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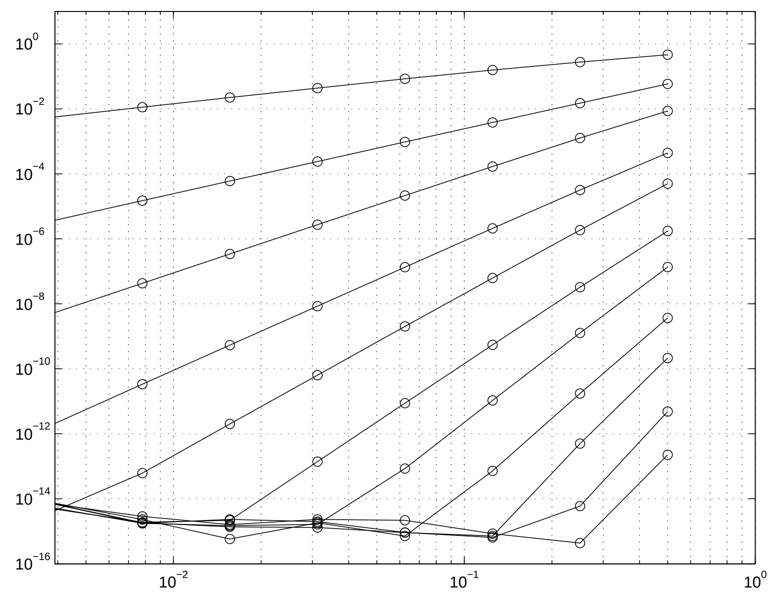

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MULTI-ADAPTIVE GALERKIN METHODS FOR ODES IV: A PRIORI ERROR ESTIMATES

ANDERS LOGG

Abstract. We prove general order a priori error estimates for the multi-adaptive continuous and discontinuous Galerkin methods mcG(q) and mdG(q). To prove the error estimates, we represent the error in terms of the residual of an interpolant of the exact solution, and a discrete dual solution. The estimates then follow from interpolation estimates, together with stability estimates for the discrete dual solution. For the general non-linear problem, we obtain exponential stability estimates, using a Grönwall argument, and for a parabolic model problem, we show that the stability factor is of unit size.

1. Introduction

This is part IV in a sequence of papers [4, 5, 8] on multi-adaptive Galerkin methods, mcG(q) and mdG(q), for approximate (numerical) solution of ODEs of the form

$$\begin{aligned} \dot{u}(t) &= f(u(t), t), \quad t \in (0, T], \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where $u : [0, T] \to \mathbb{R}^N$ is the solution to be computed, $u_0 \in \mathbb{R}^N$ a given initial condition, $T > 0$ a given final time, and $f : \mathbb{R}^N \times (0, T] \to \mathbb{R}^N$ a given function that is Lipschitz-continuous in u and bounded.

The mcG(q) and mdG(q) methods are based on piecewise polynomial approximation of degree q on partitions in time with time steps which may vary for different components $U_i(t)$ of the approximate solution $U(t)$ of (1.1). In part I and II of our series on multi-adaptive Galerkin methods, we prove a posteriori error estimates, through which the time steps are adaptively determined from residual feed-back and stability information, obtained by solving a dual linearized problem. In part III, we prove existence and stability of discrete solutions. In the current paper, we prove a priori error estimates for the mcG(q) and mdG(q) methods.

1.1. **Main results.** The main results of this paper are a priori error estimates for the mcG(q) and mdG(q) methods respectively, of the form

$$\|e(T)\|_{l_p} \leq CS(T) \|k^{2q} u^{(2q)}\|_{L_\infty([0,T], l_1)}, \tag{1.2}$$

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Key words and phrases. Multi-adaptivity, individual time steps, local time steps, ODE, continuous Galerkin, discontinuous Galerkin, mcgq, mdgq, a priori error estimates.

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and

$$\|e(T)\|_{l_p} \leq CS(T)\|k^{2q+1}u^{(2q+1)}\|_{L_\infty([0,T],l_1)}, \tag{1.3}$$

for $p = 2$ or $p = \infty$, where C is an interpolation constant, $S(T)$ is a (computable) stability factor, and $k^{2q}u^{(2q)}$ (or $k^{2q+1}u^{(2q+1)}$) combines local time steps $k = (k_{ij})$ with derivatives of the exact solution u. These estimates state that the mcG(q) method is of order $2q$ and that the mdG(q) method is of order $2q+1$ in the local time step. We refer to Section 5 for the exact results. For the general non-linear problem, we obtain exponential estimates for the stability factor $S(T)$, and for a parabolic model problem we show that the stability factor remains bounded and of unit size, independent of T (up to a logarithmic factor).

1.2. **Notation.** For a detailed description of the multi-adaptive Galerkin methods, we refer the reader to [4, 5, 8]. In particular, we refer to [4] or [8] for the definition of the methods.

The following notation is used throughout this paper: Each component $U_i(t)$, $i = 1,\ldots,N$, of the approximate m(c/d)G(q) solution $U(t)$ of (1.1) is a piecewise polynomial on a partition of $(0,T]$ into M_i subintervals. Subinterval j for component i is denoted by $I_{ij} = (t_{i,j-1}, t_{ij}]$, and the length of the subinterval is given by the local *time step* $k_{ij} = t_{ij} - t_{i,j-1}$. This is illustrated in Figure 1. On each subinterval I_{ij}, $U_i|_{I_{ij}}$ is a polynomial of degree q_{ij} and we refer to $(I_{ij}, U_i|_{I_{ij}})$ as an *element*.

