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A FINITE ELEMENT METHOD FOR A NONLINEAR STOCHASTIC PARABOLIC EQUATION

YUBIN YAN

ABSTRACT. In this paper we consider the finite element method for a stochastic parabolic partial differential equation forced by additive space-time white noise in the multi-dimensional case. Optimal strong convergence error estimates in the L_2 and $\dot{H}^{-1}$ norms with respect to spatial variable have been obtained. The proof is based on appropriate nonsmooth data error estimates for the corresponding deterministic parabolic problem.

1. INTRODUCTION

In this paper we study the finite element approximation of the nonlinear stochastic parabolic partial differential equation

$$du + Au\, dt = \sigma(u) dW, \quad \text{for } 0 < t \leq T, \quad \text{with } u(0) = u_0, \tag{1.1}$$

in a Hilbert space H, with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, where $u(t)$ is an H-valued random process, A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, σ is a nonlinear operator-valued function defined on H which we will specify later. Here $W(t)$ is a Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t \geq 0})$ and $u_0 \in H$.

For the sake of simplicity, we shall concentrate on the case $A = -\Delta$, where Δ stands for the Laplacian operator subject to homogeneous Dirichlet boundary conditions, and $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d, d = 1, 2, 3$, with a sufficiently smooth boundary $\partial \mathcal{D}$.

Such equations are common in applications. Many mathematics models in physics, chemistry, biology, population dynamics, neurophysiology, etc., are described by stochastic partial differential equations, see, Da Prato and Zabczyk [5], Walsh [17], etc. The existence, uniqueness, and properties of the solutions of such equations have been well studied, see Curtain and Falb [1], [2], Da Prato [3], Da Prato and Lunardi [4], Da Prato and Zabczyk [5], Dawson [7], Gozzi [8], Peszat and Zabczyk [13], Walsh [17], etc. However, numerical approximation of such equations has not been studied thoroughly.

This paper is closely related to [18], where we consider the finite element method for a linear stochastic parabolic partial differential equation. As in [18], we assume that $W(t)$ is a

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Wiener process with covariance operator Q. This process may be considered in terms of its Fourier series. Suppose that Q is a bounded, linear, selfadjoint, positive definite operator on H, with eigenvalues $\gamma_l > 0$ and corresponding eigenfunctions e_l. Let β_l, $l = 1, 2, \cdots$, be a sequence of real-valued independently and identically distributed Brownian motions. Then

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t),$$

is a Wiener process with covariance operator Q.

If Q is in trace class, then $W(t)$ is an H-valued process. If Q is not in trace class, for example, $Q = I$, then $W(t)$ does not belong to H, in which case $W(t)$ is called a cylindrical Wiener process.

Let $L_2^0 = HS(Q^{1/2}(H), H)$ denote the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to H, i.e.,

$$L_2^0 = \Big\{ \psi \in L(H) : \sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 < \infty \Big\},$$

with norm $\|\psi\|_{L_2^0} = \Big(\sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 \Big)^{1/2}$, where $L(H)$ is the space of bounded linear operators from H to H.

Let $\mathbf{E}$ denote the expectation. Let $\psi \in L_2^0$. Then $\int_0^t \psi(s)\, dW(s)$ can be defined and have the isometry

$$\mathbf{E}\Big\| \int_0^t \psi(s)\, dW(s) \Big\|^2 = \int_0^t \|\mathbf{E}\psi(s)\|_{L_2^0}^2\, ds. \tag{1.2}$$

Following Da Prato and Zabczyk [5, Chapter 7], we assume that $\sigma : H \to L_2^0$ satisfies the following global Lipschitz and growth conditions,

(i) $\|\sigma(x) - \sigma(y)\|_{L_2^0} \le C\|x - y\|, \quad \forall x, y \in H,$

(ii) $\|\sigma(x)\|_{L_2^0} \le C\|x\|, \quad \forall x \in H.$

Then (1.1) admits a unique mild solution which has the form,

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u(s))\, dW(s), \tag{1.3}$$

where $E(t) = e^{-tA}$ is the analytic semigroup generated by $-A$. Moreover

$$\sup_{t \in [0,T]} \mathbf{E}\|u(t)\|^2 \le C(1 + \mathbf{E}\|u_0\|^2).$$

Note that if $\mathrm{Tr}(Q) < \infty$, then the identity mapping $\sigma(u) = I$ does not satisfy the condition (ii). In order to cover this important case, we introduce a modified version of (ii), i.e.,

