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# A REDUCED $P^1$–DISCONTINUOUS GALERKIN METHOD

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Abstract. We propose a reduced linear discontinuous finite element method, with applications to convection–diffusion and Stokes' problem. We show stability and optimal *a priori* error estimates of the resulting schemes. We also discuss formulations for explicit time-stepping and give some numerical examples.

## 1. Introduction

Standard continuous Galerkin-based finite element methods have poor stability properties when applied to convection-dominated flow problems, and require delicate balancing of approximations when applied to incompressible problems. In contrast, the Discontinuous Galerkin (DG) method is known to have good stability properties when applied to first order hyperbolic problems, see, e.g., Lesaint & Raviart [13] and Johnson & Pitkäranta [11]. It also alleviates locking in (near) incompressible elasticity, cf. Hansbo & Larson [9].

In fluid mechanics applications, there are several other possibilities to stabilize the discrete problem resulting from a finite element discretization. One well known example is the Streamline Diffusion (SD) method and its relatives (e.g., Galerkin/least-squares). Nevertheless, the DG method has two important advantages over the SD method:

(1) Most importantly, the SD method is not well suited for explicit time stepping schemes in that the mass matrix cannot easily be lumped. Instead the SD method for time-dependent problems is usually formulated as an implicit space–time finite element method with space–time stabilization, cf. Hansbo [7]. It should be noted that it is not straightforward to use lumped mass with the approximation presented here; we show that it is nevertheless possible to obtain diagonal mass matrices suitable for explicit sovers (cf. Section 5.2 below).
(2) The choice of the free streamline diffusion parameter is not always clear cut, especially for non-diagonalizable convective systems (e.g., compressible flow, cf. [7]).

Unfortunately, the number of degrees of freedom in the DG method is much larger than in an SD method of the same accuracy. For this reason, we propose a combination of continuous and discontinuous elements, with the discontinuous component overlaying the

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continuous one. A typical reduction of the number of degrees of freedom is as follows: for a scalar problem in three dimensions, where a standard trilinear finite element method has $O(N)$ unknowns and the corresponding DG method has $O(8N)$ unknowns, the proposed method has $O(2N)$ unknowns. This paper aims to show that we retain the crucial advantages of the DG method for convection-dominated problems, and for avoiding locking, in spite of the considerable reduction in the number of unknowns.

## 2. Formulation of the method

To define the method we introduce a partition $\mathfrak{T} = \{T\}$ of $\Omega$ into simplices $T$ satisfying the minimal angle condition. Further, we let the mesh function $h : \Omega \to (0, \infty)$ be defined by $h|_T = h_T = \text{diam}(T)$. We let

$$P^k(T) = \{v : \; v \text{ is a polynomial of degree} \le k \text{ on } T\},$$

and define the continuous discrete space

$$V_h = \{v : \; v|_T \in P^1(T), \; \forall T \in \mathfrak{T}, \; v \in C^0(\Omega), \; v = 0 \,\text{on}\, \partial\Omega\}.$$

We shall seek approximation in the discontinuous space

$$W_h = V_h \oplus \{v \in L_2(\Omega) : \; v|_T \in P^0(T), \; \forall T \in \mathfrak{T}\},$$

i.e., the approximation is built from a piecewise linear, continuous, component and a piecewise constant component. This has the advantage of including the piecewise constants into the test space (which turns out to be crucial for stability) while keeping the number of degrees of freedom lower than in the full DG method with a discontinuous piecewise linear approximation.

We suppose that the computational mesh satisfies the following two regularity conditions. If $h$ and $\tilde{h}$ are the diameters of two neighboring elements then $\kappa^{-1} h \le \tilde{h} \le \kappa h$, and if $h_T$ is the element diameter and $h_{\partial T}^{\min}$ the length of the shortest element side then $h_T < C h_{\partial T}^{\min}$, where $C$ and $\kappa$ will be specified later.

**Remark 1.** *A special feature of our approximation is that the global constant can be represented both by the continuous and the discontinuous fields. This can be handled in different ways; we have chosen what seems to be the simplest approach from an implementation point of view: the boundary condition is enforced strongly on $V_h$ which is combined with a weak enforcement of zero boundary conditions on $W_h$ (which is expressed implicitly in the definition of the method below).*

2.1. **Model problem.** We consider the following linear convection-diffusion problem

$$\begin{aligned} \boldsymbol{\beta} \cdot \nabla u + \sigma u - \varepsilon \Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

Where $\boldsymbol{\beta}$ for simplicity is a constant unit vector, and $\varepsilon$ and $\sigma$ are both positive constants. To continue, let $\boldsymbol{n}_T$ denote the outward pointing normal to $\partial T$, and, for $\boldsymbol{x} \in \partial T$ let

$$\langle v \rangle := \begin{cases} (v^+ + v^-)/2 & \text{on } \partial T \setminus \partial\Omega, \\ 2v^+ & \text{on } \partial T \cap \partial\Omega, \end{cases} \tag{2.2}$$

and

$$[v] := \begin{cases} v^+ - v^- & \text{on } \partial T \setminus \partial\Omega, \\ v^+ & \text{on } \partial T \cap \partial\Omega, \end{cases} \tag{2.3}$$

where

$$v^\pm = \lim_{s \downarrow 0} v(\boldsymbol{x} \mp s\boldsymbol{n}_T).$$

Further, let

$$\partial T_{\text{in}} = \{\boldsymbol{x} \in \partial T : \ \boldsymbol{n}_T(\boldsymbol{x}) \cdot \boldsymbol{\beta}(\boldsymbol{x}) < 0\}, \quad v_{\text{u}} = \lim_{s \downarrow 0} v(\boldsymbol{x} - s\boldsymbol{\beta}),$$

$$\partial T_{\text{out}} = \{\boldsymbol{x} \in \partial T : \ \boldsymbol{n}_T(\boldsymbol{x}) \cdot \boldsymbol{\beta}(\boldsymbol{x}) \geq 0\}, \quad v_{\text{d}} = \lim_{s \downarrow 0} v(\boldsymbol{x} + s\boldsymbol{\beta}).$$