Furthermore, we shall assume that the interval $(0,T]$ is partitioned into blocks between certain synchronized time levels $0 = T_0 < T_1 < \ldots < T_M = T$. We refer to the set of intervals $\mathcal{T}_n$ between two synchronized time levels T_{n-1} and T_n as a *time slab*:

$$\mathcal{T}_n = \{I_{ij} : T_{n-1} \leq t_{i,j-1} < t_{ij} \leq T_n\}.$$

We denote the length of a time slab by $K_n = T_n - T_{n-1}$.

1.3. **Outline of the paper.** The outline of the paper is as follows. In Section 2, we first discuss the dual problem that forms the basic tool of the a priori error analysis, and how this differs from the dual problem we formulate in [4] for the a posteriori error analysis. We then, in Section 3, derive a representation of the error in terms of the dual solution and the residual of an interpolant of the exact solution.

In Section 4, we present interpolation results for piecewise smooth functions proved in [7, 6]. We then prove the a priori error estimates in Section 5, starting from the error representation and using the interpolation estimates together with the stability estimates from [8]. Finally, in Section 6, we present some numerical evidence for the a priori error estimates. In particular, we solve a simple model problem and show that we obtain the predicted convergence rates.

2. The dual problem

In [4], we prove *a posteriori* error estimates for the multi-adaptive methods, by deriving a representation for the error $e = U - u$, where $U : [0,T] \to \mathbb{R}^N$ is the computed approximate

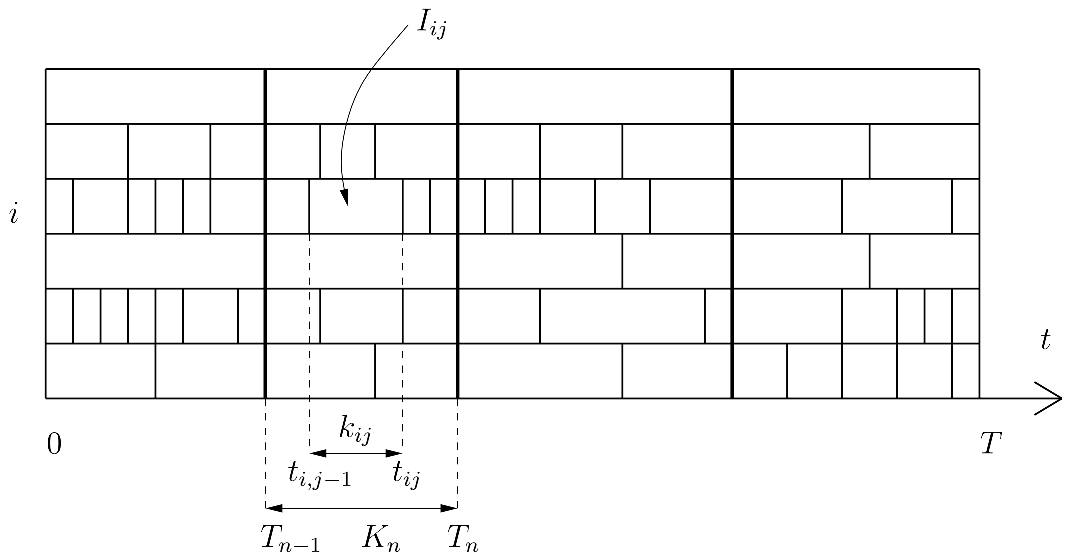

FIGURE 1. Individual partitions of the interval $(0, T]$ for different components. Elements between common synchronized time levels are organized in time slabs. In this example, we have $N = 6$ and $M = 4$.

solution of (1.1), in terms of the residual $R(U, \cdot) = \dot{U} - f(U, \cdot)$ and the solution $\phi : [0, T] \to \mathbb{R}^N$ of the continuous linearized dual problem

$$\begin{aligned} -\dot{\phi}(t) &= J^\top(u, U, t)\phi(t) + g(t), \quad t \in [0, T), \\ \phi(T) &= \psi, \end{aligned} \tag{2.1}$$

with given data $g : [0, T] \to \mathbb{R}^N$ and $\psi \in \mathbb{R}^N$, where

$$J^\top(u, U, t) = \left(\int_0^1 \frac{\partial f}{\partial u}(su(t) + (1 - s)U(t), t)\, ds \right)^\top . \tag{2.2}$$

To prove *a priori* error estimates, we derive an error representation in terms of the residual $R(\pi u, \cdot)$ of an interpolant πu of the exact solution u, and a discrete dual solution Φ, following the same approach as in [3] and [1]. The discrete dual solution Φ is defined as a Galerkin solution of the continuous linearized dual problem

$$\begin{aligned} -\dot{\phi}(t) &= J^\top(\pi u, U, t)\phi(t) + g(t), \quad t \in [0, T), \\ \phi(T) &= \psi, \end{aligned} \tag{2.3}$$

where we note that J is now evaluated at a mean value of πu and U. We will use the notation $f^*(\phi, \cdot) = J^\top(\pi u, U, \cdot)\phi + g$, to write the dual problem (2.3) in the form

$$\begin{aligned} -\dot{\phi}(t) &= f^*(\phi(t), t), \quad t \in [0, T), \\ \phi(T) &= \psi. \end{aligned} \tag{2.4}$$

We refer to [8] for the exact definition of the discrete dual solution Φ.