(ii') $\|A^{(\beta-1)/2}\sigma(x)\|_{L_2^0} \le C\|x\|, \quad$ for some $\beta \in [0, 1], \quad \forall x \in H.$

We see that (ii) is the special case $\beta = 1$ of (ii'). If $\sigma(\cdot) = I$, the condition (ii') reduces to $\|A^{(\beta-1)/2}\|_{L_2^0} \le C$ which is the condition used in [18] for the numerical approximation for linear stochastic parabolic partial differential equation.

Numerical methods for equations of the form (1.1), with various assumptions on the nonlinearity σ and the Wiener process $W(t)$, have been studied, for example, by Davie and Gaines [6], Gyöngy [9], [10], Hausenblas [11], Shardlow [15], etc. Our approach is similar to Printems [14], who considers the time discretization in an abstract framework.

In this paper we will consider error estimates for approximations of (1.1) based on the finite element method in space and the backward Euler method in time.

Let S_h be a family of finite element spaces, where S_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to the triangulation $\mathcal{T}_h$ of Ω. For simplicity, we always assume that $\{S_h\} \subset H_0^1 = H_0^1(\mathcal{D}) = \{v \in L_2(\mathcal{D}), \nabla v \in L_2(\mathcal{D}), v|_{\partial\mathcal{D}} = 0\}$. The semidiscrete problem of (1.1) is to find the process $u_h(t) = u_h(\cdot, t) \in S_h$, such that

$$du_h + A_h u_h \, dt = P_h \sigma(u_h) \, dW, \quad \text{for } 0 < t \leq T, \quad \text{with } u_h(0) = P_h u_0, \tag{1.4}$$

where P_h denotes the L_2-projection onto S_h, and $A_h : S_h \to S_h$ is the discrete analogue of A, defined by

$$(A_h \psi, \chi) = A(\psi, \chi), \quad \forall \psi, \chi \in S_h. \tag{1.5}$$

Here $A(\cdot, \cdot) = (\nabla \cdot, \nabla \cdot)$ is the bilinear form on $H_0^1(\mathcal{D})$ obtained from the operator A.

Let $E_h(t) = e^{-tA_h}$, $t \geq 0$. Then (1.2) admits a unique mild solution

$$u_h(t) = E_h(t) P_h u_0 + \int_0^t E_h(t-s) P_h \sigma(u_h(s)) \, dW(s).$$

Let $\dot{H}^s = \dot{H}^s(\mathcal{D}) = \mathcal{D}(A^{s/2})$ with norm $|v|_s = \|A^{s/2} v\|$ for any $s \in \mathbf{R}$. For any Hilbert space H, we denote

$$L_2(\Omega; H) = \Big\{ v : \mathbf{E}\|v\|_H^2 = \int_\Omega \|v(\omega)\|_H^2 \, d\mathbf{P}(\omega) < \infty \Big\},$$

with norm $\|v\|_{L_2(\Omega;H)} = (\mathbf{E}\|v\|_H^2)^{1/2}$.

Under the assumptions (i) and (ii'), we show, in Theorem 3.2, the following error estimates for $t \in [0, T]$,

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \leq C(T) h^\beta \Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0 \leq s \leq T} \mathbf{E}\|u(s)\|_{L_2(\Omega;H)} \Big).$$

We also consider error estimates in the fully discrete case. Let k be a time step and $t_n = nk$ with $n \geq 1$. We define the backward Euler method U^n,

$$\frac{U^n - U^{n-1}}{k} + A_h U^n = \frac{1}{k} \int_{t_{n-1}}^{t_n} P_h \sigma(U^n) \, dW(s), \quad n \geq 1, \quad U^0 = P_h u_0. \tag{1.7}$$

With $r(\lambda) = (1 + \lambda)^{-1}$, we can rewrite (1.5) in the form

$$\begin{aligned} U^n &= r(kA_h) U^{n-1} + \int_{t_{n-1}}^{t_n} r(kA_h) P_h \sigma(U^n) \, dW(s), \quad n \geq 1, \\ U^0 &= P_h u_0. \end{aligned} \tag{1.8}$$

Under the assumptions (i) and (ii'), in Theorem 4.2, we have, for $0 \leq \gamma < \beta \leq 1$, $t_n \in [0, T]$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \leq C(T)(k^{\gamma/2} + h^\beta)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0\leq s\leq T} \|u(s)\|_{L_2(\Omega;H)}\Big). \tag{1.9}$$

This paper is organized as follows. In Section 2, we consider the regularity of the solution of (1.1). In Section 3, we consider error estimate in semidiscrete case. In Section 4, we consider error estimate in the fully discrete case.

