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THE FINITE ELEMENT METHOD FOR A LINEAR STOCHASTIC PARABOLIC PARTIAL DIFFERENTIAL EQUATION DRIVEN BY ADDITIVE NOISE

YUBIN YAN

ABSTRACT. In this paper we consider the finite element method for a stochastic parabolic partial differential equation forced by additive space-time white noise in the multi-dimensional case. Optimal strong convergence error estimates in the L_2 and $\dot{H}^{-1}$ norms with respect to spatial variable have been obtained. The proof is based on appropriate nonsmooth data error estimates for the corresponding deterministic parabolic problem.

1. INTRODUCTION

In this paper we will study the finite element approximation of the linear stochastic parabolic partial differential equation

$$du + Au\,dt = dW, \quad \text{for } 0 < t \leq T, \quad \text{with } u(0) = u_0, \tag{1.1}$$

in a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $u(t)$ is an H-valued random process, A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, where $W(t)$ is a Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $u_0 \in H$.

For the sake of simplicity, we shall concentrate on the case $A = -\Delta$, where Δ stands for the Laplacian operator subject to homogeneous Dirichlet boundary conditions, and $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d, d = 1, 2, 3$, with a sufficiently smooth boundary $\partial\mathcal{D}$.

Such equations are common in applications. Many mathematics models in physics, chemistry, biology, population dynamics, neurophysiology, etc., are described by stochastic partial differential equations, see, Da Prato and Zabczyk [5], Walsh [18], etc. The existence, uniqueness, and properties of the solutions of such equations have been well studied, see Curtain and Falb [2], Da Prato [3], Da Prato and Lunardi [4], Da Prato and Zabczyk [5], Dawson [7], Gozzi [9], Peszat and Zabczyk [14], Walsh [18], etc. However, numerical approximation of such equations has not been studied thoroughly.

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The equation (1.1) can be written formly as

$$u_t + Au = \frac{dW}{dt} \quad \text{for } 0 < t \le T, \quad \text{with } u(0) = u_0, \tag{1.2}$$

where the derivative $\frac{dW}{dt}$ (noise) does not exist as a function of t in the usual sense. Thus the equation (1.2) is understood in the integral form.

Let $E(t) = e^{-tA}$, $t \ge 0$. Then (1.2) admits a unique mild solution, see Da Prato and Zabczyk [5, Theorem 5.2, 5.4],

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s) \quad \text{for } 0 < t \le T, \tag{1.3}$$

where the integral is understood in Itô sense.

We assume that $W(t)$ is a Wiener process with covariance operator Q. This process may be considered in terms of its Fourier series. Suppose that Q has eigenvalues $\gamma_l > 0$ and corresponding eigenfunctions e_l. Then

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t),$$

where β_l, $l = 1, 2, \cdots$, is a sequence of real-valued independent identically distributed Brownian motions.

If Q is in trace class, then $W(t)$ is an H-valued process. If Q is not in trace class, for example $Q = I$, then $W(t)$ does not belong to H, which is called a cylindrical Wiener process, but stochastic integral can be defined with respect to W, when the integral smoothes the noise process sufficiently.

Let $L_2^0 = HS(Q^{1/2}(H), H)$ denote the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to H, i.e.,

$$L_2^0 = \Big\{ \psi \in L(H) : \sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 < \infty \Big\},$$

with norm $\|\psi\|_{L_2^0} = \Big(\sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 \Big)^{1/2}$, where $L(H)$ is the space of bounded operator from H to H.

Let $\mathbf{E}$ denote the expectation. The Itô isometry for a Wiener process of covariance operator Q states that, for an integrand $\psi \in L_2^0$,

$$\mathbf{E}\Big\| \int_0^t \psi(s)\,dW(s) \Big\|^2 = \int_0^t \|\psi(s)\|_{L_2^0}^2\,ds.$$

Let S_h be a family of finite element spaces, where S_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to the triangulation $\mathcal{T}$ of Ω. For simplicity, we always assume that $\{S_h\} \subset H_0^1 = H_0^1(\mathcal{D}) = \{v \in L_2(\mathcal{D}), \nabla v \in L_2(\mathcal{D}), v|_{\partial\mathcal{D}} = 0\}$. The semidiscrete problem of (1.1) is to find the process $u_h(t) = u_h(\cdot, t) \in S_h$, such that

$$du_h + A_h u_h\,dt = P_h dW, \quad \text{for } 0 < t \le T, \quad \text{with } u_h(0) = P_h u_0, \tag{1.4}$$

where P_h denotes the L_2-projection onto S_h, and $A_h : S_h \to S_h$ is the discrete analogue of A, defined by

$$(A_h\psi, \chi) = A(\psi, \chi), \quad \forall \psi, \chi \in S_h. \tag{1.5}$$

Here $A(\cdot,\cdot)$ is the bilinear form on $H_0^1(\mathcal{D})$ obtained from the operator A in (1.1).

With $E_h(t) = e^{-tA_h}$, $t \ge 0$, then (1.4) admits a unique mild solution

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)P_h\, dW(s).$$

Let $\dot{H}^s = \dot{H}^s(\mathcal{D}) = \mathcal{D}(A^{s/2})$ for any $s \in \mathbf{R}$ and denote the norm by $|\cdot|_s = \|A^{s/2}\cdot\|$. For any Hilbert space H_1, we denote $L_2(\Omega; H_1)$ by

$$L_2(\Omega; H_1) = \Big\{ v : \mathbf{E}\|v\|_{H_1}^2 = \int_\Omega \|v(\omega)\|_{H_1}^2\, d\mathbf{P}(\omega) < \infty \Big\},$$

with the norm $\|v\|_{L_2(\Omega;H_1)} = (\mathbf{E}\|v\|_{H_1}^2)^{1/2}$.

Under the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, we show in Lemma 2.4 that $W(t) \in \dot{H}^{\beta-1} \subset \dot{H}^{-1}$, so that $P_hW(t)$ is well defined, and we obtain, in Theorems 3.2, 3.3, the error estimates in semidiscrete case,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch^\beta\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big), \tag{1.6}$$

and, with $\ell_h = \log(T/h^2)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \le Ch^{\beta+1}\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \ell_h\|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{1.7}$$

We also consider the error estimates in the fully discrete case . Let k be a time step and $t_n = nk$ with $n \ge 1$. We use the backward Euler scheme to approximate $u(t_n)$,

$$\frac{U^n - U^{n-1}}{k} + A_hU^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} P_h\, dW(s), \quad n \ge 1, \quad U^0 = P_hu_0. \tag{1.8}$$

With $r(\lambda) = (1+\lambda)^{-1}$, we can rewrite (1.8) in the form

$$U^n = r(kA_h)U^{n-1} + \int_{t_{n-1}}^{t_n} r(kA_h)P_h\, dW(s), \quad n \ge 1, \quad U^0 = P_hu_0. \tag{1.9}$$

Under the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, we obtain, in Theorems 4.2, 4.3, the error estimates in the fully discrete case,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\beta/2} + h^\beta)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big), \tag{1.10}$$

and, with $\ell_k = \log(t_n/k)$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;\dot{H}^{-1})} \le C(k^{(\beta+1)/2} + h^{\beta+1})\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \ell_k\|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{1.11}$$

We briefly recall some previous works on the numerical approximation for (1.1). Allen, Novosel, and Zhang [1] consider both finite element and finite difference methods of (1.1) in the one-dimensional case and in the cylindrical Wiener process case with $Q = I$ and

$H = L_2(0,1)$, $A = -\frac{\partial}{\partial x^2}$ with Dirichlet boundary condition. Shardlow [16] also considers the finite difference approximation of (1.1) in the one-dimensional case. Du and Zhang [8] consider the numerical approximation for (1.1) but with some special additive noises. Printems [15] considers the time discretization in more general case in abstract framework based on the θ-method. For the numerical approximation of nonlinear evolution partial differential equation, we mention Davie and Gaines [6], Gyöngy [10], [11], Hausenblas [12], etc.

This paper is organized as follows. In Section 2, we consider the regularity of the solution of (1.1). In Section 3, we consider the error estimates in semidiscrete case. In Section 4, we consider the error estimates in the fully discrete case. Finally in Section 5, we consider how to compute the approximate solution U^n numerically.

By C and c we denote large and small positive constants independent of the functions and parameters concerned, but not necessarily the same at different occurrences. When necessary for clarity we distinguish constants by subscripts.

