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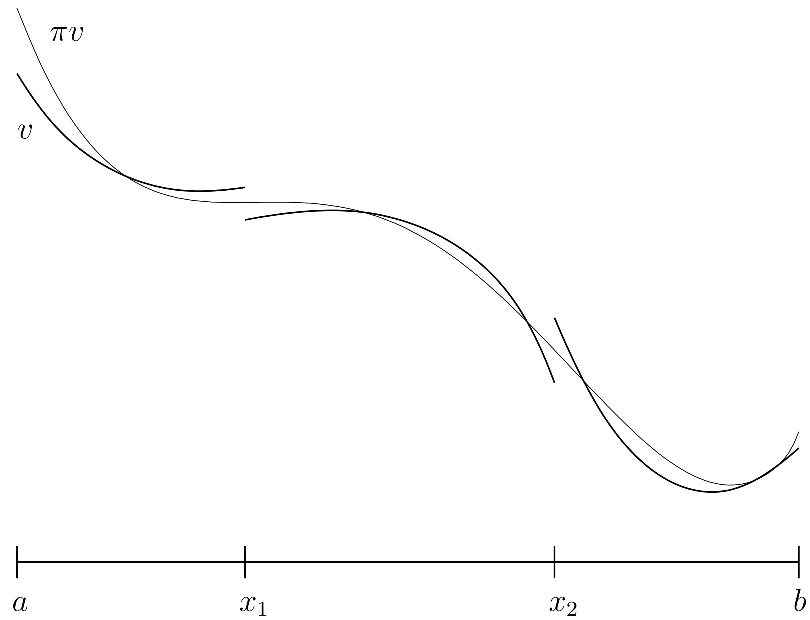

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INTERPOLATION ESTIMATES FOR PIECEWISE SMOOTH FUNCTIONS IN ONE DIMENSION

ANDERS LOGG

Abstract. In preparation for a priori error analysis of the multi-adaptive Galerkin methods mcG(q) and mdG(q) presented earlier in a series of papers, we prove basic interpolation estimates for a pair of carefully chosen non-standard interpolants. The main tool in the derivation of these estimates is the Peano kernel theorem. A large part of the paper is non-specific and applies to general interpolants on the real line.

1. Introduction

The motivation for this paper is to prepare for the a priori error analysis of the multi-adaptive Galerkin methods mcG(q) and mdG(q), presented earlier in [1, 2]. This requires a set of special interpolation estimates for piecewise smooth functions, which we prove in this paper.

Throughout this paper, V denotes the space of piecewise smooth, real-valued functions on $[a,b]$, that is, the set of functions which, for some partition $a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b$ of the interval $[a,b]$ and some $q \geq 0$, are $\mathcal{C}^{q+1}$ and bounded on each of the sub-intervals (x_{i-1}, x_i), $i = 1, \ldots, n+1$. This is illustrated in Figure 1.

For $v \in V$, we denote by πv a polynomial approximation of v on $[a,b]$, such that $\pi v \approx v$. We refer to πv as an *interpolant* of v.

We are concerned with estimating the size of the interpolation error $\pi v - v$ in the maximum norm, $\|\cdot\| = \|\cdot\|_{L_\infty([a,b])}$, in terms of the regularity of v and the length of the interval, $k = b - a$. Specifically, when $v \in \mathcal{C}^{q+1}([a,b]) \subset V$ for some $q \geq 0$, we obtain estimates of the form

$$\|(\pi v)^{(p)} - v^{(p)}\| \leq C k^{q+1-p} \|v^{(q+1)}\|, \quad p = 0, \ldots, q+1. \tag{1.1}$$

In the general case, the interpolation estimates include also the size of the jump $[v^{(p)}]_x$ in function value and derivatives at the points of discontinuity within (a,b).

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Key words and phrases. Multi-adaptivity, individual time steps, local time steps, ODE, continuous Galerkin, discontinuous Galerkin, mcgq, mdgq, a priori error estimates, Peano kernel theorem, interpolation estimates, piecewise smooth.

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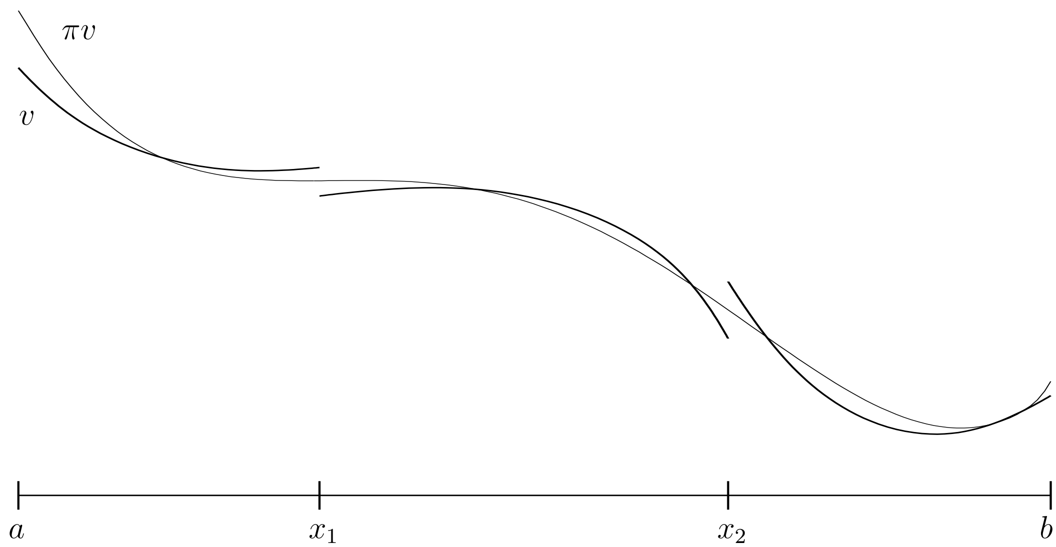

FIGURE 1. A piecewise smooth function v and its interpolant πv.

1.1. **Outline of the paper.** We first assume that $v \in \mathcal{C}^{q+1}([a,b]) \subset V$ and use the Peano kernel theorem to obtain a representation of the interpolation error $\pi v - v$ (Section 2). We then directly obtain interpolation estimates (for $v \in \mathcal{C}^{q+1}([a,b])$) in Section 3.

In Section 4, we generalize the interpolation estimates from Section 3 to v piecewise smooth by constructing a regularized version of v. Finally, in Section 5, we apply the general results of Section 4 to a pair of special interpolants that appear in the a priori error analysis of the mcG(q) and mdG(q) methods.

2. The Peano kernel theorem

The basic tool in our derivation of interpolation estimates is the Peano kernel theorem, which we discuss in this section. In its basic form, the Peano kernel theorem can be stated as follows.

Theorem 2.1. (Peano kernel theorem) *For Λ a bounded and linear functional on V that vanishes on $\mathcal{P}^q([a,b]) \subset V$, define*

$$K(t) = \frac{1}{q!}\Lambda((\cdot - t)_+^q), \tag{2.1}$$

where $v_+(t) = \max(0, v(t))$. In other words, $K(t) = \Lambda w/q!$, with $w(x) = [\max(0, x-t)]^q$. If K has bounded variation and $v \in \mathcal{C}^{q+1}([a,b])$, then

$$\Lambda v = \int_a^b K(t) v^{(q+1)}(t)\, dt. \tag{2.2}$$

Proof. See [3]. □

Let now $v \in V$ and let

$$\pi : V \to \mathcal{P}^q([a,b]) \subset V. \tag{2.3}$$

If we can show that

(1) π is linear on V,
(2) π is exact on $\mathcal{P}^q([a,b]) \subset V$, that is, $\pi v = v$ for all $v \in \mathcal{P}^q([a,b])$, and
(3) π is bounded on V, that is, $\|\pi v\| \leq C\|v\|$ for all $v \in V$ for some constant $C > 0$,

then the Peano kernel theorem directly leads to a representation of the interpolation error $\pi v - v$.

