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ADAPTIVE FEM FOR LES

JOHAN HOFFMAN

Abstract. We consider the computation of the mean drag coefficient in a turbulent flow around a surface mounted cube using an adaptive finite element method, based on a posteriori error estimates, for a LES formulation of the problem. The a posteriori error estimates are based on the solution of an associated linearized dual problem that we approximate numerically. We prove a posteriori error estimates, and we present numerical examples using adaptive mesh refinement based on these a posteriori error estimates.

1. Introduction

In this paper we compute the mean drag coefficient in a *Large Eddy Simulation LES* of a turbulent flow around a surface mounted cube, using an adaptive finite element method based on *a posteriori error estimates* in terms of the solution of an associated linearized dual problem.

The idea of using duality arguments in a posteriori error estimation goes back to Babuška and Miller [2] in the context of postprocessing 'quantities of physial interest' in elliptic model problems. A framework for more general situations has since then been systematically developed by Eriksson & Johnson and Becker & Rannacher, with coworkers, see e.g. [10, 8, 3, 4, 26, 27]. For a more detailed account of the development of this framework, including references, we refer in particular to the review papers [8, 4]. For incompressible flow, applications of adaptive finite element methods based on this framework have been increasingly advanced with computation of functionals such as the drag force for 2d stationary benchmark problems in [3, 14], and drag and lift forces and pressure differences for 3d stationary benchmark problems in [19]. In [21], time dependent problems in 3d are considered, and the extension of this framework to LES is investigated in [18].

In LES, we apply a spatial averaging operator (filter) to the Navier-Stokes equations to obtain a new set of equations for the averaged (filtered) variables. Such an averaging process involves several mathematical issues that has to be addressed; such as a possible commutation error if the filter does not commute with differentiation, finding the correct boundary conditions for the filtered variables, and the problem of *closure* due to filtering of the non linear term in the Navier-Stokes equations. There is an extensive amount of work on LES, in particular regarding the closure problem and the construction of *subgrid*

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models, and we refer to [13, 34] and the references therin for details. The commutation error is investigated in [7, 35], for example, and for work on boundary conditions (or near wall models) for LES we refer to [24, 32] and the references therin.

In [18], a posteriori error estimates in various norms and linear functionals of the filtered velocity field in a LES are presented. These estimates take into consideration both the numerical error from discretization of the filtered Navier-Stokes equations and the modeling error from unresolved subgrid scales, and are based on the solution of an associated linearized dual problem that contains information about error propagation in space-time. If we use a subgrid model in the computation, the subgrid modeling error is included in the a posteriori error estimates, which opens the possibility of comparing the error using different subgrid models. Altogether, the a posteriori error estimates open the possibility of adaptively choosing both an optimal mesh and an optimal subgrid model.

This approach to a posteriori error estimation with respect to the averaged solution, using duality teqniques, in terms of a modeling error and a discretization error was developed for convection-diffusion-reaction equations in [15, 22, 20, 16, 17]. Related approaches with a posteriori error estimates in terms of a modeling and a discretization contribution to the total error have been suggested. For example, more recently in [6] similar ideas are presented with applications to 2d convection-diffusion-reaction problems.

In this paper we consider the computation of the mean drag coefficient in a LES of a turbulent flow around a surface mounted cube, investigated experimentally in [30, 31] and computationally in e.g. [28], using an adaptive finite element method based on the teqniques developed in [18]. To the best knowledge of the author, this paper represents the first application of adaptive finite element methods, based on the solution of a dual problem, to the turbulent Navier-Stokes equations in 3d.

An outline of this paper is as follows: In Section 2 we present the Navier-Stokes equations as a model for viscous incompressible flow, and we state the discretization of the corresponding LES equations using the cG(1)cG(1) method. In Section 3 we prove a posteriori error estimates for the mean drag coefficient using the cG(1)cG(1) method, and we present a corresponding algorithm for adaptive mesh refinement, and in Section 4 we compute the mean drag coefficient using the teqniques developed in Section 3. We conclude with a summary and some remarks on future directions.

2. Turbulent flow and LES

The incompressible Navier-Stokes equations expressing conservation of momentum and incompressibility of a unit density constant temperature Newtonian fluid with constant kinematic viscosity $\nu > 0$ enclosed in a volume Ω in $\mathbb{R}^3$ with homogeneous Dirichlet boundary conditions, take the form: Find (u, p) such that

$$\begin{aligned} \dot{u} + (u \cdot \nabla)u - \nu\Delta u + \nabla p &= f && \text{in } \Omega \times I, \\ \nabla \cdot u &= 0 && \text{in } \Omega \times I, \\ u &= 0 && \text{on } \partial\Omega \times I, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where $u(x,t) = (u_i(x,t))$ is the *velocity* vector and $p(x,t)$ the *pressure* of the fluid at (x,t), and f, u_0, $I = (0,T)$, is a given driving force, initial data and time interval, respectively. The quantity $\nu\Delta u - \nabla p$ represents the total fluid force, and may alternatively be expressed as

$$\nu\Delta u - \nabla p = \operatorname{div}\sigma(u,p), \tag{2.2}$$

where $\sigma(u,p) = (\sigma_{ij}(u,p))$ is the *stress tensor*, with components $\sigma_{ij}(u,p) = 2\nu\epsilon_{ij}(u) - p\delta_{ij}$, composed of the *stress deviatoric* $2\nu\epsilon_{ij}(u)$ with zero trace and an isotropic pressure: here $\epsilon_{ij}(u) = (u_{i,j}+u_{j,i})/2$ is the *strain tensor*, with $u_{i,j} = \partial u_i/\partial x_j$, and δ_{ij} is the usual Kronecker delta, the indices i and j ranging from 1 to 3. We assume that (2.1) is normalized so that the reference velocity and typical length scale are both equal to one. The Reynolds number Re is then equal to ν^{-1}.

2.1. **The averaged Navier-Stokes equations.** In a turbulent flow we are typically not able to resolve all scales of motion computationally. We may instead aim at computing a *running average* u^h of u on a scale h, defined by

$$u^h(x,t) = \frac{1}{h^3}\int_{Q_h} u(x+y,t)\; dy, \tag{2.3}$$

where $h = h(x,t)$ is a parameter related to the local resolution of the problem and $Q_h = \{y \in \mathbb{R}^3 : |y_i| \leq h/2\}$. In the LES literature it is common to define the averaging operator through convolution by a certain filter function, and there is a multitude of filter functions being used. Though we only consider the case of the filter corresponding to (2.3) in this paper, the teqniques for a posteriori error estimation are general and apply to other filters, possibly with modifications for commutation errors associated with such filters.

By an extension of (u,p,u_0,f) to $\mathbb{R}^3$ by reflection for all $x \notin \overline{\Omega}$, the averaging operator (2.3) commutes with space and time differentiation. If we take the running average of the equations (2.1), corresponding to a LES, we obtain the following equations for u^h:

$$\begin{aligned} \dot{u}^h + (u^h\cdot\nabla)u^h - \nu\Delta u^h + \nabla p^h + F_h(u) &= f^h \qquad && \text{in } \Omega\times I,\\ \nabla\cdot u^h &= 0 && \text{in } \Omega\times I,\\ u^h &= 0 && \text{on } \partial\Omega\times I,\\ u^h(\cdot,0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{2.4}$$

where we choose homogeneous Dirichlet boundary conditions for u^h, and $F_h(u) = \nabla\cdot\tau^h(u)$, where $\tau^h_{ij}(u) = (u_iu_j)^h - u^h_iu^h_j$ is the *Reynolds stress tensor*. The closure problem of LES is how to model $F_h(u)$ in terms of u^h in a subgrid model $\hat{F}_h(u^h)$, or $\tau^h(u)$ in a model $\hat{\tau}^h(u^h)$. In this paper we focus on the computation of chosen output functionals for the problem (2.4) using adaptive finite element methods, and we refer to [13, 34] and the references therin for work on subgrid modeling for LES.

A weak formulation of (2.4) reads: find $(u^h,p^h) \in L_2(I;[H^1_0(\Omega)]^3 \times L_2(\Omega))$, with $\dot{u}^h \in L_2(I;[L_2(\Omega)]^3)$ and $u^h(\cdot,0) = u^h_0$, such that

$$\begin{aligned} &(\dot{u}^h + u^h\cdot\nabla u^h, v) + (2\nu\epsilon(u^h),\epsilon(v)) - (p^h,\nabla\cdot v)\\ &-(\tau^h(u),\nabla v) + (\nabla\cdot u^h, q) = (f^h, v), \end{aligned} \tag{2.5}$$

for all $(v, q) \in L_2(I; [H_0^1(\Omega)]^3 \times L_2(\Omega))$, where we assume that $f^h \in L_2(I; [L_2(\Omega)]^3)$.