We will also derive a priori error estimates for linear problems of the form

$$\begin{aligned} \dot{u}(t) + A(t)u(t) &= 0, \quad t \in (0, T], \\ u(0) &= u_0, \end{aligned} \tag{2.5}$$

with $A(t)$ a bounded $N \times N$-matrix, in particular for a parabolic model problem with $A(t)$ a positive semidefinite and symmetric matrix . For the linear problem (2.5), the discrete dual solution Φ is defined as a Galerkin solution of the continuous dual problem

$$\begin{aligned} -\dot{\phi}(t) + A^\top(t)\phi(t) &= g, \quad t \in [0, T), \\ \phi(T) &= \psi, \end{aligned} \tag{2.6}$$

which takes the form (2.4) with the notation $f^*(\phi, \cdot) = -A^\top \phi + g$.

3. Error representation

In this section, we derive the error representations on which the a priori error estimates are based. For each of the two methods, mcG(q) and mdG(q), we represent the error in terms of the discrete dual solution Φ and an interpolant πu of the exact solution u of (1.1), using the special interpolants $\pi u = \pi_{\mathrm{cG}}^{[q]} u$ or $\pi u = \pi_{\mathrm{dG}}^{[q]} u$ defined in Section 5 of [7]. The error representations are presented in the general non-linear case, and thus apply to the linear problem (2.5), with corresponding dual problem (2.6), in particular.

We write the error $e = U - u$ as

$$e = \bar{e} + (\pi u - u), \tag{3.1}$$

where $\bar{e} \equiv U - \pi u$ is represented in terms of the discrete dual solution and the residual of the interpolant. An estimate for the second part of the error, $\pi u - u$, follows directly from an interpolation estimate. In Theorem 3.1 below, we derive the error representation for the mcG(q) method, and then derive the corresponding representation for the mdG(q) method in Theorem 3.2.

Theorem 3.1. (Error representation for mcG(q)) *Let U be the* mcG(q) *solution of* (1.1)*, let Φ be the corresponding* mcG$(q)^*$ *solution of the dual problem* (2.4)*, and let πu be any trial space approximation of the exact solution u of* (1.1) *that interpolates u at the end-points of every local interval. Then,*

$$L_{\psi,g}(\bar{e}) \equiv (\bar{e}(T), \psi) + \int_0^T (\bar{e}, g)\, dt = -\int_0^T (R(\pi u, \cdot), \Phi)\, dt,$$

where $\bar{e} \equiv U - \pi u$.

Proof. Since $\bar{e}(0) = 0$, we can choose $\bar{e}$ as a test function for the discrete dual. By the definition of the mcG$(q)^*$ solution Φ (see [8]), we thus have

$$\int_0^T (\dot{\bar{e}}, \Phi)\, dt = \int_0^T (J(\pi u, U, \cdot)\bar{e}, \Phi)\, dt + L_{\psi,g}(\bar{e}),$$

and so, by the definition of J, we have

$$\begin{aligned} L_{\psi,g}(\bar{e}) &= \int_0^T (\dot{\bar{e}} - J(\pi u, U, \cdot)\bar{e}, \Phi)\,dt = \int_0^T (\dot{\bar{e}} - f(U,\cdot) + f(\pi u,\cdot), \Phi)\,dt \\ &= \int_0^T (R(U,\cdot) - R(\pi u,\cdot), \Phi)\,dt = -\int_0^T (R(\pi u,\cdot),\Phi)\,dt, \end{aligned}$$

since Φ is a test function for U. □

Theorem 3.2. (Error representation for mdG(q)) *Let U be the* mdG(q) *solution of* (1.1)*, let Φ be the corresponding* mdG$(q)^*$ *solution of the dual problem* (2.4)*, and let πu be any trial space approximation of the exact solution u of* (1.1) *that interpolates u at the right end-point of every local interval. Then,*

$$L_{\psi,g}(\bar{e}) \equiv (\bar{e}(T),\psi) + \int_0^T (\bar{e}, g)\,dt = -\sum_{i=1}^N \sum_{j=1}^{M_i} \left[[\pi u_i]_{i,j-1}\Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} R_i(\pi u,\cdot)\Phi_i\,dt \right],$$

where $\bar{e} \equiv U - \pi u$.

Proof. Choosing $\bar{e}$ as a test function for the discrete dual we obtain, by the definition of the mdG$(q)^*$ method (see [8]),

$$\sum_{i=1}^N \sum_{j=1}^{M_i} \left[[\bar{e}_i]_{i,j-1}\Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} \dot{\bar{e}}_i \Phi_i\,dt \right] = \int_0^T (J(\pi u, U, \cdot)\bar{e}, \Phi)\,dt + L_{\psi,g}(\bar{e}),$$

and so, by the definition of J, we have

$$\begin{aligned} L_{\psi,g}(\bar{e}) &= \sum_{i=1}^N \sum_{j=1}^{M_i} \left[[\bar{e}_i]_{i,j-1}\Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} \dot{\bar{e}}_i \Phi_i\,dt \right] - \int_0^T (J(\pi u, U, \cdot)\bar{e}, \Phi)\,dt \\ &= \sum_{i=1}^N \sum_{j=1}^{M_i} \left[[U_i - \pi u_i]_{i,j-1}\Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} (R_i(U,\cdot) - R_i(\pi u,\cdot))\Phi_i\,dt \right] \\ &= -\sum_{i=1}^N \sum_{j=1}^{M_i} \left[[\pi u_i]_{i,j-1}\Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} R_i(\pi u,\cdot)\Phi_i\,dt \right], \end{aligned}$$

since Φ is a test function for U. □

With a special choice of interpolant, $\pi u = \pi_{\mathrm{cG}}^{[q]} u$ and $\pi u = \pi_{\mathrm{dG}}^{[q]} u$ respectively, we obtain the following versions of the error representations.