Theorem 2.2. (Peano kernel theorem II) *If π is linear and bounded (with constant $C > 0$) on V, and is exact on $\mathcal{P}^q([a,b]) \subset V$, then there is a bounded function $G : [a,b] \times [a,b] \to \mathbb{R}$, with $|G(x,t)| \leq 1 + C$ for all $x, t \in [a,b]$, such that for all $v \in \mathcal{C}^{q+1}([a,b])$,*

$$\pi v(x) - v(x) = \frac{k^{q+1}}{q!} \frac{1}{k} \int_a^b v^{(q+1)}(t) G(x,t)\, dt, \tag{2.4}$$

where $k = b - a$. Furthermore, if g is an integrable, (essentially) bounded function on $[a,b]$, then there is a function $H_g : [a,b] \to \mathbb{R}$, with $|H_g(x)| \leq (1 + C)\|g\|$ for all $x \in [a,b]$, such that for all $v \in \mathcal{C}^{q+1}([a,b])$,

$$\int_a^b (\pi v(x) - v(x))\, g(x)\, dx = \frac{k^{q+1}}{q!} \int_a^b v^{(q+1)}(x) H_g(x)\, dx. \tag{2.5}$$

Proof. To prove (2.4), we define for any fixed $x \in [a,b]$,

$$\Lambda_x v = \pi v(x) - v(x),$$

which is linear, since π is linear. Furthermore, if $v \in \mathcal{P}^q([a,b])$ then $\pi v = v$ so Λ_x vanishes on $\mathcal{P}^q([a,b])$. From the estimate,

$$|\Lambda_x v| = |\pi v(x) - v(x)| \leq |\pi v(x)| + |v(x)| \leq (1 + C)\|v\|,$$

it follows that Λ_x is bounded. Now,

$$\begin{aligned} K_x(t) &= \frac{1}{q!} \Lambda_x((\cdot - t)_+^q) = \frac{1}{q!} \left[\pi((\cdot - t)_+^q)(x) - (x - t)_+^q \right] \\ &= \frac{k^q}{q!} \left[\pi\left(\left(\frac{\cdot - t}{k} \right)_+^q \right)(x) - \left(\frac{x - t}{k} \right)_+^q \right] = \frac{k^q}{q!} G(x,t), \end{aligned}$$

where $|G(x,t)| \leq 1 + C$ for $x, t \in [a,b]$. Thus, by the Peano kernel theorem,

$$\pi v(x) - v(x) = \Lambda_x v = \frac{k^{q+1}}{q!} \frac{1}{k} \int_a^b v^{(q+1)}(t) G(x,t)\, dt.$$

To prove (2.5), define Λ by

$$\Lambda v = \int_a^b (\pi v(x) - v(x)) g(x)\, dx,$$

which is linear, bounded, and vanishes on $\mathcal{P}^q([a,b])$. Now,

$$\begin{aligned} K(t) &= \frac{1}{q!}\int_a^b \left[\pi((\cdot - t)_+^q)(x) - (x-t)_+^q\right] g(x)\, dx \\ &= \frac{k^{q+1}}{q!}\frac{1}{k}\int_a^b \left[\pi\left(\left(\frac{\cdot - t}{k}\right)_+^q\right)(x) - \left(\frac{x-t}{k}\right)_+^q\right] g(x)\, dx = \frac{k^{q+1}}{q!}H_g(t), \end{aligned}$$

where $H_g(t) \leq (1+C)\|g\|$ for $t \in [a,b]$. By the Peano kernel theorem, it now follows that

$$\int_a^b (\pi v(x) - v(x))g(x)\, dx = \frac{k^{q+1}}{q!}\int_a^b v^{(q+1)}(t) H_g(t)\, dt.$$

□

In order to derive a representation for derivatives of the interpolation error, we need to investigate the differentiability of the kernel $G(x,t)$.

Lemma 2.1. (Differentiability of G) *If π is linear on V, then the kernel G, defined by*

$$G(x,t) = \pi\left(\left(\frac{\cdot - t}{k}\right)_+^q\right)(x) - \left(\frac{x-t}{k}\right)_+^q, \tag{2.6}$$

has the following properties:

(i) *For any fixed $t \in [a,b]$, $G(\cdot, t) \in \mathcal{C}^{q-1}([a,b])$ and*

$$\frac{\partial^p}{\partial x^p}G(x,t) = k^{-p}G_{x,p}(x,t), \quad p = 0,\ldots,q, \tag{2.7}$$

where each $G_{x,p}$ is bounded on $[a,b]\times[a,b]$ independent of k.

(ii) *For any fixed $x \in [a,b]$, $G(x,\cdot) \in \mathcal{C}^{q-1}([a,b])$ and*

$$\frac{\partial^p}{\partial t^p}G(x,t) = k^{-p}G_{t,p}(x,t), \quad p = 0,\ldots,q, \tag{2.8}$$

where each $G_{t,p}$ is bounded on $[a,b]\times[a,b]$ independent of k.

(iii) *For $x \neq t$ and $p_1, p_2 \geq 0$, we have*

$$\frac{\partial^{p_2}}{\partial x^{p_2}}\frac{\partial^{p_1}}{\partial t^{p_1}}G(x,t) = k^{-(p_1+p_2)}G_{t,x,p_1,p_2}(x,t), \tag{2.9}$$

where each G_{t,x,p_1,p_2} is bounded on $[a,b]\times[a,b]\backslash\{(x,t) : x = t\}$ independent of k.

Proof. Define

$$G(x,t) = \pi\left(\left(\frac{\cdot - t}{k}\right)_+^q\right)(x) - \left(\frac{x-t}{k}\right)_+^q \equiv g(x,t) - h(x,t).$$

We first note that for any fixed $t \in [a,b]$, $h(\cdot,t) \in \mathcal{C}^{q-1}([a,b])$ with

$$\frac{\partial^p}{\partial x^p}h(x,t) = \frac{\partial^p}{\partial x^p}\left(\frac{x-t}{k}\right)_+^q = k^{-p}\frac{q!}{(q-p)!}\left(\frac{x-t}{k}\right)_+^{q-p} = k^{-p}h_{x,p}(x,t),$$

where $h_{x,p}$ is bounded on $[a,b] \times [a,b]$ independent of k. Similarly, we note that for any fixed $x \in [a,b]$, $h(x,\cdot) \in \mathcal{C}^{q-1}([a,b])$ with

$$\frac{\partial^p}{\partial t^p} h(x,t) = \frac{\partial^p}{\partial t^p} \left(\frac{x-t}{k} \right)_+^q = k^{-p} \frac{q!(-1)^p}{(q-p)!} \left(\frac{x-t}{k} \right)_+^{q-p} = k^{-p} h_{t,p}(x,t),$$

where $h_{t,p}$ is bounded on $[a,b] \times [a,b]$ independent of k.