On each edge $E = T^+ \cap T^-$ in the partition $\mathfrak{E} = \{E\}$, the mesh parameter $h$ is defined by

$$h := \frac{m(T^+) + m(T^-)}{3m(E)}, \tag{2.4}$$

where $m(\cdot)$ denotes the appropriate Lebesgue measure. To each edge $E$, we associate a fixed normal vector $\boldsymbol{n} := \boldsymbol{n}_{T^+}$. We seek a function $U \in W_h$ such that

$$a(U, v) + J_0^\gamma(U, v) = L(v), \quad \forall v \in W_h \tag{2.5}$$

where, using the notation

$$(u, v)_T = \int_T u\, v\, dx, \quad (u, v)_{\partial T} = \int_{\partial T} u\, v\, ds,$$

we define

$$\begin{aligned} a(U, v) \ &:= \ \sum_{T \in \mathfrak{T}} (\varepsilon \nabla U, \nabla v)_T + \sum_{T \in \mathfrak{T}} (\boldsymbol{\beta} \cdot \nabla U + \sigma U, v)_T - \\ & \quad \frac{1}{2} \sum_{T \in \mathfrak{T}} (\langle \varepsilon \boldsymbol{n}_T \cdot \nabla U \rangle, [v])_{\partial T} \pm \frac{1}{2} \sum_{T \in \mathfrak{T}} (\langle \varepsilon \boldsymbol{n}_T \cdot \nabla v \rangle, [U])_{\partial T} + \\ & \quad \sum_{T \in \mathfrak{T}} (|\boldsymbol{n}_T \cdot \boldsymbol{\beta}|\, (U_{\text{d}} - U_{\text{u}}), v_{\text{d}})_{\partial T_{\text{in}}}, \end{aligned}$$

and

$$L(v) := \int_\Omega f v \mathrm{d}x.$$

Finally, $J_0^\gamma$ is a penalty–like term defined by

$$J_0^\gamma(U, v) := \frac{1}{2} \sum_{T \in \mathfrak{T}} (\gamma \varepsilon h^{-1}[U], [v])_{\partial T}.$$

We note the possibility of choice of sign in the bilinear term, corresponding to a symmetric or antisymmetric diffusion tensor.

Choosing $v|_T = 1$ we immediately find that the numerical scheme satisfies the following local mass conservation property

$$\int_T \sigma U \,\mathrm{d}x - \int_{\partial T} \Big( \varepsilon \nabla U \cdot \boldsymbol{n}_T - \frac{\gamma \varepsilon}{h}[U] \Big) \,\mathrm{d}s + \int_{\partial T \setminus \partial \Omega} \boldsymbol{\beta} \cdot \boldsymbol{n}_T \; U \,\mathrm{d}s = \int_T f \mathrm{d}x,$$

in the same way global conservation follows by choosing $v|_\Omega = 1$.

By use of Green's formula we readily establish that the method is consistent in the sense that

$$a(u - U, v) = 0, \quad \forall v \in W_h \tag{2.6}$$

and for $u$ a sufficiently regular solution of (2.1).

## 3. Stability of the method

The difference between the the reduced discontinuous $P^1$–element and a fully discontinuous Galerkin method is that the reduced element does not permit local stability estimates in the same sense since the $P_1$ contribution is continuous. Therefore we will prove global stability for the gradients and the jumps and for the streamline derivative when $\varepsilon < h$. To simplify the analysis below we choose the negative sign in $a(U, v)$. The analysis can be carried out in either case with only minor modifications.

**Lemma 1.** *The bilinear form $a(U, v)$ satisfies the following stability estimates*

$$\begin{aligned} a(U,U) + J_0^\gamma(U,U) &= \sum_T \Big( \|\varepsilon^{1/2} \nabla U\|_T^2 + \|\sigma^{1/2} U\|_T^2 \\ &\quad + \int_{\partial T_{in}} |\boldsymbol{\beta} \cdot \boldsymbol{n}_T| [U]^2 \; ds \Big) + J_0^\gamma(U,U) \\ &\leq C \|f\|^2 \end{aligned} \tag{3.1}$$

*and if $\varepsilon < h$, $h < \sigma^{-1}$,*

$$\sum_T \|C^{1/2} h^{1/2} \boldsymbol{\beta} \cdot \nabla U\|_T^2 + a(U,U) + J_0^\gamma(U,U) \leq C \|f\|^2. \tag{3.2}$$

**PROOF**. The first stability estimate is a direct consequence of Lemma 9.3 in [10]. The second estimate follows by choosing $v = Ch\boldsymbol{\beta} \cdot \nabla U \in W_h$ and using the fact that $\nabla v = 0$ to obtain

$$\begin{aligned} &\|C^{1/2} h^{1/2} \boldsymbol{\beta} \cdot \nabla U\|_T^2 + (\sigma U, Ch\boldsymbol{\beta} \cdot \nabla U)_T \\ &\quad + (|\boldsymbol{\beta} \cdot \boldsymbol{n}_T| (U_{\mathrm{d}} - U_{\mathrm{u}}), Ch\boldsymbol{\beta} \cdot \nabla U_{\mathrm{u}})_{\partial T_{\mathrm{in}}} \\ &\quad - \sum_{\partial T \in T} (\langle \varepsilon \boldsymbol{n}_T \cdot \nabla U \rangle, Ch\boldsymbol{\beta} \cdot \nabla U)_{\partial T} + J_0^\gamma(U, Ch\boldsymbol{\beta} \cdot \nabla U) \\ &= \int_T f Ch\boldsymbol{\beta} \cdot \nabla U \mathrm{d}x \end{aligned}$$

