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MULTI-ADAPTIVE GALERKIN METHODS FOR ODES III: EXISTENCE AND STABILITY

ANDERS LOGG

Abstract. We prove existence and stability of solutions for the multi-adaptive Galerkin methods mcG(q) and mdG(q), and their dual versions mcG(q)* and mdG(q)*, including strong stability estimates for parabolic problems. This paper is the third in a series devoted to multi-adaptive Galerkin methods. In the companion paper [7], we return to the a priori error analysis of the multi-adaptive methods. The stability estimates derived in this paper will then be essential.

1. Introduction

This is part III in a sequence of papers [4, 5] on multi-adaptive Galerkin methods, mcG(q) and mdG(q), for approximate (numerical) solution of ODEs of the form

$$\begin{aligned} \dot{u}(t) &= f(u(t), t), \quad t \in (0, T], \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where $u : [0, T] \to \mathbb{R}^N$ is the solution to be computed, $u_0 \in \mathbb{R}^N$ a given initial condition, $T > 0$ a given final time, and $f : \mathbb{R}^N \times (0, T] \to \mathbb{R}^N$ a given function that is Lipschitz-continuous in u and bounded.

The mcG(q) and mdG(q) methods are based on piecewise polynomial approximation of degree q on partitions in time with time steps which may vary for different components $U_i(t)$ of the approximate solution $U(t)$ of (1.1). In part I and II of our series on multi-adaptive Galerkin methods, we prove a posteriori error estimates, through which the time steps are adaptively determined from residual feed-back and stability information, obtained by solving a dual linearized problem. In this paper, we prove existence and stability of discrete solutions, which we later use together with special interpolation estimates to prove a priori error estimates for the mcG(q) and mdG(q) methods in part IV [7].

1.1. **Notation.** For a detailed presentation of the multi-adaptive methods, we refer to [4, 5]. Here, we only give a quick overview of the notation: Each component $U_i(t)$, $i = 1, \ldots, N$, of the approximate m(c/d)G(q) solution $U(t)$ of (1.1) is a piecewise polynomial

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on a partition of $(0,T]$ into M_i subintervals. Subinterval j for component i is denoted by $I_{ij} = (t_{i,j-1}, t_{ij}]$, and the length of the subinterval is given by the local *time step* $k_{ij} = t_{ij} - t_{i,j-1}$. This is illustrated in Figure 1. On each subinterval I_{ij}, $U_i|_{I_{ij}}$ is a polynomial of degree q_{ij} and we refer to $(I_{ij}, U_i|_{I_{ij}})$ as an *element*.

Furthermore, we shall assume that the interval $(0,T]$ is partitioned into blocks between certain synchronized time levels $0 = T_0 < T_1 < \ldots < T_M = T$. We refer to the set of intervals $\mathcal{T}_n$ between two synchronized time levels T_{n-1} and T_n as a *time slab*:

$$\mathcal{T}_n = \{I_{ij} : T_{n-1} \le t_{i,j-1} < t_{ij} \le T_n\}.$$

We denote the length of a time slab by $K_n = T_n - T_{n-1}$. We also refer to the entire collection of intervals I_{ij} as the partition $\mathcal{T}$.

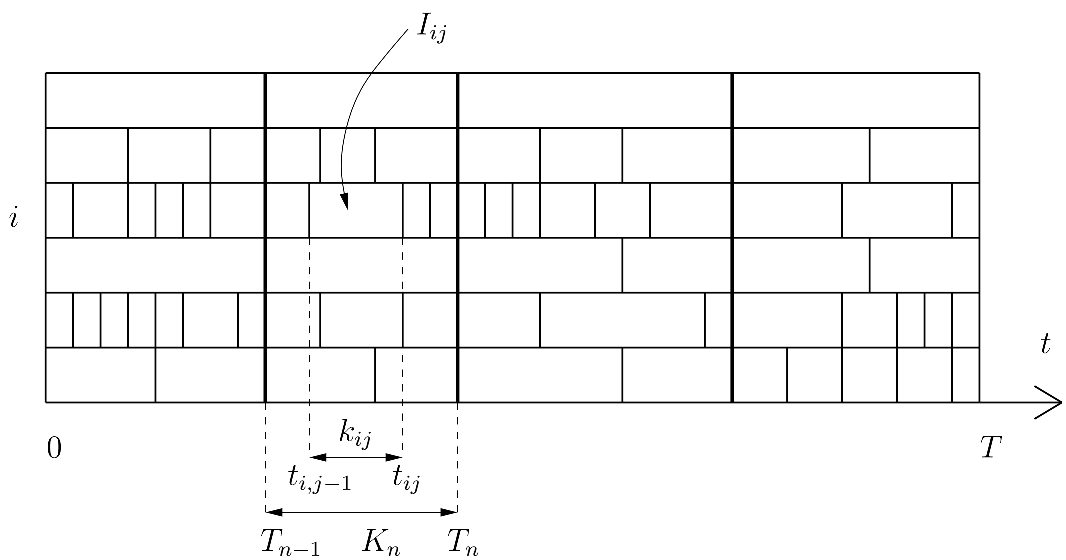

FIGURE 1. Individual partitions of the interval $(0,T]$ for different components. Elements between common synchronized time levels are organized in time slabs. In this example, we have $N = 6$ and $M = 4$.

1.2. **Outline of the paper.** The first part of this paper is devoted to proving existence of solutions for the multi-adaptive methods mcG(q) and mdG(q), including the dual methods mcG(q)* and mdG(q)* obtained by interchanging trial and test spaces, by proving (relevant) fixed point iterations assuming the time steps are sufficiently small. The proof is constructive and mimics the actual implementation of the methods. The multi-adaptive ODE-solver *Tanganyika*, presented in [5], thus repeats the proof of existence each time it computes a new solution.

In the second part of this paper, we prove stability estimates, including general exponential estimates for mcG(q), mdG(q), mcG(q)*, and mdG(q)*, and strong stability estimates for parabolic problems for the mdG(q) and mdG(q)* methods.

2. Multi-adaptive Galerkin and multi-adaptive dual Galerkin

2.1. **Multi-adaptive continuous Galerkin,** mcG(q).

To formulate the mcG(q) method, we define the *trial space* V and the *test space* $\hat{V}$ as

$$\begin{aligned} V &= \{v \in [\mathcal{C}([0,T])]^N : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij}),\ j=1,\ldots,M_i,\ i=1,\ldots,N\}, \\ \hat{V} &= \{v : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}-1}(I_{ij}),\ j=1,\ldots,M_i,\ i=1,\ldots,N\}, \end{aligned} \tag{2.1}$$

where $\mathcal{P}^q(I)$ denotes the linear space of polynomials of degree q on an interval I. In other words, V is the space of continuous piecewise polynomials of degree $q = q_i(t) = q_{ij} \geq 1$, $t \in I_{ij}$, on the partition $\mathcal{T}$, and $\hat{V}$ is the space of (possibly discontinuous) piecewise polynomials of degree $q-1$ on the same partition.

We now define the mcG(q) method for (1.1) in the following way: Find $U \in V$ with $U(0) = u_0$, such that

$$\int_0^T (\dot{U}, v)\, dt = \int_0^T (f(U,\cdot), v)\, dt \quad \forall v \in \hat{V}, \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the $\mathbb{R}^N$ inner product. If now for each local interval I_{ij} we take $v_n \equiv 0$ when $n \neq i$ and $v_i(t) = 0$ when $t \notin I_{ij}$, we can rewrite the global problem (2.2) as a sequence of successive local problems for each component: For $i = 1,\ldots,N$, $j = 1,\ldots,M_i$, find $U_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})$ with $U_i(t_{i,j-1})$ given, such that

$$\int_{I_{ij}} \dot{U}_i v\, dt = \int_{I_{ij}} f_i(U,\cdot) v\, dt \quad \forall v \in \mathcal{P}^{q_{ij}-1}(I_{ij}), \tag{2.3}$$

where the initial condition is specified for $i = 1,\ldots,N$ by $U_i(0) = u_i(0)$.

We define the *residual* R of the approximate solution U by $R_i(U,t) = \dot{U}_i(t) - f_i(U(t),t)$. In terms of the residual, we can rewrite (2.3) as

$$\int_{I_{ij}} R_i(U,\cdot) v\, dt = 0 \quad \forall v \in \mathcal{P}^{q_{ij}-1}(I_{ij}), \tag{2.4}$$

that is, the residual is orthogonal to the test space on each local interval. We refer to (2.4) as the *Galerkin orthogonality* of the mcG(q) method.