If we now let $\pi_{[0,1]}$ denote the interpolant shifted to $[0,1]$, we can write

$$\begin{aligned} g(x,t) &= \pi \left(\left(\frac{\cdot - t}{k} \right)_+^q \right)(x) = \pi \left(\left(\frac{(\cdot - a)}{k} - \frac{(t-a)}{k} \right)_+^q \right)(x) \\ &= \pi_{[0,1]} \left(\left(\cdot - \frac{(t-a)}{k} \right)_+^q \right)((x-a)/k), \end{aligned}$$

and so, for any fixed $t \in [a,b]$, $g(\cdot,t) \in \mathcal{P}^q([a,b]) \subset \mathcal{C}^{q-1}([a,b])$, with

$$\frac{\partial^p}{\partial x^p} g(x,t) = k^{-p} \pi_{[0,1]}^{(p)} \left(\left(\cdot - \frac{(t-a)}{k} \right)_+^q \right)((x-a)/k)) = k^{-p} g_{x,p}(x,t),$$

where $g_{x,p}$ is bounded on $[a,b] \times [a,b]$ independent of k. Finally, let $s = (t-a)/k$. Then,

$$g(x,t) = \pi_{[0,1]} \left((\cdot - s)_+^q \right)((x-a)/k)).$$

The degrees of freedom of the interpolant $\pi_{[0,1]} \left((\cdot - s)_+^q \right)$ are determined by the solution of a linear system, and thus each of the degrees of freedom (point values or integrals) will be a linear combination of the degrees of freedom of the function $(\cdot - s)_+^q$, each of which in turn are $\mathcal{C}^{q-1}$ in the s-variable. Hence, $g(x,\cdot) \in \mathcal{C}^{q-1}([a,b])$ for any fixed $x \in [a,b]$, with

$$\begin{aligned} \frac{\partial^p}{\partial t^p} g(x,t) &= \frac{\partial^p}{\partial t^p} \pi_{[0,1]} \left((\cdot - s)_+^q \right)((x-a)/k)) = k^{-p} \frac{\partial^p}{\partial s^p} \pi_{[0,1]} \left((\cdot - s)_+^q \right)((x-a)/k)) \\ &= k^{-p} g_{t,p}(x,t), \end{aligned}$$

where $g_{t,p}$ is bounded on $[a,b] \times [a,b]$ independent of k. We now take $G_{x,p} = g_{x,p} - h_{x,p}$ and $G_{t,p} = g_{t,p} - h_{t,p}$, which proves (2.7) and (2.8).

To prove (2.9), we note that

$$\frac{\partial^{p_1}}{\partial t^{p_1}} h(x,t) = k^{-p_1} \frac{q!(-1)^{p_1}}{(q-p_1)!} \left(\frac{x-t}{k} \right)_+^{q-p_1},$$

and so, for $x \neq t$,

$$\frac{\partial^{p_2}}{\partial x^{p_2}} \frac{\partial^{p_1}}{\partial t^{p_1}} h(x,t) = k^{-(p_1+p_2)} \frac{q!(-1)^{p_1}}{(q-(p_1+p_2))!} \left(\frac{x-t}{k} \right)_+^{q-(p_1+p_2)},$$

when $p_1 + p_2 \le q$ and $\frac{\partial^{p_2}}{\partial x^{p_2}} \frac{\partial^{p_1}}{\partial t^{p_1}} h(x,t) = 0$ for $p_1 + p_2 > q$. Furthermore, for any fixed x,

$$\frac{\partial^{p_1}}{\partial t^{p_1}} g(x,t) = k^{-p_1} \frac{\partial^{p_1}}{\partial s^{p_1}} \pi_{[0,1]} \left((\cdot - s)_+^q \right)((x-a)/k)).$$

With $y = (x-a)/k$, we thus have

$$\frac{\partial^{p_2}}{\partial x^{p_2}}\frac{\partial^{p_1}}{\partial t^{p_1}} g(x,t) = k^{-(p_1+p_2)}\frac{\partial^{p_2}}{\partial y^{p_2}}\frac{\partial^{p_1}}{\partial s^{p_1}}\pi_{[0,1]}\left((\cdot - s)_+^q\right)(y).$$

We conclude that

$$\frac{\partial^{p_2}}{\partial x^{p_2}}\frac{\partial^{p_1}}{\partial t^{p_1}} G(x,t) = k^{-(p_1+p_2)} G_{t,x,p_1,p_2}(x,t),$$

where G_{t,x,p_1,p_2} is bounded on $[a,b]\times[a,b]\backslash\{(x,t) : x = t\}$. □

By differentiating (2.4), we now obtain the following representation for derivatives of the interpolation error.

Theorem 2.3. (Peano kernel theorem III) *If π is linear and bounded on V, and is exact on $\mathcal{P}^q([a,b]) \subset V$, then there is a constant $C > 0$, depending only on the definition of the interpolant (and not on k), and functions $G_p : [a,b]\times[a,b] \to \mathbb{R}$, $p = 0,\ldots,q$, such that for all $v \in \mathcal{C}^{q+1}([a,b])$,*

$$(\pi v)^{(p)}(x) - v^{(p)}(x) = \frac{k^{q+1-p}}{q!}\frac{1}{k}\int_a^b v^{(q+1)}(t) G_p(x,t)\,dt, \quad p = 0,\ldots,q, \tag{2.10}$$

where for each p, $|G_p(x,t)| \leq C$ for all $x,t \in [a,b]\times[a,b]$.

Proof. For $p = 0$ the result follows from Theorem 2.2 with $G_0 = G$. For $p = 1,\ldots,q$, we differentiate (2.4) with respect to x and use Lemma 2.1, to obtain

$$(\pi v)^{(p)}(x) - v^{(p)}(x) = \frac{k^q}{q!}\int_a^b v^{(q+1)}(t)\frac{\partial^p}{\partial x^p}G(x,t)\,dt = \frac{k^{q-p}}{q!}\int_a^b v^{(q+1)}(t) G_{x,p}(x,t)\,dt.$$

□

3. Interpolation estimates

Using the results of the previous section, we now obtain estimates for the interpolation error $\pi v - v$. The following corollary is a simple consequence of Theorem 2.3.

Corollary 3.1. *If π is linear and bounded on V, and is exact on $\mathcal{P}^q([a,b]) \subset V$, then there is a constant $C = C(q) > 0$, such that for all $v \in \mathcal{C}^{q+1}([a,b])$,*

$$\|(\pi v)^{(p)} - v^{(p)}\| \leq C k^{r+1-p}\|v^{(r+1)}\|, \tag{3.1}$$

for $p = 0,\ldots,r+1$, $r = 0,\ldots,q$.

Proof. If π is exact on $\mathcal{P}^q([a,b])$, it is exact on $\mathcal{P}^r([a,b]) \subseteq \mathcal{P}^q([a,b])$ for $r = 0,\ldots,q$. It follows by Theorem 2.3 that for all $v \in \mathcal{C}^{(r+1)}([a,b])$, we have

$$\|(\pi v)^{(p)} - v^{(p)}\| \leq C k^{r+1-p}\|v^{(r+1)}\|,$$

for $p = 0,\ldots,r$. When $r < q$, this holds also for $p = r+1 \leq q$, and for $p = r+1 = q$, we note that $\|(\pi v)^{(p)} - v^{(p)}\| = \|v^{(p)}\| = \|v^{(r+1)}\|$. □

This leads to the following estimate for the derivative of the interpolant.

Corollary 3.2. *If π is linear and bounded on V, and is exact on $\mathcal{P}^q([a,b]) \subset V$, then there is a constant $C = C(q) > 0$, such that for all $v \in \mathcal{C}^{q+1}([a,b])$,*

$$\|(\pi v)^{(p)}\| \leq C\|v^{(p)}\|, \tag{3.2}$$

for $p = 0, \ldots, q$.