Using the inverse inequality

$$\|h^{1/2}\nabla w\|^2_{L_2(\partial T)} \le C_N \|\nabla w\|^2_{L_2(T)}, \tag{3.3}$$

valid for $w \in W_h$, we have

$$\begin{aligned}
&|(|\boldsymbol{\beta}\cdot\boldsymbol{n}_T|(U_{\rm d}-U_{\rm u}), Ch\boldsymbol{\beta}\cdot\nabla U_{\rm u})_{\partial T_{\rm in}}| \le \\
&\le \frac{1}{2}\int_{\partial T_{\rm in}} |\boldsymbol{\beta}\cdot\boldsymbol{n}_T|[U]^2 \, {\rm d}s + \frac{C_N}{2}\|C|\boldsymbol{\beta}\cdot\boldsymbol{n}_T|^{1/2}h^{1/2}\boldsymbol{\beta}\cdot\nabla U\|^2_T,
\end{aligned}$$

$$|(\varepsilon\boldsymbol{n}_T\cdot\nabla U, Ch\boldsymbol{\beta}\cdot\nabla U)_{\partial T}| \le \frac{C_N}{2}\Big(\|\varepsilon^{1/2}\nabla U\|^2_T + \|C\varepsilon^{1/2}\boldsymbol{\beta}\cdot\nabla U\|^2_T\Big)$$

and

$$J_0^\gamma(U, Ch\boldsymbol{\beta}\cdot\nabla U) = \frac{1}{2}J_0^\gamma(U,U) + \frac{1}{2}\|Ch^{1/2}\boldsymbol{\beta}\cdot\nabla U\|^2$$

Recalling that $\varepsilon < h$, $h < \sigma^{-1}$ we thus have, for $C$ sufficiently small,

$$\begin{aligned}
\|C^{1/2}h^{1/2}\boldsymbol{\beta}\cdot\nabla U\|^2 &\le \tilde{C}a(U,U) + \|h^{1/2}f\|^2 \\
&= \tilde{C}(f,U) + \|h^{1/2}f\|^2
\end{aligned}$$

and the desired stability estimate follows. □

## 4. Convergence

We define the norm $|||\cdot|||$ by

$$\begin{aligned}
|||U|||^2 &= \sum_T \Big(\|C^{1/2}h^{1/2}\boldsymbol{\beta}\cdot\nabla U\|^2_T + \|\varepsilon^{1/2}\nabla U\|^2_T \\
&\quad + \|\sigma^{1/2}U\|^2_T + \int_{\partial T}|\boldsymbol{\beta}\cdot\boldsymbol{n}_T|[U]^2 \, {\rm d}s\Big) + J_0^\gamma(U,U).
\end{aligned}$$

By arguing as in the proof of Lemma 1 it is easy to show that there exists constants $\alpha_i$ such that, for $\varepsilon < h$, $h < \sigma^{-1}$, and $v \in W_h$, there holds

$$\alpha_1|||v|||^2 \le a(v, v + Ch\boldsymbol{\beta}\cdot\nabla v) + J_0^\gamma(v, v + Ch\boldsymbol{\beta}\cdot\nabla v), \tag{4.1}$$

and

$$|||Ch\boldsymbol{\beta}\cdot\nabla v||| \le \alpha_2|||v|||. \tag{4.2}$$

Let now $\tilde{u}$ be any $P_1 \cap C_0$ interpolant of $u$ with optimal interpolation properties. As an immediate consequence of standard interpolation theory we have

$$|||u - \tilde{u}||| \le C(h^{3/2} + \varepsilon^{1/2}h + \sigma^{1/2}h^2)|u|_{H^2(\Omega)}.$$

We decompose the error into

$$|||u - U||| \le |||u - \tilde{u}||| + |||U - \tilde{u}|||.$$

For the second part we use (2.6) and (4.1) to obtain, using the notation $\tilde{e} = U - \tilde{u}$

$$
\begin{aligned}
\alpha_1 |||\tilde{e}|||^2 &\leq a(\tilde{e}, \tilde{e} + Ch\boldsymbol{\beta}\cdot\nabla\tilde{e}) + J_0^\gamma(\tilde{e}, \tilde{e} + Ch\boldsymbol{\beta}\cdot\nabla\tilde{e}) \\
&= a(u-\tilde{u}, \tilde{e} + Ch\boldsymbol{\beta}\cdot\nabla\tilde{e}) + J_0^\gamma(u-\tilde{u}, \tilde{e} + Ch\boldsymbol{\beta}\cdot\nabla\tilde{e}) \\
&= \sum_T \Big( (\varepsilon\nabla(u-\tilde{u}), \nabla\tilde{e})_T \\
&\quad + (\boldsymbol{\beta}\cdot\nabla(u-\tilde{u}) + \sigma(u-\tilde{u}), \tilde{e} + Ch\boldsymbol{\beta}\cdot\nabla\tilde{e})_T \Big) \\
&\quad + \frac{1}{2}\sum_{T\in\mathfrak{T}} (\langle \varepsilon \boldsymbol{n}_T\cdot\nabla(\tilde{u}-u)\rangle, [\tilde{e} + Ch\boldsymbol{\beta}\cdot\nabla\tilde{e}])_{\partial T} \\
&\quad + J_0^\gamma(\tilde{u}-u, \tilde{e} + Ch\boldsymbol{\beta}\cdot\nabla\tilde{e}).
\end{aligned}
$$