2.2. **Multi-adaptive discontinuous Galerkin,** mdG(q).

For the mdG(q) method, we define the trial and test spaces by

$$V = \hat{V} = \{v : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij}),\ j=1,\ldots,M_i,\ i=1,\ldots,N\}, \tag{2.5}$$

that is, both trial and test functions are (possibly discontinuous) piecewise polynomials of degree $q = q_i(t) = q_{ij} \geq 0$, $t \in I_{ij}$, on the partition $\mathcal{T}$. We define the mdG(q) solution $U \in V$ to be left-continuous.

We now define the mdG(q) method for (1.1) in the following way, similar to the definition of the continuous method: Find $U \in V$ with $U(0^-) = u_0$, such that

$$\sum_{i=1}^N \sum_{j=1}^{M_i} \left[[U_i]_{i,j-1} v_i(t_{i,j-1}^+) + \int_{I_{ij}} \dot{U}_i v_i\, dt \right] = \int_0^T (f(U,\cdot), v)\, dt \quad \forall v \in \hat{V}, \tag{2.6}$$

where $[U_i]_{i,j-1} = U_i(t_{i,j-1}^+) - U_i(t_{i,j-1}^-)$ denotes the jump in $U_i(t)$ across the node $t = t_{i,j-1}$.

The mdG(q) method in local form, corresponding to (2.3), reads: For $i = 1, \ldots, N$, $j = 1, \ldots, M_i$, find $U_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})$, such that

$$[U_i]_{i,j-1} v(t_{i,j-1}) + \int_{I_{ij}} \dot{U}_i v \, dt = \int_{I_{ij}} f_i(U, \cdot) v \, dt \quad \forall v \in \mathcal{P}^{q_{ij}}(I_{ij}), \tag{2.7}$$

where the initial condition is specified for $i = 1, \ldots, N$ by $U_i(0^-) = u_i(0)$.

In the same way as for the continuous method, we define the residual R of the approximate solution U by $R_i(U, t) = \dot{U}_i(t) - f_i(U(t), t)$, defined on the inner of each local interval I_{ij}, and rewrite (2.7) in the form

$$[U_i]_{i,j-1} v(t_{i,j-1}^+) + \int_{I_{ij}} R_i(U, \cdot) v \, dt = 0 \quad \forall v \in \mathcal{P}^{q_{ij}}(I_{ij}). \tag{2.8}$$

We refer to (2.8) as the Galerkin orthogonality of the mdG(q) method. Note that the residual has two parts: one interior part R_i and the jump term $[U_i]_{i,j-1}$.

2.3. **The dual problem.** The motivation for introducing the dual problem is for the a priori or a posteriori error analysis of the multi-adaptive methods. For the a posteriori analysis, we formulate a continuous dual problem [4]. For the a priori analysis [7], we formulate a discrete dual problem in terms of the dual multi-adaptive methods mcG$(q)^*$ and mdG$(q)^*$.

The discrete dual solution $\Phi : [0, T] \to \mathbb{R}^N$ is a Galerkin approximation of the exact solution $\phi : [0, T] \to \mathbb{R}^N$ of the continuous dual backward problem

$$\begin{aligned} -\dot{\phi}(t) &= J^\top(\pi u, U, t)\phi(t) + g(t), \quad t \in [0, T), \\ \phi(T) &= \psi, \end{aligned} \tag{2.9}$$

where πu is an interpolant or a projection of the exact solution u of (1.1), $g : [0, T] \to \mathbb{R}^N$ is a given function, $\psi \in \mathbb{R}^N$ is a given initial condition, and

$$J^\top(\pi u, U, t) = \left(\int_0^1 \frac{\partial f}{\partial u}(s\pi u(t) + (1-s)U(t), t) \, ds \right)^\top, \tag{2.10}$$

that is, an appropriate mean value of the transpose of the Jacobian of the right-hand side $f(\cdot, t)$ evaluated at $\pi u(t)$ and $U(t)$. Note that by the chain rule, we have

$$J(\pi u, U, \cdot)(U - \pi u) = f(U, \cdot) - f(\pi u, \cdot). \tag{2.11}$$

2.4. **Multi-adaptive dual continuous Galerkin,** mcG$(q)^*$**.** In the formulation of the dual method of mcG(q), we interchange the trial and test spaces of mcG(q). With the same definitions of V and $\hat{V}$ as in (2.1), we thus define the mcG$(q)^*$ method for (2.9) in the following way: Find $\Phi \in \hat{V}$ with $\Phi(T^+) = \psi$, such that

$$\int_0^T (\dot{v}, \Phi) \, dt = \int_0^T (J(\pi u, U, \cdot)v, \Phi) + L_{\psi,g}(v), \tag{2.12}$$

for all $v \in V$ with $v(0) = 0$, where

$$L_{\psi,g}(v) \equiv (v(T), \psi) + \int_0^T (v, g)\, dt. \tag{2.13}$$

Notice the extra condition that the test functions should vanish at $t = 0$, which is introduced to make the dimension of the test space equal to the dimension of the trial space. Integrating by parts, (2.12) can alternatively be expressed in the form

$$\sum_{i=1}^{N} \sum_{j=1}^{M_i} \left[-[\Phi_i]_{ij} v_i(t_{ij}) - \int_{I_{ij}} \dot{\Phi}_i v_i \, dt \right] = \int_0^T (J^\top(\pi u, U, \cdot)\Phi + g, v)\, dt. \tag{2.14}$$

2.5. **Multi-adaptive dual discontinuous Galerkin,** mdG$(q)^*$**.** Interchanging trial and test spaces does not make any difference for the discontinuous method since the trial and test spaces are identical. With the same definitions of V and $\hat{V}$ as in (2.5), we define the mdG$(q)^*$ method for (2.9) in the following way: Find $\Phi \in \hat{V}$ with $\Phi(T^+) = \psi$, such that

$$\sum_{i=1}^{N} \sum_{j=1}^{M_i} \left[[v_i]_{i,j-1} \Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} \dot{v}_i \Phi_i \, dt \right] = \int_0^T (J(\pi u, U, \cdot) v, \Phi)\, dt + L_{\psi,g}(v), \tag{2.15}$$

for all $v \in V$ with $v(0^-) = 0$. Integrating by parts, (2.15) can alternatively be expressed in the form

$$\sum_{i=1}^{N} \sum_{j=1}^{M_i} \left[-[\Phi_i]_{ij} v_i(t_{ij}^-) - \int_{I_{ij}} \dot{\Phi}_i v_i \, dt \right] = \int_0^T (J^\top(\pi u, U, \cdot)\Phi + g, v)\, dt. \tag{2.16}$$

3. Existence of solutions

To prove existence of the discrete mcG(q), mdG(q), mcG$(q)^*$, and mdG$(q)^*$ solutions defined in the previous section, we formulate fixed point iterations for the construction of solutions. Existence then follows from the Banach fixed point theorem, if the time steps are sufficiently small. The proof is thus constructive and gives a method for computing solutions (see [5]).

3.1. **Multi-adaptive Galerkin in fixed point form.** We start by proving the following simple lemma.

Lemma 3.1. *Let A be a $d \times d$ matrix with elements $A_{mn} = \frac{n}{m+n-1}$, and let B be a $d \times d$ matrix with elements $B_{mn} = \frac{n}{m+n}$, for $m, n = 1, \ldots, d$. Then,* $\det A \neq 0$ *and* $\det B \neq 0$.

Proof. To prove that A is nonsingular, we let $p(t) = \sum_{n=1}^{d} x_n n t^{n-1}$ be a polynomial of degree $d-1$ on $[0, 1]$. If for $m = 1, \ldots, d$, we have $\int_0^1 p(t) t^{m-1}\, dt = 0$, it follows that $p \equiv 0$. Thus, $\sum_{n=1}^{d} x_n \frac{n}{m+n-1} = 0$ for $m = 1, \ldots, d$ implies $x = 0$, which proves that $\det A \neq 0$. To prove that B is nonsingular, let again $p(t) = \sum_{n=1}^{d} x_n n t^{n-1}$. If for $m = 1, \ldots, d$ we have $\int_0^1 p(t) t^m\, dt = 0$, take $q(t) = \sum_{m=1}^{d} y_m t^m$, such that p and q have the same zeros on $[0, 1]$

and $pq \geq 0$. Then, $\int_0^1 pq\,dt = 0$ but $pq \geq 0$ on $[0,1]$ and so $p \equiv 0$. Thus, $\sum_{n=1}^{n} x_n \frac{n}{m+n} = 0$ for $m = 1, \ldots, d$ implies $x = 0$, which proves that $\det B \neq 0$. $\square$

To rewrite the methods in explicit fixed point form, we introduce a simple basis for the trial and test spaces and solve for the degrees of freedom on each local interval.