Here $L_2(\Omega)$ is the Hilbert space of Lebesgue square integrable functions on Ω, with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and $H^s(\Omega)$ is the standard Hilbert space of functions in $L_2(\Omega)$ with also partial derivatives of order $\leq s$ in $L_2(\Omega)$. $H_w^s(\Omega)$ denotes the functions $v \in H^s(\Omega)$ that satisfies the Dirichlet boundary condition $v|_{\partial\Omega} = w$ (in the sense of traces), and in particular $H_0^s(\Omega)$ denotes the functions in $H^s(\Omega)$ that vanish on $\partial\Omega$. We let $\mathcal{C}(I; X)$ denote the space of all continuous functions $v : I \to X$ with $\max_{t\in I} \|v(t)\|_X < \infty$, where X denotes a Banach space with norm $\|\cdot\|_X$. For further details on the function spaces above we refer to [1, 11].

Assuming we have also Neumann boundary conditions, we denote Γ_D the part of the boundary $\partial\Omega$ where Dirichlet boundary conditions are specified, and $\Gamma_N = \partial\Omega \setminus \Gamma_D$ the part with Neumann boundary conditions. Now $H_w^s(\Omega)$ and $H_0^s(\Omega)$ denote the spaces of functions in $H^s(\Omega)$ that satisfies the Dirichlet boundary conditions on Γ_D.

The viscous term $(2\nu\epsilon(u^h), \epsilon(v))$ in the weak formulation (2.5) may alternatively be expressed as $(\nu\nabla u^h, \nabla v)$, where we have used that $(\epsilon(u^h), \nabla v) = (\epsilon(u^h), \epsilon(v))$ by the symmetry of the strain tensor. In the case of pure Dirichlet boundary conditions the two forms are equivalent, but in the case of Neumann boundary conditions on part of the boundary the difference is a boundary integral over Γ_N of the normal derivative of u multiplied by ν and the test function v.

2.2. **Discretization: the cG(1)cG(1) method.** The cG(1)cG(1) method is a variant of the G^2 method [25, 21, 18] using the continuous Galerkin method cG(1) in time instead of a discontinuous Galerkin method. With cG(1) in time the trial functions are continuous piecewise linear and the test functions piecewise constant. cG(1) in space corresponds to both test functions and trial functions being continuous piecewise linear. Let $0 = t_0 < t_1 < ... < t_N = T$ be a sequence of discrete time steps with associated time intervals $I_n = (t_{n-1}, t_n]$ of length $k_n = t_n - t_{n-1}$ and space-time slabs $S_n = \Omega \times I_n$, and let $W^n \subset H^1(\Omega)$ be a finite element space consisting of continuous piecewise linear functions on a mesh $\mathcal{T}_n = \{K\}$ of mesh size $h_n(x)$ with W_w^n the functions $v \in W^n$ satisfying the Dirichlet boundary condition $v|_{\Gamma_D} = w$.

We now seek functions (U_h, P_h), continuous piecewise linear in space and time, and the cG(1)cG(1) method for the averaged Navier-Stokes equations (2.4), with homogeneous Dirichlet boundary conditions and subgrid model $\hat{\tau}^h$, reads: For $n = 1, ..., N$, find $(U_h^n, P_h^n) \equiv (U_h(t_n), P_h(t_n))$ with $U_h^n \in V_0^n \equiv [W_0^n]^3$ and $P_h^n \in W^n$, such that

$$
\begin{aligned}
&((U_h^n - U_h^{n-1})k_n^{-1} + \hat{U}_h^n \cdot \nabla \hat{U}_h^n, v) + (2\nu\epsilon(\hat{U}_h^n), \epsilon(v)) \\
&- (P_h^n, \nabla \cdot v) - (\hat{\tau}_n^h(\hat{U}_h^n), \nabla v) + (\nabla \cdot \hat{U}_h^n, q) \\
&+ \delta_1(\hat{U}_h^n \cdot \nabla \hat{U}_h^n + \nabla P_h^n, \hat{U}_h^n \cdot \nabla v + \nabla q) + \delta_2(\nabla \cdot \hat{U}_h^n, \nabla \cdot v) \\
&= (f^h, v + \delta_1(\hat{U}_h^n \cdot \nabla v + \nabla q)) \quad \forall (v, q) \in V_0^n \times W^n,
\end{aligned} \tag{2.6}
$$

where $\hat{U}_h^n = \frac{1}{2}(U_h^n + U_h^{n-1})$, $\delta_1 = \frac{1}{2}(k_n^{-2} + |U|^2 h_n^{-2})^{-1/2}$ in the convection-dominated case $\nu < \hat{U}_h^n h_n$ and $\delta_1 = \kappa_1 h^2$ otherwise, $\delta_2 = \kappa_2 h$ if $\nu < \hat{U}_h^n h_n$ and $\delta_2 = \kappa_2 h^2$ otherwise, with

κ_1 and κ_2 positive constants of unit size, and

$$(v, w) = \sum_{K \in \mathcal{T}_n} \int_K v \cdot w \, dx,$$

$$(\epsilon(v), \epsilon(w)) = \sum_{i,j=1}^{3} (\epsilon_{ij}(v), \epsilon_{ij}(w)),$$

$$(\hat{\tau}^h(v), \nabla w) = \sum_{i,j=1}^{3} (\hat{\tau}^h_{ij}(v), \partial w_i / \partial x_j),$$

where we assume that the subgrid model $\hat{\tau}^h$ is sufficiently regular for the above integrals to make sense.

Again we note that the viscous term $(2\nu\epsilon(U_h), \epsilon(v))$ may alternatively occur in the form $(\nu\nabla U_h, \nabla v) = \sum_{i=1}^{3}(\nu\nabla(U_h)_i, \nabla v_i)$. In the case of Dirichlet boundary conditions the corresponding variational formulations are equivalent, but not so in the case of Neumann boundary conditions.

If we have a Neumann boundary conditions $\sigma \cdot n = g$ on $\Gamma_N \subset \partial\Omega$, then the right hand side of (2.6) is supplemented with an integral of $g \cdot v$ over Γ_N. This implements the Neumann boundary condition in weak form through the presence of a term $(-P_h, \nabla \cdot v) + (2\nu\epsilon(U_h), \epsilon(v)) = (\sigma, \epsilon(v))$ on the left hand side, which when integrated by parts generates an integral of $(\sigma \cdot n) \cdot v$ over Γ_N. If the viscous term appears in the form $(\nu\nabla U_h, \nabla v)$, the corresponding Neumann boundary condition has the form $\nu\nabla U_h \cdot n - P_h n = g$, where $\nabla U_h \cdot n$ is the derivative in the unit outward normal direction n. To simulate an outflow boundary condition we may use a Neumann boundary condition with $g = 0$ corresponding to a zero force at outflow, simulating outflow into a large empty reservoir. The alternative condition $\nu\nabla U_h \cdot n - P_h n = 0$ acts slightly differently as an approximation of a *transparent* outflow boundary condition, see [33].

2.3. **Computation of the mean drag force.** We want to compute an approximation of the quantity

$$N(\sigma(u^h, p^h)) = \frac{1}{|I|} \int_I \int_{\Gamma_0} \sum_{i,j=1}^{3} \sigma_{ij}(u^h, p^h) n_j \phi_i \; ds \; dt, \tag{2.7}$$

where (u^h, p^h) solves (2.4), ϕ is the trace on Γ_0 of a function in $H^1(\Omega)$, and $\Gamma_0 \subset \Gamma_D$ is a closed surface representing the boundary of a body immersed in the flow. If ϕ is a unit vector in the direction of the mean flow, (2.7) represents the mean of the drag force due to (u^h, p^h) on Γ_0 over a time interval I, and if ϕ is a unit vector in a direction perpendicular to the mean flow, (2.7) is the temporal mean of the lift force on Γ_0 due to (u^h, p^h) in that direction. With the idea of increasing the precision, see [14], we may use (2.4) and integration by parts to rewrite the surface integral in (2.7) as a volume integral, leading to the following expression for (2.7):

$$\begin{aligned} N(\sigma(u^h,p^h)) &= \frac{1}{|I|}\int_I (\dot{u}^h + u^h\cdot\nabla u^h - f^h,\varphi) - (p^h,\nabla\cdot\varphi) \\ &\quad +(2\nu\epsilon(u^h),\epsilon(\varphi)) - (\tau^h(u),\nabla\varphi) + (\nabla\cdot u^h,\theta)\ dt, \end{aligned} \tag{2.8}$$

for any $\varphi \in L_2(I;[H^1_{\phi,0}(\Omega)]^3)$, where $H^1_{\phi,0}(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_0} = \phi, v|_{\Gamma_1} = 0\}$, $\Gamma_1 = \Gamma_D \setminus \Gamma_0$, and $\theta \in L_2(I;L_2(\Omega))$. We note that due to (2.4), this representation does neither depend on the choice of θ, nor the particular extension φ of ϕ being used. We are thus led to approximate $N(\sigma(u^h,p^h))$ by the quantity

$$\begin{aligned} N^h(\sigma(U_h,P_h)) &= \frac{1}{|I|}\int_I (\dot{U}_h + U_h\cdot\nabla U_h - f^h,\Phi) - (P_h,\nabla\cdot\Phi) \\ &\quad +(2\nu\epsilon(U_h),\epsilon(\Phi)) - (\hat{\tau}^h(U_h),\nabla\Phi) + (\nabla\cdot U_h,\Theta)\ dt, \end{aligned} \tag{2.9}$$

where $(U_h,P_h) \in L_2(I;V_0^n\times W^n)$ and $(\Phi,\Theta) \in L_2(I;V^n_{\phi,0}\times W^n)$, with $V^n_{\phi,0} = \{v \in [W^n]^3 : v|_{\Gamma_0} = \phi, v|_{\Gamma_1} = 0\}$.