2. Regularity of the mild solution

In this section we will consider the regularity of the mild solution of (1.1). We have the following theorem.

Theorem 2.1. *Assume that σ satisfies (i) and (ii'). Let $u(t)$ be the mild solution* (**??**) *of* (1.1)*. Then we have, for $u_0 \in L_2(\Omega; \dot{H}^\beta)$,*

$$\|u(t)\|_{L_2(\Omega;\dot{H}^\beta)} \leq C\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0\leq s\leq t} \|u(s)\|_{L_2(\Omega;H)}\Big). \tag{2.1}$$

In particular, if σ satisfies (i) and (ii), then we have, for $u_0 \in L_2(\Omega; \dot{H}^1)$,

$$\|u(t)\|_{L_2(\Omega;\dot{H}^1)} \leq C\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \sup_{0\leq s\leq t} \|u(s)\|_{L_2(\Omega;H)}\Big). \tag{2.2}$$

To prove this theorem, we need some regularity results which are related to the fact that $E(t) = e^{-tA}$ is an analytic semigroup on H. For later use, we collect some results in the next two lemmas, see Thomée [16] or Pazy [12].

Lemma 2.2. *For any $\mu, \nu \in \mathbf{R}$ and $l \geq 0$, there is $C > 0$ such that*

$$|D_t^l E(t) v|_\nu \leq C t^{-(\nu-\mu)/2 - l} |v|_\mu, \quad for\, t > 0, \quad 2l + \nu \geq \mu, \tag{2.3}$$

and

$$\int_0^t s^\mu |D_t^l E(s) v|_\nu^2 \, ds \leq C |v|_{2l+\nu-\mu-1}^2, \quad for\, t \geq 0, \quad \mu \geq 0. \tag{2.4}$$

Lemma 2.3. *For any $\mu \geq 0$, $0 \leq \nu \leq 1$, there is $C > 0$ such that*

$$\|A^\mu E(t)\| \leq C t^{-\mu}, \quad for\, t > 0, \tag{2.5}$$

and

$$\|A^{-\nu}(I - E(t))\| \leq C t^\nu, \quad for\, t \geq 0. \tag{2.6}$$

Proof of Theorem 2.1. Recall that the mild solution has the form

$$u(t) = E(t) u_0 + \int_0^t E(t-s) \sigma(u(s)) \, dW(s).$$

Thus, for any $\beta \geq 0$, using the stability of $E(t)$ and the isometry (**??**),

$$\begin{aligned}\mathbf{E}|u(t)|_\beta^2 &\leq 2\mathbf{E}|E(t)u_0|_\beta^2 + 2\mathbf{E}\Big\| \int_0^t A^{\beta/2}E(t-s)\,\sigma(u(s))dW(s)\Big\|^2 \\ &= 2\mathbf{E}|u_0|_\beta^2 + 2\mathbf{E}\int_0^t \|A^{\beta/2}E(t-s)\sigma(u(s))\|_{L_2^0}^2\,ds \\ &= 2\mathbf{E}|u_0|_\beta^2 + 2\mathbf{E}\int_0^t \|A^{1/2}E(t-s)A^{(\beta-1)/2}\sigma(u(s))\|_{L_2^0}^2\,ds.\end{aligned}$$

By (ii') and Lemma 2.2, we have

$$\begin{aligned}&\mathbf{E}\int_0^t \|A^{1/2}E(t-s)A^{(\beta-1)/2}\sigma(u(s))\|_{L_2^0}^2\,ds \\ &\leq \Big(\int_0^t \|A^{1/2}E(t-s)\|^2\,ds\Big)\sup_{0\leq s\leq t}\mathbf{E}\|u(s)\|^2 \\ &\leq C\sup_{0\leq s\leq t}\mathbf{E}\|u(s)\|^2.\end{aligned}$$