Lemma 3.2. *The* mcG(q) *method for* (1.1) *in fixed point form reads: For* $i = 1, \ldots, N$, $j = 1, \ldots, M_i$, *find* $\{\xi_{ijn}\}_{n=1}^{q_{ij}}$, *such that*

$$\xi_{ijn} = \xi_{ij0} + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t)) f_i(U(t), t)\,dt, \tag{3.1}$$

with

$$\xi_{ij0} = \begin{cases} \xi_{i,j-1,q_{i,j-1}}, & j > 1, \\ u_i(0), & j = 1, \end{cases}$$

where $\{w_n^{[q_{ij}]}\}_{n=1}^{q_{ij}} \subset \mathcal{P}^{q_{ij}-1}([0,1])$, $w_{q_{ij}}^{[q_{ij}]} \equiv 1$, *and* $\tau_{ij}(t) = (t - t_{i,j-1})/(t_{ij} - t_{i,j-1})$. *A component* $U_i(t)$ *of the solution is given on* I_{ij} *by*

$$U_i(t) = \sum_{n=0}^{q_{ij}} \xi_{ijn} \lambda_n^{[q_{ij}]}(\tau_{ij}(t)),$$

where $\{\lambda_n^{[q_{ij}]}\}_{n=0}^{q_{ij}} \subset \mathcal{P}^{q_{ij}}([0,1])$ *is the standard Lagrange basis on* $[0,1]$ *with* $t = 0$ *and* $t = 1$ *as two of its* $q_{ij} + 1 \geq 2$ *nodal points.*

Proof. Our starting point is the local formulation (2.3). Dropping indices for ease of notation and rescaling to the interval $[0,1]$, we have

$$\int_0^1 \dot{U} v\,dt = \int_0^1 f v\,dt \quad \forall v \in \mathcal{P}^{q-1}([0,1]),$$

with $U \in \mathcal{P}^q([0,1])$. Let now $\{\lambda_n^{[q]}\}_{n=0}^{q}$ be a basis for $\mathcal{P}^q([0,1])$. In terms of this basis, we have $U(t) = \sum_{n=0}^{q} \xi_n \lambda_n^{[q]}(t)$, and so

$$\sum_{n=0}^{q} \xi_n \int_0^1 \dot{\lambda}_n^{[q]} \lambda_m^{[q-1]}\,dt = \int_0^1 f \lambda_m^{[q-1]}\,dt, \quad m = 0, \ldots, q-1.$$

Since the solution is continuous, the value at $t = 0$ is known from the previous interval (or from the initial condition). This gives

$$U(0) = \sum_{n=0}^{q} \xi_n \lambda_n^{[q]}(0) = \xi_0,$$

if we assume that $\lambda_n^{[q]}(0) = \delta_{0n}$. The remaining q degrees of freedom are then determined by

$$\sum_{n=1}^{q} \xi_n \int_0^1 \dot{\lambda}_n^{[q]} \lambda_{m-1}^{[q-1]}\,dt = \int_0^1 f \lambda_{m-1}^{[q-1]} - \xi_0 \int_0^1 \dot{\lambda}_0^{[q]} \lambda_{m-1}^{[q-1]}\,dt, \quad m = 1, \ldots, q.$$

If $\det\left(\int_0^1 \dot\lambda_n^{[q]}\lambda_{m-1}^{[q-1]}\,dt\right) \neq 0$, this system can be solved for the degrees of freedom $(\xi_1,\ldots,\xi_n)$. With $\lambda_n^{[q]}(t)=t^n$, we have

$$\det\left(\int_0^1 \dot\lambda_n^{[q]}\lambda_{m-1}^{[q-1]}\,dt\right) = \det\left(\int_0^1 nt^{n-1}t^{m-1}\,dt\right) = \det\left(\frac{n}{m+n-1}\right) \neq 0,$$

by Lemma 3.1. Solving for $(\xi_1,\ldots,\xi_n)$, we obtain

$$\xi_n = \alpha_n^{[q]}\xi_0 + \int_0^1 w_n^{[q]} f\,dt, \quad n=1,\ldots,q,$$

for some constants $\{\alpha_n^{[q]}\}_{n=1}^q$, where $\{w_n^{[q]}\}_{n=1}^q \subset \mathcal{P}^{q-1}([0,1])$ and ξ_0 is determined from the continuity requirement. For any other basis $\{\lambda_n^{[q]}\}_{n=0}^q$ with $\lambda_n^{[q]}(0)=\delta_{0n}$, we obtain a similar expression for the degrees of freedom by a linear transformation. In particular, let $\{\lambda_n^{[q]}\}_{n=0}^q$ be the Lagrange basis functions for a partition of $[0,1]$ with $t=0$ as a nodal point. For $f\equiv 0$ it is easy to see that the mcG(q) solution is constant and equal to its initial value. It follows that $\alpha_n^{[q]}=1$, $n=1,\ldots,q$, and so

$$\xi_n = \xi_0 + \int_0^1 w_n^{[q]} f\,dt, \quad n=1,\ldots,q,$$

with $U(1)=\xi_q$ if also $t=1$ is a nodal point. To see that $w_q^{[q]}\equiv 1$, take $v\equiv 1$ in (2.3). The result now follows by rescaling to I_{ij}. □

Lemma 3.3. *The* mdG(q) *method for* (1.1) *in fixed point form reads: For* $i=1,\ldots,N$, $j=1,\ldots,M_i$, *find* $\{\xi_{ijn}\}_{n=0}^{q_{ij}}$, *such that*

$$\xi_{ijn} = \xi_{ij0}^- + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t)) f_i(U(t),t)\,dt, \tag{3.2}$$

with

$$\xi_{ij0}^- = \begin{cases} \xi_{i,j-1,q_{i,j-1}}, & j>1, \\ u_i(0), & j=1, \end{cases}$$

where $\{w_n^{[q_{ij}]}\}_{n=0}^{q_{ij}} \subset \mathcal{P}^{q_{ij}}([0,1])$, $w_{q_{ij}}^{[q_{ij}]}\equiv 1$, *and* $\tau_{ij}(t)=(t-t_{i,j-1})/(t_{ij}-t_{i,j-1})$. *A component* $U_i(t)$ *of the solution is given on* I_{ij} *by*

$$U_i(t) = \sum_{n=0}^{q_{ij}} \xi_{ijn}\lambda_n^{[q_{ij}]}(\tau_{ij}(t)),$$

where $\{\lambda_n^{[q_{ij}]}\}_{n=0}^{q_{ij}} \subset \mathcal{P}^{q_{ij}}([0,1])$ *is the standard Lagrange basis on* $[0,1]$ *with* $t=1$ *as one of its* $q_{ij}+1\geq 1$ *nodal points.*

Proof. In as similar way as in the proof of Lemma 3.2, we use (2.7) to obtain

$$(\xi_0-\xi_0^-)\lambda_m^{[q]}(0) + \sum_{n=0}^{q}\xi_n\int_0^1 \dot\lambda_n^{[q]}\lambda_m^{[q]}\,dt = \int_0^1 f\lambda_m^{[q]}\,dt, \quad m=0,\ldots,q.$$

With $\lambda_n^{[q]}(t) = t^n$, these $1+q$ equations can be written in the form

$$\begin{aligned} \xi_0 + \sum_{n=1}^{q} \xi_n \int_0^1 n t^{n-1}\,dt &= \int_0^1 f\,dt + \xi_0^-, \\ \sum_{n=1}^{q} \xi_n \int_0^1 n t^{n-1} t^m\,dt &= \int_0^1 f t^m\,dt, \quad m = 1,\ldots,q, \end{aligned}$$

which by Lemma 3.1 has a solution, since

$$\det\left(\int_0^1 n t^{n-1} t^m\,dt\right) = \det\left(\frac{n}{m+n}\right) \neq 0.$$