Remark 1. *Ultimately, we are of course interested in $N(\sigma(u,p))$ rather than $N(\sigma(u^h,p^h))$. The error $|N(\sigma(u,p)) - N(\sigma(u^h,p^h))|$ is typically of the order $\mathcal{O}(h)$, and in this paper we assume this error to be small compared to the error $|N(\sigma(u^h,p^h)) - N^h(\sigma(U_h,P_h))|$. For example, if we want to compute the mean drag force, so that $\phi = (1,0,0)$ in (2.7) with the positive x_1-direction being the mean flow direction, in the case of a rectangular box with upstream and downstream boundaries Γ_u and Γ_d respectively, corresponding unit outward normals $n_u = (-1,0,0)$ and $n_d = (1,0,0)$, and $n = (n_1,n_2,n_3)$ the unit outward normal of the whole box, we have*

$$\begin{aligned} &|N(\sigma(u,p)) - N(\sigma(u^h,p^h))| = |N(\sigma(u-u^h,p-p^h))| \\ &= |\frac{1}{|I|}\int_I\int_{\Gamma_0}(2\nu\epsilon_{11}(u-u^h) - (p-p^h))n_1\ ds\ dt| \equiv |I_u + I_p|, \end{aligned}$$

where

$$\begin{aligned} |I_p| = \quad & |\frac{1}{|I|}\int_I\int_{\Gamma_u} p(s)-p^h(s)\ ds - \int_{\Gamma_d} p(s)-p^h(s)\ ds\ dt| \\ = \quad & |\frac{1}{|I|}\int_I\int_{\Gamma_u}\frac{1}{h^3}\int_{Q_h} p(s)-p(s+y)\ dy\ ds - \int_{\Gamma_d}\frac{1}{h^3}\int_{Q_h} p(s)-p(s+y)\ dy\ ds\ dt| \\ = \quad & |\frac{1}{|I|}\int_I\int_{\Gamma_u}\frac{1}{h^3}\int_{Q_h}\nabla p(\xi_1)\cdot(-y)\ dy\ ds - \int_{\Gamma_d}\frac{1}{h^3}\int_{Q_h}\nabla p(\xi_2)\cdot(-y)\ dy\ ds\ dt| \\ & (\text{with } \xi_1,\xi_2 \in s+Q_h) \\ \leq \quad & 2\frac{1}{|I|}\int_I\int_{\Gamma_u\cup\Gamma_d}\max_{\xi\in s+Q_h}|\nabla p(\xi)|\ ds\ dt\ \frac{1}{h^3}\int_{Q_h}|y|\ dy \\ \leq \quad & 2\frac{1}{|I|}\int_I\int_{\Gamma_u\cup\Gamma_d}\max_{\xi\in s+Q_h}|\nabla p(\xi)|\ ds\ dt\ \frac{\sqrt{3}}{2}h \equiv Ch, \end{aligned}$$

and in a similar way we get that $|I_u| \leq \nu C h$, where we have assumed sufficient regularity of (u, p) in the formal calculations above.

3. Adaptive finite element methods for LES

An adaptive algorithm includes feed-back from computation to achieve the computational goal with minimal computational cost. In an adaptive finite element method this feed-back from computation relies on a posteriori error estimates. In Algorithm 2, an adaptive algorithm for the computation of the mean drag force $N(\sigma(u^h, p^h))$ is presented, which is based on a posteriori error estimates of the form

$$|N(\sigma(u^h, p^h)) - N^h(\sigma(U_h, P_h))| \leq \sum_{K \in \mathcal{T}_n^k} \mathcal{E}_K^k, \tag{3.1}$$

where $\mathcal{E}_K^k$ is an *error indicator* for element K. We have here chosen the computational mesh $\mathcal{T}_n^k$ to be constant in time for each iteration k in the adaptive algorithm, and we have also chosen the time step length k_n to be constant in time, namely

$$k_n = \min_{K \in \mathcal{T}_n^k} \mathrm{diam}(K), \tag{3.2}$$

where $\mathrm{diam}(K)$ is the diameter of element K. A time dependent mesh could be used based on Theorem 5, but the possible gain must be compared to the increased computational cost of administrating a time dependent mesh. In this case, where we have chosen the mesh to be constant in time, the error indicators $\mathcal{E}_K^k$ contain the total contribution to the error from element K over the whole time interval I.

Algorithm 2 (Adaptive mesh refinement)**.** *Start at $k = 0$, then do*

(1) *compute approximation to the primal problem (2.4) on $\mathcal{T}_n^k$*
(2) *compute approximation to the dual problem (3.3) on $\mathcal{T}_n^k$*
(3) *if $\sum_{K \in \mathcal{T}_k} \mathcal{E}_K^k < TOL$ then STOP, else*
(4) *refine a fixed fraction of the elements in $\mathcal{T}_n^k$ with largest $\mathcal{E}_K^k \to \mathcal{T}_n^{k+1}$*
(5) *set $k = k + 1$, then goto (1)*

3.1. **A posteriori error estimation.** Algorithm 2 is based on a posteriori error estimates of the form (3.1), which we derive by introducing the following linearized dual problem: Find $(\varphi, \theta) \in L_2(I; [H^2_{\psi_3}(\Omega)]^3 \times H^2(\Omega))$, with $(\dot{\varphi}, \dot{\theta}) \in \mathcal{C}(I; [H^1(\Omega)]^3 \times L_2(\Omega))$ and $\varphi(T) = 0$, such that

$$\begin{aligned} \int_I -(v, \dot{\varphi}) + ((u^h \cdot \nabla) v + (v \cdot \nabla) U_h, \varphi) + (2\nu \epsilon(v), \epsilon(\varphi)) \\ -(q, \nabla \cdot \varphi) + (\nabla \cdot v, \theta)\, dt = \int_I (v, \psi_1) + (q, \psi_2)\, dt, \end{aligned} \tag{3.3}$$

for all $(v, q) \in L_2(I; [H^1_0(\Omega)]^3 \times L_2(\Omega))$ with $v(0) = 0$, given the data $\psi_1 \in L_2(I; [L_2(\Omega)]^3)$, $\psi_2 \in L_2(I; L_2(\Omega))$, and $\psi_3 \in L_2(I; [L_2(\partial\Omega)]^3)$, and where $(\nabla U_h \cdot \varphi)_j = (U_h)_{,j} \cdot \varphi$. We assume that there exists a unique solution to (3.3), and we note that for (3.3) to make sense we would only need that $(\varphi,\theta) \in L_2(I;[H^1_{\psi_3}(\Omega)]^3 \times L_2(\Omega))$ and $\dot\varphi \in L_2(I;[L_2(\Omega)]^3)$. The extra regularity is used for interpolation error estimates in the proof of Theorem 5.