2. Regularity of the mild solution

In this section we will consider the regularity of the mild solution of (1.1). We have

Theorem 2.1. *Let $u(t)$ be the mild solution* (1.3) *of* (1.1)*. If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$. Then we have, for fixed $t \in [0,T]$,*

$$\|u(t)\|_{L_2(\Omega;\dot{H}^\beta)} \le C\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big), \quad \text{for } u_0 \in L_2(\Omega;\dot{H}^\beta). \tag{2.1}$$

In particular, if $W(t)$ is an H-valued Wiener process with covariance operator $Q, \mathrm{Tr}(Q) < \infty$, then we have

$$\|u(t)\|_{L_2(\Omega;\dot{H}^1)} \le C\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \mathrm{Tr}(Q)^{1/2}\Big), \quad \text{for } u_0 \in L_2(\Omega;\dot{H}^1). \tag{2.2}$$

To prove this theorem, we need some regularity results which are related to the fact that $E(t) = e^{-tA}$ is an analytic semigroup on H. For later use, we collect some results in the next two lemmas, see Thomée [17] or Pazy [13].

Lemma 2.2. *Let $\alpha, \beta \in \mathbf{R}$ and let $l \ge 0$ be any integer. We have*

$$|D_t^l E(t)v|_\beta \le Ct^{-(\beta-\alpha)/2-l}|v|_\alpha, \quad \text{for } t > 0, \quad 2l+\beta \ge \alpha, \tag{2.3}$$

and

$$\int_0^t s^\alpha |D_t^l E(s)v|_\beta^2\, ds \le C|v|_{2l+\beta-\alpha-1}^2, \quad \text{for } t \ge 0, \quad \alpha \ge 0. \tag{2.4}$$

Lemma 2.3. *For arbitrary $\alpha \ge 0$, $0 \le \beta \le 1$, we have*

$$\|A^\alpha E(t)\| \le Ct^{-\alpha}, \quad \text{for } t > 0, \tag{2.5}$$

and

$$\|A^{-\beta}(I - E(t))\| \le Ct^\beta, \quad \text{for } t \ge 0. \tag{2.6}$$

PROOF OF THEOREM 2.1. Since the mild solution has the form

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s).$$

Thus, for arbitrary $\beta \geq 0$, using stability property of $E(t)$ and isometry property,

$$\begin{aligned}\mathbf{E}(|u(t)|_\beta^2) &\leq 2\mathbf{E}(|E(t)u_0|_\beta^2) + 2\mathbf{E}\Big\| \int_0^t A^{\beta/2}E(t-s)\,dW(s)\Big\|^2 \\ &= 2\mathbf{E}(|u_0|_\beta^2) + 2\mathbf{E}\int_0^t \|A^{\beta/2}E(t-s)\|_{L_2^0}^2\,ds.\end{aligned} \tag{2.7}$$

With $\{e_l\}_{l=1}^\infty$ an arbitrary orthonormal basis on H, we have, using Lemma 2.2,

$$\begin{aligned}\int_0^t \|A^{\beta/2}E(t-s)\|_{L_2^0}^2\,ds &= \sum_{j=1}^\infty \int_0^t \|A^{\beta/2}E(t-s)Q^{1/2}e_l\|^2\,ds \\ = \sum_{j=1}^\infty \int_0^t |E(s)Q^{1/2}e_l|_\beta^2\,ds &\leq C\sum_{j=1}^\infty |Q^{1/2}e_l|_{\beta-1}^2 = C\|A^{(\beta-1)/2}\|_{L_2^0}^2.\end{aligned}$$

Together with (2.7) this shows (2.1).

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ because

$$\|I\|_{L_2^0}^2 = \sum_{j=1}^\infty \|Q^{1/2}e_j\|^2 = \sum_{j=1}^\infty \gamma_j = \mathrm{Tr}(Q).$$

Corollary 2.1. Let $u(t)$ be the solution of (1.1) and $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. Assume that $W(t)$ is a cylindrical Wiener process with $Q = I$. Then we have, for every $\beta \in [0, 1/2)$,

$$\|u(t)\|_{L_2(\Omega;\dot H^\beta)} \leq C(1 + \|u_0\|_{L_2(\Omega;\dot H^\beta)}), \quad \text{for } u_0 \in L_2(\Omega; \dot H^\beta).$$

PROOF. By (2.1), it suffices to check that in what case $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$. It is well known that A has eigenvalues $\lambda_j = j^2\pi^2, j = 1, 2, \cdots$, and corresponding eigenfunctions $\varphi_j = \sqrt{2}\sin j\pi x, j = 1, 2, \cdots$, which form an orthonormal basis in $H = L_2(0, 1)$. Thus, we have

$$\|A^{(\beta-1)/2}\|_{L_2^0}^2 = \sum_{j=1}^\infty \|A^{(\beta-1)/2}\varphi_j\|^2 = \sum_{j=1}^\infty \lambda_j^{\beta-1},$$

which is convergent if $\beta \in [0, 1/2)$. The proof is complete.

We note that in Theorem 2.1, we require the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for $\beta \in [0, 1]$. The following lemma shows that this condition is equivalent to saying tht $W(t)$ is $\dot H^{\beta-1}$-valued. In particular, $W(t) \in \dot H^{-1}$, which is important when applying the finite element method.

Lemma 2.4. *Assume that $W(t)$ is a Wiener process with covariance operator Q. Assume that A and Q have the same eigenvectors. Then the following statements hold.*

(i) If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, then

$$W(t) = \sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t), \quad t \geq 0,$$

defines an $\dot{H}^{\beta-1}$-valued Wiener process with covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$. In particular, $\tilde{Q} = Q$ if $\mathrm{Tr}(Q) < \infty$;

(ii) If $W(t) = \sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t)$, $t \geq 0$, is an $\dot{H}^{\beta-1}$-valued Wiener process with the covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$, then

$$\|A^{(\beta-1)/2}\|_{L_2^0} < \infty, \quad \textit{for some } \beta \in [0,1].$$

Proof. We first prove (i). With $\{\gamma_l, e_l\}_{l=1}^{\infty}$ the eigensystem of Q in H, it is easy to show that $g_l = Q^{1/2} e_l = \gamma_l^{1/2} e_l$ is an orthonormal basis of $Q^{1/2}(H)$. In fact,

$$(g_l, g_k)_{Q^{1/2}(H)} = (Q^{-1/2} g_l, Q^{1/2} g_k) = (e_l, e_k) = \delta_{k,l}.$$

Note that

$$\sum_{l=1}^{\infty} |g_l|_{\beta-1}^2 = \sum_{l=1}^{\infty} \|A^{(\beta-1)/2} Q^{1/2} e_l\|^2 = \|A^{(\beta-1)/2}\|_{L_2^0} < \infty,$$

which means that the embedding of $Q^{1/2}(H)$ into $\dot{H}^{\beta-1}$ is Hilbert-Schmidt. By Lemma 4.11 in Da Prato and Zabczyk [5], $W(t)$ defines an $\dot{H}^{\beta-1}$-valued Wiener process with covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$. It is obvious that $\tilde{Q} = Q$ if $\mathrm{Tr}(Q) < \infty$.

We now turn to (ii). Since $W(t) = \sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t)$, $t \geq 0$, is an $\dot{H}^{\beta-1}$-valued Wiener process with the covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$, we have

$$\mathbf{E}|W(t)|_{\beta-1}^2 < \infty.$$

With $\{\lambda_l, e_l\}_{l=1}^{\infty}$ the eigensystem of A in H, we have

$$\begin{aligned}
\mathbf{E}|W(t)|_{\beta-1}^2 &= \mathbf{E}\Big| \sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t) \Big|_{\beta-1}^2 \\
&= \mathbf{E} \sum_{l=1}^{\infty} \lambda_l^{\beta-1} \gamma_l \beta_l(t)^2 = t \|A^{(\beta-1)/2}\|_{L_2^0},
\end{aligned}$$

which implies that $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for $\beta \in [0,1]$. The proof is complete.

3. Error estimates in the semidiscrete case

In this section we will consider the error estimates for stochastic partial differential equation in semidiscrete case.

3.1. **Error estimates for deterministic problem.** In order to prove our error estimates for stochastic partial differential equation, we need some nonsmooth data error estimates for homogeneous deterministic parabolic problem.

Let us first consider the stationary problem

$$-\Delta u = f \text{ in } \mathcal{D}, \quad \text{with } u = 0 \text{ on } \partial\mathcal{D}, \tag{3.1}$$

where $f \in \dot{H}^{-1}$.

The variational form of (3.1) is to find $u \in H_0^1$ such that

$$(\nabla u, \nabla \phi) = \langle f, \phi \rangle, \quad \forall \phi \in H_0^1, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\dot{H}^{-1}$ and H_0^1.