Proof. It is clear that (3.2) holds when $p = 0$ since π is bounded. For $0 < p \leq q$, we add and subtract $v^{(p)}$, to obtain

$$\|(\pi v)^{(p)}\| \leq \|(\pi v)^{(p)} - v^{(p)}\| + \|v^{(p)}\|.$$

Since π is exact on $\mathcal{P}^q([a,b])$, it is exact on $\mathcal{P}^{p-1}([a,b]) \subset \mathcal{P}^q([a,b])$ for $p \leq q$. It follows by Corollary 3.1, that

$$\|(\pi v)^{(p)}\| \leq Ck^{(p-1)+1-p}\|v^{((p-1)+1)}\| + \|v^{(p)}\| = (C+1)\|v^{(p)}\| = C'\|v^{(p)}\|.$$

□

Finally, we show that it is often enough to show that π is linear on V and exact on $\mathcal{P}^{(q)}([a,b])$, that is, we do not have to show that π is bounded.

Lemma 3.1. *If $\pi : V \to \mathcal{P}^q([a,b])$ is linear on V, and is uniquely determined by n_1 interpolation conditions and $n_2 = q + 1 - n_1$ projection conditions,*

$$\pi v(x_i) = v(x_i), \quad i = 1, \ldots, n_1, \tag{3.3}$$

where each $x_i \in [a,b]$, and

$$\int_a^b \pi v(x) w_i(x)\, dx = \int_a^b v(x) w_i(x)\, dx, \quad i = 1, \ldots, n_2, \tag{3.4}$$

where each w_i is bounded, then π is bounded on V, that is,

$$\|\pi v\| \leq C\|v\| \quad \forall v \in V, \tag{3.5}$$

for some constant $C = C(q) > 0$.

Proof. Define $\|\cdot\|_*$ on $\mathcal{P}^q([a,b])$ by

$$\|v\|_* = \sum_{i=1}^{n_1} |v(x_i)| + \sum_{i=1}^{n_2} \left| \frac{1}{k} \int_a^b v(x) w_i(x)\, dx \right|.$$

Clearly, the triangle inequality holds and $\|\alpha v\|_* = |\alpha|\|v\|_*$ for all $\alpha \in \mathbb{R}$. Furthermore, if $v = 0$ then $\|v\|_* = 0$. Conversely, if $\|v\|_* = 0$ then $v \in \mathcal{P}^q([a,b])$ is the unique interpolant of 0, and so $v = \pi 0 = 0$ by linearity. Thus, $\|\cdot\|_*$ is a norm. Since all norms on a finite dimensional vector space are equivalent, we obtain

$$\|\pi v\| \leq C\|\pi v\|_* = C\|v\|_* \leq C(n_1 + C'n_2)\|v\| = C''\|v\|.$$

□

4. Interpolation of piecewise smooth functions

We now extend the interpolation results of the previous two sections to piecewise smooth functions.

4.1. **Extensions of the Peano kernel theorem.** To extend the Peano kernel theorem to piecewise smooth functions, we construct a smooth version v_ϵ of the discontinuous function v, with $v_\epsilon \to v$ as $\epsilon \to 0$. For the construction of v_ϵ, we introduce the function

$$w(x) = Cx \sum_{m=0}^{q+1} \binom{q+1}{m} \frac{(-1)^m x^{2(q+1)-2m}}{2(q+1)-2m+1}, \tag{4.1}$$

where

$$C = \left(\sum_{m=0}^{q+1} \binom{q+1}{m} \frac{(-1)^m}{2(q+1)-2m+1} \right)^{-1}. \tag{4.2}$$

In Figure 2, we plot the function $w(x)$ on $[-1,1]$ for $q = 0, \dots, 4$.

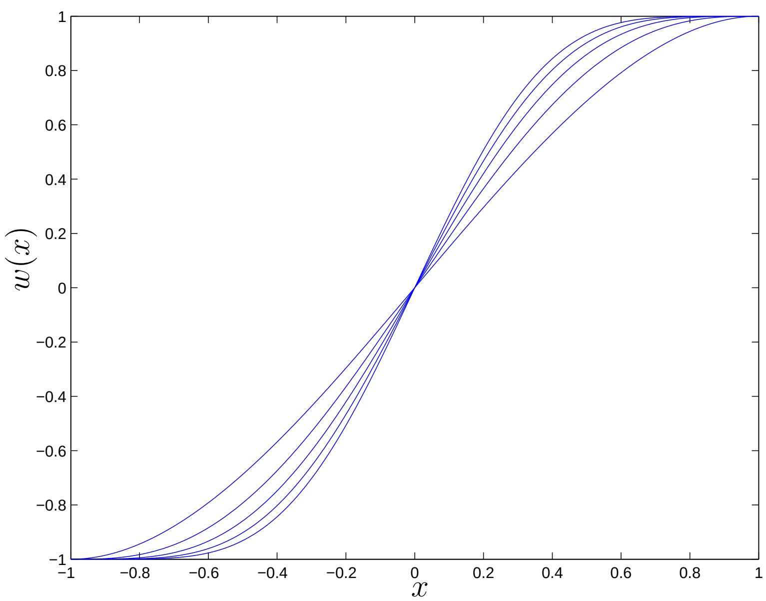

Figure 2. The function $w(x)$ on $[-1,1]$ for $q = 0, \dots, 4$.

Let also $T_y v$ denote the Taylor expansion of order $q \geq 0$ around a given point $y \in (a,b)$. For $\epsilon > 0$ sufficiently small, we then define

$$v_\epsilon(x) = \begin{cases} \left(1 - \frac{w((x-x_1)/\epsilon)+1}{2}\right) T_{x_1-\epsilon} v(x) + \frac{w((x-x_1)/\epsilon)+1}{2} T_{x_1+\epsilon} v(x), & x \in [x_1-\epsilon, x_1+\epsilon], \\ \dots \\ \left(1 - \frac{w((x-x_n)/\epsilon)+1}{2}\right) T_{x_n-\epsilon} v(x) + \frac{w((x-x_n)/\epsilon)+1}{2} T_{x_n+\epsilon} v(x), & x \in [x_n-\epsilon, x_n+\epsilon], \end{cases} \tag{4.3}$$

with $v_\epsilon = v$ on $[a,b] \setminus ([x_1-\epsilon, x_1+\epsilon] \cup \ldots \cup [x_n-\epsilon, x_n+\epsilon])$. As a consequence of the following Lemma, v_ϵ has $q+1$ continuous derivatives on $[a,b]$.

Lemma 4.1. *The weight function w defined by (4.1) and (4.2) has the following properties:*

$$-1 = w(-1) \le w(x) \le w(1) = 1, \quad x \in [-1,1], \tag{4.4}$$

and

$$w^{(p)}(-1) = w^{(p)}(1) = 0, \quad p = 1, \ldots, q+1. \tag{4.5}$$

Proof. It is clear from the definition that $w(1)=1$ and $w(-1)=-1$. Taking the derivative of w, we obtain

$$\begin{aligned} \frac{dw}{dx}(x) &= C\sum_{m=0}^{q+1}\binom{q+1}{m}\frac{(-1)^m \frac{d}{dx} x^{2(q+1)-2m+1}}{2(q+1)-2m+1} = C\sum_{m=0}^{q+1}\binom{q+1}{m}(-1)^m x^{2(q+1)-2m} \\ &= C(x^2-1)^{q+1} = C(x+1)^{q+1}(x-1)^{q+1}, \end{aligned}$$

and so $w^{(p)}$ is zero at $x = \pm 1$ for $p = 1, \ldots, q+1$. Furthermore, since $\frac{dw}{dx}$ has no zeros within $(-1,1)$, w attains its maximum and minimum at $x=1$ and $x=-1$ respectively. □

This leads to the following extension of Theorem 2.2.