We thus obtain

$$\xi_n = \alpha_n^{[q]} \xi_0^- + \int_0^1 w_n^{[q]} f\,dt, \quad n = 0,\ldots,q.$$

By the same argument as in the proof of Lemma 3.2, we conclude that when $\{\lambda_n^{[q]}\}_{n=0}^{q}$ is the Lagrange basis for a partition of $[0,1]$, we have

$$\xi_n = \xi_0^- + \int_0^1 w_n^{[q]} f\,dt, \quad n = 0,\ldots,q,$$

with $U(1) = \xi_q = \xi_0^- + \int_0^1 f\,dt$ if $t=1$ is a nodal point. The result now follows by rescaling to I_{ij}. □

Lemma 3.4. *The* mcG$(q)^*$ *method for* (2.9) *in fixed point form reads: For* $i = 1,\ldots,N$, $j = M_i,\ldots,1$, *find* $\{\xi_{ijn}\}_{n=0}^{q_{ij}-1}$, *such that*

$$\xi_{ijn} = \psi_i + \int_{t_{ij}}^{T} f_i^*(\Phi,\cdot)\,dt + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t)) f_i^*(\Phi(t),t)\,dt, \tag{3.3}$$

where $f^*(\Phi,\cdot) = J^\top(\pi u, U, \cdot)\Phi + g$, $\{w_n^{[q_{ij}]}\}_{n=0}^{q_{ij}-1} \subset \mathcal{P}^{q_{ij}}([0,1])$, $w_n^{[q_{ij}]}(0) = 0$, $w_n^{[q_{ij}]}(1) = 1$, $n = 0,\ldots,q_{ij}-1$, *and* $\tau_{ij}(t) = (t - t_{i,j-1})/(t_{ij} - t_{i,j-1})$. *A component* $\Phi_i(t)$ *of the solution is given on* I_{ij} *by*

$$\Phi_i(t) = \sum_{n=0}^{q_{ij}-1} \xi_{ijn} \lambda_n^{[q_{ij}-1]}(\tau_{ij}(t)),$$

where $\{\lambda_n^{[q_{ij}-1]}\}_{n=0}^{q_{ij}-1} \subset \mathcal{P}^{q_{ij}-1}([0,1])$ *is the standard Lagrange basis on* $[0,1]$.

Proof. Our starting point is the definition (2.14). For any $1 \le i \le N$, take $v_n \equiv 0$ when $n \neq i$ and let v_i be a continuous piecewise polynomial that vanishes on $[0, t_{i,j-1}]$ with $v_i \equiv 1$ on $[t_{ij}, T]$, see Figure 3.4. With $f^*(\Phi,\cdot) = J^\top(\pi u, U, \cdot)\Phi + g$, we then have

$$\sum_{l=j}^{M_i}\left[-[\Phi_i]_{il} v_i(t_{il}) - \int_{I_{il}} \dot{\Phi}_i v_i\,dt\right] = \int_{t_{i,j-1}}^{T} f_i^*(\Phi,\cdot) v_i\,dt.$$

We integrate by parts, moving the derivative onto the test function, to get

$$\begin{aligned} -[\Phi_i]_{il} v_i(t_{il}) - \int_{I_{il}} \dot{\Phi}_i v_i \, dt &= -[\Phi_i]_{il} v_i(t_{il}) - [\Phi_i v_i]_{t_{i,l-1}^+}^{t_{il}^-} + \int_{I_{il}} \Phi_i \dot{v}_i \, dt \\ &= \Phi_i(t_{i,l-1}^+) v_i(t_{i,l-1}) - \Phi_i(t_{il}^+) v_i(t_{il}) + \int_{I_{il}} \Phi_i \dot{v}_i \, dt. \end{aligned}$$

Summing up, noting that $v_i(t_{i,j-1}) = 0$, $\dot{v}_i = 0$ on $[t_{ij}, T]$ and $\Phi_i(t_{M_i}^+) = \psi_i$, we have

$$-\psi_i + \int_{I_{ij}} \Phi_i \dot{v}_i \, dt = \int_{t_{i,j-1}}^T f_i^*(\Phi, \cdot) v_i \, dt,$$

or

$$\int_{I_{ij}} \Phi_i \dot{v}_i \, dt = \tilde{\Phi}_{ij} + \int_{I_{ij}} f_i^*(\Phi, \cdot) v_i \, dt,$$

with $\tilde{\Phi}_{ij} = \psi_i + \int_{t_{ij}}^T f_i^*(\Phi, \cdot) \, dt$. Dropping indices and rescaling to the interval $[0, 1]$, we obtain

$$\int_0^1 \Phi \dot{v} \, dt = \tilde{\Phi}(1) + \int_0^1 f^* v \, dt,$$

for all $v \in \mathcal{P}^q([0, 1])$ with $v(0) = 0$ and $v(1) = 1$. Let now $\{\lambda_n^{[q-1]}\}_{n=0}^{q-1}$ be a basis of $\mathcal{P}^{q-1}([0, 1])$ and write $\Phi(t) = \sum_{n=1}^q \xi_n \lambda_{n-1}^{[q-1]}(t)$. For $m = 1, \dots, q$, we then have

$$\sum_{n=1}^q \xi_n \int_0^1 \lambda_{n-1}^{[q-1]}(t) \, m t^{m-1} \, dt = \tilde{\Phi}(1) + \int_0^1 f^* \, t^m \, dt.$$

If now $\det \left(\int_0^1 \lambda_{n-1}^{[q-1]}(t) m t^{m-1} \, dt \right) \neq 0$, we can solve for $(\xi_1, \dots, \xi_n)$. With $\lambda_{n-1}^{[q-1]}(t) = t^{n-1}$, we have

$$\det \left(\int_0^1 \lambda_{n-1}^{[q-1]}(t) m t^{m-1} \, dt \right) = \det \left(\int_0^1 m t^{n-1} t^{m-1} \, dt \right) = \det \left(\frac{m}{m+n-1} \right) \neq 0,$$

by Lemma 3.1. Solving for the degrees of freedom, we obtain

$$\xi_n = \alpha_n^{[q]} \tilde{\Phi}(1) + \int_0^1 w_n^{[q]} f^* \, dt, \quad n = 1, \dots, q.$$

By a linear transformation, we obtain a similar expression for any other basis of $\mathcal{P}^q([0, 1])$. For $f^* \equiv 0$, it is easy to see that the mcG$(q)^*$ solution is constant and equal to its initial value. Thus, when $\{\lambda_n^{[q-1]}\}_{n=0}^{q-1}$ is the standard Lagrange basis for a partition of $[0, 1]$, it follows that $\alpha_n^{[q]} = 1$, $n = 1, \dots, q$, and so

$$\xi_n = \tilde{\Phi}(1) + \int_0^1 w_n^{[q]} f^* \, dt, \quad n = 1, \dots, q.$$

We note that $w_n^{[q]}(0) = 0$, $n = 1, \dots, q$, since each $w_n^{[q]}$ is a linear combination of the functions $\{t^m\}_{m=1}^q$. We also conclude that $w_n^{[q]}(1) = 1$, since $w_n^{[q]}(1) = \alpha_n^{[q]} = 1$, $n = 1, \dots, q$.

The result now follows by rescaling to I_{ij}. We also relabel the degrees of freedom from $(\xi_1, \dots, \xi_q)$ to $(\xi_0, \dots, \xi_{q-1})$. ☐

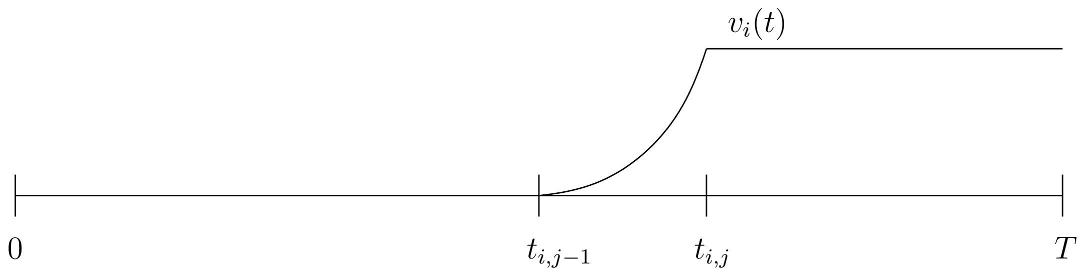

FIGURE 2. The special choice of test function used in the proof of Lemma 3.4.