Let $S_h \subset H_0^1$ be the finite element space. The semidiscrete problem of (3.2) is to find $u_h \in S_h$ such that

$$(\nabla u_h, \nabla \chi) = \langle f, \chi \rangle, \quad \forall \chi \in S_h. \tag{3.3}$$

By Lax-Milgram lemma, there exist unique solutions $u \in H_0^1$ and $u_h \in S_h$ such that (3.2) and (3.3) hold. Moreover the following stability result holds:

$$|u|_1 \le C|f|_{-1}, \quad \forall f \in \dot{H}^{-1}. \tag{3.4}$$

The standard error estimates read:

$$\|u_h - u\| \le Ch^s|u|_s, \quad s = 1, 2. \tag{3.5}$$

Let $G : \dot{H}^{-1} \to H_0^1$ denote the exact solution operator of (3.1), i.e., $u = Gf$. We define the linear operator $G_h : \dot{H}^{-1} \to S_h$ by $G_h f = u_h$, so that $u_h = G_h f \in S_h$ is the approximate solution of (3.2). It is easy to see that G_h is selfadjoint, positive semidefinite on H, and positive definite on S_h. Introducing the elliptic projection $R_h : H_0^1 \to S_h$ by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \forall v \in H_0^1.$$

We see that $G_h = R_h G$, and $R_h v$ is the finite element approximation of the solution of the corresponding elliptic problem with exact solution v. By (3.5), we get

$$\|R_h v - v\| \le Ch^s|v|_s, \quad s = 1, 2.$$

Hence, using (3.4) and the elliptic regularity estimate, we have

$$\big\|\big(G_h - G\big)f\big\| = \|(R_h - I)Gf\| \le Ch^s|Gf|_s = Ch^s|f|_{s-2}, \quad \text{for } s = 1, 2, \tag{3.6}$$

which we need below.

Let $E_h(t) = e^{-tA_h}$ with $A_h = G_h^{-1}$, and let $E(t) = e^{-tA}$ with $A = G^{-1}$. We then have the following error estimates for deterministic parabolic problem.

Lemma 3.1. *Let $F_h(t) = E_h(t)P_h - E(t)$. Then*

$$\|F_h v\|_{L_\infty([0,T];H)} \le Ch^\beta |v|_\beta, \quad \textit{for } v \in \dot{H}^\beta,\ 0 \le \beta \le 1, \tag{3.7}$$

and

$$\|F_h v\|_{L_2([0,T];H)} \le Ch^\beta |v|_{\beta-1}, \quad \textit{for } v \in \dot{H}^{\beta-1},\ 0 \le \beta \le 1. \tag{3.8}$$

Further, in the weak norm,

$$\|F_h v\|_{L_\infty([0,T];\dot{H}^{-1})} \le Ch^\beta |v|_{\beta-1}, \quad for\, v \in \dot{H}^{\beta-1},\ 1 \le \beta \le 2, \tag{3.9}$$

and, with $\ell_h = \log(T/h^2)$,

$$\|F_h v\|_{L_2([0,T];\dot{H}^{-1})} \le Ch^\beta \ell_h |v|_{\beta-2}, \quad for\, v \in \dot{H}^{\beta-2},\ 1 \le \beta \le 2. \tag{3.10}$$

PROOF. We denote $u(t) = E(t)v$, $u_h(t) = E_h(t)P_h v$, and $e(t) = u_h(t) - u(t) = F_h(t)v$. We first show (3.7). By the stability properties of the L_2 projection operator P_h and the solution operators $E_h(t)$ and $E(t)$, we have

$$\|e(t)\| = \|E_h(t)P_h v - E(t)v\| \le 2\|v\|, \quad \text{for } t \ge 0,\ v \in H. \tag{3.11}$$

We will show that

$$\|e(t)\| \le Ch|v|_1, \quad \text{for } t \ge 0,\ v \in \dot{H}^1. \tag{3.12}$$

Combining this with interpolation theory, we get (3.7).

To show (3.12), let us consider the error equation

$$G_h e_t + e = \rho, \tag{3.13}$$

where $\rho = (G_h - G)u_t$. We note that $G_h e(0) = 0$ for

$$(G_h e(0), w) = (P_h v - v, G_h w) = 0, \quad \text{for } w \in H, \tag{3.14}$$

since $G_h w \in S_h$.

By the energy method, we can show, see Thomée [17, Lemma 3.3],

$$\|e(t)\| \le C \sup_{s \le t} \Big(s\|\rho_t(s)\| + \|\rho(s)\| \Big), \quad t \ge 0.$$

Obviously, by (3.6) and Lemma 2.2,

$$\|\rho(s)\| = \|(G_h - G)u_t\| \le Ch|u_t|_{-1} \le Ch|v|_1,$$

and

$$s\|\rho_t(s)\| \le Chs|u_t(s)|_1 \le Ch|v|_1.$$

Hence (3.12) follows and therefore we get (3.7).

We next show (3.8). By interpolation theory, it suffices to show that

$$\|e\|_{L_2([0,T];H)} \le C|v|_{-1}, \tag{3.15}$$

and

$$\|e\|_{L_2([0,T];H)} \le Ch\|v\|. \tag{3.16}$$

Taking the inner product of (3.13) with e, we get

$$(G_h e_t, e) + (e, e) = (\rho, e).$$

Integrating with respect to t, we get, noting that $G_h e(0) = 0$ and using the inequality $(\rho, e) \le \frac{1}{2}(\|\rho\|^2 + \|e\|^2)$,

$$(G_h e(T), e(T)) + \int_0^T \|e\|^2\, dt \le \int_0^T \|\rho\|^2\, dt. \tag{3.17}$$

Obviously, by (3.6) and Lemma 2.2,

$$\int_0^T \|\rho\|^2\,dt \le \int_0^T \|(G_h-G)u_t\|^2\,dt \le Ch^2\int_0^T |u|_1^2\,dt \le Ch^2\|v\|^2, \tag{3.18}$$

which implies that (3.16) holds.

To show (3.15), we note that, by Lemma 2.2 and its discrete counterpart,

$$\int_0^T \|e\|^2\,dt \le 2\int_0^T \Big(\|u_h\|^2+\|u\|^2\Big)\,dt \le 2|v|_{-1,h}^2 + 2|v|_{-1}^2, \tag{3.19}$$

where $|v|_{-1,h}$ is a discrete seminorm defined by

$$|v|_{-1,h} = (G_h v, v)^{1/2} = \|G_h^{1/2} v\|.$$

Since $|v|_{-1} = \sup\{(v,w)/|w|_1 : w \in \dot H^1\}$, see Thomée [17, Chapter 6], we thus have, with $w = G_h v,\ v \in \dot H^{-1}$,

$$|v|_{-1} = \sup_{w\in \dot H^1} \frac{(v,w)}{|w|_1} \ge \frac{(v,G_h v)}{|G_h v|_1} = \frac{(v,G_h v)}{(v,G_h v)^{1/2}} = |v|_{-1,h},$$

since

$$|G_h v|_1^2 = (AG_h v, G_h v) = A(G_h v, G_h v) = (A_h G_h v, G_h v) = (v, G_h v),$$

where $A_h = G_h^{-1}$. Hence by (3.19), we get $\int_0^T \|e\|^2\,dt \le 4|v|_{-1}^2$, which implies that (3.15) holds.

We now turn to (3.9). It suffices to show that

$$|e(t)|_{-1} \le Ch\|v\|, \tag{3.21}$$

and

$$|e(t)|_{-1} \le Ch^2|v|_1. \tag{3.22}$$

By (3.17) and (3.18), we have

$$(G_h e, e) = |e|_{-1,h}^2 \le Ch^2\|v\|^2. \tag{3.23}$$

Using

$$|e|_{-1} \le |e|_{-1,h} + Ch\|e\|, \tag{3.24}$$

which follows from, by (3.6),

$$|e|_{-1}^2 = (G_h e, e) + ((G-G_h)e, e) \le |e|_{-1,h}^2 + Ch^2\|e\|^2.$$

We obtain, by (3.11)

$$|e|_{-1} \le |e|_{-1,h} + Ch\|e\| \le Ch\|v\|,$$

which is (3.21).

By (3.17) and (3.6), we obtain

$$|e(t)|_{-1,h}^2 = (G_h e(t), e(t)) \le \frac{1}{2}\int_0^t \|\rho\|^2\,ds \le Ch^4\int_0^t |u_t|^2\,ds \le Ch^4|v|_1^2.$$

Combining this with (3.12) and (3.24), we get (3.22).