Thus we get

$$\mathbf{E}|u(t)|_\beta^2 \leq C(\mathbf{E}|u_0|_\beta^2 + \sup_{0\leq s\leq t}\mathbf{E}\|u(s)\|^2),$$

which implies (2.1) by noting that

$$\Big(\sup_{0\leq s\leq t}\mathbf{E}\|u(s)\|^2\Big)^{1/2} \leq \sup_{0\leq s\leq t}\Big(\mathbf{E}\|u(s)\|^2\Big)^{1/2} = \sup_{0\leq s\leq t}\mathbf{E}\|u(s)\|_{L_2(\Omega;H)}.$$

In particular, if (ii) holds, then $\beta = 1$ and we get (2.2).

Remark 2.1. In Theorem 2.1, if $\sigma(\cdot) = I$, the condition (ii') reduces to $\|A^{(\beta-1)/2}\|_{L_2^0} \leq C$ which is the condition used in [18] for the numerical approximation for linear stochastic parabolic partial differential equation.

3. Error estimates in the semidiscrete case

In this section we consider error estimates for stochastic partial differential equation in the semidiscrete case. In order to prove our error estimates, we need some nonsmooth data error estimates for the homogeneous deterministic parabolic problem.

Let $E_h(t) = e^{-tA_h}$ and $E(t) = e^{-tA}$. We then have the following error estimates for deterministic parabolic problem, see [18].

Lemma 3.1. *Let $F_h(t) = E_h(t)P_h - E(t)$. Then*

$$\|F_h v\|_{L_\infty([0,T];H)} \leq Ch^\beta |v|_\beta, \quad \text{for } v \in \dot{H}^\beta,\ 0 \leq \beta \leq 1, \tag{3.1}$$

and

$$\|F_h v\|_{L_2([0,T];H)} \leq Ch^\beta |v|_{\beta-1}, \quad \text{for } v \in \dot{H}^{\beta-1},\ 0 \leq \beta \leq 1. \tag{3.2}$$

Our main result in this section is the following.

Theorem 3.2. *Assume that σ satisfies (i) and (ii′). Let u_h and u be the solutions of (1.2) and (1.1), respectively. Then there is $C = C(T)$ such that, for $t \in [0,T]$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch^\beta \Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|_{L_2(\Omega;H)} \Big). \tag{3.3}$$

In particular, if σ satisfies (i) and (ii), then we have

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \le Ch \Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|_{L_2(\Omega;H)} \Big). \tag{3.4}$$

Proof. We have, with $E(t) = e^{-tA}$,

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u(s))\, dW(s),$$

and, with $E_h(t) = e^{-tA_h}$,

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)\, P_h \sigma(u_h(s))\, dW(s).$$

Denoting $e(t) = u_h(t) - u(t)$ and $F_h(t) = E_h(t)P_h - E(t)$, we write

$$\begin{aligned} e(t) &= F_h(t)u_0 + \int_0^t F_h(t-s)\sigma(u(s))\, dW(s) \\ &\quad + \int_0^t E_h(t-s)P_h \Big(\sigma(u_h(s) - \sigma(u(s)) \Big)\, dW(s) \\ &= I + II + III. \end{aligned}$$

Thus

$$\|e(t)\|_{L_2(\Omega;H)} \le C \Big(\|I\|_{L_2(\Omega;H)} + \|II\|_{L_2(\Omega;H)} + \|III\|_{L_2(\Omega;H)} \Big).$$

For I, we have, by (3.1) with $v = u_0$,

$$\|I\| = \|F_h(t)u_0\| \le Ch^\beta |u_0|_\beta,$$

which implies that $\|I\|_{L_2(\Omega;H)} \le Ch^\beta \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}$.