Lemma 3.5. *The* mdG$(q)^*$ *method for* (2.9) *in fixed point form reads: For* $i = 1, \dots, N$, $j = M_i, \dots, 1$, *find* $\{\xi_{ijn}\}_{n=0}^{q_{ij}}$, *such that*

$$\xi_{ijn} = \xi_{ijq_{ij}}^{+} + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t)) f_i^*(\Phi(t), t)\, dt, \tag{3.4}$$

with

$$\xi_{ijq_{ij}}^{+} = \begin{cases} \xi_{i,j+1,0}, & j < M_i, \\ \psi_i, & j = M_i, \end{cases}$$

where $f^*(\Phi, \cdot) = J^\top(\pi u, U, \cdot)\Phi + g$, $\{w_n^{[q_{ij}]}\}_{n=0}^{q_{ij}} \subset \mathcal{P}^{q_{ij}}([0,1])$, *and* $\tau_{ij}(t) = (t - t_{i,j-1})/(t_{ij} - t_{i,j-1})$. *A component* $\Phi_i(t)$ *of the solution is given on* I_{ij} *by*

$$\Phi_i(t) = \sum_{n=0}^{q_{ij}} \xi_{ijn} \lambda_n^{[q_{ij}]}(\tau_{ij}(t)),$$

where $\{\lambda_n^{[q_{ij}]}\}_{n=0}^{q_{ij}} \subset \mathcal{P}^{q_{ij}}([0,1])$ *is the standard Lagrange basis on* $[0,1]$ *with* $t = 0$ *as one of its* $q_{ij} + 1 \geq 1$ *nodal points.*

Proof. The mdG$(q)^*$ method is identical to the mdG(q) method with time reversed. ☐

Corollary 3.1. *Let* $\mathcal{T}_n$ *be a time slab with synchronized time levels* T_{n-1} *and* T_n. *With time reversed for the dual methods (to simplify the notation), the* mcG(q), mdG(q), mcG$(q)^*$, *and* mdG$(q)^*$ *methods can all be expressed in the form: For all* $I_{ij} \in \mathcal{T}_n$, *find* $\{\xi_{ijn}\}$, *such that*

$$\xi_{ijn} = \tilde{U}_i(T_{n-1}^-) + \int_{T_{n-1}}^{t_{i,j-1}} f_i(U, \cdot)\, dt + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t)) f_i(U, \cdot)\, dt, \tag{3.5}$$

with a suitable definition of $\tilde{U}_i(T_{n-1}^-)$. *As before,* $\tau_{ij}(t) = (t - t_{i,j-1})/(t_{ij} - t_{i,j-1})$ *and* $\{w_n^{[q_{ij}]}\}$ *is a set of polynomial weight functions on* $[0,1]$.

Proof. For mcG(q), mdG(q), and mdG$(q)^*$, the result follows if we take $\tilde{U}(T_{n-1}^-) = U(T_{n-1}^-)$ and note that $w_q^{[q]} \equiv 1$. For mcG$(q)^*$, the result follows if we define $\tilde{U}(T_{n-1}^-) = u_i(0) + \int_0^{T_{n-1}} f_i(U, \cdot)\, dt$. ☐

3.2. **Fixed point iteration.** We now prove that for each of the four methods, mcG(q), mdG(q), mcG(q)*, and mdG(q)*, the fixed point iterations of Corollary 3.1 converge, proving existence of the discrete solutions.

Theorem 3.1. (Existence of solutions) *Let $K = \max K_n$ be the maximum time slab length and define the Lipschitz constant $L_f > 0$ by*

$$\|f(x,t) - f(y,t)\|_{l_\infty} \leq L_f \|x - y\|_{l_\infty} \quad \forall t \in [0,T]\ \forall x, y \in \mathbb{R}^N. \tag{3.6}$$

If now

$$KC_q L_f < 1, \tag{3.7}$$

where C_q is a constant of moderate size, depending only on the order and method, then each of the fixed point iterations, (3.1), (3.2), (3.3), *and* (3.4), *converge to the unique solution of* (2.2), (2.6), (2.12), *and* (2.15), *respectively.*

Proof. Let $x = (\ldots, \xi_{ijn}, \ldots)$ be the set of values for the degrees of freedom of $U(t)$ on the time slab $\mathcal{T}_n$ of length $K_n = T_n - T_{n-1} \leq K$. Then, by Corollary 3.1, we can write the fixed point iteration on the time slab in the form

$$\xi_{ijn} = g_{ijn}(x) = \tilde{U}_i(T_{n-1}^-) + \int_{T_{n-1}}^{t_{i,j-1}} f_i(U,\cdot)\,dt + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t)) f_i(U,\cdot)\,dt.$$

Let $V(t)$ be another trial space function on the time slab with degrees of freedom $y = (\ldots, \eta_{ijn}, \ldots)$. Then,

$$g_{ijn}(x) - g_{ijn}(y) = \int_{T_{n-1}}^{t_{i,j-1}} (f_i(U,\cdot) - f_i(V,\cdot))\,dt + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t))(f_i(U,\cdot) - f_i(V,\cdot))\,dt,$$

and so

$$\|g(x) - g(y)\|_{l_\infty} \leq CL_f \int_{T_{n-1}}^{T_n} \|U(t) - V(t)\|_{l_\infty}\,dt \leq CL_f K \sup_{(T_{n-1},T_n]} \|U(t) - V(t)\|_{l_\infty}.$$

Noting now that

$$|U_i(t) - V_i(t)| \leq \sum_n |\xi_{ijn} - \eta_{ijn}||\lambda_n^{[q_{ij}]}(t)| \leq C'\|x - y\|_\infty,$$

for $t \in I_{ij}$, we thus obtain

$$\|g(x) - g(y)\|_{l_\infty} \leq CC'L_f K \|x - y\|_{l_\infty}.$$

By Banach's fixed point theorem, we conclude that the fixed point iteration converges to a unique fixed point if $CC'L_f K < 1$. □

4. Stability estimates

In this section, we prove stability estimates for the multi-adaptive methods and the corresponding multi-adaptive dual methods. We consider the linear model problem

$$\begin{aligned} \dot u(t) + A(t)u(t) &= 0, \quad t \in (0,T], \\ u(0) &= u_0, \end{aligned} \tag{4.1}$$

where $A = A(t)$ is a piecewise smooth $N \times N$ matrix on $(0,T]$. The dual backward problem of (4.1) for $\phi = \phi(t)$ is then given by

$$\begin{aligned} -\dot\phi(t) + A^\top(t)\phi(t) &= 0, \quad t \in [0,T), \\ \phi(T) &= \psi. \end{aligned} \tag{4.2}$$

With $w(t) = \phi(T-t)$, we have $\dot w(t) = -\dot\phi(T-t) = -A^\top(T-t)w(t)$, and so (4.2) can be written as a forward problem for w in the form

$$\begin{aligned} \dot w(t) + B(t)w(t) &= 0, \quad t \in (0,T], \\ w(0) &= w_0, \end{aligned} \tag{4.3}$$

where $w_0 = \psi$ and $B(t) = A^\top(T-t)$. In the following discussion, w represents either u or $\phi(T-\cdot)$ and, correspondingly, W represents either the discrete mc/dG(q) approximation U of u or the discrete mc/dG(q)* approximation Φ of ϕ.

4.1. **Exponential stability estimates.** The stability estimates are based on the following version of the discrete Grönwall inequality.

Lemma 4.1. (A discrete Grönwall inequality) *Assume that $z, a : \mathbb{N} \to \mathbb{R}$ are non-negative, $a(m) \le 1/2$ for all m, and $z(n) \le C + \sum_{m=1}^n a(m)z(m)$ for all n. Then, for $n = 1, 2, \ldots$, we have*

$$z(n) \le 2C \exp\left(\sum_{m=1}^{n-1} 2a(m)\right). \tag{4.4}$$

Proof. By a standard discrete Grönwall inequality, we have $z(n) \le C \exp\left(\sum_{m=0}^{n-1} a(m)\right)$, if $z(n) \le C + \sum_{m=0}^{n-1} a(m)z(m)$ for $n \ge 1$ and $z(0) \le C$, see [8]. Here, $(1-a(n))z(n) \le C + \sum_{m=1}^{n-1} a(m)z(m)$, and so $z(n) \le 2C + \sum_{m=1}^{n-1} 2a(m)z(m)$, since $1 - a(n) \ge 1/2$. The result now follows if we take $a(0) = z(0) = 0$. □

Theorem 4.1. (Stability estimate) *Let W be the* mcG(q), mdG(q), mcG(q)*, *or* mdG(q)* *solution of* (4.3). *Then, there is a constant C_q of moderate size, depending only on the highest order* $\max q_{ij}$, *such that if*

$$K_n C_q \|B\|_{L_\infty([T_{n-1},T_n],l_p)} \le 1, \quad n = 1, \ldots, M, \tag{4.5}$$

then

$$\|W\|_{L_\infty([T_{n-1},T_n],l_p)} \le C_q \|w_0\|_{l_p} \exp\left(\sum_{m=1}^{n-1} K_m C_q \|B\|_{L_\infty([T_{m-1},T_m],l_p)}\right), \tag{4.6}$$

for $n = 1, \dots, M$, $1 \le p \le \infty$.