Corollary 3.1. (Error representation for mcG(q)) *Let U be the* mcG(q) *solution of* (1.1)*, let Φ be the corresponding* mcG$(q)^*$ *solution of the dual problem* (2.4)*, and let $\pi_{\mathrm{cG}}^{[q]} u$ be an interpolant, as defined in* [7]*, of the exact solution u of* (1.1)*. Then,*

$$L_{\psi,g}(\bar{e}) = \int_0^T (f(\pi_{\mathrm{cG}}^{[q]} u, \cdot) - f(u,\cdot), \Phi)\,dt.$$

Proof. The residual of the exact solution is zero and so, by Theorem 3.1, we have

$$\begin{aligned} L_{\psi,g}(\bar{e}) &= \int_0^T (R(u,\cdot) - R(\pi_{\mathrm{cG}}^{[q]}u,\cdot),\Phi)\,dt \\ &= \int_0^T (f(\pi_{\mathrm{cG}}^{[q]}u,\cdot) - f(u,\cdot),\Phi)\,dt + \int_0^T (\frac{d}{dt}(u - \pi_{\mathrm{cG}}^{[q]}u),\Phi)\,dt, \end{aligned}$$

where we note that

$$\int_0^T (\frac{d}{dt}(u - \pi_{\mathrm{cG}}^{[q]}u),\Phi)\,dt = 0,$$

by the construction of the interpolant $\pi_{\mathrm{cG}}^{[q]}u$ (Lemma 5.2 in [7]). □

Corollary 3.2. (Error representation for mdG(q)) *Let* U *be the* mdG(q) *solution of* (1.1)*, let* Φ *be the corresponding* mdG$(q)^*$ *solution of the dual problem* (2.4)*, and let* $\pi_{\mathrm{dG}}^{[q]}u$ *be an interpolant, as defined in* [7]*, of the exact solution* u *of* (1.1)*. Then,*

$$L_{\psi,g}(\bar{e}) = \int_0^T (f(\pi_{\mathrm{dG}}^{[q]}u,\cdot) - f(u,\cdot),\Phi)\,dt.$$

Proof. The residual of the exact solution is zero, and the jump of the exact solution is zero at every node. Thus, by Theorem 3.2,

$$\begin{aligned} L_{\psi,g}(\bar{e}) &= \sum_{i=1}^{N}\sum_{j=1}^{M_i}\left[[u_i - \pi_{\mathrm{dG}}^{[q_{ij}]}u_i]_{i,j-1}\Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} (R_i(u,\cdot) - R_i(\pi_{\mathrm{dG}}^{[q]}u,\cdot))\Phi_i\,dt \right] \\ &= \int_0^T (f(\pi_{\mathrm{dG}}^{[q]}u,\cdot) - f(u,\cdot),\Phi)\,dt, \end{aligned}$$

where we have used the fact that $\pi_{\mathrm{dG}}^{[q]}u$ interpolates u at the right end-point of every local interval, and thus that

$$\sum_{i=1}^{N}\sum_{j=1}^{M_i}\left[[u_i(t_{i,j-1}^+) - \pi_{\mathrm{dG}}^{[q_{ij}]}u_i(t_{i,j-1}^+)]\Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} (\frac{d}{dt}(u_i - \pi_{\mathrm{dG}}^{[q_{ij}]}u_i))\Phi_i\,dt \right] = 0,$$

by the construction of the interpolant $\pi_{\mathrm{dG}}^{[q]}u$ (Lemma 5.3 in [7]). □

3.1. **A note on quadrature errors.** In the derivation of the error representations, we have used the Galerkin orthogonalities for the mcG(q) and mdG(q) solutions. For the mcG(q) method, we have assumed that

$$\int_0^T (R(U,\cdot),\Phi)\,dt = 0$$

in the proof of Theorem 3.1, and for the mdG(q)method, we have assumed that

$$\sum_{i=1}^{N}\sum_{j=1}^{M_i}\left[[U_i]_{i,j-1}\Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} R_i(U,\cdot)\Phi_i\,dt \right] = 0$$

in the proof of Theorem 3.2. In the presence of quadrature errors, these terms are nonzero. As a result, we obtain additional terms of the form

$$\int_0^T (\tilde{f}(U,\cdot) - f(U,\cdot), \Phi)\, dt,$$

where $\tilde{f}$ is the interpolant of f corresponding the quadrature rule that is used. Typically, Lobatto quadrature (with $q+1$ nodal points) is used for the mcG(q) method, which means that the quadrature error is of order $2(q+1)-2 = 2q$ and so (super-) convergence of order $2q$ is obtained also in the presence of quadrature errors. Similarly for the mdG(q) method, we use Radau quadrature with $q+1$ nodal points, which means that the quadrature error is of order $2(q+1)-1 = 2q+1$, and so the $2q+1$ convergence order of mdG(q) is also maintained under quadrature.

4. Interpolation estimates

To prove the a priori error estimates, starting from the error representations derived in the previous section, we need special interpolation estimates. These estimates are proved in [6], based on the interpolation estimates of [7]. In this section, we present the interpolation estimates, first for the general non-linear problem and then for linear problems, and refer to [7, 6] for the proofs.