Using the notation $w := \tilde{e} + Ch\boldsymbol{\beta}\cdot\nabla\tilde{e}$, we integrate by parts on each element in the second term on the right hand side and apply the Cauchy-Schwartz inequality to obtain

$$
\begin{aligned}
&\sum_T (\boldsymbol{\beta}\cdot\nabla(u-\tilde{u}), w)_T \leq \sum_T \Big( -(\boldsymbol{\beta}\cdot\nabla w, u-\tilde{u})_T \\
&\quad + \left( \int_{\partial T_{\mathrm{in}}} |\boldsymbol{\beta}\cdot n| [w]^2 \,\mathrm{d}s \right)^{1/2} |\boldsymbol{\beta}\cdot n|^{1/2} \|u-\tilde{u}\|_{\partial T_{\mathrm{in}}} \Big) \\
&\leq C|||w||| h^{-1/2} \|u-\tilde{u}\| \\
&\leq C|||w||| h^{3/2} |u|_{H^2(\Omega)}
\end{aligned}
$$

where we have used

$$
\sum_T |(\boldsymbol{\beta}\cdot\nabla w, u-\tilde{u})_T| \leq \sum_T \|h^{1/2}\boldsymbol{\beta}\cdot\nabla w\| h^{-1/2} \|u-\tilde{u}\|_T \leq C|||w||| h^{3/2} |u|_{H^2(\Omega)}
$$

and

$$
\begin{aligned}
&\sum_T |(\boldsymbol{\beta}\cdot\boldsymbol{n}_T(w_{\mathrm{u}} - w_{\mathrm{d}}), u_{\mathrm{u}} - \tilde{u}_{\mathrm{u}})_{\partial T_{\mathrm{in}}}| \leq \\
&\leq \left( \int_{\partial T_{\mathrm{in}}} |\boldsymbol{\beta}\cdot n| [w]^2 \,\mathrm{d}s \right)^{1/2} |\boldsymbol{\beta}\cdot n|^{1/2} \|u-\tilde{u}\|_{\partial T_{\mathrm{in}}} \\
&\leq |||w||| C h^{3/2} |u|_{H^2(\Omega)}.
\end{aligned}
$$

The sum over the element edges is controlled by an application of Cauchy-Schwartz inequality and a trace inequality yielding

$$
\begin{aligned}
&\frac{1}{2}\sum_{T\in\mathfrak{T}}(\langle\varepsilon\boldsymbol{n}_T\cdot\nabla(\tilde{u}-u)\rangle,[w])_{\partial T}\leq\\
&\leq c\left(\sum_{T\in\mathfrak{T}}\|h^{1/2}\gamma^{-1/2}\varepsilon^{1/2}\nabla(\tilde{u}-u)\|^2_{\partial T}\right)^{1/2} J_0^\gamma(w,w)^{1/2}\\
&\leq c\left(\sum_{T\in\mathfrak{T}}\|h^{1/2}\gamma^{-1/2}\varepsilon^{1/2}\nabla(\tilde{u}-u)\|^2_{\partial T}\right)^{1/2} |||w|||\\
&\leq c\sum_{T\in\mathfrak{T}}\left(h^{-1}\|h^{1/2}\gamma^{-1/2}\varepsilon^{1/2}\nabla(\tilde{u}-u)\|^2_{T}+|h\gamma^{-1/2}\varepsilon^{1/2}u|^2_{H^2(T)}\right)^{1/2}|||w|||\\
&\leq c\,c_i\,h\,\varepsilon^{1/2}\,\gamma^{-1/2}\,|u|_{H^2(\Omega)}\,|||w|||
\end{aligned}
$$

The remaining terms may be bounded in the following fashion

$$
\begin{aligned}
\sum_T|(\varepsilon\nabla(u-\tilde{u}),\nabla w)_T+(\sigma(u-\tilde{u}),w)_T| &\leq |||u-\tilde{u}|||\;|||w|||\\
&\leq |||w|||C(\varepsilon^{1/2}h+\sigma^{1/2}h^2)|u|_{H^2(\Omega)}
\end{aligned}
$$

and $J_0^\gamma(\tilde{u}-u,w)\leq J_0^\gamma(\tilde{u}-u,\tilde{u}-u)^{1/2}J_0^\gamma(w,w)^{1/2}$. Hence, by virtue of (4.2), we have proved the following

**Theorem 1.** *Suppose that the assumptions of Lemma 1 are satisfied; then the solution $U$ of (2.5) satisfies the following a priori estimate*

$$
|||u-U|||\leq C(\varepsilon^{1/2}h+h^{3/2}+\sigma^{1/2}h^2)|u|_{H^2(\Omega)} \tag{4.3}
$$

**Remark 2.** *The next natural step is to consider the case of low Peclet number, which is easier since the problem can be considered elliptic and we only need the first stability estimate for the $H^1$ a priori estimate then we get $L_2$ by duality following* [16].

4.1. **Numerical example.** We consider the case of a convection–diffusion–reaction problem with $\sigma=1$, $\boldsymbol{\beta}=(1,0)$ and $\varepsilon=10^{-5}$, corresponding to the convection dominated case. We choose $\Omega=[0,1]\times[0,1]$ and use a source terms $f$ corresponding to the exact solution $u=e^{(-5\,(x-0.5)^2-15\,(y-0.5)^2)}$. In Figure 1 we show an elevation of the approximate solutions, and in Figure 2 we show the convergence in $L_2$. This example gives second order convergence which is typical for DG methods; a better rate than the theoretical one is usually obtained on structured meshes.