It remains to show (3.10). Integrating (3.13) with respect to t, we have, with $\tilde{e}(t) = \int_0^t e(s)\,ds,\ \tilde{\rho}(t) = \int_0^t \rho(s)\,ds$,

$$G_h e + \tilde{e} = \tilde{\rho}, \quad \tilde{e}(0) = 0. \tag{3.25}$$

Taking the inner product of (3.25) with e, we get, since $e = \tilde{e}_t$,

$$(G_h e, e) + \frac{1}{2}\frac{d}{dt}\|\tilde{e}\|^2 = (\tilde{\rho}, e) = \frac{d}{dt}(\tilde{\rho}, \tilde{e}) - (\rho, \tilde{e}).$$

After integration, we have, noting that $\tilde{e}(0) = 0$,

$$\begin{aligned}
\int_0^T |e|_{-1,h}^2\,ds + \frac{1}{2}\|\tilde{e}(T)\|^2 &= \int_0^T (\tilde{\rho}, e)\,ds = \Big[(\tilde{\rho}, \tilde{e})\Big]_0^T - \int_0^T (\rho, \tilde{e})\,ds \\
&\le \|\tilde{\rho}(T)\|\|\tilde{e}(T)\| + \Big(\int_0^T \|\rho\|\,ds\Big) \sup_{0\le s\le T} \|\tilde{e}(s)\| \\
&\le 2\Big(\int_0^T \|\rho\|\,ds\Big) \sup_{0\le s\le T} \|\tilde{e}(s)\|.
\end{aligned}$$

By a kick-back argument, we obtain

$$\int_0^T |e|_{-1,h}^2\,ds \le C\Big(\int_0^T \|\rho\|\,ds\Big)^2.$$

Noting that

$$\begin{aligned}
\int_0^T \|\rho\|\,ds &= \int_0^{h^2} \|\rho\|\,ds + \int_{h^2}^T \|\rho\|\,ds \\
&\le C\int_0^{h^2} s^{-1/2}|v|_{-1}\,ds + C\int_{h^2}^T h|u|_1\,ds \le Ch\ell_h|v|_{-1},
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\int_0^T \|\rho\|\,ds &= \int_0^{h^2} \|\rho\|\,ds + \int_{h^2}^T \|\rho\|\,ds \\
&\le Ch^2\|v\| + Ch^2\int_{h^2}^T |u|_2\,ds \\
&\le Ch^2\|v\| + Ch^2\log(T/h^2)\|v\| \le Ch^2\ell_h\|v\|,
\end{aligned}$$

we therefore get

$$\int_0^T |e|_{-1,h}^2\,ds \le Ch^2\ell_h^2|v|_{-1}^2,$$

and

$$\int_0^T |e|_{-1,h}^2\,ds \le Ch^4\ell_h^2\|v\|^2.$$

By (3.18), (3.19) and (3.24), we obtain

$$\int_0^T |e|_{-1}^2\,ds \le C\int_0^T |e|_{-1,h}^2\,ds + Ch^2\int_0^T \|e\|^2\,ds$$
$$\le Ch^2\ell_h^2|v|_{-1}^2 + Ch^2|v|_{-1}^2 \le Ch^2\ell_h^2|v|_{-1}^2,$$

and

$$\int_0^T |e|_{-1}^2\,ds \le Ch^4\ell_h^2\|v\|^2 + Ch^4\|v\|^2 \le Ch^4\ell_h^2\|v\|^2.$$

Now (3.10) follows from the interpolation theory. The proof is complete.

3.2. **Strong norm convergence.** In this subsection, we will consider the error estimate for (1.1) in semidiscrete case with respect to strong norm. We have

Theorem 3.2. *Let u_h and u be the solutions of* (1.4) *and* (1.1). *If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, then we have, for $t \ge 0$ and $u_0 \in L_2(\Omega; \dot H^\beta)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch^\beta\Big(\|u_0\|_{L_2(\Omega;\dot H^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{3.26}$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, *then we have, for $t \ge 0$ and $u_0 \in L_2(\Omega; \dot H^1)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch\Big(\|u_0\|_{L_2(\Omega;\dot H^1)} + \mathrm{Tr}(Q)^{1/2}\Big). \tag{3.27}$$

Proof. By definition of the mild solution, we have, with $E(t) = e^{-tA}$,

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,dW(s),$$

and, with $E_h(t) = e^{-tA_h}$,

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)\,P_h dW(s).$$

Denoting $e(t) = u_h(t) - u(t)$ and $F_h(t) = E_h(t)P_h - E(t)$, we write

$$e(t) = E_h(t)P_h u_0 - E(t)u_0 + \int_0^t \Big(E_h(t-s)P_h - E(t-s)\Big)\,dW(s)$$
$$= F_h(t)u_0 + \int_0^t F_h(t-s)\,dW(s) = I + II.$$

Thus

$$\|e(t)\|_{L_2(\Omega;H)} \le 2\Big(\|I\|_{L_2(\Omega;H)} + \|II\|_{L_2(\Omega;H)}\Big).$$

For I, we have, by (3.7) with $v = u_0$,

$$\|I\| = \|F_h(t)u_0\| \le Ch^\beta|u_0|_\beta, \quad \text{for } 0 \le \beta \le 1,$$

which implies that $\|I\|_{L_2(\Omega;H)} \le Ch^\beta\|u_0\|_{L_2(\Omega;\dot H^\beta)}$, for $0 \le \beta \le 1$.

For II, we have, by the isometry property,

$$\|II\|^2_{L_2(\Omega;H)} = \Big\|\mathbf{E}\int_0^t F_h(t-s)\,dW(s)\Big\|^2 = \int_0^t \|F_h(t-s)\|^2_{L_2^0}\,ds$$
$$= \sum_{l=1}^{\infty}\int_0^t \|F_h(t-s)Q^{1/2}e_l\|^2\,ds,$$

where $\{e_l\}$ is any orthonormal basis in H.

Using (3.8) with $v = Q^{1/2}e_l$, we obtain

$$\|II\|^2_{L_2(\Omega;H)} \le C\sum_{l=1}^{\infty} h^{2\beta}\|Q^{1/2}e_l\|^2_{\beta-1} = C\sum_{l=1}^{\infty} h^{2\beta}\|A^{(\beta-1)/2}Q^{1/2}e_l\|^2$$
$$= Ch^{2\beta}\|A^{(\beta-1)/2}\|^2_{L_2^0},$$

which completes the proof of (3.26).

In particular, if $W(t)$ is a Wiener process with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ in (3.26) and obtain (3.27), since $\|I\|^2_{L_2^0} = \mathrm{Tr}(Q)$.

Corollary 3.1. Let u_h and u be the solutions of (1.4) and (1.1), respectively. Assume that $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) \subset H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $t \ge 0$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch^\beta\big(1 + \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}\big), \quad \text{for } 0 \le \beta < 1/2.$$

Proof. The proof is similar to the proof of Corollary 2.1.

3.3. **Weak norm convergence.** In this subsection we state our weak norm convergence error estimate.

Theorem 3.3. *Let u_h and u be the solutions of* (1.4) *and* (1.1). *If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, then we have, for $0 \le t \le T$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$, with $\ell_h = \log(T/h^2)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \le Ch^{\beta+1}\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \ell_h\|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{3.28}$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, then we have, for $0 \le t \le T$ and $u_0 \in L_2(\Omega; \dot{H}^1)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \le Ch^2\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \ell_h\,\mathrm{Tr}(Q)^{1/2}\Big). \tag{3.29}$$

Proof. Using the same notation as in Theorem 3.2, we have, by (3.9),

$$\|I\|_{L_2(\Omega;\dot{H}^{-1})} \le Ch^{\beta+1}\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}, \quad \text{for } 0 \le \beta \le 1.$$

For II, we have, by the isometry property, and (3.10) with $v = Q^{1/2}e_l$,

$$\begin{aligned}\|II\|^2_{L_2(\Omega;\dot{H}^{-1})} &= \mathbf{E}\Big|\int_0^t F_h(t-s)\,dW(s)\Big|^2_{-1} = \mathbf{E}\Big\|\int_0^t A^{-1/2}F_h(t-s)\,dW(s)\Big\|^2 \\ &= \int_0^t \|A^{-1/2}F_h(t-s)\|^2_{L_2^0}\,ds \\ &\le Ch^{2\beta}\ell_h^2\sum_{l=1}^{\infty}\|A^{(\beta-1)/2}Q^{1/2}e_l\|^2 \le Ch^{2(\beta+1)}\ell_h^2\|A^{(\beta-1)/2}\|^2_{L_2^0},\end{aligned}$$

which completes the proof of (3.28).

In particular, if $W(t)$ is a Wiener process on H with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ in (3.28) and obtain (3.29).

Corollary 3.2. Let u_h and u be the solutions of (1.4) and (1.1), respectively. Assume that $A = -\frac{\partial^2}{\partial x^2}$ and $\mathcal{D}(A) = H_0^1(0,1)\cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $0 \le t \le T$ and $u_0 \in L_2(\Omega;\dot{H}^\beta)$, with $\ell_h = \log(T/h^2)$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \le Ch^{\beta+1}(1 + \ell_h\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}), \quad \text{for } 0 \le \beta < 1/2.$$

4. Error estimates in the fully discrete case

In this section we will consider the error estimates for (1.1) in the fully discrete case.

4.1. **Error estimates for deterministic problem.**

As in the semidiscrete case, in order to prove error estimates for the stochastic partial differential equation in the fully discrete case, we need some error estimates for deterministic parabolic problem.