4.1. **The general non-linear problem.**

In order to prove the interpolation estimates for the general non-linear problem, we need to make the following assumptions: Given a time slab $\mathcal{T}$, assume that for each pair of local intervals I_{ij} and I_{mn} within the time slab, we have

$$q_{ij} = q_{mn} = \bar{q}, \tag{A1}$$

and

$$k_{ij} > \alpha\, k_{mn}, \tag{A2}$$

for some $\bar{q} \geq 0$ and some $\alpha \in (0,1)$. We also assume that the problem (1.1) is autonomous,

$$\frac{\partial f_i}{\partial t} = 0, \quad i = 1,\ldots,N, \tag{A3}$$

noting that the dual problem nevertheless is non-autonomous in general. Furthermore, we assume that

$$\|f_i\|_{D^{\bar{q}+1}(\mathcal{T})} < \infty, \quad i = 1,\ldots,N, \tag{A4}$$

where $\|\cdot\|_{D^p(\mathcal{T})}$ is defined for $v : \mathbb{R}^N \to \mathbb{R}$ and $p \geq 0$ by $\|v\|_{D^p(\mathcal{T})} = \max_{n=0,\ldots,p} \|D^n v\|_{L_\infty(\mathcal{T},l_\infty)}$, with

$$\|D^n v\, w^1 \cdots w^n\|_{L_\infty(\mathcal{T})} \leq \|D^n v\|_{L_\infty(\mathcal{T},l_\infty)} \|w^1\|_{l_\infty} \cdots \|w^n\|_{l_\infty} \quad \forall\, w^1,\ldots,w^n \in \mathbb{R}^N, \tag{4.5}$$

and $D^n v$ the nth-order tensor given by

$$D^n v\, w^1 \cdots w^n = \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N \frac{\partial^n v}{\partial x_{i_1} \cdots \partial x_{i_n}}\, w^1_{i_1} \cdots w^n_{i_n}.$$

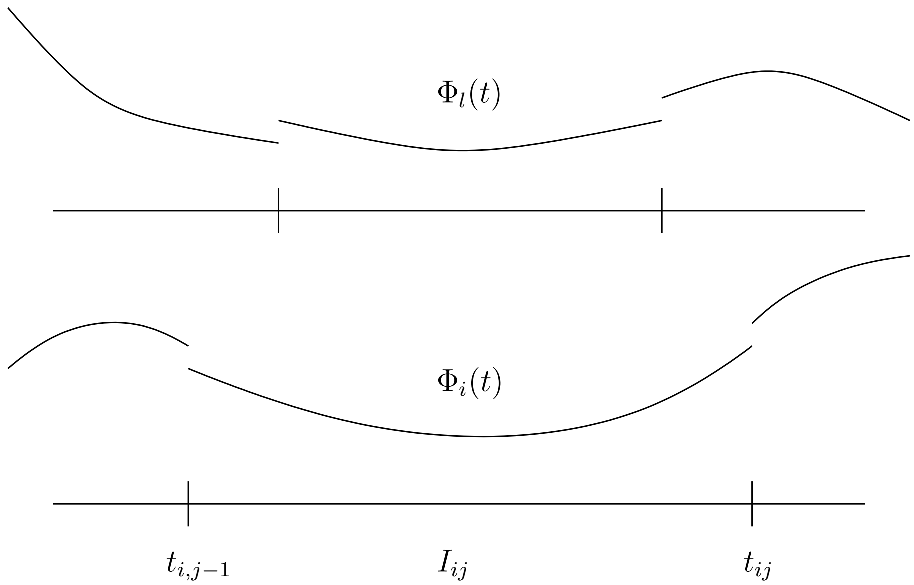

FIGURE 2. If some other component $l \neq i$ has a node within I_{ij}, then Φ_l may be discontinuous within I_{ij}, causing φ_i to be discontinuous within I_{ij}.

Furthermore, we choose $\|f\|_{\mathcal{T}} \geq \max_{i=1,\ldots,N} \|f_i\|_{D^{\bar{q}+1}(\mathcal{T})}$, such that

$$\|d^p/dt^p(\partial f/\partial u)^\top(x(t))\|_{l_\infty} \leq \|f\|_{\mathcal{T}} C_x^p, \tag{4.6}$$

for $p = 0, \ldots, \bar{q}$, and

$$\|[d^p/dt^p(\partial f/\partial u)^\top(x(t))]_t\|_{l_\infty} \leq \|f\|_{\mathcal{T}} \sum_{n=0}^{p} C_x^{p-n} \|[x^{(n)}]_t\|_{l_\infty}, \tag{4.7}$$

for $p = 0, \ldots, \bar{q}-1$ and any given $x : \mathbb{R} \to \mathbb{R}^N$, where $C_x > 0$ denotes a constant, such that $\|x^{(n)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C_x^n$, for $n = 1, \ldots, p$. Note that assumption (A4) implies that each f_i is bounded by $\|f\|_{\mathcal{T}}$. We further assume that there is a constant $c_k > 0$, such that

$$k_{ij} \|f\|_{\mathcal{T}} \leq c_k, \tag{A5}$$

for each local interval I_{ij}. We summarize the list of assumptions as follows:

(A1) the local orders q_{ij} are equal within each time slab;
(A2) the local time steps k_{ij} are semi-uniform within each time slab;
(A3) f is autonomous;
(A4) f and its derivatives are bounded;
(A5) the local time steps k_{ij} are small.

To derive a priori error estimates for the non-linear problem (1.1), we need to estimate the interpolation error $\pi\varphi_i - \varphi_i$ on a local interval I_{ij}, where φ_i is defined by

$$\varphi_i = (J^\top(\pi u, u)\Phi)_i = \sum_{l=1}^{N} J_{li}(\pi u, u)\Phi_l, \quad i = 1, \ldots, N. \tag{4.9}$$

We note that φ_i may be discontinuous within I_{ij}, if I_{ij} contains a node for some other component, which is generally the case with multi-adaptive time-stepping. This is illustrated in Figure 2. Using assumptions (A1)–(A5), we obtain the following interpolation estimates for the function φ.