For II, we have, by the isometry (**??**),

$$\begin{aligned} \|II\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\| \int_0^t F_h(t-s)\sigma(u(s))\, dW(s) \Big\|^2 \\ &= \int_0^t \mathbf{E}\|F_h(t-s)A^{(1-\beta)/2}A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0}\, ds \\ &\le \Big(\int_0^t \|F_h(t-s)A^{(1-\beta)/2}\|^2\, ds \Big) \sup_{0\le s\le t} \mathbf{E}\|A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0}. \end{aligned}$$

We will show that

$$\int_0^t \|F_h(t-s)A^{-(\beta-1)/2}\|^2\, ds \le Ch^{2\beta}. \tag{3.5}$$

Assuming this for the moment, we have, by the growth condition (ii'),

$$\|II\|^2_{L_2(\Omega;H)} \le Ch^{2\beta} \sup_{0\le s\le t} \mathbf{E}\|A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0} \le Ch^{2\beta} \sup_{0\le s\le t} \mathbf{E}\|u(s)\|^2.$$

For III, we have, by the isometry property and the Lipschitz condition (i),

$$\begin{aligned}\|III\|^2_{L_2(\Omega;H)} &= \mathbf{E}\int_0^t \Big\|E_h(t-s)P_h\Big(\sigma(u_h(s)-\sigma(u(s))\Big)\Big\|^2_{L_2^0} ds \\ &\le \mathbf{E}\int_0^t \|E_h(t-s)P_h\|^2\|u_h(s)-u(s)\|^2_{L_2^0} ds \\ &\le \int_0^t \mathbf{E}\|e(s)\|^2 ds.\end{aligned}$$

Hence

$$\|e(t)\|^2_{L_2(\Omega;H)} \le Ch^{2\beta}\Big(|u_0|^2_\beta + \sup_{0\le s\le t}\mathbf{E}\|u(s)\|\Big) + C\int_0^t \|e(s)\|^2_{L_2(\Omega;H)} ds.$$

Then (3.3) follows from Gronwall's lemma.

It remains to show (3.5). In fact, by the definition of the operator norm and the monotone convergence theorem, we have

$$\begin{aligned}\int_0^t \|F_h(t-s)A^{-(\beta-1)/2}\|^2 ds &= \int_0^t \sup_{v\ne 0}\frac{\|F_h(t-s)A^{-(\beta-1)/2}v\|^2}{\|v\|^2} ds \\ &= \sup_{v\ne 0}\frac{\int_0^t \|F_h(t-s)A^{-(\beta-1)/2}v\|^2 ds}{\|v\|^2}.\end{aligned}$$

Combining this with (3.2), we show (3.5) and therefore (3.3) holds.

In particular, if (ii) holds then $\beta = 1$ and we obtain (3.4).

4. Error estimates in the fully discrete case

In this section we will consider the error estimates in the fully discrete case. As in the semidiscrete case, we need some error estimates for the deterministic parabolic problem.

Let $E_{kh} = r(kA_h)$ and $E(t) = e^{-tA}$, where $r(\lambda) = 1/(1+\lambda)$ is introduced in (1.6). We have, see [18],

Lemma 4.1. *Let $F_n = E^n_{kh}P_h - E(t_n)$. Then*

(4.1) $$\|F_n v\| \le C(k^{\beta/2}+h^\beta)|v|_\beta, \quad \textit{for } v\in \dot H^\beta,\ 0\le\beta\le 1,$$

and

(4.2) $$\Big(k\sum_{j=1}^n \|F_j v\|^2\Big)^{1/2} \le C(k^{\beta/2}+h^\beta)|v|_{\beta-1}, \quad \textit{for } v\in \dot H^{\beta-1},\ 0\le\beta\le 1.$$

Our main result in this section is the following.

Theorem 4.2. *Assume that σ satisfies (i) and (ii′). Let U^n and $u(t_n)$ be the solutions of* (1.6) *and* (1.1), *respectively. Let $0 \le \gamma < \beta$. Then there is $C = C(T)$ such that, for $t_n \in [0, T]$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$,*

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\gamma/2} + h^\beta)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0\le s\le T} \|u(s)\|_{L_2(\Omega;H)}\Big) \tag{4.3}$$

In particular, if σ satisfies (i) and (ii), then we have, for $u_0 \in L_2(\Omega; \dot{H}^1)$, and $0 \le \gamma < 1$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \le C(k^{\gamma/2} + h)\Big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \sup_{0\le s\le T} \|u(s)\|_{L_2(\Omega;H)}\Big). \tag{4.4}$$

To prove this theorem we need the following regularity result for the solution of (1.1).