Let $E_{kh} = r(kA_h)$ and $E(t_n) = e^{-t_nA}$, where $r(\lambda) = 1/(1+\lambda)$ is introduced in (1.9). We have

Lemma 4.1. *Let* $F_n = E_{kh}^nP_h - E(t_n)$. *Then*

$$\|F_nv\| \le C(k^{\beta/2} + h^\beta)|v|_\beta, \quad \textit{for } v \in \dot{H}^\beta,\ 0 \le \beta \le 1, \tag{4.1}$$

and

$$\Big(k\sum_{j=1}^n \|F_jv\|^2\Big)^{1/2} \le C(k^{\beta/2} + h^\beta)|v|_{\beta-1}, \quad \textit{for } v \in \dot{H}^{\beta-1},\ 0 \le \beta \le 1. \tag{4.2}$$

Further, in the weak norm,

$$|F_nv|_{-1} \le C(k^{\beta/2} + h^\beta)|v|_{\beta-1}, \quad \textit{for } v \in \dot{H}^{\beta-1},\ 1 \le \beta \le 2, \tag{4.3}$$

and, with $\ell_k = \log(T/k)$ *where* $T = t_n$,

$$\Big(k\sum_{j=1}^n |F_jv|^2_{-1}\Big)^{1/2} \le C(k^{\beta/2} + h^\beta)\ell_k|v|_{\beta-2}, \quad \textit{for } v \in \dot{H}^{\beta-2},\ 1 \le \beta \le 2. \tag{4.4}$$

PROOF. We denote $u(t_n) = u^n = E(t_n)v$, $U^n = E_{kh}^n P_h v$, and $e^n = F_n v$. We first show (4.1). By the stability properties of the L_2 projection operator P_h and the solution operators $E_{kh}(t)$ and $E(t)$, we have

$$\|e^n\| = \|E_{kh}^n P_h v - E(t_n)v\| \le 2\|v\|, \quad \text{for } t \ge 0,\ v \in H. \tag{4.5}$$

We will show that

$$\|e^n\| \le C(k^{1/2} + h)|v|_1, \quad \text{for } v \in \dot{H}^1. \tag{4.6}$$

Combining this with interpolation theory, we get (4.1).

To show (4.6), let us consider the error equation, with $\partial_t e^n = (e^n - e^{n-1})/k$,

$$G_h \partial_t e^n + e^n = \rho^n + G_h \tau^n, \tag{4.7}$$

where $\rho^n = (G_h - G)u_t(t_n)$ and $\tau^n = u_t(t_n) - \partial_t u^n$.

By the energy method, we have

$$t_n\|e^n\|^2 \le t_n\|\rho^n\|^2 + k\sum_{j=1}^n \Big(\|\rho^j\|^2 + t_{j-1}^2\|\partial_t\rho^j\|^2 + \|G_h\tau^j\|^2 + t_{j-1}^2\|\tau^j\|^2\Big).$$

Here, by (3.6) and Lemma 2.2, we have

$$\|\rho^j\| = \|(G_h - G)u_t(t_j)\| \le Ch|u_t(t_j)|_{-1} \le Ch|v|_1,$$

and

$$\begin{aligned} t_{j-1}\|\partial_t\rho^j\| &= \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j} t_{j-1}\rho_t(s)\,ds\Big\| \le \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j} s\rho_t(s)\,ds\Big\| \\ &\le \sup_{0\le s\le t_n}\|s\rho_t(s)\| \le Ch\sup_{0\le s\le t_n}|su_t(s)|_1 \le Ch|v|_1. \end{aligned}$$

Further, we write

$$\|G_h\tau^j\| = \|(G_h - G)\tau^j\| + \|G\tau^j\|,$$

where, using (3.6) and Lemma 2.2,

$$\|(G_h - G)\tau^j\| \le Ch|\tau^j|_{-1} \le Ch\sup_{0\le s\le t_n}|u_t(s)|_{-1} \le Ch|v|_1.$$

Hence we obtain

$$\|e^n\|^2 \le Ch^2|v|_1^2 + Ckt_n^{-1}\sum_{j=1}^n\Big(\|G\tau^j\|^2 + t_{j-1}^2\|\tau^j\|^2\Big).$$

By Taylor's formula, we have

$$\begin{aligned} \|G\tau^j\|^2 &= \Big\|G\frac{1}{k}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_{tt}(s)\,ds\Big\|^2 = \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_t(s)\,ds\Big\|^2 \\ &\le \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s - t_{j-1})^{1/2}k^{1/2}u_t(s)\,ds\Big\|^2 \le \int_{t_{j-1}}^{t_j}(s - t_{j-1})\|u_t(s)\|^2\,ds \\ &\le t_n\int_{t_{j-1}}^{t_j}\|u_t(s)\|^2\,ds, \end{aligned}$$

and

$$t_{j-1}^2\|\tau^j\|^2 = t_{j-1}^2\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j} u_{tt}(s)\,ds\Big\|^2 \le t_{j-1}^2\int_{t_{j-1}}^{t_j}(s-t_{j-1})\|u_{tt}(s)\|^2\,ds$$
$$\le t_n\int_{t_{j-1}}^{t_j} s^2\|u_{tt}(s)\|^2\,ds.$$

Applying Lemma 2.2, we have

$$kt_n^{-1}\sum_{j=1}^n t_{j-1}^2\|\tau^j\|^2 \le k\sum_{j=1}^n\int_{t_{j-1}}^{t_j}\|u_t(s)\|^2\,ds$$
$$= k\int_0^{t_n}\|u_t(s)\|^2\,ds \le Ck|v|_1,$$

and

$$kt_n^{-1}\sum_{j=1}^n t_{j-1}^2\|\tau^j\|^2 \le k\sum_{j=1}^n\int_{t_{j-1}}^{t_j} s^2\|u_{tt}(s)\|^2\,ds$$
$$= k\int_0^{t_n} s^2\|u_{tt}(s)\|^2\,ds \le Ck|v|_1.$$

Hence (4.6) follows and therefore we get (4.1).

We next show (4.2). By interpolation theory, it suffices to show that

$$\Big(k\sum_{j=1}^n\|F_jv\|^2\Big)^{1/2} \le C|v|_{-1}, \tag{4.8}$$

and

$$\Big(k\sum_{j=1}^n\|F_jv\|^2\Big)^{1/2} \le C(k^{1/2}+h)\|v\|. \tag{4.9}$$

Taking the inner product of (4.7) with e^n, we get

$$(G_h\partial_t e^n, e^n) + (e^n, e^n) = (\rho^n, e^n) + (G_h\tau^n, e^n).$$

By summation on n, using the inequality $(\rho^n, e^n) \le \frac{1}{2}(\|\rho^n\|^2 + \|e^n\|^2)$, and noting that $G_he^0 = 0$, we have

$$(G_he_n, e_n) + k\sum_{j=1}^n\|e_j\|^2 \le Ck\sum_{j=1}^n\|\rho_j\|^2 + Ck\sum_{j=1}^n\|G\tau^j\|^2 + Ck\sum_{j=1}^n\|(G_h-G)\tau^j\|^2. \tag{4.10}$$

Here, using Lemma 2.2, we have, since $\rho^j = \rho(s) + \int_s^{t_j} \rho_t(\tau)\, d\tau$,

$$
\begin{aligned}
k\sum_{j=1}^{n} \|\rho^j\|^2 &= k\|\rho\|^2 + \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \|\rho^j\|^2\, ds \\
&\le k\|\rho\|^2 + 2\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \Big(\|\rho(s)\|^2 + \|\int_s^{t_j} \rho_t(\tau)\, d\tau\|^2\Big)\, ds \\
&\le k\|\rho\|^2 + 2\int_{t_1}^{t_n} \|\rho(s)\|^2\, ds + 2\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \Big((t_j - s)\int_s^{t_j} \|\rho_t(\tau)\|^2\, d\tau\Big)\, ds \\
&\le k\|\rho\|^2 + 2\int_{t_1}^{t_n} \|\rho(s)\|^2\, ds + 2k\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \tau\|\rho_t(\tau)\|^2\, d\tau \\
&\le k\|\rho\|^2 + 2\int_{t_1}^{t_n} \|\rho(s)\|^2\, ds + 2k\int_{t_1}^{t_n} \tau\|\rho_t(\tau)\|^2\, d\tau \\
&\le Ck\|u\|^2 + Ch^2\int_0^{t_n} |u(s)|_1^2\, ds + Ck\int_0^{t_n} \tau\|u_t(\tau)\|^2\, d\tau \le C(k+h^2)\|v\|^2,
\end{aligned}
\tag{4.11}
$$