Lemma 4.1. (Interpolation estimates for φ) *Let φ be defined as in* (4.9)*. If assumptions* (A1)–(A5) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|\pi_{\mathrm{cG}}^{[q_{ij}-2]}\varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}-1} \|f\|_{\mathcal{T}}^{q_{ij}} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad q_{ij} = \bar{q} \geq 2, \tag{4.10}$$

and

$$\|\pi_{\mathrm{dG}}^{[q_{ij}-1]}\varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}} \|f\|_{\mathcal{T}}^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad q_{ij} = \bar{q} \geq 1, \tag{4.11}$$

for each local interval I_{ij} within the time slab $\mathcal{T}$.

Proof. See [6]. □

4.2. **Linear problems.** For the linear problem (2.5), we make the following basic assumptions: Given a time slab $\mathcal{T}$, assume that for each pair of local intervals I_{ij} and I_{mn} within the time slab, we have

$$q_{ij} = q_{mn} = \bar{q}, \tag{B1}$$

and

$$k_{ij} > \alpha\ k_{mn}, \tag{B2}$$

for some $\bar{q} \geq 0$ and some $\alpha \in (0, 1)$. Furthermore, assume that A has $\bar{q} - 1$ continuous derivatives and let $C_A > 0$ be constant, such that

$$\max\left(\|A^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)}, \|A^{\top(p)}\|_{L_\infty(\mathcal{T},l_\infty)}\right) \leq C_A^{p+1}, \quad p = 0, \ldots, \bar{q}, \tag{B3}$$

for all time slabs $\mathcal{T}$. We further assume that there is a constant $c_k > 0$, such that

$$k_{ij} C_A \leq c_k. \tag{B4}$$

We summarize the list of assumptions as follows:

(B1) the local orders q_{ij} are equal within each time slab;
(B2) the local time steps k_{ij} are semi-uniform within each time slab;
(B3) A and its derivatives are bounded;
(B4) the local time steps k_{ij} are small.

As for the general non-linear problem, we need to estimate the interpolation error $\pi\varphi_i - \varphi_i$ on a local interval I_{ij}, where φ_i is now defined by

$$\varphi_i = (A^\top \Phi)_i = \sum_{l=1}^{N} A_{li} \Phi_l, \quad i = 1, \ldots, N. \tag{4.16}$$

Using assumptions (B1)–(B4), we obtain the following interpolation estimates for the function φ.

Lemma 4.2. (Interpolation estimates for φ) *Let φ be defined as in* (4.16)*. If assumptions* (B1)–(B4) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|\pi_{\mathrm{cG}}^{[q_{ij}-2]}\varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}-1} C_A^{q_{ij}} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}, \quad q_{ij} = \bar{q} \geq 2, \tag{4.17}$$

and

$$\|\pi_{\mathrm{dG}}^{[q_{ij}-1]}\varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}} C_A^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}, \quad q_{ij} = \bar{q} \geq 1, \tag{4.18}$$

for each local interval I_{ij} within the time slab $\mathcal{T}$.

Proof. See [6]. □

5. A priori error estimates

Using the error representations derived in Section 3, the interpolation estimates of the previous section, and the stability estimates from [8], we now derive our main results: a priori error estimates for general order mcG(q) and mdG(q). The estimates are derived first for the general non-linear problem (1.1), then for the general linear problem (2.5), and finally for a parabolic model problem.

5.1. **The general non-linear problem.**

Theorem 5.1. (A priori error estimate for mcG(q)) *Let U be the* mcG(q) *solution of* (1.1)*, and let Φ be the corresponding* mcG(q)* *solution of the dual problem* (2.4)*. Then, there is a constant $C = C(q) > 0$, such that*

$$|L_{\psi,g}(\bar{e})| \leq C S(T) \|k^{q+1}\bar{u}^{q+1}\|_{L_\infty([0,T],l_2)}, \tag{5.1}$$

where $(k^{q+1}\bar{u}^{(q+1)})_i(t) = k_{ij}^{q_{ij}+1}\|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ for $t \in I_{ij}$, and where the stability factor $S(T)$ is given by $S(T) = \int_0^T \|J^\top(\pi_{\mathrm{cG}}^{[q]}u, u, \cdot)\Phi\|_{l_2}\,dt$. Furthermore, if assumptions (A1)–(A5) *hold and $g = 0$ in (2.4), then there is a constant $C = C(q, c_k, \alpha) > 0$, such that*

$$|L_{\psi,g}(\bar{e})| \leq C \bar{S}(T) \|k^{2q}\bar{\bar{u}}^{(2q)}\|_{L_\infty([0,T],l_1)}, \tag{5.2}$$

where $(k^{2q}\bar{\bar{u}}^{(2q)})_i(t) = k_{ij}^{2q_{ij}}\|f\|_{\mathcal{T}}^{q_{ij}-1}\|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ for $t \in I_{ij}$, and where the stability factor $\bar{S}(T)$ is given by

$$\bar{S}(T) = \int_0^T \|f\|_{\mathcal{T}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\,dt = \sum_{n=1}^M K_n\|f\|_{\mathcal{T}_n}\|\Phi\|_{L_\infty(\mathcal{T}_n,l_\infty)}.$$

Proof. By Corollary 3.1, we obtain

$$\begin{aligned} L_{\psi,g}(\bar{e}) &= \int_0^T (f(\pi_{\mathrm{cG}}^{[q]}u, \cdot) - f(u, \cdot), \Phi)\,dt = \int_0^T (J(\pi_{\mathrm{cG}}^{[q]}u, u, \cdot)(\pi_{\mathrm{cG}}^{[q]}u - u), \Phi)\,dt \\ &= \int_0^T (\pi_{\mathrm{cG}}^{[q]}u - u, J^\top(\pi_{\mathrm{cG}}^{[q]}u, u, \cdot)\Phi)\,dt. \end{aligned}$$