Lemma 4.3. *Assume that (ii′) holds. Let u be the mild solution of* (1.1). *Then we have, for $0 \le \gamma < \beta \le 1$,*

$$\begin{aligned} \mathbf{E}\|u(t_2) - u(t_1)\|^2 &\le C(t_2 - t_1)^\gamma \mathbf{E}|u_0|_\gamma^2 \\ &\quad + C(t_2 - t_1)^\gamma \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2. \end{aligned} \tag{4.5}$$

Proof. The weak solution of (1.1) has the form, with $E(t) = e^{-tA}$,

$$u(t) = E(t)u_0 + \int_0^T E(t-s)\sigma(u(s))\,dW(s).$$

Thus we have

$$\begin{aligned} u(t_2) - u(t_1) &= \Big(E(t_2)u_0 - E(t_1)u_0\Big) \\ &\quad + \Big(\int_0^{t_2} E(t_2 - s)\sigma(u(s))\,dW(s) - \int_0^{t_1} E(t_1 - s)\sigma(u(s))\,dW(s)\Big), \\ &= I + II. \end{aligned}$$

and therefore

$$\mathbf{E}\|u(t_2) - u(t_1)\|^2 \le 2\mathbf{E}\|I\|^2 + 2\mathbf{E}\|II\|^2.$$

For I, we have, by Lemma 2.3, for $0 \le \gamma \le 2$, with $t_1 \ne 0$,

$$\begin{aligned} \|I\| &= \|E(t_1)A^{-\gamma/2}(E(t_2) - E(t_1))A^{\gamma/2}u_0\| \\ &\le C(t_2 - t_1)^{\gamma/2}|u_0|_\gamma, \end{aligned} \tag{4.6}$$

which implies that $\mathbf{E}\|I\|^2 \le C(t_2 - t_1)^\gamma \mathbf{E}|u_0|_\gamma^2$.

For II, we have

$$\begin{aligned} II &= \int_0^{t_1} \Big(E(t_2 - s) - E(t_1 - s)\Big)\sigma(u(s))\,dW(s) \\ &\quad + \int_{t_1}^{t_2} E(t_2 - s)\sigma(u(s))\,dW(s) \\ &= II_1 + II_2. \end{aligned}$$

Using (ii'), isometry, and Lemma 2.3, we have, for $0 \le \gamma < \beta \le 1$,

$$\begin{aligned}
\mathbf{E}\|II_1\|^2 &= \mathbf{E}\Big\| \int_0^{t_1} \Big(E(t_2-s)-E(t_1-s)\Big)\sigma(u(s))\,dW(s)\Big\|^2 \\
&= \int_0^{t_1} \mathbf{E}\Big\|\Big(E(t_2-s)-E(t_1-s)\Big)A^{(1-\beta)/2}A^{(\beta-1)/2}\sigma(u(s))\Big\|^2_{L_2^0}\,ds \\
&\le \int_0^{t_1} \Big\|A^{(1-\beta)/2}\Big(E(t_2-s)-E(t_1-s)\Big)\Big\|^2\,ds \sup_{0\le s\le t_1}\mathbf{E}\|u(s)\|^2 \\
&= \int_0^{t_1} \Big\|A^{(1-\beta)/2+\gamma/2}E(t_1-s)A^{-\gamma/2}\Big(I-E(t_2-t_1)\Big)\Big\|^2\,ds \sup_{0\le s\le t_1}\mathbf{E}\|u(s)\|^2 \\
&\le C(t_2-t_1)^\gamma\Big(\int_0^{t_1}(t_1-s)^{-(1-\beta)-\gamma}\,ds\Big)\sup_{0\le s\le t_1}\mathbf{E}\|u(s)\|^2 \\
&\le C(t_2-t_1)^\gamma \sup_{0\le s\le t_1}\mathbf{E}\|u(s)\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}\|II_2\|^2 &= \int_{t_1}^{t_2}\mathbf{E}\|A^{(1-\beta)/2}E(t_2-s)A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0}\,ds \\
&\le C\int_{t_1}^{t_2}\|A^{(1-\beta)/2}E(t_2-s)\|^2\cdot\mathbf{E}\|A^{(\beta-1)/2}\sigma(u(s))\|^2_{L_2^0}\,ds \\
&\le C\Big(\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\,ds\Big)\sup_{t_1\le s\le t_2}\mathbf{E}\|u(s)\|^2 \\
&\le C(t_2-t_1)^\beta\sup_{t_1\le s\le t_2}\mathbf{E}\|u(s)\|^2, \quad \text{for } \beta>0.
\end{aligned}$$