The data ψ_i correspond to error estimates of different output functionals. For example, non zero ψ_1 and ψ_2 typically corresponds to point values or mean values of the velocity or the pressure respectively, see e.g. [19] for various examples of data corresponding to error estimates of different functionals. Here we want to compute the force on a body immersed in the fluid, and we are thus interested in computing an integral over the surface of this body. The corresponding data for the dual problem (3.3) is a non zero boundary condition ψ_3 on this surface $\Gamma_0 \subset \Gamma_D$. We introduce the following definitions:

$$
\begin{aligned}
&(v,w)_K = \int_K v\cdot w\ dx, \quad (v,w)_{\partial K} = \int_{\partial K} v\cdot w\ ds,\\
&\|v\|_K = (v,v)_K^{1/2}, \quad \|v\|_{\partial K} = (v,v)_{\partial K}^{1/2},\\
&|v|_K = (\|v_1\|_K, \|v_2\|_K, \|v_3\|_K), \quad |v|_{\partial K} = (\|v_1\|_{\partial K}, \|v_2\|_{\partial K}, \|v_3\|_{\partial K}),\\
&|v|_{K,\infty} = (\max_{\eta\in I_n}\|v_1(\eta)\|_K, \max_{\eta\in I_n}\|v_2(\eta)\|_K, \max_{\eta\in I_n}\|v_3(\eta)\|_K),\\
&|v|_{\partial K,\infty} = (\max_{\eta\in I_n}\|v_1(\eta)\|_{\partial K}, \max_{\eta\in I_n}\|v_2(\eta)\|_{\partial K}, \max_{\eta\in I_n}\|v_3(\eta)\|_{\partial K}),
\end{aligned} \tag{3.4}
$$

with the obvious simplifications for scalar functions v and w. To estimate interpolation errors over the space-time slabs $S_n = \Omega \times I_n$ in the proof of Theorem 5, we recall the following two lemmas from [18]:

Lemma 3. *For $(v,w) \in L_2(I_n;[L_2(\Omega)]^3 \times L_2(\Omega))$, $(\varphi,\theta) \in L_2(I_n;[H^2_0(\Omega)]^3 \times H^2(\Omega))$, with $(\dot\varphi,\dot\theta) \in \mathcal{C}(I_n;[L_2(\Omega)]^3 \times L_2(\Omega))$, and $(\Phi,\Theta) \in V_0^n \times W^n$ constant in time, we have*

$$
|\int_{I_n} (v,\varphi-\Phi)\ dt| \le \int_{I_n} \sum_{K\in\mathcal{T}_n} |v|_K \cdot (C^k_{n,K} k_n |\dot\varphi|_{K,\infty} + C^h_{n,K} h^2_{n,K} |D^2\varphi|_K)\ dt,
$$

$$
|\int_{I_n} (w,\theta-\Theta)\ dt| \le \int_{I_n} \sum_{K\in\mathcal{T}_n} |w|_K (C^k_{n,K} k_n |\dot\theta|_{K,\infty} + C^h_{n,K} h^2_{n,K} |D^2\theta|_K)\ dt,
$$

where $h_{n,K}$ is the diameter of element $K \in \mathcal{T}_n$, and D^2 measures second order derivatives with respect to x.

Lemma 4. *For $w \in L_2(I_n;[L_2(\Omega)]^3)$, $\varphi \in L_2(I_n;[H^2_0(\Omega)]^3)$, $\dot\varphi \in \mathcal{C}(I_n;[H^1(\Omega)]^3)$, and $\Phi \in V_0^n$ constant in time, we have*

$$
\begin{aligned}
&|\int_{I_n} \sum_{K\in\mathcal{T}_n} \int_{\partial K\setminus\partial\Omega} w\cdot(\varphi-\Phi)\ ds\ dt|\\
&\le \int_{I_n} \sum_{K\in\mathcal{T}_n} |w|_{\partial K\setminus\partial\Omega} \cdot (C^k_{n,K} k_n |\dot\varphi|_{\partial K\setminus\partial\Omega,\infty} + C^h_{n,K} h^{3/2}_{n,k} |D^2\varphi|_K)\ dt,
\end{aligned}
$$

where $h_{n,K}$ is the diameter of element $K \in \mathcal{T}_n$, and D^2 measures second order derivatives with respect to x.

In the rest of this paper we will refer to the *discretization error* as the error we get when approximating (2.5) by (2.6) assuming the subgrid model to be exact, and the *modeling error* as the error from the approximation $\hat{\tau}^h(U_h) \approx \tau^h(u)$ in (2.6). We note that for the discretization error we have a *Galerkin orthogonality property* (see e.g. [8]), which enables us to sharpen the a posteriori error estimates for this error. This is not the case for the modeling error, which may result in less sharp estimates for this error.

For simplicity, we present the a posteriori error estimate for the cG(1)cG(1) method with $\delta_1 = \delta_2 = 0$, which thus introduces an error in the a posteriori error estimates depending on the stabilization parameters δ_1 and δ_2. For the case $\delta_1, \delta_2 \neq 0$, we would adjust the dual problem (3.3) to be the transposition of the linearized variational form corresponding to the stabilized method, which is beyond the scope of this paper. In [23], the choices of different dual problems for stabilized finite element methods are investigated in the case of linear problems.

Theorem 5. *If u^h solves (2.4), (U_h, P_h) solves (2.6) with $\delta_1 = \delta_2 = 0$, and (φ, θ) solves (3.3) with data $\psi_1 = \psi_2 = \psi_3|_{\Gamma_1} = 0$ and $\psi_3|_{\Gamma_0} = \phi$, then*

$$|N(\sigma(u^h, p^h)) - N^h(\sigma(U_h, P_h))| \leq \sum_{K \in \mathcal{T}_n} \mathcal{E}_K = \sum_{K \in \mathcal{T}_n} (e_D^K + e_M^K) = e_D + e_M,$$

where e_D and e_M are the discretization and modeling errors respectively, defined by

$$e_D^K = \frac{1}{|I|} \sum_{n=1}^{N} \int_{I_n} |R_1(U_h, P_h)|_K \cdot \omega_1 + |R_2(U_h)|_K \ \omega_2 + R_3(U_h) \cdot \omega_3 \ dt,$$
$$e_M^K = \frac{1}{|I|} \sum_{n=1}^{N} \int_{I_n} |R_4(u, U_h)|_K \cdot \omega_4 + R_5(U_h) \cdot \omega_5 \ dt,$$

with the residuals

$$\begin{aligned}
R_1(U_h, P_h) &= \dot{U}_h + (U_h \cdot \nabla) U_h + \nabla P_h - \nu \Delta U_h + \nabla \cdot \hat{\tau}^h(U_h) - f^h, \\
R_2(U_h) &= \nabla \cdot U_h, \\
R_3(U_h) &= \frac{1}{2} \max_{S \subset \partial K} (|[(\nu \nabla U_h - \hat{\tau}^h(U_h))_1 \cdot n_S]|, ..., |[(\nu \nabla U_h - \hat{\tau}^h(U_h))_3 \cdot n_S]|), \\
R_4(u, U_h) &= \nabla \cdot (\tau^h(u) - \hat{\tau}^h(U_h)), \\
R_5(U_h) &= \frac{1}{2} \max_{S \subset \partial K} (|[(\hat{\tau}^h(U_h))_1 \cdot n_S]|, ..., |[(\hat{\tau}^h(U_h))_3 \cdot n_S]|),
\end{aligned}$$

where $(M)_i$ denotes the i:th row of the matrix M and $[\cdot]$ denotes the jump over interior element boundaries $\partial K \setminus \partial\Omega$, and the dual weights

$$\begin{aligned}
\omega_1 &= C^k_{n,K} k_n |\dot\varphi|_{K,\infty} + C^h_{n,K} h^2_{n,K} |D^2\varphi|_K,\\
\omega_2 &= C^k_{n,K} k_n |\dot\theta|_{K,\infty} + C^h_{n,K} h^2_{n,K} |D^2\theta|_K,\\
\omega_3 &= C^k_{n,K} k_n |\dot\varphi|_{\partial K\setminus\partial\Omega,\infty} + C^h_{n,K} h^{3/2}_{n,k} |D^2\varphi|_K,\\
\omega_4 &= |\varphi|_K,\\
\omega_5 &= |\varphi|_{\partial K\setminus\partial\Omega,\infty},
\end{aligned}$$

where $h_{n,K}$ is the diameter of element $K \in \mathcal{T}_n$, D^2 measures second order derivatives with respect to x, and $C^h_{n,K}, C^k_{n,K}$ represent interpolation constants.

Proof. To derive a posteriori error estimates for $N(\sigma(u^h,p^h))$, the natural quantity to consider is the difference between (2.8) and (2.9), see [14, 19]. If we set $(\varphi,\theta) = (\Phi,\Theta) \in L_2(I; V^n_{\phi,0} \times W^n)$ in (2.8) and then subtract (2.9), we get

$$\begin{aligned}
(3.5)\quad & N(\sigma(u^h,p^h)) - N^h(\sigma(U_h,P_h))\\
= \quad & \frac{1}{|I|}\int_I (\dot u^h + u^h\cdot\nabla u^h, \Phi) - (p^h, \nabla\cdot\Phi) + (2\nu\epsilon(u^h), \epsilon(\Phi)) - (\tau^h(u), \nabla\Phi) + (\nabla\cdot u^h, \Theta)\\
& -((\dot U_h + U_h\cdot\nabla U_h, \Phi) - (P_h, \nabla\cdot\Phi) + (2\nu\epsilon(U_h), \epsilon(\Phi)) - (\hat\tau^h(U_h, \nabla\Phi) + (\nabla\cdot U_h, \Theta))\; dt.
\end{aligned}$$