Proof. By Corollary 3.1, we can write the mcG(q), mdG(q), mcG(q)*, and mdG(q)* methods in the form

$$\xi_{ijn'} = w_i(0) + \int_0^{t_{i,j-1}} f_i(W,\cdot)\,dt + \int_{I_{ij}} w_{n'}^{[q_{ij}]}(\tau_{ij}(t)) f_i(W,\cdot)\,dt.$$

Applied to the linear model problem (4.3), we have

$$\xi_{ijn'} = w_i(0) - \int_0^{t_{i,j-1}} (BW)_i\,dt - \int_{I_{ij}} w_{n'}^{[q_{ij}]}(\tau_{ij}(t))(BW)_i\,dt,$$

and so

$$\begin{aligned} |\xi_{ijn'}| &\le |w_i(0)| + \left|\int_0^{t_{i,j-1}} (BW)_i\,dt\right| + \left|\int_{I_{ij}} w_{n'}^{[q_{ij}]}(\tau_{ij}(t))(BW)_i\,dt\right| \\ &\le |w_i(0)| + C\int_0^{t_{ij}} |(BW)_i|\,dt \le |w_i(0)| + C\int_0^{T_n} |(BW)_i|\,dt, \end{aligned}$$

where T_n is smallest synchronized time level for which $t_{ij} \le T_n$. It now follows that for all $t \in [T_{n-1}, T_n]$, we have $|W_i(t)| \le C|w_i(0)| + C\int_0^{T_n} |(BW)_i|\,dt$, and so

$$\|W(t)\|_{l_p} \le C\|w_0\|_{l_p} + C\int_0^{T_n} \|BW\|_{l_p}\,dt = C\|w_0\|_{l_p} + C\sum_{m=1}^{n}\int_{T_{m-1}}^{T_m} \|BW\|_{l_p}\,dt.$$

With $\bar{W}_n = \|W\|_{L_\infty([T_{n-1},T_n],l_p)}$, this means that

$$\begin{aligned} \bar{W}_n &\le C\|w_0\|_{l_p} + C\sum_{m=1}^{n} K_m \|B\|_{L_\infty([T_{m-1},T_m],l_p)} \bar{W}_m \\ &\equiv (C_q/2)\|w_0\|_{l_p} + \sum_{m=1}^{n} K_m (C_q/2)\|B\|_{L_\infty([T_{m-1},T_m],l_p)} \bar{W}_m. \end{aligned}$$

By assumption, $K_m C_q \|B\|_{L_\infty([T_{m-1},T_m],l_p)} \le 1$ for all m, and so the result follows by Lemma 4.1. □

4.2. **Stability estimates for parabolic problems.** We consider now the parabolic model problem $\dot{u}(t) + Au(t) = 0$, with A a symmetric, positive semidefinite, and constant $N \times N$ matrix, and prove stability estimates for the mdG(q) and mdG(q)* methods. As before, we write the problem in the form (4.3), and note that $B = A = A^\top$. We thus consider the problem: Find $w : [0,T] \to \mathbb{R}^N$, such that

$$\begin{aligned} \dot{w}(t) + Aw(t) &= 0, \quad t \in (0,T], \\ w(0) &= w_0. \end{aligned} \tag{4.7}$$

For the continuous problem (4.7), we have the following standard strong stability estimates, where "strong" indicates control of Aw (or $\dot{w}$) in terms of (the l_2-norm of) the initial data w_0.

Theorem 4.2. (Strong stability for the continuous problem) *The solution w of* (4.7) *satisfies for $T > 0$ and $0 < \epsilon < T$ with $\|\cdot\| = \|\cdot\|_{l_2}$,*

$$\|w(T)\|^2 + 2\int_0^T (Aw, w)\,dt = \|w_0\|^2, \tag{4.8}$$

$$\int_0^T t\,\|Aw\|^2\,dt \le \frac{1}{4}\|w_0\|^2, \tag{4.9}$$

$$\int_\epsilon^T \|Aw\|\,dt \le \frac{1}{2}(\log(T/\epsilon))^{1/2}\,\|w_0\|. \tag{4.10}$$

Proof. Multiply (4.7) with $v = w$, $v = tAw$, and $v = t^2A^2w$, respectively. See [2] for a full proof. □

We now prove an extension to multi-adaptive time-stepping of the strong stability estimate Lemma 6.1 in [1]. See also Lemma 1 in [3] for a similar estimate. In the proof, we use a special interpolant π, defined on the partition $\mathcal{T}$ as follows. On each local interval, the component $(\pi\varphi)_i$ of the interpolant $\pi\varphi$ of a given function $\varphi : [0,T] \to \mathbb{R}^N$, is defined by the following conditions: $(\pi\varphi_i)|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})$ interpolates φ_i at the left end-point $t^+_{i,j-1}$ of I_{ij} and $\pi\varphi_i - \varphi_i$ is orthogonal to $\mathcal{P}^{q_{ij}-1}(I_{ij})$. (This is the interpolant denoted by $\pi^{[q]}_{\mathrm{dG}^*}$ in [6].) We also introduce the left-continuous piecewise constant function $\bar{t} = \bar{t}(t)$ defined by $\bar{t}(t) = \min_{ij}\{t_{ij} : t \le t_{ij}\}$. With $\{t_m\}$ the ordered sequence of individual time levels $\{t_{ij}\}$, as illustrated in Figure 3, $\bar{t} = \bar{t}(t)$ is thus the piecewise constant function that takes the value t_m on $(t_{m-1}, t_m]$. We make the following assumption on the partition $\mathcal{T}$:

$$T_{n-1}\int_{T_{n-1}}^{T_n} (Av, Av)\,dt \le \gamma \int_{T_{n-1}}^{T_n} (Av, \pi(\bar{t}Av))\,dt, \quad n = 2,\ldots,M, \tag{4.11}$$

for all functions v in the trial (and test) space V of the mdG(q) and mdG(q)* methods, where $\gamma \ge 1$ is a constant of moderate size if Av is not close to being orthogonal to V. In the case of equal time steps for all components, this estimate is trivially true, because then $\pi Av = Av$ since $Av \in V$ if $v \in V$. Note that we may computationally test the validity of (4.11), see [7].

Theorem 4.3. (Strong stability for the discrete problem) *Let W be the* mdG(q) *or* mdG(q)* *solution of* (4.7)*, computed with the same time step and order for all components on the first time slab $\mathcal{T}_1$. Assume that* (4.11) *holds and that $\sigma K_n \le T_{n-1}$, $n = 2,\ldots,M$, for some* *constant $\sigma > 1$. Then, there is a constant $C = C(q, \gamma, \sigma)$, such that*

$$\|W(T)\|^2 + 2\int_0^T (AW, W)\,dt + \sum_{i=1}^N \sum_{j=1}^{M_i} [W_i]_{i,j-1}^2 = \|w_0\|^2, \tag{4.12}$$

$$\sum_{n=1}^M T_n \int_{T_{n-1}}^{T_n} \left\{\|\dot{W}\|^2 + \|AW\|^2\right\} dt + \sum_{n=1}^M T_n \sum_{ij} [W_i]_{i,j-1}^2 / k_{ij} \leq C\,\|w_0\|^2, \tag{4.13}$$

$$\int_0^T \left\{\|\dot{W}\| + \|AW\|\right\} dt + \sum_{n=1}^N \left(\sum_{ij} |[W_i]_{i,j-1}|^2\right)^{1/2} \leq C \left(\log\frac{T}{K_1} + 1\right)^{1/2} \|w_0\|, \tag{4.14}$$

where $\|\cdot\| = \|\cdot\|_{l_2}$, $\sum_{ij}$ denotes the sum over all elements within the current time slab $\mathcal{T}_n$, and where in all integrals the domain of integration does not include points where the integrand is discontinuous.