Hence we get, for $0 \le \gamma < \beta \le 1$,

$$\mathbf{E}\|II\|^2 \le 2\mathbf{E}\|II_1\|^2 + 2\mathbf{E}\|II_2\|^2 \le C(t_2-t_1)^\gamma \sup_{t_1\le s\le t_2}\mathbf{E}\|u(s)\|^2.$$

Together these estimates complete the proof.

PROOF OF THEOREM 4.2. We have, by (1.6), with $E_{kh}^n = r(kA_h)^n$,

$$U^n = E_{kh}^n P_h u_0 + \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E_{kh}^{n-j+1}P_h\sigma(U^j)\,dW(s),$$

and, by the definition of the mild solution of (1.1), with $E(t)=e^{-tA}$,

$$u(t_n) = E(t_n)u_0 + \int_0^{t_n} E(t_n-s)\sigma(u(s))\,dW(s).$$

Denoting $e^n = U^n - u(t_n)$ and $F_n = E_{kh}^n P_h - E(t_n)$, we write

$$\begin{aligned} e^n &= F_n u_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} r(kA_h)^{n-j+1} P_h \Big(\sigma(U^j) - \sigma(u(t_j)) \Big) \, dW(s) \\ &\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} r(kA_h)^{n-j+1} P_h \Big(\sigma(u(t_j)) - \sigma(u(s)) \Big) \, dW(s) \\ &\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big(r(kA_h)^{n-j+1} P_h - E(t_n - t_{j-1}) \Big) \sigma(u(s)) \, dW(s) \\ &\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big(E(t_n - t_{j-1}) - E(t_n - s) \Big) \sigma(u(s)) \, dW(s) \\ &= \sum_{j=1}^{5} I_j. \end{aligned}$$

Thus

$$\|e^n\|_{L_2(\Omega;H)} \leq C \sum_{j=1}^{5} \|I_j\|_{L_2(\Omega;H)}.$$

For I_1, we have, by (4.1) with $v = u_0$,

$$\|I_1\| = \|F_n u_0\| \leq C(k^{\beta/2} + h^{\beta}) |u_0|_{\beta},$$

which implies that $\|I_1\|_{L_2(\Omega;H)} \leq C(k^{\beta/2} + h^{\beta}) \|u_0\|_{L_2(\Omega;\dot{H}^{\beta})}$.

For I_2, we have, by isometry and the stability of $r(\lambda)$ and the Lipschitz condition (i),

$$\begin{aligned} \|I_2\|_{L_2(\Omega;H)}^2 &= \mathbf{E} \Big\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} r(kA_h)^{n-j+1} P_h \Big(\sigma(U^j) - \sigma(u(t_j)) \Big) \, dW(s) \Big\|^2 \\ &= k \sum_{j=1}^{n} \mathbf{E} \Big\| r(kA_h)^{n-j+1} P_h \Big(\sigma(U^j) - \sigma(u(t_j)) \Big) \Big\|_{L_2^0}^2 \\ &\leq k \sum_{j=1}^{n} \|r(kA_h)^{n-j+1} P_h\|^2 \, \mathbf{E} \|\sigma(U^j) - \sigma(u(t_j))\|_{L_2^0}^2 \\ &\leq Ck \sum_{j=1}^{n} \mathbf{E} \|U^j - u(t_j)\|^2 = C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbf{E} \|e^j\|^2 \, ds. \end{aligned}$$

For I_3, we have, by Lemma 4.3, for $0 \le \gamma < \beta \le 1$,

$$\begin{aligned}\|I_3\|^2_{L_2(\Omega;H)} &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathbf{E}\Big\| r(kA_h)^{n-j+1} P_h \Big(\sigma(u(t_j)) - \sigma(u(s))\Big)\Big\|^2_{L_2^0} ds \\ &\le C \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathbf{E}\|u(t_j) - u(s)\|^2 \, ds \\ &\le C\Big(\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (t_j - s)^\gamma \, ds\Big)\big(\mathbf{E}|u_0|^2_\gamma + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\big) \\ &\le Ck^\gamma \big(\mathbf{E}|u_0|^2_\gamma + \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\big).\end{aligned}$$