By Theorem 5.1 in [7], it now follows that

$$|L_{\psi,g}(\bar{e})| \leq C\|k^{q+1}\bar{u}^{q+1}\|_{L_\infty([0,T],l_2)} \int_0^T \|J^\top(\pi_{\mathrm{cG}}^{[q]}u,u,\cdot)\Phi\|_{l_2}\,dt,$$

which proves (5.1). To prove (5.2), we note that by definition, $\pi_{\mathrm{cG}}^{[q_{ij}]}u_i - u_i$ is orthogonal to $\mathcal{P}^{q_{ij}-2}(I_{ij})$ for each local interval I_{ij}, and so, recalling that $\varphi = J^\top(\pi_{\mathrm{cG}}^{[q]}u,u,\cdot)\Phi$,

$$L_{\psi,g}(\bar{e}) = \sum_{i=1}^N \sum_{j=1}^{M_i} \int_{I_{ij}} (\pi_{\mathrm{cG}}^{[q_{ij}]}u_i - u_i)\varphi_i\,dt = \sum_{i=1}^N \sum_{j=1}^{M_i} \int_{I_{ij}} (\pi_{\mathrm{cG}}^{[q_{ij}]}u_i - u_i)(\varphi_i - \pi_{\mathrm{cG}}^{[q_{ij}-2]}\varphi_i)\,dt,$$

where we take $\pi_{\mathrm{cG}}^{[q_{ij}-2]}\varphi_i \equiv 0$ for $q_{ij} = 1$. By Theorem 5.1 in [7] and Lemma 4.1, it now follows that

$$\begin{aligned}
|L_{\psi,g}(\bar{e})| &\leq \int_0^T |(\pi_{\mathrm{cG}}^{[q]}u - u, \varphi - \pi_{\mathrm{cG}}^{[q-2]}\varphi)|\,dt \\
&= \int_0^T |(k^{q-1}\|f\|_{\mathcal{T}}^{q-1}(\pi_{\mathrm{cG}}^{[q]}u - u), k^{-(q-1)}\|f\|_{\mathcal{T}}^{-(q-1)}(\varphi - \pi_{\mathrm{cG}}^{[q-2]}\varphi))|\,dt \\
&\leq C\|k^{2q}\bar{\bar{u}}^{(2q)}\|_{L_\infty([0,T],l_1)} \int_0^T \|f\|_{\mathcal{T}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\,dt = C\bar{S}(T)\|k^{2q}\bar{\bar{u}}^{(2q)}\|_{L_\infty([0,T],l_1)},
\end{aligned}$$

where the stability factor $\bar{S}(T)$ is given by

$$\bar{S}(T) = \int_0^T \|f\|_{\mathcal{T}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\,dt = \sum_{n=1}^M K_n\|f\|_{\mathcal{T}_n}\|\Phi\|_{L_\infty(\mathcal{T}_n,l_\infty)}.$$

□

Theorem 5.2. (A priori error estimate for mdG(q)) *Let U be the* mdG(q) *solution of* (1.1)*, and let Φ be the corresponding* mdG(q)* *solution of the dual problem* (2.4)*. Then, there is a constant $C = C(q) > 0$, such that*

$$|L_{\psi,g}(\bar{e})| \leq CS(T)\|k^{q+1}\bar{u}^{q+1}\|_{L_\infty([0,T],l_2)}, \tag{5.3}$$

where $(k^{q+1}\bar{u}^{(q+1)})_i(t) = k_{ij}^{q_{ij}+1}\|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ for $t \in I_{ij}$, and where the stability factor $S(T)$ is given by $S(T) = \int_0^T \|J^\top(\pi_{\mathrm{dG}}^{[q]}u,u,\cdot)\Phi\|_{l_2}\,dt$. Furthermore, if assumptions (A1)–(A5) *hold and $g = 0$ in (2.4), then there is a constant $C = C(q, c_k, \alpha) > 0$, such that*

$$|L_{\psi,g}(\bar{e})| \leq C\bar{S}(T)\|k^{2q+1}\bar{\bar{u}}^{(2q+1)}\|_{L_\infty([0,T],l_1)}, \tag{5.4}$$

where $(k^{2q+1}\bar{\bar{u}}^{(2q+1)})_i(t) = k_{ij}^{2q_{ij}+1}\|f\|_{\mathcal{T}}^{q_{ij}}\|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ for $t \in I_{ij}$, and where the stability factor $\bar{S}(T)$ is given by

$$\bar{S}(T) = \int_0^T \|f\|_{\mathcal{T}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\,dt = \sum_{n=1}^M K_n\|f\|_{\mathcal{T}_n}\|\Phi\|_{L_\infty(\mathcal{T}_n,l_\infty)}.$$

Proof. The estimate (5.3) is obtained in the same way as we obtained the estimate (5.1). To prove (5.4), we note that as in the proof of Theorem 5.1, we obtain $L_{\psi,g}(\bar{e}) = \int_0^T (\pi_{\mathrm{dG}}^{[q]} u - u, \varphi)\, dt$. By definition, $\pi_{\mathrm{dG}}^{[q_{ij}]} u_i - u_i$ is orthogonal to $\mathcal{P}^{q_{ij}-1}(I_{ij})$ for each local interval I_{ij}, and so

$$L_{\psi,g}(\bar{e}) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{M_i}\int_{I_{ij}} (\pi_{\mathrm{dG}}^{[q_{ij}]} u_i - u_i)\varphi_i \, dt = \sum_{i=1}^{N}\sum_{j=1}^{M_i}\int_{I_{ij}} (\pi_{\mathrm{dG}}^{[q_{ij}]} u_i - u_i)(\varphi_i - \pi_{\mathrm{dG}}^{[q_{ij}-1]}\varphi_i)\, dt,$$