Theorem 4.1. (Peano kernel theorem for piecewise smooth functions) *If π is linear and bounded (with constant $C > 0$) on V, and is exact on $\mathcal{P}^q([a,b]) \subset V$, then there is a bounded function $G : [a,b]\times[a,b] \to \mathbb{R}$, with $|G(x,t)| \le 1 + C$ for all $x,t \in [a,b]$, such that for v piecewise $\mathcal{C}^{q+1}$ on $[a,b]$ with discontinuities at $a < x_1 < x_2 < \ldots < x_n < b$, we have*

$$\pi v(x) - v(x) = \frac{k^{q+1}}{q!}\frac{1}{k}\int_a^b v^{(q+1)}(t) G(x,t)\, dt + \sum_{j=1}^{n}\sum_{m=0}^{q} C_{qjm}(x) k^m [v^{(m)}]_{x_j}, \tag{4.6}$$

where $k = b - a$ and for each C_{qjm} we have $|C_{qjm}(x)| \le C$ for all $x \in [a,b]$. Furthermore, if g is an integrable, (essentially) bounded function on $[a,b]$, then there is a function $H_g : [a,b] \to \mathbb{R}$, with $|H_g(x)| \le (1+C)\|g\|$ for all $x \in [a,b]$, such that

$$\int_a^b (\pi v(x) - v(x)) g(x)\, dx = \frac{k^{q+1}}{q!}\int_a^b v^{(q+1)}(x) H_g(x)\, dx + \sum_{j=1}^{n}\sum_{m=0}^{q} D'_{qjm}(x) k^{m+1} [v^{(m)}]_{x_j}, \tag{4.7}$$

where for each D'_{qjm} we have $|D'_{qjm}(x)| \le C$ for all $x \in [a,b]$.

Proof. Without loss of generality, we can assume that v is discontinuous only at $x_1 \in (a,b)$. Fix $x \in [a,b]\setminus\{x_1\}$, take $0 < \epsilon < |x - x_1|$, and define v_ϵ as in (4.3). Then,

$$\pi v - v = (\pi v - \pi v_\epsilon) + (\pi v_\epsilon - v_\epsilon) + (v_\epsilon - v),$$

where $|\pi v(x) - \pi v_\epsilon(x)| \to 0$ and $|v_\epsilon(x) - v(x)| \to 0$ when $\epsilon \to 0$. Since $v_\epsilon \in \mathcal{C}^{q+1}([a,b])$, we have by Theorem 2.2,

$$\pi v_\epsilon(x) - v_\epsilon(x) = \frac{k^q}{q!}\int_a^b v_\epsilon^{(q+1)}(t) G(x,t)\, dt = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{k^q}{q!}\int_a^{x_1-\epsilon} v^{(q+1)}(t)G(x,t)\,dt,$$
$$I_2 = \frac{k^q}{q!}\int_{x_1-\epsilon}^{x_1+\epsilon} v_\epsilon^{(q+1)}(t)G(x,t)\,dt,$$
$$I_3 = \frac{k^q}{q!}\int_{x_1+\epsilon}^{b} v^{(q+1)}(t)G(x,t)\,dt.$$

Since $v|_{[a,x_1)} \in \mathcal{C}^{[q+1]}([a,x_1))$ and $v|_{(x_1,b]} \in \mathcal{C}^{[q+1]}((x_1,b])$, it is clear that

$$I_1 + I_3 = \frac{k^q}{q!}\int_a^b v^{(q+1)}(t)G(x,t)\,dt + \mathcal{O}(\epsilon).$$

We therefore focus on the remaining term I_2. With $w_\epsilon(t) = \frac{w((t-x_1)/\epsilon)+1}{2}$, we have

$$\begin{aligned}
I_2 &= \frac{k^q}{q!}\int_{x_1-\epsilon}^{x_1+\epsilon} v_\epsilon^{(q+1)}(t)G(x,t)\,dt \\
&= \frac{k^q}{q!}\int_{x_1-\epsilon}^{x_1+\epsilon} \left[(1-w_\epsilon)T_{x_1-\epsilon}v + w_\epsilon T_{x_1+\epsilon}v\right]^{(q+1)}(t)G(x,t)\,dt \\
&= \sum_{m=0}^{q+1}\frac{k^q}{q!}\binom{q+1}{m}\int_{x_1-\epsilon}^{x_1+\epsilon}\left[(1-w_\epsilon)^{(m)}T_{x_1-\epsilon}^{(q+1-m)}v + w_\epsilon^{(m)}T_{x_1+\epsilon}^{(q+1-m)}v\right](t)G(x,t)\,dt \\
&= \sum_{m=0}^{q+1}\frac{k^q}{q!}\binom{q+1}{m} I_{2m},
\end{aligned}$$

with obvious notation. Evaluating the integrals I_{2m}, we have for $m=0$,

$$I_{20} = \int_{x_1-\epsilon}^{x_1+\epsilon}\left[(1-w_\epsilon)^{(0)}T_{x_1-\epsilon}^{(q+1-0)}v + w_\epsilon^{(0)}T_{x_1+\epsilon}^{(q+1-0)}v\right](t)G(x,t)\,dt = \mathcal{O}(\epsilon),$$

while for $m = 1,\ldots,q+1$, we obtain

$$\begin{aligned}
I_{2m} &= \int_{x_1-\epsilon}^{x_1+\epsilon}\left[(1-w_\epsilon)^{(m)}T_{x_1-\epsilon}^{(q+1-m)}v + w_\epsilon^{(m)}T_{x_1+\epsilon}^{(q+1-m)}v\right](t)G(x,t)\,dt \\
&= \int_{x_1-\epsilon}^{x_1+\epsilon} w_\epsilon^{(m)}(t)(T_{x_1+\epsilon}^{(q+1-m)}v(t) - T_{x_1-\epsilon}^{(q+1-m)}v(t))G(x,t)\,dt.
\end{aligned}$$

Let now $h = (T_{x_1+\epsilon}^{(q+1-m)}v - T_{x_1-\epsilon}^{(q+1-m)}v)G(x,\cdot)$ and note that by Lemma 2.1, $h \in \mathcal{C}^{q-1}([x_1 - \epsilon, x_1+\epsilon])$. Noting also that $w_\epsilon^{(m)}(x_1 \pm \epsilon) = 0$ for $m = 1,\ldots,q+1$, we integrate by parts to obtain

$$\begin{aligned} I_{2m} &= \int_{x_1-\epsilon}^{x_1+\epsilon} w_\epsilon^{(m)}(t)h(t)\,dt \\ &= (-1)^{m-1}\int_{x_1-\epsilon}^{x_1+\epsilon} w_\epsilon'(t)h^{(m-1)}(t)\,dt = \frac{(-1)^{m-1}}{2\epsilon}\int_{x_1-\epsilon}^{x_1+\epsilon} w'((t-x_1)/\epsilon)h^{(m-1)}(t)\,dt \\ &= \frac{C(-1)^{m-1}}{2\epsilon}\int_{x_1-\epsilon}^{x_1+\epsilon} (((t-x_1)/\epsilon)^2-1)^{q+1}h^{(m-1)}(t)\,dt, \\ &= \frac{C(-1)^{m-1}}{2}\int_{-1}^{1}(s^2-1)^{q+1}h^{(m-1)}(x_1+\epsilon s)\,ds, \end{aligned}$$

where C is the constant defined in (4.2). Evaluating the derivatives of h, we obtain