The dual problem (3.3) with data

$$\psi_1 = \psi_2 = \psi_3|_{\Gamma_1} = 0, \quad \psi_3|_{\Gamma_0} = \phi, \tag{3.6}$$

and ϕ from (2.7), gives that

$$\begin{aligned}
& \frac{1}{|I|}\int_I (\dot u^h + u^h\cdot\nabla u^h, \varphi) - (p^h, \nabla\cdot\varphi) + (2\nu\epsilon(u^h), \epsilon(\varphi)) - (\tau^h(u), \nabla\varphi) + (\nabla\cdot u^h, \theta)\\
& -((\dot U_h + U_h\cdot\nabla U_h, \varphi) - (P_h, \nabla\cdot\varphi) + (2\nu\epsilon(U_h), \epsilon(\varphi)) - (\hat\tau^h(U_h), \nabla\varphi) + (\nabla\cdot U_h, \theta)\; dt\\
= \quad & \frac{1}{|I|}\int_I -(\dot\varphi, e) + (u^h\cdot\nabla e + e\cdot\nabla U_h, \varphi) - (p^h - P_h, \nabla\cdot\varphi) + (2\nu\epsilon(e), \epsilon(\varphi))\\
(3.7)\quad & +(\nabla\cdot e, \theta) + (\hat\tau^h(U_h) - \tau^h(u), \nabla\varphi)\; dt = \frac{1}{|I|}\int_I (\hat\tau^h(U_h) - \tau^h(u), \nabla\varphi)\; dt,
\end{aligned}$$

using partial integration with $\varphi(T) = e(0) = 0$, where $e = u^h - U_h$, and that $(u^h\cdot\nabla)u^h - (U_h\cdot\nabla)U_h = (u^h\cdot\nabla)e + (e\cdot\nabla)U_h$. By (2.5), (3.5), and (3.7), we then have that

$$\begin{aligned}
& N(\sigma(u^h,p^h)) - N^h(\sigma(U_h,P_h))\\
= \quad & \frac{1}{|I|}\int_I ((\dot U_h + U_h\cdot\nabla)U_h - f^h, \varphi - \Phi) - (P_h, \nabla\cdot(\varphi-\Phi)) + (\nabla\cdot U_h, \theta - \Theta)\\
(3.8)\quad & +(2\nu\epsilon(U_h) - \hat\tau^h(U_h), \nabla(\varphi-\Phi)) + (\hat\tau^h(U_h) - \tau^h(u), \nabla\varphi)\; dt.
\end{aligned}$$

From this *error representation formula* there are various possibilities to estimate the integrals in (3.8).

Integration by parts in the viscous term results in non zero boundary integrals over interior element boundaries $\partial K \setminus \partial\Omega$, for each t, since ∇U is piecewise constant in x over the elements and thus discontinuous over interior element boundaries, and we have the same problem for the subgrid model $\hat{\tau}^h(U_h)$. This is not the case for the pressure term since the pressure is continuous in x over element boundaries, and so is $\varphi - \Phi$ and $\tau^h(u)$.

To estimate these element boundary integrals we use a standard finite element teqnique, see e.g. [9], where we first rewrite the sum of interior element boundary integrals as a sum of jumps in normal derivative of the form $[\nu\nabla U_h \cdot n_S]$ over all interior faces S in $\mathcal{T}_n$, with n_S being a globally defined unit normal vector associated with the face S. We then attribute half of the jump to each of the two elements sharing the face and rewrite the sum again over the elements $K \in \mathcal{T}_n$, to get

$$\begin{aligned}
&|N(\sigma(u^h,p^h)) - N^h(\sigma(U_h,P_h))| \\
&\leq \frac{1}{|I|}\sum_{n=1}^{N}\int_I \{\ |(\dot{U}_h + (U_h\cdot\nabla)U_h + \nabla P_h - \nu\Delta U_h + \nabla\cdot\hat{\tau}^h(U_h) - f^h, \varphi - \Phi)| \\
&+ |(\nabla\cdot U_h, \theta - \Theta)| + |\sum_{K\in\mathcal{T}_n}\int_{\partial K\setminus\partial\Omega}\frac{1}{2}[(\nu\nabla U_h - \hat{\tau}^h(U_h))\cdot n_S]\cdot(\varphi - \Phi)\ ds| \\
&+ |(\nabla\cdot(\tau^h(u) - \hat{\tau}^h(U_h)), \varphi)| + |\sum_{K\in\mathcal{T}_n}\int_{\partial K\setminus\partial\Omega}\frac{1}{2}[\hat{\tau}^h(U_h)\cdot n_S]\cdot\varphi\ ds|\ \}\ dt.
\end{aligned}$$

We finally use Cauchy-Schwarz inequality for each element, and then Lemma 3 and Lemma 4 to estimate the interpolation errors.

□

Remark 6. *The modification of Theorem 5 for the case of inhomogeneous Dirichlet boundary conditions on part of the boundary, such as a given inflow velocity, is straight forward by incorporating this boundary condition in the corresponding trial spaces.*

3.2. **Remarks on the linearization error in the dual problem.** The a posteriori error estimate in Theorem 5 is designed to be useful in an adaptive algorithm as a stopping criterion, a refinement criterion for the space and time discretization, and an error indicator for the subgrid model. To evaluate the error bounds in Theorem 5 we approximate the dual weights ω_i numerically, by computing approximate solutions to the dual problem (3.3). In this paper we solve the dual problem using the cG(1)cG(1) method on the same computational mesh as we use for the primal problem, which is neither necessary nor optimal but is chosen here for reasons of simplicity. In the computation of the dual problem we do not have access to u^h, the solution of (2.4). Instead we approximate u^h by U_h, a finite element approximation of u^h, which thus introduces a linearization error $u^h - U_h$ in the dual problem.

Here we make the assumption that U_h converges to u^h pointwise and that thus the linearization error converges pointwise to zero. Such an assumption gives some justification for using Theorem 5, although there may still be problems when the computational mesh is not fine enough, and/or the subgrid model is not accurate enough. Practical experience from using this type of a posteriori error estimates for adaptive mesh refinement for various problems has been positive, with effective mesh refinement criterions and sharp a posteriori error estimates, see e.g. [19, 4] for examples of incompressible flow.

Alternative forms of the linearized dual problem (3.3) are possible. For example, one may linearize the dual problem at u, the exact solution of (2.1). Although, this is not practical for numerical approximation of the dual problem since the corresponding linearization error $u-U_h$ can never be pointwise small in a LES. Typically the error $u-U_h$ is large pointwise, since u contains finer scales than U_h.

4. Numerical results

We now use Algorithm 2 to compute the mean drag coefficient for a surface mounted cube in a turbulent channel flow. We use the cG(1)cG(1) method for both the primal and the dual problem, on tetrahedral meshes $\mathcal{T}_n^k$. In the definition of the averaging operator (2.3) corresponding to the LES for the adaptive step k, we let $h = h(x)$ be defined to be the piecewise constant function that equals the diameters of the tetrahedrons in $\mathcal{T}_n^k$.

We use no subgrid model in the computations, but we use the following scale similarity subgrid model from [29] to estimate the size of the modeling residual $R_4(u,U_h) = \nabla\cdot\tau^h(u)$:

$$\tau_{ij}^h(u) \approx C_L\tau_{ij}^{2h}(u^h) = C_L((u_i^h u_j^h)^{2h} - (u_i^h)^{2h}(u_j^h)^{2h}), \tag{4.1}$$

with $C_L = 1$, and u^h approximated by U_h in the computations.

4.1. **Flow around a surface mounted cube.**

We consider the problem of a turbulent flow around a surface mounted cube, investigated in [30, 31, 28]. In our computational model we use the Navier-Stokes equations to model the incompressible fluid around a cubic body of dimension $H\times H\times H$ that sits on the floor of a rectangular channel of length $15H$, height $2H$, and width $7H$, centered at $(3.5H, 0.5H, 3.5H)$. At the inlet we use a velocity profile interpolated from experiments, we use no slip boundary conditions on the body and the vertical boundaries, slip boundary conditions on the lateral boundaries, and a transparent outflow boundary condition. The viscosity ν is chosen to give a Reynolds number $Re = U_bH/\nu = 40.000$, where we have used $U_b = 1.0$.