and, by Taylor's formula,

$$
\begin{aligned}
k\sum_{j=1}^{n} \|(G_h - G)\tau^j\|^2 &\le Ckh^2|\tau^1|_{-1}^2 + Ckh^2\sum_{j=2}^{n} |\tau^j|_{-1}^2 \\
&= Ckh^2\Big|u_t(k) - \frac{1}{k}\int_0^k u_t(\tau)\, d\tau\Big|_{-1}^2 + Ckh^2\sum_{j=2}^{n} \Big|\frac{1}{k}\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)\, ds\Big|_{-1}^2 \\
&\le Ch^2\Big(k|u_t(k)|_{-1}^2 + \int_0^k |u_t(\tau)|_{-1}^2\, d\tau\Big) + Ch^2\sum_{j=2}^{n} \Big|\int_{t_{j-1}}^{t_j} (s - t_{j-1})^{1/2}u_{tt}(s)\, ds\Big|_{-1}^2 \\
&\le Ch^2\|v\|^2 + Ch^2\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} k(s - t_{j-1})|u_{tt}(s)|_{-1}^2\, ds \\
&\le Ch^2\|v\|^2 + Ch^2\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} s^2|u_{tt}(s)|_{-1}^2\, ds \le Ch^2\|v\|^2,
\end{aligned}
$$

and

$$k\sum_{j=1}^{n}\|G\tau^j\|^2 = k\sum_{j=1}^{n}\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_t(s)\,ds\Big\|^2$$
$$\le k\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\|u_t(s)\|^2$$
$$\le Ck\int_0^{t_n} s\|u_t(s)\|^2\,ds \le k\|v\|^2.$$

We therefore obtain

$$(G_he^n,e^n)^{1/2}+\Big(k\sum_{j=1}^{n}\|e^j\|^2\Big)^{1/2}\le C(k^{1/2}+h)\|v\|, \tag{4.12}$$

which implies that (4.9) holds.

To show (4.8), we note that,

$$k\sum_{j=1}^{n}\|e^j\|^2\le Ck\sum_{j=1}^{n}\|U^j\|^2+Ck\sum_{j=1}^{n}\|u(t_j)\|^2. \tag{4.13}$$

Here, we have, following (4.11) with ρ replaced by u,

$$k\sum_{j=1}^{n}\|u(t_j)\|^2\le k\|u(t_1)\|^2+2\int_{t_1}^{t_n}\|u(s)\|^2\,ds+2\int_{t_1}^{t_n}s^2\|u_t(s)\|^2\,ds\le C|v|_{-1}^2, \tag{4.14}$$

and, by (3.20),

$$k\sum_{j=1}^{n}\|U^j\|^2\le C|v|_{-1,h}^2\le C|v|_{-1}^2,$$

which imply that (4.8) holds and the proof of (4.2) is complete.

We now turn to (4.3). It suffices to show that

$$|e^n|_{-1}\le C(k^{1/2}+h)\|v\|, \tag{4.15}$$

and

$$|e^n|_{-1}\le C(k+h^2)|v|_1. \tag{4.16}$$

Obviously (4.15) follows by (3.11), (3.24), and (4.12). Note that, by Lemma 2.2,

$$\begin{aligned}
k\sum_{j=1}^{n}\|\rho_j\|^2 &\le C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\Big(\|\rho(s)\|^2+\Big\|\int_s^{t_j}\rho_t(\tau)\,d\tau\Big\|^2\Big)\,ds\\
&\le C\int_0^{t_n}\|\rho(s)\|^2\,ds+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}k^2\|\rho_t(\tau)\|^2\,d\tau\\
&\le \int_0^{t_n}\|\rho(s)\|^2+Ck^2\int_0^{t_n}\|\rho_t(\tau)\|^2\,d\tau\\
&\le Ch^4\int_0^{t_n}|u|_2^2\,ds+Ck^2\int_0^{t_n}\|u_t\|^2\,ds\le C(h^4+k^2)|v|_1^2,
\end{aligned}$$

and

$$\begin{aligned}
k\sum_{j=1}^{n}\|(G_h-G)\tau^j\|^2 &\le Ckh^4\sum_{j=1}^{n}\|\tau^j\|^2\\
&= Ckh^4\sum_{j=1}^{n}\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_{tt}(s)\,ds\Big\|^2\\
&\le Ch^4\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(s-t_{j-1})^2\|u_{tt}(s)\|^2\,ds\\
&\le Ch^4\int_0^{t_n}s^2\|u_{tt}(s)\|^2\,ds\le Ch^4|v|_1^2,
\end{aligned}$$

and

$$\begin{aligned}
k\sum_{j=1}^{n}\|G\tau^j\|^2 &= k\sum_{j=1}^{n}\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_t(s)\,ds\Big\|^2\\
&\le k^{-1}\sum_{j=1}^{n}\Big(\int_{t_{j-1}}^{t_j}(s-t_{j-1})^2\,ds\Big)\int_{t_{j-1}}^{t_j}\|u_t(s)\|^2\,ds\\
&\le Ck^2\int_0^{t_n}\|u_t(s)\|^2\,ds\le Ck^2|v|_1^2.
\end{aligned}$$

Combining these estimates with (4.10), we get (4.16)

It remains to show (4.4). As in the proof of (3.10), it suffices to show

$$\Big(k\sum_{j=1}^{n}|e_j|_{-1}^2\Big)^{1/2}\le C(k+h^2)\ell_k\|v\|, \tag{4.17}$$

and

$$\Big(k\sum_{j=1}^{n}|e_j|_{-1}^2\Big)^{1/2}\le C(k^{1/2}+h)\ell_k|v|_{-1}. \tag{4.18}$$

Let $\tilde{e}^n = k\sum_{j=1}^n e^j$, $\tilde{e}^0 = 0$, and $\partial_t \tilde{e}^n = (\tilde{e}^n - \tilde{e}^{n-1})/k = e^n$ for $n \ge 1$. We have the error equation

$$G_h \partial_t \tilde{e}^n + \tilde{e}^n = \tilde{\rho^n} + G_h \tilde{\tau}^n, \quad \text{for } n \ge 1, \tag{4.19}$$

where $\tilde{\tau}^n = k\sum_{j=1}^n \tau^j$, and $\tilde{\rho}^n = k\sum_{j=1}^n \rho^j$, where τ^j and ρ^j are defined as before.

Taking the inner product of (4.19) with $\partial_t \tilde{e}^n$, we get, since $\partial_t \tilde{e}^n = e^n$,

$$\begin{aligned}(G_h \partial_t \tilde{e}^n, \partial_t \tilde{e}^n) + \frac{1}{2}\partial_t(\tilde{e}^n, \tilde{e}^n) + \frac{1}{k}(\partial_t \tilde{e}^n, \partial_t \tilde{e}^n) &= (\tilde{\rho}^n, \partial_t \tilde{e}^n) + (G_h \partial_t \tilde{\tau}^n, \partial_t \tilde{e}^n)\\ = \partial_t(\tilde{\rho}^n, \tilde{e}^n) - (\partial_t \tilde{\rho}^n, \tilde{e}^{n-1}) + \partial_t(G_h \tilde{\tau}^n, \tilde{e}^n) - (\partial_t(G_h \tilde{\tau}^n), \tilde{e}^{n-1}).\end{aligned}$$

By summation on n, noting that $\tilde{e}^0 = 0$, we have

$$\begin{aligned}&k\sum_{j=1}^n (G_h \partial_t \tilde{e}^j, \partial_t \tilde{e}^j) + \frac{1}{2}(\tilde{e}^n, \tilde{e}^n)\\ &\le \|\tilde{\rho}^n\|\|\tilde{e}^n\| + k\sum_{j=1}^n |(\rho^n, \tilde{e}^{n-1})| + \|G_h \tilde{\tau}^n\|\|\tilde{e}^n\| + k\sum_{j=1}^n |(G_h \tau^n, \tilde{e}^{n-1})|\\ &\le \max_j \|\tilde{e}^j\|\Big(\|\tilde{\rho}^n + k\sum_{j=1}^n \|\rho^j\| + k\sum_{j=1}^n \|G_h \tau^j\| + \|G_h \tilde{\tau}^n\|\Big).\end{aligned}$$

By a kick-back argument, we obtain

$$\Big(k\sum_{j=1}^n (G_h \partial_t \tilde{e}^j, \partial_t \tilde{e}^j)\Big)^{1/2} \le Ck\sum_{j=1}^n \|\rho^j\| + Ck\sum_{j=1}^n \|(G_h - G)\tau^j\| + Ck\|G\tau^j\|.$$

Here, with $\ell_k = \log(T/k)$ where $T = t_n$, we have

$$\begin{aligned}k\sum_{j=1}^n \|\rho^j\| = k\|\rho\| + k\sum_{j=2}^n \|\rho^j\| &\le Ck\|v\| + Ck\sum_{j=2}^n t_j^{-1}\|v\|\\ &\le Ck\|v\| + Ck\ell_k\|v\| \le Ck\ell_k\|v\|,\end{aligned}$$

and

$$\begin{aligned}k\sum_{j=1}^n \|(G_h - G)\tau^j\| &\le Ckh^2\sum_{j=1}^n \|\tau^j\| = Ckh^2\|\tau^1\| + Ckh^2\sum_{j=2}^n \|\tau^j\|\\ &= Ckh^2\Big\|u_t(k) - \partial_t u^1\Big\| + Ch^2\sum_{j=2}^n \Big\|\int_{t_{j-1}}^{t_j} (s - t_j)u_{tt}(s)\, ds\Big\|\\ &\le Ch^2\big(\|ku_t(k)\| + \|u(k)\| + \|v\|\big) + Ch^2\sum_{j=2}^n \int_{t_{j-1}}^{t_j} \|su_{tt}(s)\|\, ds\\ &\le Ch^2\|v\| + Ch^2\int_{t_1}^{t_n} \|su_{tt}(s)\|\, ds \le Ch^2\ell_k\|v\|,\end{aligned}$$

and

$$k\sum_{j=1}^{n}\|G\tau^j\| = k\|\tau^1\| + k\sum_{j=2}^{n}\|G\tau^j\| \le Ck\ell_k\|v\|,$$

which imply that (4.17) holds. Similarly we can show (4.18). Hence (4.4) follows.