For I_4, we have

$$\begin{aligned}\|I_4\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} F_{n-j+1}\sigma(u(s)) \, dW(s)\Big\|^2 \\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathbf{E}\|F_{n-j+1} A^{(1-\beta)/2} A^{(\beta-1)/2} \sigma(u(s))\|^2_{L_2^0} \, ds \\ &\le C\Big(k \sum_{j=1}^n \|F_j A^{(1-\beta)/2}\|^2\Big) \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2.\end{aligned}$$

We will show that

$$k \sum_{j=1}^n \|F_j A^{(1-\beta)/2}\|^2 \le C(k^\beta + h^{2\beta}). \tag{4.7}$$

Assuming this for the moment, we get

$$\|I_4\|^2_{L_2(\Omega;H)} \le C(k^\beta + h^{2\beta}) \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2.$$

For I_5, we have

$$\begin{aligned}\|I_5\|^2_{L_2(\Omega;H)} &= \mathbf{E}\Big\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (E(t_n - t_{j-1}) - E(t_n - s))\sigma(u(s)) \, dW(s)\Big\|^2 \\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathbf{E}\|(E(t_n - t_{j-1}) - E(t_n - s))A^{(1-\beta)/2} A^{(\beta-1)/2} \sigma(u(s))\|^2_{L_2^0} \, ds \\ &\le C\Big(\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|(E(t_n - t_{j-1}) - E(t_n - s))A^{(1-\beta)/2}\|^2 \, ds\Big) \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2.\end{aligned}$$

Noting that, by Lemmas 2.2 and 2.3,

$$\begin{aligned}
&\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|(E(t_n-t_{j-1})-E(t_n-s))A^{(1-\beta)/2}\|^2\,ds\\
&=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|A^{1/2}E(t_n-s)A^{-\beta/2}(I-E(s-t_{j-1}))\|^2\,ds\\
&\le Ck^\beta\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|A^{1/2}E(t_n-s)\|^2\,ds\\
&=Ck^\beta\int_0^{t_n}\|A^{1/2}E(s)\|^2\,ds\le Ck^\beta,
\end{aligned}$$

we have

$$\|I_5\|^2_{L_2(\Omega;H)}\le Ck^\beta\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2.$$

It remains to show (4.7). In fact, by (4.2),

$$\begin{aligned}
k\sum_{j=1}^{n}\|F_jA^{(1-\beta)/2}\|^2 &= k\sum_{j=1}^{n}\Big(\sup_{v\neq 0}\frac{\|F_jA^{(1-\beta)/2}v\|}{\|v\|}\Big)^2\\
&=\sup_{v\neq 0}\frac{k\sum_{j=1}^{n}\|F_jA^{(1-\beta)/2}v\|^2}{\|v\|^2}\\
&\le\sup_{v\neq 0}\frac{C(k^\beta+h^{2\beta})|A^{(1-\beta)/2}v|^2_{\beta-1}}{\|v\|^2}\le C(k^\beta+h^{2\beta}).
\end{aligned}$$

Together these estimates show, for $0\le\gamma<\beta\le 1$,

$$\begin{aligned}
\mathbf{E}\|e^n\|^2 &\le C(k^\gamma+h^{2\beta})\mathbf{E}|u_0|^2_\beta+Ck\sum_{j=1}^{n}\mathbf{E}\|e^j\|^2\\
&\quad +C(k^\gamma+h^{2\beta})\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2.
\end{aligned}\tag{4.8}$$

By the discrete Gronwall lemma, we get

$$\mathbf{E}\|e^n\|^2\le C(k^\gamma+h^{2\beta})\big(\mathbf{E}|u_0|^2_\beta+\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\big),\tag{4.9}$$

which implies that,

$$\|e^n\|_{L_2(\Omega;H)}\le C(k^{\gamma/2}+h^\beta)\big(\mathbf{E}|u_0|_{L_2(\Omega;\dot H^\beta)}+\sup_{0\le s\le t_n}\|u(s)\|_{L_2(\Omega;H)}\big).\tag{4.10}$$

The proof is now complete.

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