## 5. Further developments

5.1. **The incompressible Stokes problem.** The linear convection–diffusion problem is of limited engineering interest; a more important class of problems in CFD is described by the incompressible Navier–Stokes equations. To indicate the usefulness of the proposed

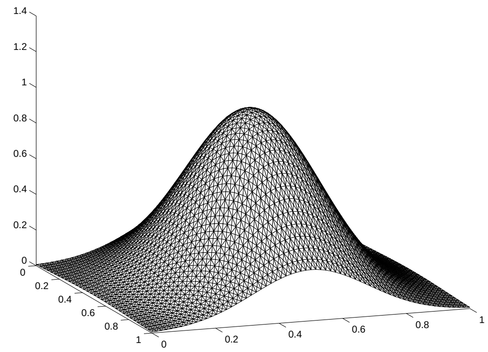

FIGURE 1. Elevation of the approximate eolution on the last mesh in the sequence used to check convergence.

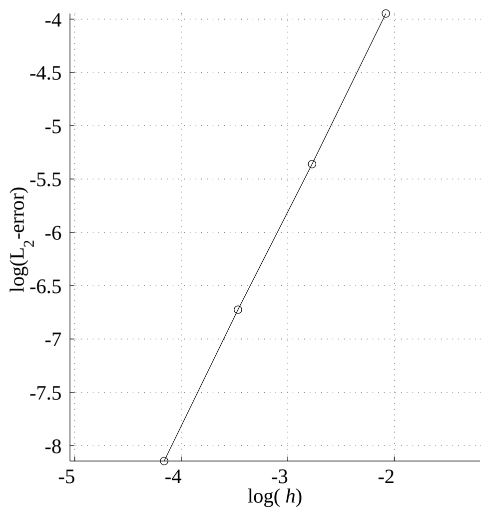


FIGURE 2. Second order convergence in $L_2$.

approach method in a more general setting we give a simple stability proof for the incompressible Stokes equations using velocities approximated by $W_h$ in combination with piecewise linear, continuous, pressures.

Consider thus the problem of finding

$$\boldsymbol{u} \in V = \{[H^1(\Omega)]^2 : \ \boldsymbol{v}|_{\partial\Omega} = 0\}$$

and $p \in L_2(\Omega)/\mathbb{R}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(p, \boldsymbol{v}) + b(q, \boldsymbol{u}) = L(\boldsymbol{v}), \quad \forall(\boldsymbol{v}, q) \in V \times L_2(\Omega)/\mathbb{R}, \tag{5.1}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad b(p, v) := -\int_\Omega p\nabla \cdot \boldsymbol{v}\, dx,$$

and

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

Here, $\mu$ is a constant, $\boldsymbol{f}$ is a given body force, the components of the strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are given by

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),$$

and $\boldsymbol{\varepsilon} : \boldsymbol{\tau} = \sum_{ij} \varepsilon_{ij}\tau_{ij}$ for tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\tau}$.

It is well known that standard Galerkin-based continuous finite element methods for (5.1) are unstable unless and inf-sup condition is satisfied for the discrete spaces. We will now define a discontinuos/continuous element that is a discontinuous relative of the MINI element of Arnold, Brezzi, and Fortin [1], and thus is stable.

We shall seek velocities $\boldsymbol{U} \in [W_h]^2$ and pressures $P \in Q_h$, where

$$Q_h = \{v :\ v|_T \in P^1(T),\ \forall T \in \mathfrak{T},\ v \in C^0(\Omega)\},$$

i.e., the approximation of the pressure is continuous, piecewise linear, and the velocities have one continuous and one discontinuous component.

Consider now the problem of finding $(\boldsymbol{U}, P) \in [W_h]^2 \times Q_h$ such that

$$a_h(\boldsymbol{U},\boldsymbol{v}) + b_h(P,\boldsymbol{v}) + b_h(q,\boldsymbol{U}) = L(\boldsymbol{v}) \quad \forall(\boldsymbol{v},q) \in [W_h]^2 \times Q_h, \tag{5.2}$$

where

$$\begin{aligned} a_h(\boldsymbol{U},\boldsymbol{v}) \ &:= \ \sum_{T\in\mathfrak{T}}(2\mu\boldsymbol{\varepsilon}(\boldsymbol{U}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_T + \frac{1}{2}\sum_{T\in\mathfrak{T}}(2\mu\,\gamma\, h^{-1}[\boldsymbol{U}],[\boldsymbol{v}])_{\partial T} \\ &\quad -\frac{1}{2}\sum_{T\in\mathfrak{T}}(2\mu\langle \boldsymbol{n}_T\cdot\boldsymbol{\varepsilon}(\boldsymbol{U}))\rangle,[\boldsymbol{v}])_{\partial T} \\ &\quad -\frac{1}{2}\sum_{T\in\mathfrak{T}}(2\mu\langle \boldsymbol{n}_T\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}))\rangle,[\boldsymbol{U}])_{\partial T}, \end{aligned}$$

$$b_h(P,\boldsymbol{v}) = -\sum_{T\in\mathfrak{T}}(P,\nabla\cdot v)_T + \frac{1}{2}\sum_{T\in\mathfrak{T}}(P,[\boldsymbol{v}\cdot\boldsymbol{n}])_{\partial T},$$

and

$$L(\boldsymbol{v}) := \sum_{T\in\mathfrak{T}}(\boldsymbol{f},\boldsymbol{v})_T.$$

