) \int_0^k \int_{\tau/k}^1 kAe^{-(1-s)kA} f'(\tau)\, ds\, d\tau \Big\| \\ &\quad + Ck^3 \Big\| kAP(kA) \int_0^{2k} f'(\tau)\, d\tau \Big\| \\ &\quad + Ck^3 \Big\| kA \int_0^{2k} \int_{\tau/2k}^1 e^{-2(1-s)kA} f'(\tau)\, ds\, d\tau \Big\| \\ &\le Ck^3 \int_0^{2k} \|f'(\tau)\|\, d\tau. \end{aligned}$$

Now we turn to I.

$$\begin{aligned} I &= k^3\|Aq_1(kA)(U^1-u^1)\| \le k^3\|Aq_1(kA)(r(kA)-e^{-kA}v\| \\ &\quad + k^3\Big\|Aq_1(kA)\Big(kr(kA)-k\int_0^1 e^{-(1-s)kA}\,ds\Big)f(0)\Big\| \\ &\quad + k^3\Big\|Aq_1(kA)\Big(kr(kA)\int_0^k f'(\tau)\,d\tau - k\int_0^1 e^{-(1-s)kA}\int_0^{ks} f'(\tau)\,d\tau\,ds\Big)\Big\| \\ &= I_1+I_2+I_3. \end{aligned}$$

It is easy to show that

$$I_1 \le Ck^2\|v\|, \quad I_2 \le Ck^3\|f(0)\|, \quad \text{and} \quad I_3 \le 2k^3\int_0^k \|f'\|\,d\tau.$$

Thus we get

$$I \le Ck^2\Big(\|v\| + k\|f(0)\| + k\int_0^k \|f'\|\,d\tau\Big).$$

Combining this with the estimates for II, III and IV, we obtain (4.14)

We next show that

$$|A(U^2-u^2)|_{-6} \le Ck^2\Big(\|v\| + \int_0^{2k}\|f'(\tau)\|\,d\tau\Big). \tag{4.15}$$

In fact, by (4.12) and (4.13),

$$\begin{aligned} |A(U^2-u^2)|_{-6} &\le |Aq_1(kA)(U^1-u^1)|_{-6} + |AQ(kA)v|_{-6} \\ &\quad + |kAb_0(kA)f(0)|_{-6} + |kAR(f)|_{-6} \\ &= I' + II' + III' + IV'. \end{aligned} \tag{4.16}$$

We first estimate the terms II', III', and IV', then we turn to the term I'.

For II', we have, since $|\lambda^{-2}Q(\lambda)| < C$ for $0<\lambda<\infty$,

$$II' = |kAQ(kA)v|_{-6} = k^2\|(kA)^{-2}Q(kA)v\| \le Ck^2\|v\|.$$

For III', we have, since $|\lambda^{-1}b_0(\lambda)| \le C$ for $0<\lambda<\infty$,

$$III' = |kAb_0(kA)f(0)|_{-6} = k^2\|(kA)^{-1}b_0(kA)A^{-1}\| \le Ck^2\|f(0)\|.$$

For IV', we have

$$\begin{aligned} IV' &\leq C\Big\|kA^{-2}q_1(kA)\int_0^k\int_{\tau/k}^1 kAe^{-(1-s)kA}f'(\tau)\,ds\,d\tau\Big\| \\ &\quad + C\Big\|kA^{-2}P(kA)\int_0^{2k} f'(\tau)\,d\tau\Big\| \\ &\quad + C\Big\|kA^{-2}\int_0^{2k}\int_{\tau/2k}^1 e^{-2(1-s)kA}f'(\tau)\,ds\,d\tau\Big\| \\ &\leq Ck^2\int_0^{2k}\|f'(\tau)\|\,d\tau. \end{aligned}$$

Now we turn to I'.

$$\begin{aligned} I' &= |Aq_1(kA)(U^1-u^1)|_{-6} \leq |Aq_1(kA)(r(kA)-e^{-kA}v|_{-6} \\ &\quad + \Big|Aq_1(kA)\Big(kr(kA)-k\int_0^1 e^{-(1-s)kA}\,ds\Big)f(0)\Big|_{-6} \\ &\quad + \Big|Aq_1(kA)\Big(kr(kA)\int_0^k f'(\tau)\,d\tau - k\int_0^1 e^{-(1-s)kA}\int_0^{ks} f'(\tau)\,d\tau\,ds\Big)\Big|_{-6} \\ &= I_1' + I_2' + I_3'. \end{aligned}$$

It is easy to show that

$$I_1' \leq Ck^2\|v\|, \quad I_2' \leq Ck^2\|f(0)\|, \quad \text{and} \quad I_3' \leq Ck^2\int_0^k \|f'\|\,d\tau.$$

Thus we get

$$I' \leq Ck^2\Big(\|v\| + \|f(0)\| + \int_0^k \|f'\|\,d\tau\Big).$$

Combining this with the estimates for II', III' and IV', we obtain (4.15).

Together these estimates we show (4.11). The proof is complete. □

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