Together these estimates complete the proof.

4.2. **Strong norm convergence.** We have the following strong norm convergence result in the fully discrete case.

Theorem 4.2. *Let U^n and $u(t_n)$ be the solutions of* (1.9) *and* (1.1), *respectively. If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, then we have, for $u_0 \in L_2(\Omega, \dot{H}^\beta)$,*

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\beta/2} + h^\beta)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{4.20}$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, then we have, for $u_0 \in L_2(\Omega; \dot{H}^1)$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{1/2} + h)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \mathrm{Tr}(Q)^{1/2}\Big). \tag{4.21}$$

Proof. We have, by (1.9), with $E_{kh}^n = r(kA_h)^n$,

$$U^n = E_{kh}^n P_h u_0 + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_{kh}^{n-j+1} P_h\, dW(s),$$

and, by the definition of the mild solution of (1.1), with $E(t) = e^{-tA}$,

$$u(t_n) = E(t_n)u_0 + \int_0^{t_n} E(t_n - s)\, dW(s).$$

Denoting $e^n = U^n - u(t_n)$ and $F_n = E_{kh}^n P_h - E(t_n)$, we write

$$\begin{aligned} e^n &= F_n u_0 + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} F_{n-j+1}\, dW(s) \\ &\quad + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \Big(E(t_n - t_{j-1}) - E(t_n - s)\Big)\, dW(s) \\ &= I + II + III. \end{aligned}$$

Thus

$$\|e^n\|_{L_2(\Omega;H)} \le C\Big(\|I\|_{L_2(\Omega;H)} + \|II\|_{L_2(\Omega;H)} + \|III\|_{L_2(\Omega;H)}\Big).$$

For I, we have, by (4.1) with $v = u_0$,

$$\|I\| = \|F_n u_0\| \le C(k^{\beta/2} + h^\beta)|u_0|_\beta,$$

which implies that $\|I\|_{L_2(\Omega;H)} \le C(k^{\beta/2} + h^\beta)\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}$.

For II, we have, by the isometry property,

$$\begin{aligned}\|II\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} F_{n-j+1}\, dW(s)\Big\|^2 = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|F_{n-j+1}\|^2_{L_2^0}\, ds \\ &= \sum_{l=1}^{\infty} \Big(k \sum_{j=1}^{n} \|F_{n-j+1} Q^{1/2} e_l\|^2 \Big),\end{aligned}$$

where $\{e_l\}$ is any orthonormal basis in H. Using (4.2) with $v = Q^{1/2} e_l$, we obtain

$$\begin{aligned}\|II\|^2_{L_2(\Omega;H)} &\le C \sum_{l=1}^{\infty} (k^\beta + h^{2\beta}) |Q^{1/2} e_l|^2_{\beta-1} \\ &= C \sum_{l=1}^{\infty} (k^\beta + h^{2\beta}) \|A^{(\beta-1)/2} Q^{1/2} e_l\|^2 \\ &= C (k^\beta + h^{2\beta}) \|A^{(\beta-1)/2}\|^2_{L_2^0}.\end{aligned}$$

For III, we have, by the isometry property,

$$\begin{aligned}\|III\|^2_{L_2(\Omega;H)} &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big\| \Big(E(t_n - t_{j-1}) - E(t_n - s) \Big) \Big\|^2_{L_2^0} ds \\ &= \sum_{l=1}^{\infty} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big\| A^{-\beta/2} \Big(E(s - t_{j-1}) - I \Big) A^{\beta/2} E(t_n - s) Q^{1/2} e_l \Big\|^2 ds.\end{aligned}$$

Using (2.6), and (2.4) with $v = A^{(\beta-1)/2} Q^{1/2} e_l$, we obtain

$$\begin{aligned}\|III\|^2_{L_2(\Omega;H)} &\le C k^\beta \sum_{l=1}^{\infty} \int_0^{t_n} \|A^{1/2} E(t_n - s) A^{(\beta-1)/2} Q^{1/2} e_l\|^2\, ds \\ &\le C k^\beta \sum_{l=1}^{\infty} \|A^{(\beta-1)/2} Q^{1/2} e_l\|^2 = C k^\beta \|A^{(\beta-1)/2}\|^2_{L_2^0},\end{aligned} \tag{4.22}$$

which completes the proof of (4.20).

In particular, if $W(t)$ is a Wiener process with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ in the proof of (3.26) and obtain (3.27) since $\|I\|_{L_2^0} = \mathrm{Tr}(Q)$.

Corollary 4.1. Let U^n and $u(t_n)$ be the solutions of (1.9) and (1.1), respectively. Assume that $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) \subset H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $u_0 \in L_2(\Omega; \dot{H}^\beta)$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\beta/2} + h^\beta)(1 + \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}), \quad \text{for } 0 \le \beta < 1/2.$$

4.3. **Weak norm convergence.** In this subsection we show the weak norm convergence error estimate.

Theorem 4.3. *Let U^n and $u(t_n)$ be the solutions of* (1.9) *and* (1.1)*, respectively. If* $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ *for some* $\beta \in [0,1]$*, then we have, for* $u_0 \in L_2(\Omega; \dot{H}^\beta)$*, with* $\ell_k = \log(T/k)$ *where* $T = t_n$*,*

$$\|U^n - u(t_n)\|_{L_2(\Omega;\dot{H}^{-1})} \le C(k^{(\beta+1)/2} + h^{\beta+1})\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \ell_k \|A^{(\beta-1)/2}\|_{L_2^0}\Big). \tag{4.23}$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$*, then we have, for* $u_0 \in L_2(\Omega; \dot{H}^1)$*,*

$$\|U^n - u(t_n)\|_{L_2(\Omega;\dot{H}^{-1})} \le C(k + h^2)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \ell_k \,\mathrm{Tr}(Q)^{1/2}\Big). \tag{4.24}$$

Proof. Using the same notation as in Theorem 4.2, we have, by (4.3),

$$\|I\|_{L_2(\Omega;\dot{H}^{-1})} \le C h^{\beta+1} \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}, \quad \text{for } 0 \le \beta \le 1.$$

For II, we have, by the isometry property, and (3.10) with $v = Q^{1/2} e_l$,

$$\begin{aligned}
\|II\|^2_{L_2(\Omega;\dot{H}^{-1})} &= \mathbf{E}\Big\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} A^{-1/2} F_{n-j+1}\, dW(s) \Big\|^2 \\
&= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|A^{-1/2} F_{n-j+1}\|^2_{L_2^0}\, ds \\
&= \sum_{l=1}^\infty \Big(k \sum_{j=1}^n \|A^{-1/2} F_{n-j+1} Q^{1/2} e_l\|^2 \Big) \\
&\le C(k^{\beta+1} + h^{2(\beta+1)}) \ell_k^2 \sum_{l=1}^\infty \|A^{(\beta-1)/2} Q^{1/2} e_l\|^2 \\
&\le C(k^{\beta+1} + h^{2(\beta+1)}) \ell_k^2 \|A^{(\beta-1)/2}\|^2_{L_2^0}.
\end{aligned}$$

For III, we have, by the isometry property,

$$\begin{aligned}
\|III\|^2_{L_2(\Omega;\dot{H}^{-1})} &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \Big\| A^{-1/2}\Big(E(t_n - t_{j-1}) - E(t_n - s) \Big) \Big\|^2_{L_2^0} ds \\
&= \sum_{l=1}^\infty \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \Big\| A^{-(\beta+1)/2} \Big(E(s - t_{j-1}) - I \Big) A^{1/2} E(t_n - s) A^{(\beta-1)/2} Q^{1/2} e_l \Big\|^2 ds.
\end{aligned}$$

Following the proof of (4.22), we get

$$\|III\|^2_{L_2(\Omega;\dot{H}^{-1})} \le C k^\beta \|A^{(\beta-2)/2}\|^2_{L_2^0},$$

which completes the proof of (4.23).