We now introduce the energy-like norm

$$|||\boldsymbol{v}|||^2 := \sum_{T\in\mathfrak{T}}(2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_T + \frac{1}{2}\sum_{T\in\mathfrak{T}}(2\mu\,\gamma\, h^{-1}[\boldsymbol{v}],[\boldsymbol{v}])_{\partial T},$$

and the quotient norm

$$\|q\|_0 := \|q\|_{L_2(\Omega)\backslash\mathbb{R}},$$

and recall that a discrete method for Stokes can be stable only if the approximating spaces fulfill the inf-sup condition, which in this case reads

$$\inf_{q\in Q_h}\sup_{\boldsymbol{v}\in[W_h]^2}\frac{b_h(q,\boldsymbol{v})}{|||\boldsymbol{v}|||\|q\|_0}\geq c_0, \tag{5.3}$$

where $c_0$ is a fixed positive constant. This condition can be verified by constructing an interpolant $\pi_h : V \to [W_h]^2$ such that

$$b_h(q,\boldsymbol{v}-\pi_h\boldsymbol{v})=0 \quad \forall(\boldsymbol{v},q)\in V\times Q_h, \tag{5.4}$$

and

$$|||\boldsymbol{v}|||\leq c\,|||\pi_h\boldsymbol{v}||| \quad \forall \boldsymbol{v}\in V, \tag{5.5}$$

see Brezzi and Fortin [3]. Since the pressure is piecewise linear and continuous, we find, by integration by parts, that (5.4) can be written

$$\int_\Omega(\boldsymbol{v}-\pi_h\boldsymbol{v})\cdot\nabla q\,dx=0 \quad \forall(\boldsymbol{v},q)\in V\times Q_h,$$

which can be written as

$$\int_T(\boldsymbol{v}-\pi_h\boldsymbol{v})\,dx=0 \quad \forall\boldsymbol{v}\in V, \quad \forall T\in\mathfrak{T}. \tag{5.6}$$

This condition can be verified in the same way as for the MINI element [1]: first we introduce the Clément interpolant $\pi_h^{\mathrm{C}} : V \to [W_h]^2$ satisfying

$$\sum_{T\in\mathfrak{T}}h_T^{-2}\|\pi_h^{\mathrm{C}}\boldsymbol{v}-\boldsymbol{v}\|^2_{L_2(T)}\leq C|||\boldsymbol{v}|||^2 \tag{5.7}$$

and

$$\sum_{T\in\mathfrak{T}}h_T^{-1}\|\pi_h^{\mathrm{C}}\boldsymbol{v}-\boldsymbol{v}\|^2_{H^1(T)}\leq C|||\boldsymbol{v}|||^2. \tag{5.8}$$

We next verify (5.6) by writing

$$\pi_h\boldsymbol{v}:=\pi_h^{\mathrm{C}}\boldsymbol{v}+\boldsymbol{\alpha}_T,\ \text{where}\ \boldsymbol{\alpha}_T:=\frac{1}{m(T)}\int_T(\pi_h^{\mathrm{C}}\boldsymbol{v}-\boldsymbol{v})\,dx$$

is the mean value over $T$ of the difference between the Clément interpolant of $\boldsymbol{v}$ and $\boldsymbol{v}$ itself. In order to verify (5.4), we note that, by Hölder's inequality,

$$|\boldsymbol{\alpha}_T|\leq\frac{1}{m(T)}\|\pi_h^{\mathrm{C}}\boldsymbol{v}-\boldsymbol{v}\|_{L_1(T)}\leq\frac{C}{h_T}\|\pi_h^{\mathrm{C}}\boldsymbol{v}-\boldsymbol{v}\|_{L_2(T)} \tag{5.9}$$

so that

$$
\begin{aligned}
|||\pi_h \boldsymbol{v}|||^2 &\le |||\pi_h^{\mathrm{C}} \boldsymbol{v}|||^2 + \frac{1}{2}\sum_{T\in\mathfrak{T}} \langle 2\mu\,\gamma\, h^{-1}[\boldsymbol{\alpha}_T], [\boldsymbol{\alpha}_T]\rangle_{\partial T} \\
&\le |||\pi_h^{\mathrm{C}} \boldsymbol{v}|||^2 + C\sum_{T\in\mathfrak{T}} |\boldsymbol{\alpha}_T|^2 \\
&\le |||\pi_h^{\mathrm{C}} \boldsymbol{v}|||^2 + C\sum_{T\in\mathfrak{T}} h_T^{-2}\|\pi_h^{\mathrm{C}}\boldsymbol{v} - \boldsymbol{v}\|^2_{L_2(T)} \\
&\le C|||\boldsymbol{v}|||^2,
\end{aligned} \tag{5.10}
$$

where the last inequality follows from (5.9) and the stability of the Clément operator.

5.1.1. *Numerical example.* We consider the unit square with exact flow solution (from [15]) given by $u = (20\,x\,y^3,\ 5\,x^4 - 5\,y^4)$ and $p = 60\,x^2y - 20\,y^3 + C$. Choosing $\gamma = 10\mu$ and imposing zero mean pressure ($C = -5$), we obtain the convergence shown in Figure 3; second order for the velocity and approximately $O(h^{3/2})$ for the pressure in $L_2$–norm.

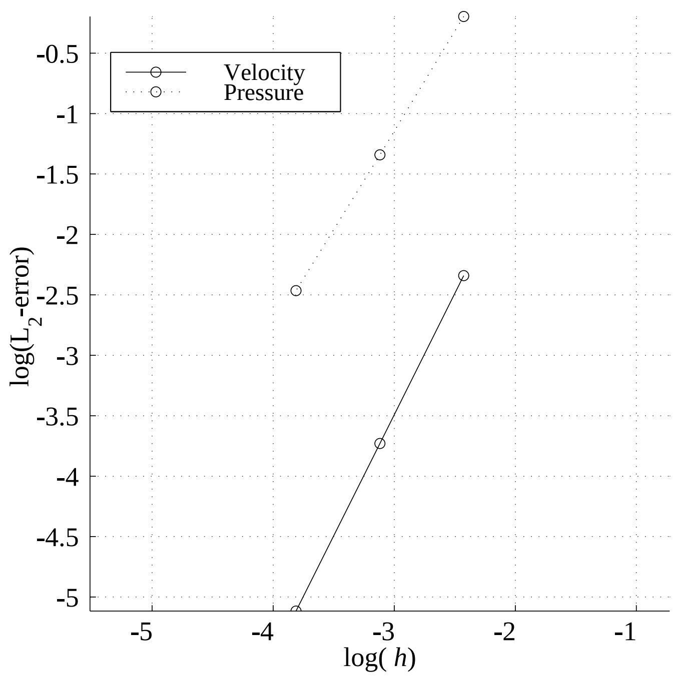


FIGURE 3. $L_2$-norm convergence of the velocity and of the pressure for Stokes.