Proof. We follow the proof presented in [1] and make extensions where necessary. With V the trial (and test) space for the mdG(q) method defined in Section 2, the mdG(q) (or mdG(q)*) approximation W of w on a time slab $\mathcal{T}_n$ is defined as follows: Find $W \in V$, such that

$$\sum_{ij} \left([W_i]_{i,j-1} v_i(t_{i,j-1}^+) + \int_{I_{ij}} \dot{W}_i v_i\,dt \right) + \int_{T_{n-1}}^{T_n} (AW, v)\,dt = 0, \tag{4.15}$$

for all test functions $v \in V$, where the sum is taken over all intervals I_{ij} within the time slab $\mathcal{T}_n$. To prove the basic stability estimate (4.12), we take $v = W$ in (4.15) to get

$$\frac{1}{2}\sum_{ij}[W_i]_{i,j-1}^2 + \frac{1}{2}\|W(T_n^-)\|^2 - \frac{1}{2}\|W(T_{n-1}^-)\|^2 + \int_{T_{n-1}}^{T_n} (AW, W)\,dt = 0.$$

The estimate now follows by summation over all time slabs $\mathcal{T}_n$.

For the proof of (4.13), we would like to take $v = tAW$ in (4.15), but this is not a valid test function. In the proof of Lemma 6.1 in [1], the test function is chosen as $v = T_n AW$, which is not possible in the multi-adaptive case, since A mixes the components of W and as a result, $v_i = T_n(AW)_i$ may not be a test function for component W_i. Instead, we take $v = \pi(\bar{t}AW)$, with π and $\bar{t}$ defined as above, to obtain

$$\sum_{ij} \left([W_i]_{i,j-1}(\bar{t}(AW)_i)(t_{i,j-1}^+) + \int_{I_{ij}} \dot{W}_i\,\bar{t}(AW)_i\,dt \right) + \int_{T_{n-1}}^{T_n} (AW, \pi(\bar{t}AW))\,dt = 0,$$

where we have used the orthogonality condition of the interpolant for $\dot{W}_i \in \mathcal{P}^{q_{ij}-1}(I_{ij})$. Noting that $[W_i]_m = W_i(t_m^+) - W_i(t_m^-) = 0$ if component i has no node at time $t = t_m$, we rewrite the sum as a sum over all intervals $(t_{m-1}, t_m]$ within $(T_{n-1}, T_n]$, in the form

$$\sum_i \sum_m [W_i]_{m-1} t_m (AW(t_{m-1}^+))_i + \int_{I_m} \dot{W}_i\, t_m (AW)_i\,dt$$

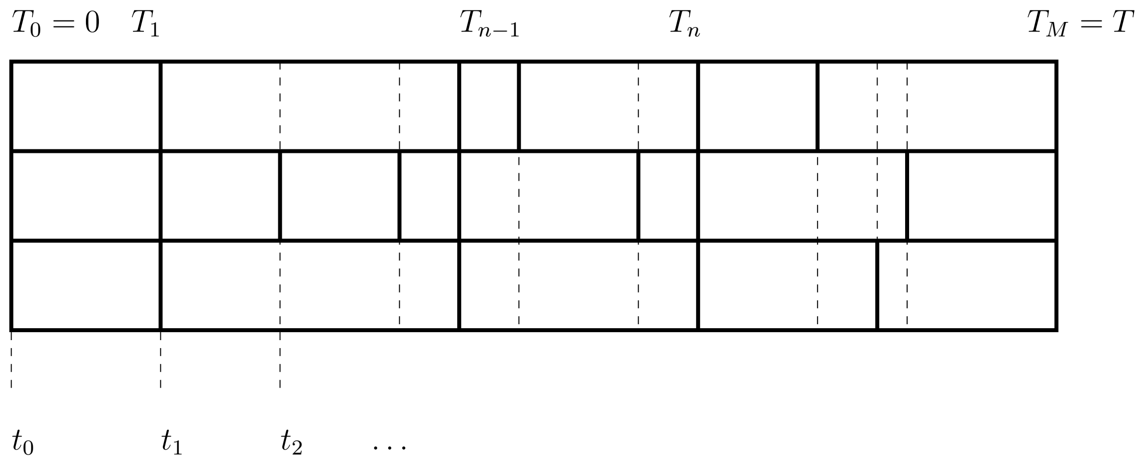

FIGURE 3. The sequence $\{t_m\}$ of individual time levels.

$$
\begin{aligned}
&= \sum_m t_m([W]_{m-1}, AW(t_{m-1}^+)) + \frac{t_m}{2} \int_{I_m} \frac{d}{dt}(W, AW)\, dt \\
&= \sum_m \frac{t_m}{2}([W]_{m-1}, A[W]_{m-1}) + \frac{t_m}{2} \left[(W(t_m^-), AW(t_m^-)) - (W(t_{m-1}^-), AW(t_{m-1}^-))\right],
\end{aligned}
$$

where, using the notation $k_m = t_m - t_{m-1}$, we note that

$$
\sum_m \frac{t_m}{2} \left[(W(t_m^-), AW(t_m^-)) - (W(t_{m-1}^-), AW(t_{m-1}^-))\right]
$$

$$
\begin{aligned}
&= \sum_m \frac{t_m}{2}(W(t_m^-), AW(t_m^-)) - \frac{t_{m-1}}{2}(W(t_{m-1}^-), AW(t_{m-1}^-)) - \frac{k_m}{2}(W(t_{m-1}^-), AW(t_{m-1}^-)) \\
&= \frac{T_n}{2}(W(T_n^-), AW(T_n^-)) - \frac{T_{n-1}}{2}(W(T_{n-1}^-), AW(T_{n-1}^-)) - \sum_m \frac{k_m}{2}(W(t_{m-1}^-), AW(t_{m-1}^-)).
\end{aligned}
$$

Collecting the terms, we thus have

(4.16)

$$
\begin{aligned}
\sum_m t_m([W]_{m-1}, A[W]_{m-1}) + T_n(W(T_n^-), AW(T_n^-)) - T_{n-1}(W(T_{n-1}^-), AW(T_{n-1}^-)) \quad + \\
+ \quad 2\int_{T_{n-1}}^{T_n} (AW, \pi(\bar{t}AW))\, dt = \sum_m k_m(W(t_{m-1}^-), AW(t_{m-1}^-)).
\end{aligned}
$$

For $n = 1$, we have

$$
\int_{T_{n-1}}^{T_n} (AW, \pi(\bar{t}AW))\, dt = T_1 \int_0^{T_1} (AW, \pi(AW))\, dt = T_1 \int_0^{T_1} \|AW\|^2\, dt,
$$

since $\pi(AW) = AW$ on $[0, T_1]$, where the same time steps and orders are used for all components. We further estimate the right-hand side of (4.16) as follows:

$$\begin{aligned} K_1(w_0, Aw_0) &= K_1([W]_0, A[W]_0) - K_1(W(0^+), AW(0^+)) + 2K_1(w_0, AW(0^+)) \\ &\leq T_1([W]_0, A[W]_0) + 2K_1(w_0, AW(0^+)) \\ &\leq T_1([W]_0, A[W]_0) + \frac{1}{\epsilon}\|w_0\|^2 + \epsilon K_1^2 \|AW(0^+)\|^2 \\ &\leq T_1([W]_0, A[W]_0) + \frac{1}{\epsilon}\|w_0\|^2 + \epsilon C_q T_1 \int_0^{T_1} \|AW\|^2 \, dt, \end{aligned}$$

where we have used an inverse estimate for AW on $[0, T_1]$. With $\epsilon = 1/C_q$, we thus obtain the estimate

$$T_1(W(T_1^-), AW(T_1^-)) + T_1 \int_0^{T_1} \|AW\|^2 \leq C_q \|w_0\|^2. \tag{4.17}$$