4.2. **Adaptive computation of the mean drag coefficient.**

In Figure 1 we show a snapshot of the solution and the corresponding computational mesh after 13 adaptive mesh refinements, using Algorithm 2 to compute an approximation of the mean drag coefficient $\bar{c}_D$ over a time interval $I = [T_0, T]$, with $T_0 = 10$ and $T = 20$, defined by

$$\bar{c}_D = \frac{1}{|T-T_0|}\int_{T_0}^{T} c_D(t)\ dt, \tag{4.2}$$

where $c_D(t)$ is the drag coefficient at time t, and $\bar{c}_D = N(\sigma(u^h,p^h))\times 2/(H^2U_b^2)$. To estimate the computational cost of computing an approximation of $\bar{c}_D$ we study the solution of the corresponding dual problem with data according to (3.6), and $\phi = (2/(H^2U_b^2), 0, 0)$ on the surface of the cube. One way to estimate the computational cost is through the

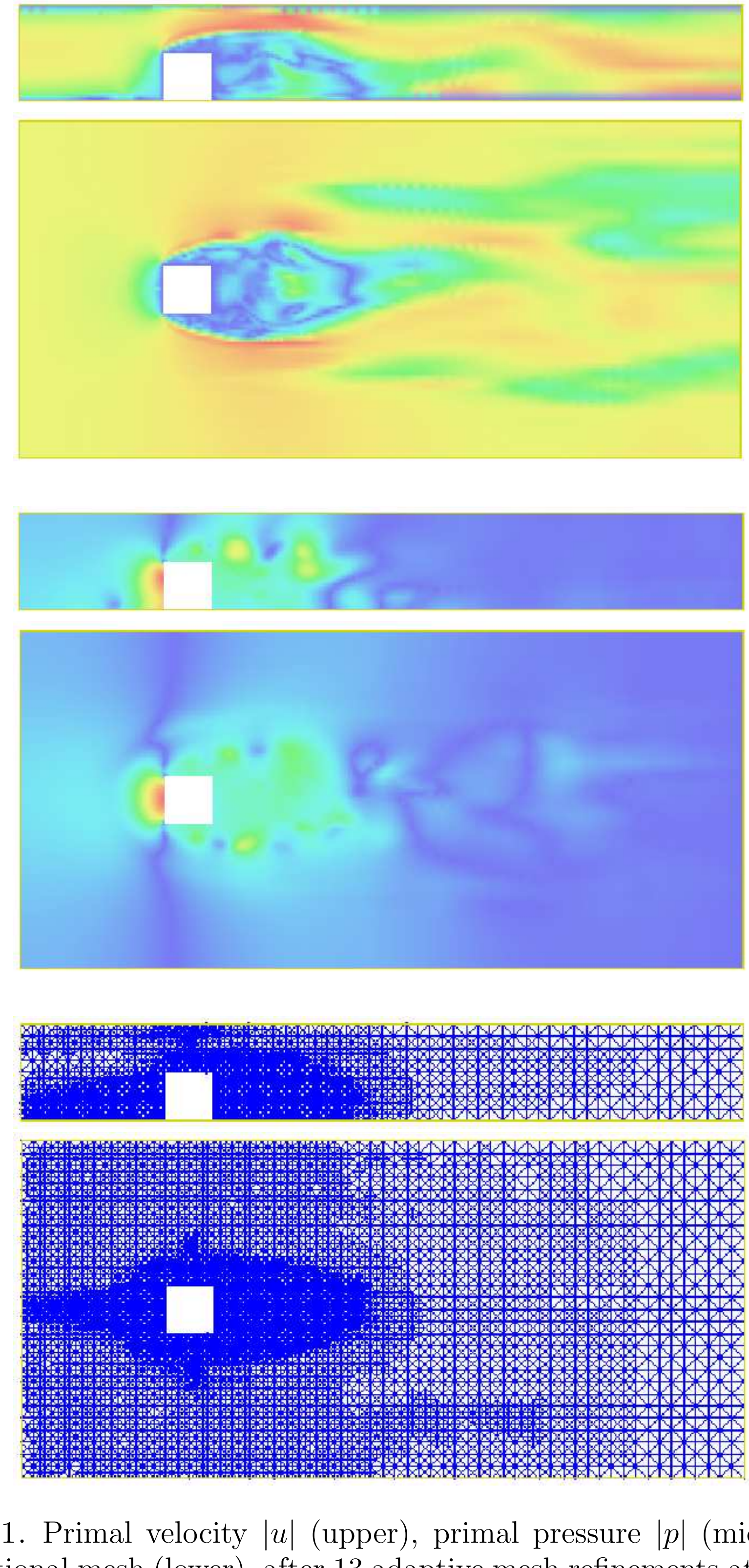

FIGURE 1. Primal velocity $|u|$ (upper), primal pressure $|p|$ (middle), and computational mesh (lower), after 13 adaptive mesh refinements at $z = 3.5H$ and $y = 0.5H$.

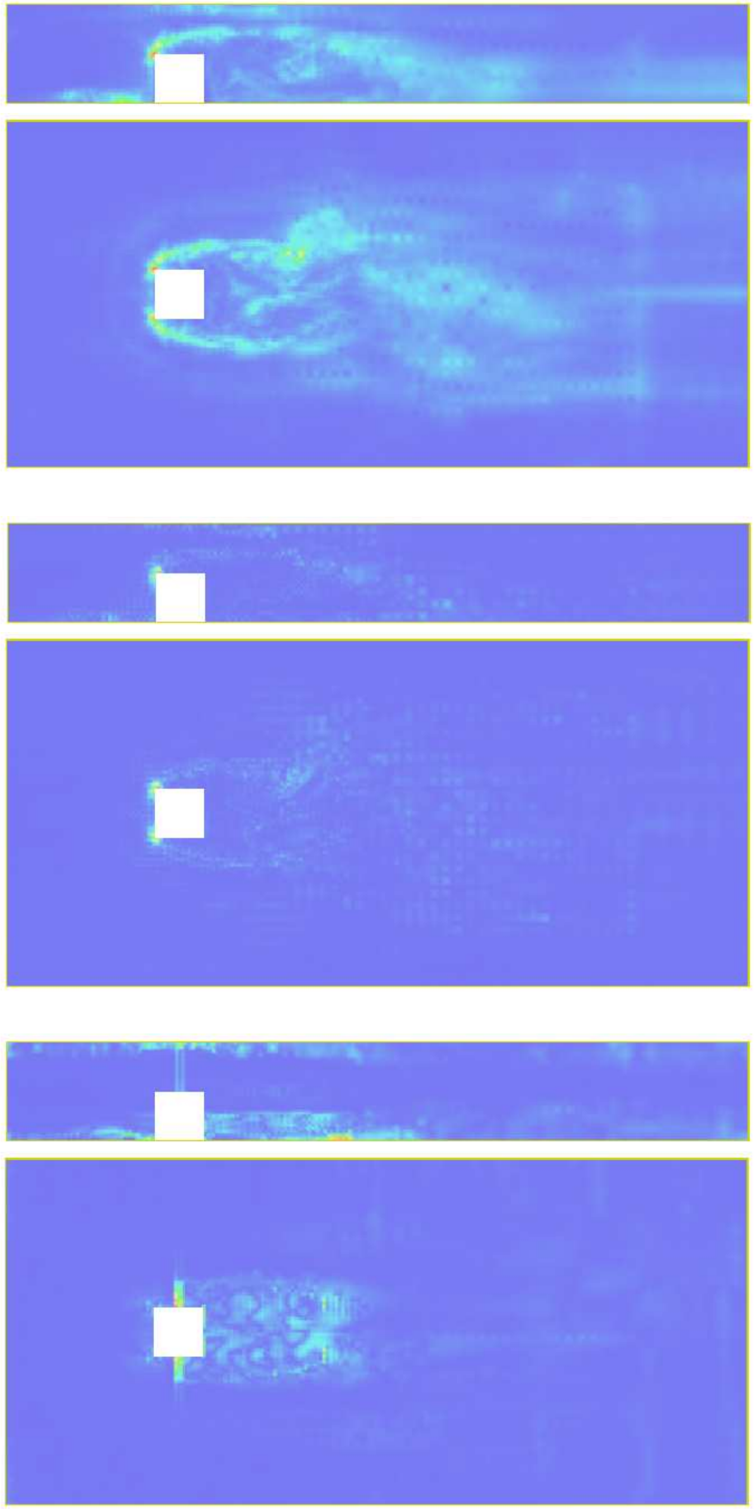

FIGURE 2. Residuals $|R_1(U_h, P_h)|$ (upper), $|R_2(U_h)|$ (middle), and $|R_4(u, U_h)|$ (lower), after 13 adaptive mesh refinements at $z = 3.5H$ and $y = 0.5H$.