Pressure isolines and velocity vectors on the final mesh in the sequence used to obtain the convergence plot are shown in Figures 4 and 5. Note the oscillations in the pressure near the boundaries. Our experience is that these cannot be gotten rid of by other choices of $\gamma$.

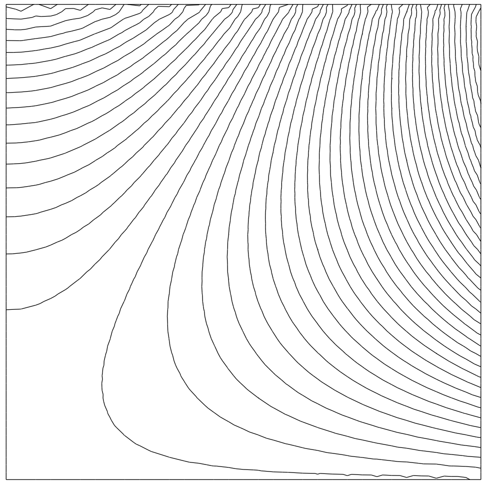

FIGURE 4. Pressure isolines.

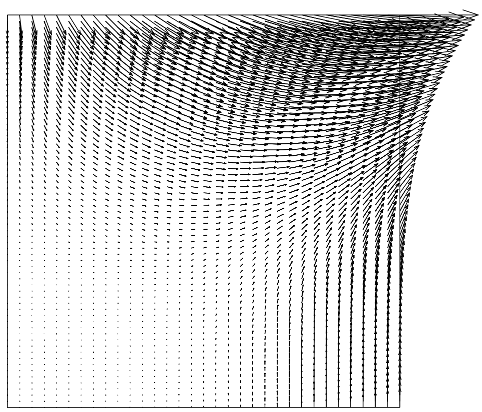

FIGURE 5. Finite element solution for the velocity.

5.2. **Explicit time-stepping.** We noted in the Introduction that one motivation for the DG method in general is the ease with which explicit time-stepping algorithms can be used. Indeed, in a DG method one can use $L_2-$orthogonal bases so that the mass matrix is diagonal by default (see, e.g., [12]). This is not the case with functions in $W^h$, but there is a way around this. To illustrate this, we simply consider an $L_2-$projection of a function $g$ onto $W^h$: Find $U \in W^h$ such that

$$\int_\Omega U\, v\, d\Omega = \int_\Omega g\, v\, d\Omega, \quad \forall v \in W^h.$$

The case of an explicit time-stepping scheme yields a similiar problem with right-hand side emanating from the solution at previous time-levels (plus boundary terms which are of no consequence for the current discussion). We consider the case when the solution is split (hierarchically) into two parts, one continuous, $U_c$, and one discontinuous, $U_d$. The nodal values corresponding to the continuous part are denoted $\boldsymbol{u}_c$ and the discontinuous part $\boldsymbol{u}_d$, and thus we have the following linear problem to solve:

$$\begin{bmatrix} \boldsymbol{M}_c & \boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{B} & \boldsymbol{M}_d \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_c \\ \boldsymbol{u}_d \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}_c \\ \boldsymbol{g}_d \end{bmatrix}.$$

Since the constant function can be represented both by the continuous and the discontinuous fields, the problem is in fact underdetermined. In order to settle this problem, we may (for instance) choose to require that the discontinuous field fulfills

$$\boldsymbol{B}^{\mathrm{T}}\boldsymbol{u}_d = \boldsymbol{0}. \tag{5.11}$$

We lump the matrix $\boldsymbol{M}_c$ using row-sum to obtain $\boldsymbol{M}_{\mathrm{L}}$ and solve

$$\boldsymbol{M}_{\mathrm{L}}\boldsymbol{u}_c = \boldsymbol{g}_c$$

and

$$\boldsymbol{M}_d\boldsymbol{u}_d = \boldsymbol{g}_d - \boldsymbol{B}\,\boldsymbol{M}_{\mathrm{L}}^{-1}\boldsymbol{g}_c,$$

with both $\boldsymbol{M}_{\mathrm{L}}$ and $\boldsymbol{M}_d$ diagonal matrices. It is well known (cf. [8]) that the row-sum lumped mass matrix conserves mass, so we have that

$$\int_\Omega U_c\, d\Omega = \int_\Omega g\, d\Omega;$$

thus the condition (5.11) makes the discontinuous field *massless*.

5.2.1. *Numerical example.* We consider the domain $(0,1)\times(0,1)$ on which we convect the initial data

$$u = e^{-100\,((x-1/2)^2+(y-1/4)^2)},$$

using the velocity field $\boldsymbol{\beta} = (1/2-y, x-1/2)$. The initial data are interpolated in the linear part of the approximation and the piecewise constant part is put to zero. In Figures 6 and 7 we give the initial solution and the solution after one full rotation using the `ode45` solver in MATLAB® (without default options). The maximum value after one rotation is $U \approx 0.8521$, a loss of 15%. Finally, In Figure 8, we also show the corresponding results when instead prescribing the *continuous* field to have zero mass (which is done analogously). Then the approximation has better local conservation properties, which may explain that the maximum value after one rotation is $U \approx 0.9060$, a loss of only 9%.