For $n > 1$, it follows by the assumption (4.11), that

$$\int_{T_{n-1}}^{T_n} (AW, \pi(\bar{t}AW)) \, dt \geq \gamma^{-1} T_{n-1} \int_{T_{n-1}}^{T_n} \|AW\|^2 \, dt \geq \frac{\gamma^{-1}\sigma}{\sigma + 1} T_n \int_{T_{n-1}}^{T_n} \|AW\|^2 \, dt,$$

where we have also used the assumption $T_{n-1} \geq \sigma K_n$. The terms on the right-hand side of (4.16) are now estimated as follows:

$$k_m(W(t_{m-1}^-), AW(t_{m-1}^-)) = k_m(W(t_{m-1}^+) - [W]_{m-1}, A(W(t_{m-1}^+) - [W]_{m-1}))$$

$$\begin{aligned} &= k_m \left[(W(t_{m-1}^+), AW(t_{m-1}^+)) + ([W]_{m-1}, A[W]_{m-1}) - 2(W(t_{m-1}^+), A[W]_{m-1}) \right] \\ &\leq k_m \left[(1 + \beta)(W(t_{m-1}^+), AW(t_{m-1}^+)) + (1 + \beta^{-1})([W]_{m-1}, A[W]_{m-1}) \right], \end{aligned}$$

for any $\beta > 0$. We choose $\beta = 1/(\sigma - 1)$, to obtain

$$\begin{aligned} k_m(W(t_{m-1}^-), AW(t_{m-1}^-)) &\leq \frac{\sigma k_m}{\sigma - 1}(W(t_{m-1}^+), AW(t_{m-1}^+)) + \sigma k_m([W]_{m-1}, A[W]_{m-1}) \\ &\leq \frac{C_q \sigma}{\sigma - 1} \int_{I_m} (W, AW) \, dt + \sigma k_m([W]_{m-1}, A[W]_{m-1}), \end{aligned}$$

where we have again used an inverse estimate, for W on I_m. From the assumption that $\sigma K_n \leq T_{n-1}$ for $n > 1$, it follows that $\sigma k_m \leq \sigma K_n \leq T_{n-1} \leq t_m$, and so

$$\begin{aligned} T_n(W(T_n^-), AW(T_n^-)) - T_{n-1}(W(T_{n-1}^-), AW(T_{n-1}^-)) + \frac{2\gamma^{-1}\sigma}{\sigma + 1} T_n \int_{T_{n-1}}^{T_n} \|AW\|^2 \, dt \\ \leq \frac{C_q \sigma}{\sigma - 1} \int_{T_{n-1}}^{T_n} (W, AW) \, dt. \end{aligned}$$

Summing over $n > 1$ and using the estimate (4.17) for $n = 1$, we obtain by (4.12),

$$T\left(W(T^-), AW(T^-)\right) + T_1 \int_0^{T_1} \|AW\|^2\, dt + \frac{2\gamma^{-1}\sigma}{\sigma+1} \sum_{n=2}^{M} T_n \int_{T_{n-1}}^{T_n} \|AW\|^2\, dt$$
$$\leq C_q \|w_0\|^2 + \frac{C_q \sigma}{\sigma - 1} \int_{T_1}^{T} (W, AW)\, dt \leq C_q \left(1 + \frac{\sigma/2}{\sigma - 1}\right) \|w_0\|^2,$$

which we write as

$$\sum_{n=1}^{M} T_n \int_{T_{n-1}}^{T_n} \|AW\|^2\, dt \leq C\, \|w_0\|^2,$$

noting that $T(W(T^-), AW(T^-)) \geq 0$. For the proof of (4.13), it now suffices to prove that

$$\int_{T_{n-1}}^{T_n} \|\dot{W}\|\, dt \leq C \int_{T_{n-1}}^{T_n} \|AW\|^2\, dt, \tag{4.18}$$

and

$$\sum_{ij} [W_i]_{i,j-1}^2 / k_{ij} \leq C \int_{T_{n-1}}^{T_n} \|AW\|^2\, dt. \tag{4.19}$$

To prove (4.18), we take $v_i = (t - t_{i,j-1})\dot{W}_i / k_{ij}$ on each local interval I_{ij} in (4.15), which gives

$$\sum_{ij} \int_{I_{ij}} \frac{t - t_{i,j-1}}{k_{ij}} \dot{W}_i^2\, dt = -\sum_{ij} \int_{I_{ij}} (AW)_i \frac{t - t_{i,j-1}}{k_{ij}} \dot{W}_i\, dt$$
$$\leq \sum_{ij} \left(\int_{I_{ij}} \frac{t - t_{i,j-1}}{k_{ij}} (AW)_i^2\, dt \right)^{1/2} \left(\int_{I_{ij}} \frac{t - t_{i,j-1}}{k_{ij}} \dot{W}_i^2\, dt \right)^{1/2}$$
$$\leq \left(\sum_{ij} \int_{I_{ij}} \frac{t - t_{i,j-1}}{k_{ij}} (AW)_i^2\, dt \right)^{1/2} \left(\sum_{ij} \int_{I_{ij}} \frac{t - t_{i,j-1}}{k_{ij}} \dot{W}_i^2\, dt \right)^{1/2},$$

where we have used Cauchy's inequality twice; first on $L_2(I_{ij})$ and then on l_2. Using an inverse estimate for $\dot{W}_i^2$, we obtain

$$\int_{T_{n-1}}^{T_n} \|\dot{W}\|^2\, dt = \sum_{ij} \int_{I_{ij}} \dot{W}_i^2\, dt \leq C_q \sum_{ij} \int_{I_{ij}} \frac{t - t_{i,j-1}}{k_{ij}} \dot{W}_i^2\, dt$$
$$\leq C_q \sum_{ij} \int_{I_{ij}} \frac{t - t_{i,j-1}}{k_{ij}} (AW)_i^2\, dt \leq C_q \sum_{ij} \int_{I_{ij}} (AW)_i^2\, dt = C_q \int_{T_{n-1}}^{T_n} \|AW\|\, dt,$$

which proves (4.18).

To prove (4.19), we take $v_i = [W_i]_{i,j-1}/k_{ij}$ on each local interval I_{ij} in (4.15), which gives

$$\sum_{ij} [W_i]_{i,j-1}^2/k_{ij} = -\sum_{ij} \int_{I_{ij}} (\dot{W}_i + (AW)_i)\, [W_i]_{i,j-1}/k_{ij}\, dt$$

$$\leq \left(\sum_{ij} \int_{I_{ij}} (\dot{W}_i + (AW)_i)^2\, dt \right)^{1/2} \left(\sum_{ij} [W_i]_{i,j-1}^2/k_{ij} \right)^{1/2},$$

where we have again used Cauchy's inequality twice. We thus have

$$\sum_{ij} [W_i]_{i,j-1}^2/k_{ij} \leq 2 \int_{T_{n-1}}^{T_n} \|\dot{W}\|^2\, dt + 2 \int_{T_{n-1}}^{T_n} \|AW\|^2\, dt,$$

and so (4.19) follows, using (4.18). This also proves (4.13).

Finally, to prove (4.14), we use Cauchy's inequality with (4.13) to get

$$\sum_{n=1}^{M} \int_{T_{n-1}}^{T_n} \|\dot{W}\|\, dt = \sum_{n=1}^{M} \sqrt{K_n/T_n}\, \sqrt{T_n/K_n} \int_{T_{n-1}}^{T_n} \|\dot{W}\|\, dt$$

$$\leq \left(\sum_{n=1}^{M} K_n/T_n \right)^{1/2} \left(\sum_{n=1}^{M} T_n \int_{T_{n-1}}^{T_n} \|\dot{W}\|^2\, dt \right)^{1/2}$$

$$\leq \left(1 + \int_{T_1}^{T} \frac{1}{t}\, dt \right)^{1/2} C\|w_0\| \leq C\left(\log(T/K_1) + 1\right)^{1/2} \|w_0\|,$$

with a similar estimate for AW. The proof is now complete, noting that

$$\sum_{n=1}^{M} \left(\sum_{ij} |[W_i]_{i,j-1}|^2 \right)^{1/2} \leq \sum_{n=1}^{M} \left(\frac{K_n}{T_n} T_n \sum_{ij} |[W_i]_{i,j-1}|^2/k_{ij} \right)^{1/2}$$

$$\leq \left(\sum_{n=1}^{M} K_n/T_n \right)^{1/2} \left(\sum_{n=1}^{M} T_n \sum_{ij} |[W_i]_{i,j-1}|^2/k_{ij} \right)^{1/2}$$

$$\leq C(\log(T/K_1) + 1)^{1/2} \|w_0\|.$$

□

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