In particular, if $W(t)$ is a Wiener process, then we can choose $\beta = 1$ in (4.23) and obtain (4.24).

Corollary 4.2. Let U^n and $u(t_n)$ be the solutions of (1.9) and (1.1), respectively. Assume that $A = -\frac{\partial^2}{\partial x^2}$ and $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $u_0 \in L_2(\Omega; \dot{H}^\beta)$, with $\ell_k = \log(T/k)$ where $T = t_n$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;\dot{H}^{-1})} \leq C\big(k^{(\beta+1)/2} + h^{(\beta+1)}\big)\big(1 + \ell_k \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}\big), \quad \text{for } 0 \leq \beta < 1/2.$$

5. Computational analysis

In this section we consider how to compute the approximate solution U^n of the solution u of (1.1). Recall that the Wiener process $W(t)$ with covariance operator Q has the form, see Da Prato and Zabczyk [5, Chapter 4],

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} e_j \beta_j(t),$$

where $\{\gamma_j, e_j\}_{j=1}^\infty$ is eigensystem of Q, and $\{\beta_j(t)\}_{j=1}^\infty$ are independently and identically distributed (iid) real-valued Brownian motions. If $\mathrm{Tr}(Q) < \infty$, then $W(t)$ is an H-valued process. In fact

$$\mathbf{E}\|W(t)\|^2 = \mathbf{E}\sum_{j=1}^{\infty} \gamma_j \beta_j(t)^2 = \sum_{j=1}^{\infty} \gamma_j \big(\mathbf{E}\beta_j(t)^2\big) = t\,\mathrm{Tr}(Q) < \infty.$$

If $\mathrm{Tr}(Q) = \infty$, for example $Q = I$, then $W(t)$ is not H-valued.

Let U^n be the approximation in S_h of $u(t)$ at $t = t_n = nk$. The backward Euler method is to find $U^n \in S_h$, s.t., with $\bar{\partial} U^n = (U^n - U^{n-1})/k$, $n \geq 1$, $U^0 = P_h u_0$,

$$(\bar{\partial} U^n, \chi) + (A_h U^n, \chi) = \Big(\frac{1}{k}\int_{t_{n-1}}^{t_n} P_h\, dW(s), \chi\Big), \quad \forall \chi \in S_h, \tag{5.1}$$

where A_h, P_h are defined in the introduction.

If $W(t)$ is H-valued, then $P_h W(t)$ is well-defined. We therefore can write

$$\int_{t_{n-1}}^{t_n} P_h\, dW(s) = P_h\big(W(t_n) - W(t_{n-1})\big) = P_h \sum_{j=1}^{\infty} \gamma_j^{1/2}\big(\beta_j(t_n) - \beta_j(t_{n-1})\big).$$

Here

$$\frac{1}{\sqrt{k}}\Big(\beta_j(t_n) - \beta_j(t_{n-1})\Big) = \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ is the real-valued Gaussian random variable.

Thus the right hand side of (5.1) can be computed by truncating the following series to J terms, i.e.,

$$
\begin{aligned}
\Big(\frac{1}{k}\int_{t_{n-1}}^{t_n} P_h\, dW(s), \chi\Big) &= \Big(\frac{1}{k}\sum_{j=1}^{\infty} \gamma_j^{1/2} e_j\big(\beta_j(t_n) - \beta_j(t_{n-1})\big), \chi\Big) \\
&= \frac{1}{k}\sum_{j=1}^{\infty} \gamma_j^{1/2}\big(\beta_j(t_n) - \beta_j(t_{n-1})\big)(e_j, \chi) \\
&\approx \frac{1}{k}\sum_{j=1}^{J} \gamma_j^{1/2}\big(\beta_j(t_n) - \beta_j(t_{n-1})\big)(e_j, \chi).
\end{aligned} \tag{5.2}
$$

If $W(t)$ is not H-valued, then we see that, from Lemma 2.4, $W(t)$ is $\dot{H}^{\beta-1}$-valued with $\beta \in [0,1]$. In this case we may introduce the $\dot{H}^{-1}$-projection $P_h : \dot{H}^{-1} \to S_h$ defined by

$$
(P_h v, \chi) = \langle v, \chi \rangle, \quad \forall v \in \dot{H}^{-1},\ \chi \in S_h \subset \dot{H}^1,
$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\dot{H}^{-1}$ and $\dot{H}^1$.

Below we will show that it is sufficient to choose $J = N_h$ in order to achieve the required convergence order. To see this, let us consider the semidiscrete approximation solution u_h of u of (1.1). Recall that the semidiscrete solution u_h satisfies

$$
\begin{aligned}
u_h(t) &= E_h(t)P_h u_0 + \int_0^t E_h(t-s)P_h\, dW(s) \\
&= E_h(t)P_h u_0 + \sum_{j=1}^{\infty}\int_0^t E_h(t-s)P_h e_j \gamma_j^{1/2}\, d\beta_j(s).
\end{aligned} \tag{5.3}
$$

Truncating the series in the right side of (5.3), we have

$$
u_h^J(t) = E_h(t)P_h u_0 + \sum_{j=1}^{J}\int_0^t E_h(t-s)P_h e_j \gamma_j^{1/2}\, d\beta_j(s). \tag{5.4}
$$

We then have the following lemma with respect to L_2 norm in space.

Lemma 5.1. *Let u_h^J and u_h be defined by (5.3) and (5.4), respectively. If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$. Assume that $\{S_h\}$ is defined on a quasi-uniform family of triangulations and let N_h be the dimension of S_h. If $J \geq N_h$, then we have, for $t > 0$,*

$$
\|u_h^J(t) - u_h(t)\|_{L_2(\Omega,H)} \leq C h^{\beta} \|A^{(\beta-1)/2}\|_{L_2^0}. \tag{5.5}
$$

PROOF. Using the same notation as in the proof of Theorem 3.2, we have, by isometry property,

$$\begin{aligned}\mathbf{E}\|u_h^J(t)-u_h(t)\|^2 &= \mathbf{E}\Big\|\sum_{j=J+1}^{\infty}\int_0^t E_h(t-s)P_h e_j\gamma_j^{1/2}\,d\beta_j(s)\Big\|^2\\ &= \sum_{j=J+1}^{\infty}\gamma_j\int_0^t \|E_h(t-s)P_h e_j\|^2\,ds\\ &\le 2\sum_{j=J+1}^{\infty}\gamma_j\int_0^t \|E(t-s)e_j\|^2\,ds\\ &\quad + 2\sum_{j=J+1}^{\infty}\gamma_j\int_0^t \|F_h(t-s)e_j\|^2\,ds\\ &= I+II.\end{aligned}$$

For I, we have

$$\begin{aligned}I &= 2\sum_{j=J+1}^{\infty}\gamma_j\int_0^t e^{-2(t-s)\lambda_j}\,ds \le \sum_{j=J+1}^{\infty}\gamma_j\lambda_j^{-1}\\ &= \sum_{j=J+1}^{\infty}\lambda_j^{-\beta}\lambda_j^{\beta-1}\gamma_j \le \lambda_{J+1}^{-\beta}\|A^{(\beta-1)/2}\|^2_{L_2^0}.\end{aligned}$$

For II, we have, by (3.8),

$$\begin{aligned}II &\le Ch^{2\beta}\sum_{j=J+1}^{\infty}\gamma_j|e_j|^2_{\beta-1} \le Ch^{2\beta}\sum_{j=1}^{\infty}|Q^{1/2}e_j|^2_{\beta-1}\\ &= Ch^{2\beta}\|A^{(\beta-1)/2}\|^2_{L_2^0}.\end{aligned}$$

Thus we get

$$\mathbf{E}\|u_h^J(t)-u_h(t)\|^2 \le C(\lambda_{J+1}^{-\beta}+h^{2\beta})\|A^{(\beta-1)/2}\|^2_{L_2^0}.$$

Hence (5.5) follows from the following obvious facts: with some constant C which may be different in different inequalities,

$$\lambda_{J+1}^{-1} \le CJ^{-2/d} \le CN_h^{-2/d} \le Ch^2,$$

where d is the dimension of the spatial domain $\mathcal{D}$.

Under the same assumptions as in Lemma 5.1, we can also show the following results with respect to weak norm in space,

$$\|u_h^J(t)-u_h(t)\|_{L_2(\Omega,\dot H^{-1})} \le Ch^{\beta+1}\ell_h\|A^{(\beta-1)/2}\|_{L_2^0}.$$

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