$$\begin{aligned} h^{(m-1)} &= \frac{d^{m-1}}{dt^{m-1}}(T_{x_1+\epsilon}^{(q+1-m)}v - T_{x_1-\epsilon}^{(q+1-m)}v)G(x,\cdot) \\ &= \sum_{j=0}^{m-1}\binom{m-1}{j}(T_{x_1+\epsilon}^{(q+1-m+j)}v - T_{x_1-\epsilon}^{(q+1-m+j)}v)G_t^{(m-1-j)}(x,\cdot) \\ &= \sum_{j=0}^{m-1}\binom{m-1}{j}(v^{(q+1-m+j)}(x_1^+) - v^{(q+1-m+j)}(x_1^-) + \mathcal{O}(\epsilon))G_t^{(m-1-j)}(x,\cdot) \\ &= \sum_{j=0}^{m-1}\binom{m-1}{j}\left[v^{(q-(m-1-j))}\right]_{x_1} G_t^{(m-1-j)}(x,\cdot) + \mathcal{O}(\epsilon), \end{aligned}$$

where $G_t^{(p)}(x,t)$ denotes $\frac{\partial^p}{\partial t^p}G(x,t)$. Consequently,

$$\begin{aligned} I_2 &= \sum_{m=1}^{q+1}\sum_{j=0}^{m-1} c_{qmj}k^q\left[v^{(q-(m-1-j))}\right]_{x_1}\int_{-1}^{1}(s^2-1)^{q+1}G_t^{(m-1-j)}(x,x_1+\epsilon s)\,ds + \mathcal{O}(\epsilon) \\ &= \sum_{m=1}^{q+1}\sum_{j=0}^{m-1} c_{qmj}k^q\left[v^{(q-(m-1-j))}\right]_{x_1} G_t^{(m-1-j)}(x,x_1)\int_{-1}^{1}(s^2-1)^{q+1}\,ds + \mathcal{O}(\epsilon) \\ &= \sum_{m=0}^{q} c_{qm}k^q G_t^{(q-m)}(x,x_1)\left[v^{(m)}\right]_{x_1} + \mathcal{O}(\epsilon). \end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\pi v(x) - v(x) = \frac{k^q}{q!}\int_a^b v^{(q+1)}(t)G(x,t)\,dt + \sum_{m=0}^{q} c_{qm}k^q G_t^{(q-m)}(x,x_1)\left[v^{(m)}\right]_{x_1},$$

for $x \in [a,b]\backslash\{x_1\}$. By continuity this holds also when $x = x_1$. From Lemma 2.1, we know that $G_t^{(q-m)}(x,x_1) = k^{-(q-m)} G_{t,q-m}(x,x_1)$ where $G_{t,q-m}$ is bounded. Hence,

$$\begin{aligned}\pi v(x) - v(x) &= \frac{k^q}{q!}\int_a^b v^{(q+1)}(t)G(x,t)\,dt + \sum_{m=0}^{q} c_{qm} k^{q-(q-m)} G_{t,q-m}(x,x_1)\left[v^{(m)}\right]_{x_1} \\ &= \frac{k^q}{q!}\int_a^b v^{(q+1)}(t)G(x,t)\,dt + \sum_{m=0}^{q} C_{qm}(x,x_1) k^m \left[v^{(m)}\right]_{x_1},\end{aligned}$$

where $C_{qm}(x,x_1)$ is bounded on $[a,b]\times[a,b]$ independent of k. The second result, (4.7), is proved similarly. □

We now extend this to a representation of derivatives of the interpolation error, corresponding to Theorem 2.3.

Theorem 4.2. (Peano kernel theorem for piecewise smooth functions II) *If π is linear and bounded on V, and is exact on $\mathcal{P}^q([a,b]) \subset V$, then there is a constant $C > 0$, depending only on the definition of the interpolant (and not on k), and functions $G_p : [a,b]\times[a,b] \to \mathbb{R}$, $p = 0,\ldots,q$, such that for v piecewise $\mathcal{C}^{q+1}$ on $[a,b]$ with discontinuities at $a < x_1 < x_2 < \ldots < x_n < b$, we have*

$$(4.8)\quad (\pi v)^{(p)}(x) - v^{(p)}(x) = \frac{k^{q+1-p}}{q!}\frac{1}{k}\int_a^b v^{(q+1)}(t)G_p(x,t)\,dt + \sum_{j=1}^{n}\sum_{m=0}^{q} C_{qjmp}(x)k^{m-p}[v^{(m)}]_{x_j},$$

for $p = 0,\ldots,q$, with $|G_p(x,t)| \le C$ for all $x,t \in [a,b]\times[a,b]$, and $|C_{qjmp}(x)| \le C$ for all $x \in [a,b]\setminus\{x_1,\ldots,x_n\}$.

Proof. For $p = 0$, the result follows from Theorem 4.1 with $G_0 = G$ and $C_{qjm0} = C_{qjm}$. For $p = 1,\ldots,q$, we differentiate (4.6) with respect to x, to obtain

$$(\pi v)^{(p)}(x) - v^{(p)}(x) = \frac{k^q}{q!}\int_a^b v^{(q+1)}(t)\frac{d^p}{dx^p}G(x,t)\,dt + \sum_{j=1}^{n}\sum_{m=0}^{q} C_{qjm}^{(p)}(x)k^m[v^{(m)}]_{x_j}.$$

From Lemma 2.1, we know that $\frac{d^p}{dx^p}G(x,t) = k^{-p}G_{x,p}(x,t)$, where $G_{x,p}$ is bounded on $[a,b]\times[a,b]$. Furthermore, from the proof of Theorem 4.1, we know that

$$C_{qjm}^{(p)}(x) = \frac{d^p}{dx^p}C_{qjm}(x) = \frac{d^p}{dx^p}c_{qm}G_{t,q-m}(x,x_j) = c_{qm}k^{q-m}\frac{\partial^p}{\partial x^p}\frac{\partial^{q-m}}{\partial t^{q-m}}G(x,x_j),$$

and so, by Lemma 2.1,

$$C_{qjm}^{(p)}(x) = c_{qm}k^{q-m}k^{-(q-m+p)}G_{t,x,q-m,p}(x,x_j) = C_{qjmp}(x)k^{-p},$$

where each C_{qjmp} is bounded on $[a,b]\backslash\{x_1,\ldots,x_n\}$. □

4.2. **Interpolation estimates.** The following corollary, corresponding to Corollary 3.1, is a simple consequence of Theorem 4.2.

Corollary 4.1. *If π is linear and bounded on V, and is exact on $\mathcal{P}^q([a,b]) \subset V$, then there is a constant $C = C(q) > 0$, such that for all v piecewise $\mathcal{C}^{q+1}$ on $[a,b]$ with discontinuities at $a < x_1 < \ldots < x_n < b$,*

$$\|(\pi v)^{(p)} - v^{(p)}\| \leq C k^{r+1-p} \|v^{(r+1)}\| + C \sum_{j=1}^{n} \sum_{m=0}^{r} k^{m-p} \left| \left[v^{(m)} \right]_{x_j} \right|, \tag{4.9}$$

for $p = 0, \ldots, r+1$, $r = 0, \ldots, q$.

Proof. See proof of Corollary 3.1. □

We also obtain the following estimate for derivatives of the interpolant, corresponding to Corollary 3.2.