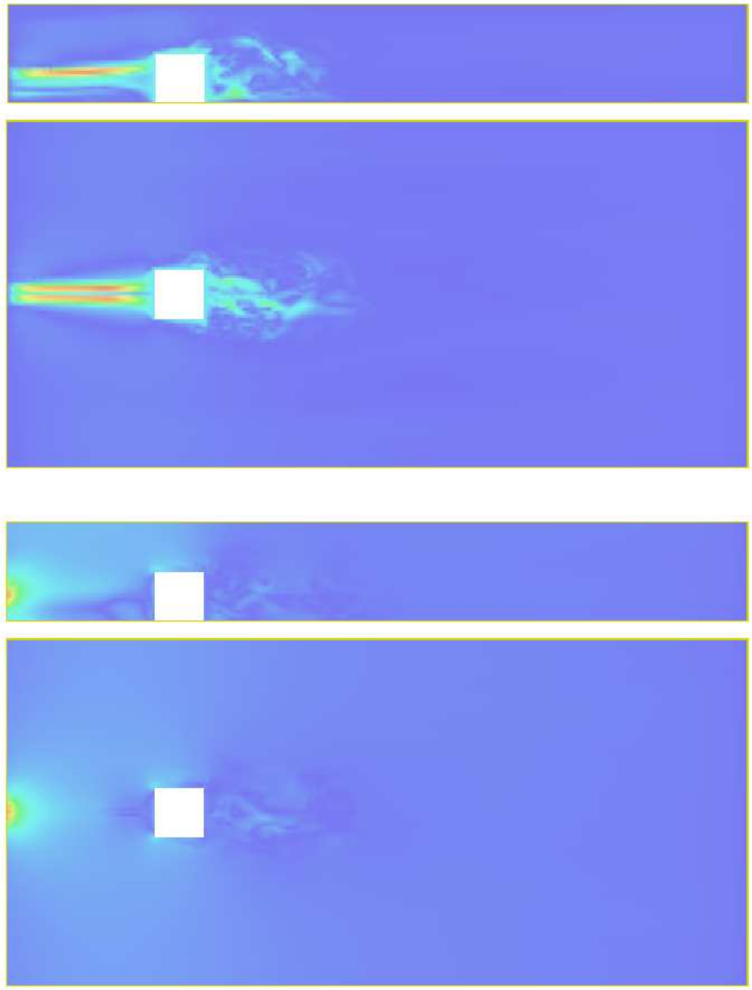

FIGURE 3. Dual velocity $|\varphi|$ (upper) and dual pressure $|\theta|$ (lower), after 13 adaptive mesh refinements at $z = 3.5H$ (upper) and $y = 0.5H$ (lower).

computation of *stability factors*, see e.g. [19]. For example, in Figure 5 we plot the stability factor $S_{1,1}(T_0)$, defined by

$$S_{1,1}(T_0) = \frac{1}{|T - T_0|} \int_{T_0}^{T} \int_{\Omega} |\varphi(x,t)| \; dx \; dt, \tag{4.3}$$

where φ is the solution of the dual problem (3.3) with data as above. We find that the computational cost at first increases with the length of the time interval $[T_0, T]$ (T fix, T_0 varying), but when the interval exceeds a certain length the computational cost does not increase significally beyond a certain level. Thus the computational cost of computing $\bar{c}_D$ is relatively constant for time intervals longer than a certain length in this problem.

The error indicator $\mathcal{E}_K^k$ in Algorithm 2, giving the computational mesh in Figure 1, depends on the product of the residuals and the dual solution. In Figure 2 we show a

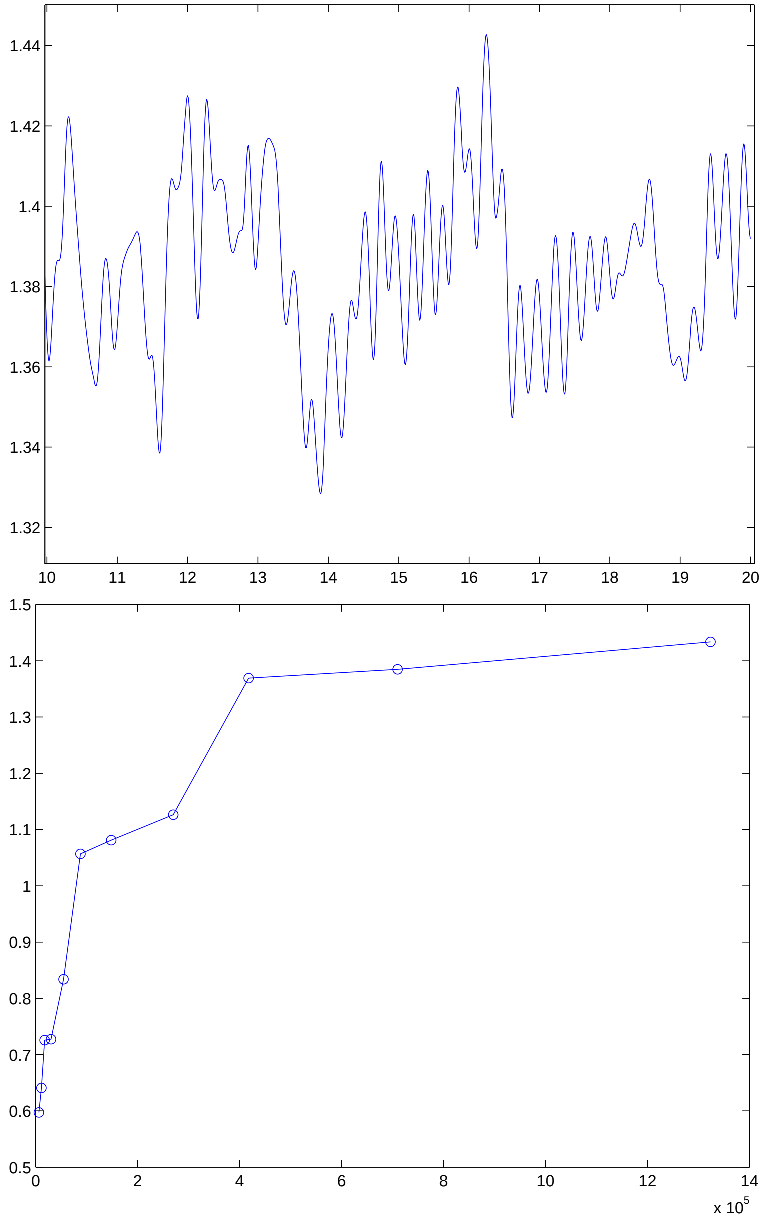

FIGURE 4. Drag coefficient $\bar{c}_D$ as a function of time, after 13 adaptive mesh refinements (upper), and mean drag coefficient $\bar{c}_D$ over the time interval $[10, 20]$, as a function of the number of degrees of freedom (lower).

snapshot of the residuals after 13 adaptive mesh refinements, and in Figure 3 we show a snapshot of the corresponding dual solution.

We find that the discretization residuals $R_1(U_h, P_h)$, corresponding to the momentum equation, and $R_2(U_h)$, corresponding to the continuity equation, are large at the upstream corners of the cube and along the high velocity streaks around the cube, and the modeling

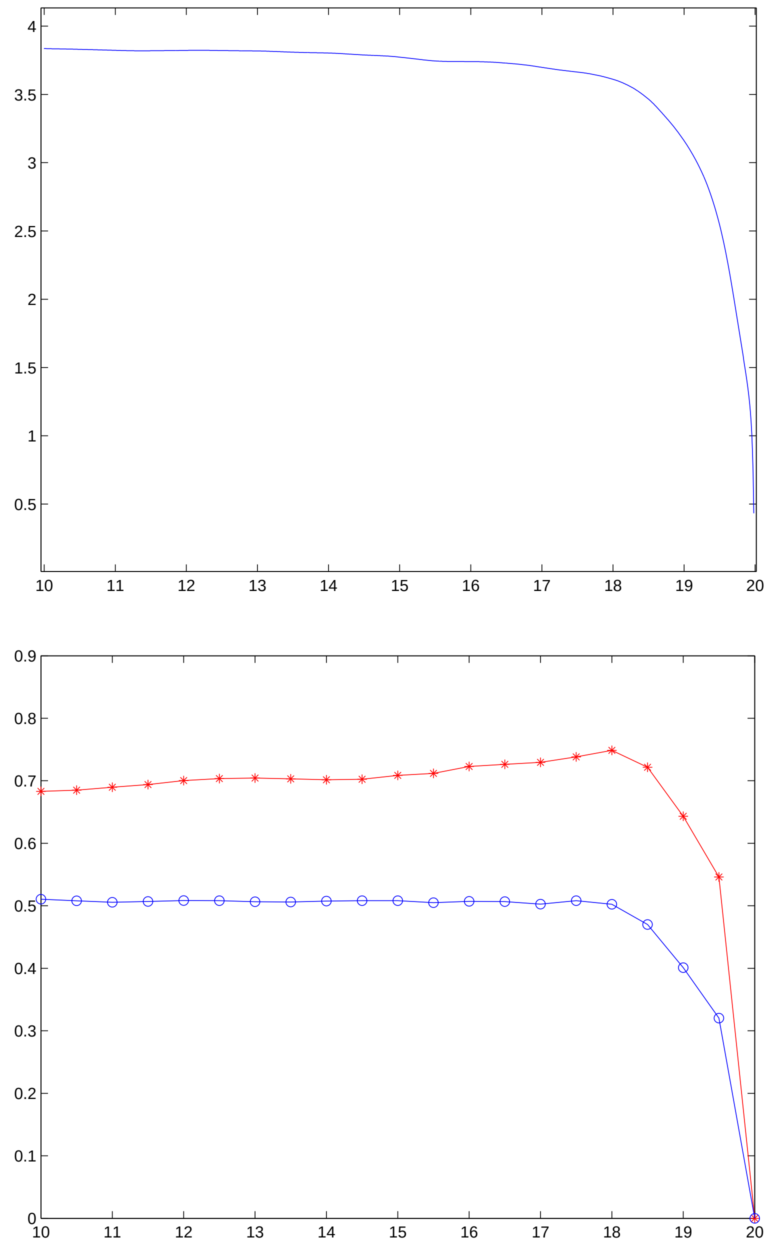

FIGURE 5. Stability factor $S_{1,1}(T_0)$ (upper), and discretization error e_D ($'o'$) and modeling error e_M ($'*'$) (lower), after 13 adaptive mesh refinements as functions of the length of the time interval $[T_0, T]$, with T fix and T_0 varying, assuming $U_h(T_0) = u^h(T_0)$.