## References


[1] Arnold D, Brezzi F, Fortin A stable element for the Stokes equation. *Calcolo* 1984; **23**: 337–344.

[2] Baumann CE, Oden JT. A discontinuous *hp* finite element method for convection-diffusion problems. *Computer Methods in Applied Mechanics an Engineering* 1999; **175**(3-4):311–341.

[3] Brezzi F, Fortin M. *Mixed and Hybrid Finite Element Methods.* Springer-Verlag: New York, 1991.

[4] Brezzi F, Franca LP, Hughes TJR, Russo A. $b = \int g$. *Computer Methods in Applied Mechanics an Engineering* 1997; **145**:329–339.

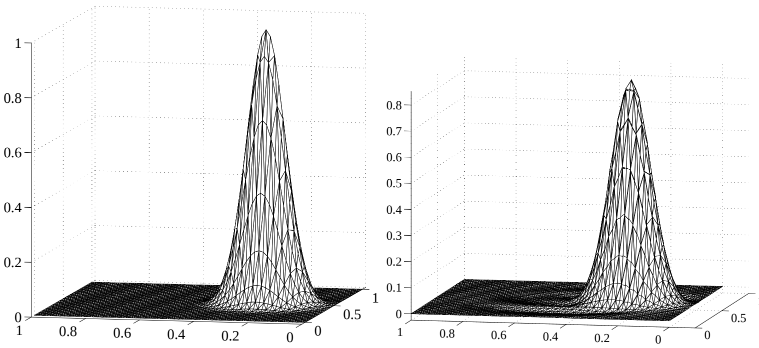

FIGURE 6. Initial solution and solution after one rotation.

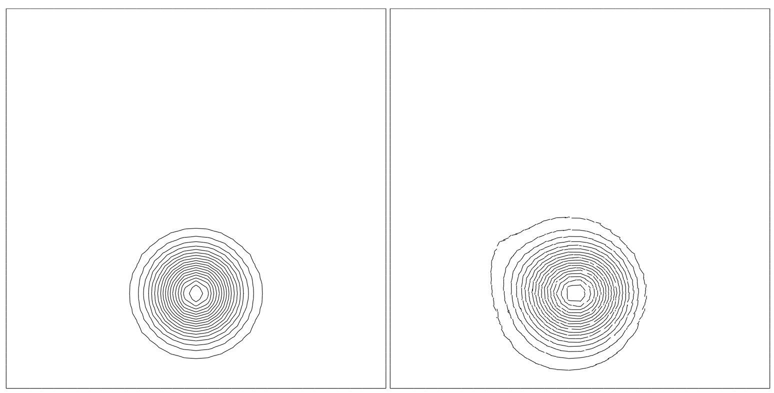

FIGURE 7. Isolines corresponding to Fig. 6.

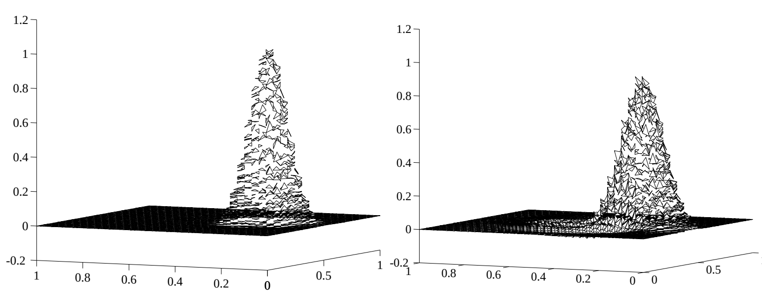

FIGURE 8. Initial solution and solution after one rotation, zero mean continuous field.


[5] Burman E, Ern A. Nonlinear diffusion and discrete maximum principle for stabilized Galerkin approximations of the advection-diffusion-reaction equation. *Computer Methods in Mechanics and Engineering* **191**:3822–3855 (2002).

[6] Burman E, Ern A. Discrete maximum principle for galerkin approximations of elliptic problems: circumventing the strictly acute condition, in preparation (2003).

[7] Hansbo P. Explicit streamline diffusion finite-element methods for the compressible Euler equations in conservation variables. *Journal of Computational Physics* 1993; **109**(2):274–288.

[8] Hansbo P. Aspects of conservation in finite element flow computations. *Computer Methods in Applied Mechanics an Engineering* 1994; **117**:423–437.

[9] Hansbo P, Larson MG. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Computer Methods in Applied Mechanics an Engineering* 2002; **191**(17-18):1895–1908.

[10] Johnson C. *Numerical Solutions of Partial Differential Equations by the Finite Element Method* Cambridge University Press, 1987.

[11] Johnson C, Pitkäranta J. An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation, *Mathematics of Computation* 1986; **46**(173):1–26.

[12] Karniadakis GE, Sherwin SJ. *Spectral/hp Element Methods for CFD.* Oxford University Press, 1998.

[13] Lesaint P, Raviart PA. On a finite element method for solving the neutron transport equation, in: *Mathematical Aspects of Finite Elements in Partial Differential Equations*, de Boor, C (ed.) Academic Press, 1974.

[14] Nitsche J. Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 1971; **36**:9–15.

[15] S. Norburn and D. Silvester, Stabilised vs stable mixed methods for incompressible flow, Comput. Methods Appl. Mech. Eng. 1998; **166**:131–141.

[16] Rivière B, Wheeler M, Girault V. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM Journal of Numerical Analysis* 2001; **39**(3):902–931

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