Corollary 4.2. *If π is linear and bounded on V, and is exact on $\mathcal{P}^q([a,b]) \subset V$, then there is a constant $C = C(q) > 0$ and a constant $C' = C'(q,n) > 0$, such that for all v piecewise $\mathcal{C}^{q+1}$ on $[a,b]$ with discontinuities at $a < x_1 < \ldots < x_n < b$,*

$$\|(\pi v)^{(p)}\| \leq C \|v^{(p)}\| + C \sum_{j=1}^{n} \sum_{m=0}^{p-1} k^{m-p} \left| \left[v^{(m)} \right]_{x_j} \right| \leq C' \left(\|v^{(p)}\| + \sum_{m=0}^{p-1} k^{m-p} \|v^{(m)}\| \right), \tag{4.10}$$

for $p = 0, \ldots, q$.

Proof. It is clear that (4.10) holds when $p = 0$, since π is bounded. As in the proof of Corollary 3.2, we add and subtract $v^{(p)}$ for $0 < p \leq q$, to obtain

$$\|(\pi v)^{(p)}\| \leq \|(\pi v)^{(p)} - v^{(p)}\| + \|v^{(p)}\|.$$

Since π is exact on $\mathcal{P}^q([a,b])$, it is exact on $\mathcal{P}^{p-1}([a,b]) \subset \mathcal{P}^q([a,b])$ for $p \leq q$. It follows by Corollary 4.1 that

$$\begin{aligned}
\|(\pi v)^{(p)}\| &\leq C k^{(p-1)+1-p} \|v^{((p-1)+1)}\| + C \sum_{j=1}^{n} \sum_{m=0}^{p-1} k^{m-p} \left| \left[v^{(m)} \right]_{x_j} \right| + \|v^{(p)}\| \\
&\leq C \|v^{(p)}\| + C \sum_{j=1}^{n} \sum_{m=0}^{p-1} k^{m-p} \left| \left[v^{(m)} \right]_{x_j} \right| \\
&\leq C \|v^{(p)}\| + 2Cn \sum_{m=0}^{p-1} k^{m-p} \|v^{(m)}\| \leq C' \left(\|v^{(p)}\| + \sum_{m=0}^{p-1} k^{m-p} \|v^{(m)}\| \right).
\end{aligned}$$

□

5. Two special interpolants

In this section, we use the results of Sections 2–4 to prove interpolation estimates for two special interpolants.

For the mcG(q) method, we define the following interpolant:

$$
\begin{aligned}
&\pi_{\mathrm{cG}}^{[q]} : V \to \mathcal{P}^q([a,b]),\\
&\pi_{\mathrm{cG}}^{[q]} v(a) = v(a) \text{ and } \pi_{\mathrm{cG}}^{[q]} v(b) = v(b),\\
&\int_a^b (v - \pi_{\mathrm{cG}}^{[q]} v) w \, dx = 0 \quad \forall w \in \mathcal{P}^{q-2}([a,b]),
\end{aligned}
\tag{5.1}
$$

where V denotes the set of functions that are piecewise $\mathcal{C}^{q+1}$ and bounded on $[a,b]$. In other words, $\pi_{\mathrm{cG}}^{[q]} v$ is the polynomial of degree q that interpolates v at the end-points of the interval $[a,b]$ and additionally satisfies $q-1$ projection conditions. This is illustrated in Figure 3. We also define the dual interpolant $\pi_{\mathrm{cG}^*}^{[q]}$ as the standard L_2-projection onto $\mathcal{P}^{q-1}([a,b])$.

For the mdG(q) method, we define the following interpolant:

$$
\begin{aligned}
&\pi_{\mathrm{dG}}^{[q]} : V \to \mathcal{P}^q([a,b]),\\
&\pi_{\mathrm{dG}}^{[q]} v(b) = v(b),\\
&\int_a^b (v - \pi_{\mathrm{dG}}^{[q]} v) w \, dx = 0 \quad \forall w \in \mathcal{P}^{q-1}([a,b]),
\end{aligned}
\tag{5.2}
$$

that is, $\pi_{\mathrm{dG}}^{[q]} v$ is the polynomial of degree q that interpolates v at the right end-point of the interval $[a,b]$ and additionally satisfies q projection conditions. This is illustrated in Figure 4. The dual interpolant $\pi_{\mathrm{dG}^*}^{[q]}$ is defined similarly, with the only difference that we use the left end-point $x = a$ for interpolation.

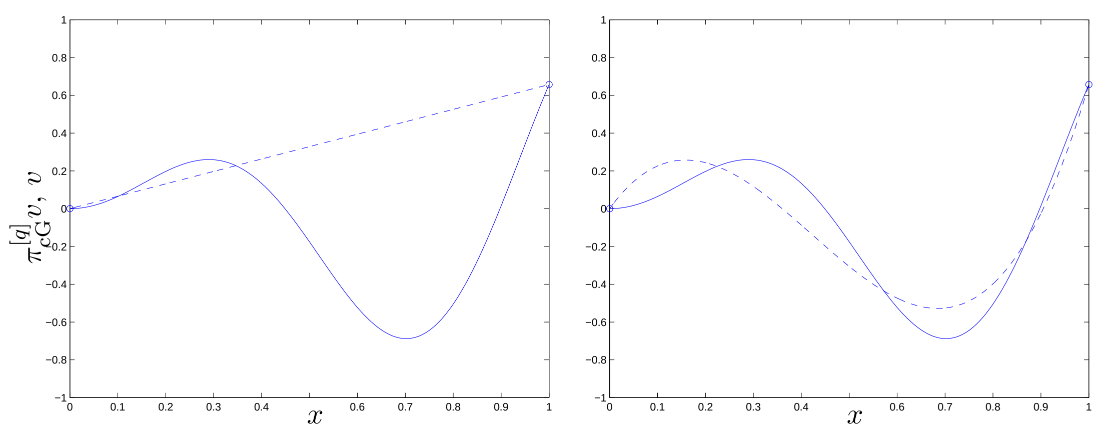

FIGURE 3. The interpolant $\pi_{\mathrm{cG}}^{[q]} v$ (dashed) of the function $v(x) = x\sin(7x)$ (solid) on $[0,1]$ for $q = 1$ (left) and $q = 3$ (right).

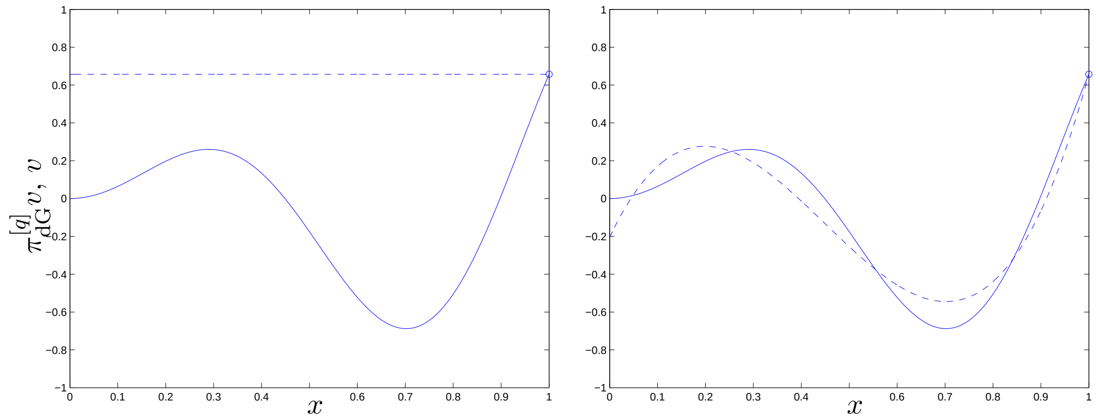

FIGURE 4. The interpolant $\pi_{\mathrm{dG}}^{[q]}v$ (dashed) of the function $v(x) = x\sin(7x)$ (solid) on $[0,1]$ for $q = 0$ (left) and $q = 3$ (right).