residual $R_4(u, U_h)$ is large on the top and the bottom of the channel and in the recirculation zone downstream the cube. The dual velocity, indicating domain of influence for $R_1(U_h, P_h)$ and $R_4(u, U_h)$, is large upstream the cube, along the boundary of the cube, and

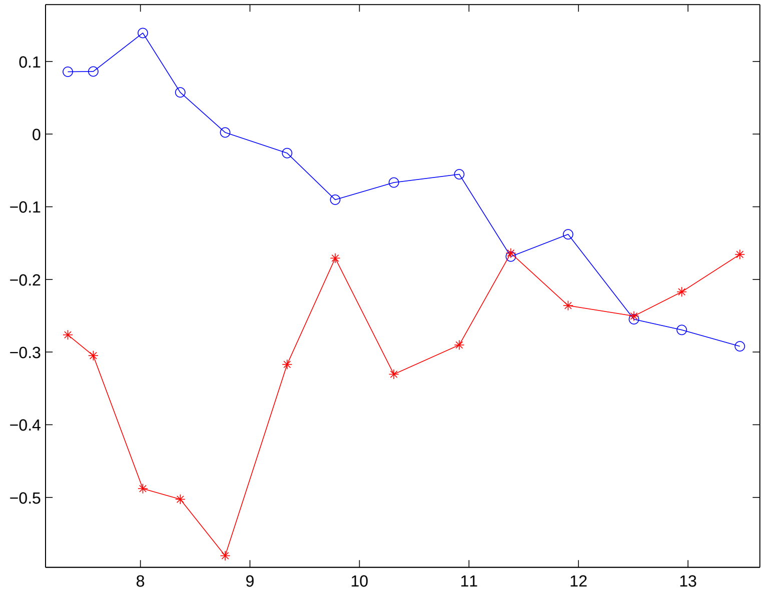

FIGURE 6. A posteriori error estimates of the discretization error e_D ($'o'$) and the modeling error e_M ($'*'$) for the time interval $[10, 20]$, as functions of the number of degrees of freedom in a log_{10}-log_{10} plot.

in the recirculation zone downstream the cube, and the dual pressure, indicating domain of influence for $R_2(U_h)$, is large near the inlet as well as near the upstream corners of the cube.

In Figure 4 we plot the corresponding drag coefficient as a function of time, and the mean drag coefficient over the time interval $[10, 20]$ as a function of the number of degrees of freedom. We find that even though we do not reach full convergence using the avaliable number of degrees of freedom, the value for the mean drag coefficient seems to asymptotically approach a value between 1.45-1.5.

We know of no experimental reference values of $\bar{c}_D$, but in [28] $\bar{c}_D$ is approximated computationally. The computational setup is similar to the one in this paper except the numerical method, a different length of the time interval, and that we in this paper use a channel of length $15H$, compared to a channel of length $10H$ in [28]. Using different meshes and subgrid models, approximations of $\bar{c}_D$ in the interval $[1.14, 1.24]$ are presented in [28].

After 13 adaptive mesh refinements we have $\sim 1.3 \cdot 10^6$ degrees of freedom, using about 1.5 GB of memory on a regular PC. For $H = 0.1$, the diameter of the smallest element in the mesh $\mathcal{T}_n^{14}$ is about 10^{-3}, which corresponds to a local Reynolds number $Re_{loc} \approx (2H/h)^{4/3} \approx 1200$ (with channel height $2H$), using standard Kolmogorov arguments of turbulent flow [12], or $Re_{loc} \approx h^{-1} = 1000$, assuming the numerical viscosity of the cG(1)cG(1) method is acting as a term $\delta_1(\nabla U_h, \nabla U_h)$ with $\delta_1 \sim h$. That is, we are locally able to resolve scales corresponding to a Reynolds number of about 1000, even though it would be impossible to do globally to a similar computational cost. Since turbulence often is a local phenomena, adaptive methods are ideal for computation of turbulence. In theory, if we refine the same elements in each step of the algorithm we would get a finest $h \approx H \times (1/2)^{13} \approx 10^{-5}$, corresponding to $Re_{loc} \approx 10^5$. *That is, we would be able to locally resolve flows corresponding to a Reynolds number of* 10^5 *in a Direct Numerical Simulation using an ordinary PC or laptop computer.*

4.3. **A posteriori error estimates.** After 13 adaptive mesh refinements we plot the a posteriori error estimates of the discretization and the modeling errors in Figure 5 as functions of the length of the time interval $[T_0, T]$ (T fix, T_0 varying), where we have assumed that the initial solution is exact for each T_0, so that $U_h(T_0) = u^h(T_0)$. We note the similarity to the plot of the stability factor $S_{1,1}(T_0)$, and we find that the error estimates give rather large bounds for the error in the case of longer time averages.

In Figure 6 we plot the discretization and the modeling errors for $\bar{c}_D$ over the time interval $[10, 20]$ as functions of the number of degrees of freedom, where we note an expected decrease in the estimate of the discretization error as we refine the mesh. We also note that the estimate of the modeling error on the other hand increases. This might at first seem alarming, but is in fact to be expected since in this case we have used the simple model (4.1) to estimate the Reynolds stresses in the modeling residual. Even though the true Reynolds stresses are smaller for a finer resolution h of the problem, the model (4.1) will in fact first increase as we resolve more scales of motion since it is solely based on the resolved velocity fluctuations on the scale $2h$, and since we are not using any subgrid model in the computations the estimate of the modeling residual will also increase. This is of course a problem, and in a continuation of this study we seek sharper estimates of the Reynolds stresses based on scale extrapolation.

Remark 7. *The use of a stabilized Galerkin finite element method in the computations may be viewed as a type of subgrid model in itself, since we then in fact solve a modified set of equations using a standard Galerkin method. We will further investigate this relation between numerical stabilization and subgrid modeling in a continuation of this work. In this paper we only consider the stabilization to be part of the numerical method and not an explicit subgrid model. This means in particular that in fact the modeling residual may be overestimated since we do not subtract the contribution from this implicit subgrid model.*

Remark 8. *In each step of Algorithm 2 we mark* $\sim 50\%$ *of the elements for refinement. To get a consistent mesh we have to refine additional elements, so the exact fraction of the elements that are refined each step vary. For more details on mesh refinement algorithms we refer to* [5]*, and the references therin.*

Remark 9. *We have assumed* $R_3(U, P)$ *to be neglible compared to the other residuals in the computations, due to the multiplication by* ν.

Remark 10. *In the computations we use a slightly different form of the dual problem (3.3), obtained by partial integration in the term* $((u^h \cdot \nabla)v, \varphi)$, *giving instead* $-((u^h \cdot \nabla)\varphi, v)$. *For pure Dirichlet boundary conditions the two forms are equivalent, but in the case of Neumann boundary conditions the difference is a surface integral of* $(u\varphi) \cdot n$ *multiplied by*

v over the Neumann part of the boundary. In the computations we have a Neumann type outflow boundary condition, but we find that φ is close to zero on this outflow boundary, see Figure 3, and we thus consider our computational approximation of the dual problem to be justified.

Remark 11. *We have used $C^k_{n,K} = 1/2$ and $C^h_{n,K} = 1/8$ as constant approximations of the interpolation constants in Theorem 5. These values are motivated by simple analysis on reference elements.*

5. Summary

In this paper we have proved a posteriori error estimates, and used a corresponding adaptive finite element method to compute approximations of the temporal mean of the drag coefficient in a turbulent flow around a surface mounted cube. The a posteriori error estimates, based on the solution of an associated linearized dual problem, are used to estimate the computational cost associated with the approximation of the mean drag coefficient and as error indicators for the adaptive mesh refinement algorithm.

We emphasize the local nature of turbulence in this problem that makes adaptive methods ideal for efficient and accurate computations. Due to the computational goal of approximating the mean drag coefficient we refine the mesh according to the corresponding a posteriori error estimate, resolving scales of motion corresponding to local Reynolds numbers of about 1000, and in theory we would be able to resolve local scales of motion corresponding to local Reynolds numbers of the order 10^5 to a similar computational cost.

In continuations of this study we will address methods for sharp estimation of the modeling residual, as well as adaptive strategies to combine numerical stabilization with subgrid modeling for LES.

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