It is clear that both $\pi_{\mathrm{cG}}^{[q]}$ and $\pi_{\mathrm{dG}}^{[q]}$ are linear and so, by Lemma 3.1, we only have to show that $\pi_{\mathrm{cG}}^{[q]}$ and $\pi_{\mathrm{dG}}^{[q]}$ are exact on $\mathcal{P}^q([a,b])$.

Lemma 5.1. *The two interpolants* $\pi_{\mathrm{cG}}^{[q]}$ *and* $\pi_{\mathrm{dG}}^{[q]}$ *are exact on* $\mathcal{P}^q([a,b])$*, that is,*

$$\pi_{\mathrm{cG}}^{[q]}v = v \quad \forall v \in \mathcal{P}^q([a,b]), \tag{5.3}$$

and

$$\pi_{\mathrm{dG}}^{[q]}v = v \quad \forall v \in \mathcal{P}^q([a,b]). \tag{5.4}$$

Proof. To prove (5.3), take $v \in \mathcal{P}^q([a,b])$ and note that $p = \pi_{\mathrm{cG}}^{[q]}v - v \in \mathcal{P}^q([a,b])$. Since $p(a) = p(b) = 0$, p has at most $q-2$ zeros within (a,b) and so we can take $w \in \mathcal{P}^{q-2}([a,b])$ with $pw \geq 0$ on $[a,b]$. By definition, $\int_a^b pw\,dx = 0$, and so we conclude that $p = 0$.

To prove (5.4), take $p = \pi_{\mathrm{dG}}^{[q]}v - v \in \mathcal{P}^q([a,b])$. Then, $p(b) = 0$ and so p has at most $q-1$ zeros within (a,b). Take now $w \in \mathcal{P}^{q-1}([a,b])$ with $pw \geq 0$ on $[a,b]$. By definition, $\int_a^b pw\,dt = 0$, and so again we conclude that $p = 0$. □

The desired interpolation estimates now follow Corollaries 3.1, 3.2, 4.1, and 4.2.

Theorem 5.1. (Estimates for $\pi_{\mathrm{cG}}^{[q]}$ and $\pi_{\mathrm{dG}}^{[q]}$) *For any* $q \geq 1$*, there is a constant* $C = C(q) > 0$*, such that for all* $v \in \mathcal{C}^{r+1}([a,b])$*,*

$$\|(\pi_{\mathrm{cG}}^{[q]}v)^{(p)} - v^{(p)}\| \leq Ck^{r+1-p}\|v^{(r+1)}\|, \tag{5.5}$$

for $p = 0,\dots,r+1$*,* $r = 0,\dots,q$*, and*

$$\|(\pi_{\mathrm{cG}}^{[q]}v)^{(p)}\| \leq C\|v^{(p)}\|, \tag{5.6}$$

for $p = 0,\dots,q$*. Furthermore, for any* $q \geq 0$*, there is a constant* $C = C(q) > 0$*, such that for all* $v \in \mathcal{C}^{r+1}([a,b])$*,*

$$\|(\pi_{\mathrm{dG}}^{[q]}v)^{(p)} - v^{(p)}\| \leq Ck^{r+1-p}\|v^{(r+1)}\|, \tag{5.7}$$

for $p = 0, \dots, r+1$, $r = 0, \dots, q$, *and*

$$\|(\pi_{\rm dG}^{[q]} v)^{(p)}\| \leq C \|v^{(p)}\|, \tag{5.8}$$

for $p = 0, \dots, q$.

Theorem 5.2. (Estimates for $\pi_{\rm cG}^{[q]}$ and $\pi_{\rm dG}^{[q]}$ II) *For any* $q \geq 1$ *and any* $n \geq 0$, *there is a constant* $C = C(q) > 0$, *such that for all* v *piecewise* $\mathcal{C}^{r+1}$ *on* $[a, b]$ *with discontinuities at* $a < x_1 < \ldots < x_n < b$,

$$\|(\pi_{\rm cG}^{[q]} v)^{(p)} - v^{(p)}\| \leq C k^{r+1-p} \|v^{(r+1)}\| + C \sum_{j=1}^{n} \sum_{m=0}^{r} k^{m-p} \left| \left[v^{(m)} \right]_{x_j} \right|, \tag{5.9}$$

for $p = 0, \dots, r+1$, $r = 0, \dots, q$, *and*

$$\|(\pi_{\rm cG}^{[q]} v)^{(p)}\| \leq C \|v^{(p)}\| + C \sum_{j=1}^{n} \sum_{m=0}^{p-1} k^{m-p} \left| \left[v^{(m)} \right]_{x_j} \right|, \tag{5.10}$$

for $p = 0, \dots, q$. *Furthermore, for any* $q \geq 0$ *and any* $n \geq 0$, *there is a constant* $C = C(q) > 0$, *such that for all* v *piecewise* $\mathcal{C}^{r+1}$ *on* $[a, b]$ *with discontinuities at* $a < x_1 < \ldots < x_n < b$,

$$\|(\pi_{\rm dG}^{[q]} v)^{(p)} - v^{(p)}\| \leq C k^{r+1-p} \|v^{(r+1)}\| + C \sum_{j=1}^{n} \sum_{m=0}^{r} k^{m-p} \left| \left[v^{(m)} \right]_{x_j} \right|, \tag{5.11}$$

for $p = 0, \dots, r+1$, $r = 0, \dots, q$, *and*

$$\|(\pi_{\rm dG}^{[q]} v)^{(p)}\| \leq C \|v^{(p)}\| + C \sum_{j=1}^{n} \sum_{m=0}^{p-1} k^{m-p} \left| \left[v^{(m)} \right]_{x_j} \right|, \tag{5.12}$$

for $p = 0, \dots, q$.

Note that the corresponding estimates hold for the dual versions of the two interpolants, $\pi_{\rm cG^*}^{[q]}$ and $\pi_{\rm dG^*}^{[q]}$, the only difference being that $r \leq q-1$ for $\pi_{\rm cG^*}^{[q]}$.

Finally, we note the following properties of the two interpolants, which is of importance for the a priori error analysis.

Lemma 5.2. *For any* $v \in \mathcal{C}([a, b])$ *and any* $q \geq 1$, *we have*

$$\int_a^b \left(\frac{d}{dx}(v - \pi_{\rm cG}^{[q]} v)\right) w \, dx = 0 \quad \forall w \in \mathcal{P}^{q-1}([a, b]). \tag{5.13}$$

Proof. For any $w \in \mathcal{P}^{q-1}([a, b])$, we integrate by parts to get

$$\int_a^b \left(\frac{d}{dx}(v - \pi_{\rm cG}^{[q]} v)\right) w \, dx = \Big[(v - \pi_{\rm cG}^{[q]} v) w \Big]_a^b - \int_a^b (v - \pi_{\rm cG}^{[q]} v) \, w' \, dx = 0,$$

by the definition of $\pi_{\rm cG}^{[q]}$. □

Lemma 5.3. *For any* $v \in \mathcal{C}([a,b])$ *and any* $q \geq 0$*, we have*

$$[v(a) - \pi_{\mathrm{dG}}^{[q]} v(a)]w(a) + \int_a^b \left(\frac{d}{dx}(v - \pi_{\mathrm{dG}}^{[q]} v)\right) w \, dx = 0 \quad \forall w \in \mathcal{P}^q([a,b]). \tag{5.14}$$

Proof. Integrate by parts as in the proof of Lemma 5.2 and use the definition of $\pi_{\mathrm{dG}}^{[q]}$. ☐

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