where we take $\pi_{\mathrm{dG}}^{[q_{ij}-1]}\varphi_i = 0$ for $q_{ij} = 0$. By Theorem 5.1 in [7] and Lemma 4.1, it now follows that

$$\begin{aligned} |L_{\psi,g}(\bar{e})| &\le \int_0^T |(\pi_{\mathrm{dG}}^{[q]} u - u, \varphi - \pi_{\mathrm{dG}}^{[q-1]}\varphi)|\, dt \\ &= \int_0^T |(k^q \|f\|_{\mathcal{T}}^q (\pi_{\mathrm{dG}}^{[q]} u - u), k^{-q}\|f\|_{\mathcal{T}}^{-q}(\varphi - \pi_{\mathrm{dG}}^{[q-1]}\varphi))|\, dt \\ &\le C\|k^{2q+1}\bar{\bar{u}}^{(2q+1)}\|_{L_\infty([0,T],l_1)} \int_0^T \|f\|_{\mathcal{T}} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\, dt. \end{aligned}$$

□

Using the stability estimates proved in [8], we obtain the following bound for the stability factor $\bar{S}(T)$.

Lemma 5.1. *Assume that $K_n C_q \|f\|_{\mathcal{T}_n} \le 1$ for all time slabs $\mathcal{T}_n$, with $C_q = C_q(q) > 0$ the constant in Theorem 4.1 of* [8]*, and take $g = 0$. Then,*

$$\bar{S}(T) \le \|\psi\|_{l_\infty} e^{C_q \|f\|_{[0,T]} T}, \tag{5.5}$$

where $\|f\|_{[0,T]} = \max_{n=1,\ldots,M} \|f\|_{\mathcal{T}_n}$.

Proof. By Theorem 4.1 in [8], we obtain

$$\|\Phi\|_{L_\infty(\mathcal{T}_n,l_\infty)} \le C_q \|\psi\|_{l_\infty} \exp\left(\sum_{m=n+1}^{M} K_m C_q \|f\|_{\mathcal{T}_m}\right) \le C_q \|\psi\|_{l_\infty} e^{C_q \|f\|_{[0,T]}(T-T_n)},$$

and so

$$\begin{aligned} \bar{S}(T) &= \sum_{n=1}^{M} K_n \|f\|_{\mathcal{T}_n} \|\Phi\|_{L_\infty(\mathcal{T}_n,l_\infty)}\, dt \le \|\psi\|_{l_\infty} \sum_{n=1}^{M} K_n C_q \|f\|_{[0,T]} e^{C_q\|f\|_{[0,T]}(T-T_n)} \\ &\le \|\psi\|_{l_\infty} \int_0^T C_q \|f\|_{[0,T]} e^{C_q \|f\|_{[0,T]} t}\, dt \le \|\psi\|_{l_\infty} e^{C_q\|f\|_{[0,T]}T}. \end{aligned}$$

□

Finally, we rewrite the estimates 5.1 and 5.2 for special choices of data ψ and g. We first take $\psi = 0$. With $g_n = 0$ for $n \ne i$, $g_i(t) = 0$ for $t \notin I_{ij}$, and

$$g_i(t) = \mathrm{sgn}(\bar{e}_i(t))/k_{ij}, \quad t \in I_{ij},$$

we obtain $L_{\psi,g}(\bar{e}) = \frac{1}{k_{ij}} \int_{I_{ij}} |\bar{e}_i(t)|\, dt$ and so $\|\bar{e}_i\|_{L_\infty(I_{ij})} \leq C L_{\psi,g}(\bar{e})$ by an inverse estimate. By the definition of $\bar{e}$, it follows that $\|e_i\|_{L_\infty(I_{ij})} \leq C L_{\psi,g}(\bar{e}) + C k_{ij}^{q_{ij}+1} \|u_i^{q_{ij}+1}\|_{L_\infty(I_{ij})}$. Note that for this choice of g, we have $\|g\|_{L_1([0,T],l_2)} = \|g\|_{L_1([0,T],l_\infty)} = 1$.

We also make the choice $g = 0$. Noting that $\bar{e}(T) = e(T)$, since $\pi u(T) = u(T)$, we obtain

$$L_{\psi,g}(\bar{e}) = (e(T), \psi) = |e_i(T)|,$$

for $\psi_i = \mathrm{sgn}(e_i(T))$ and $\psi_n = 0$ for $n \neq i$, and

$$L_{\psi,g}(\bar{e}) = (e(T), \psi) = \|e(T)\|_{l_2},$$

for $\psi = e(T)/\|e(T)\|_{l_2}$. Note that for both choices of ψ, we have $\|\psi\|_{l_\infty} \leq 1$.

With these choices of data, we obtain the following versions of the a priori error estimates.

Corollary 5.1. (A priori error estimate for mcG(q)) *Let U be the* mcG(q) *solution of* (1.1). *Then, there is a constant $C = C(q) > 0$, such that*

$$\|e\|_{L_\infty([0,T],l_\infty)} \leq C S(T) \|k^{q+1} \bar{u}^{q+1}\|_{L_\infty([0,T],l_2)}, \tag{5.6}$$

where the stability factor $S(T) = \int_0^T \|J^\top(\pi_{\mathrm{cG}}^{[q]} u, u, \cdot)\Phi\|_{l_2}\, dt$ is taken as the maximum over $\psi = 0$ and $\|g\|_{L_1([0,T],l_\infty)} = 1$. Furthermore, if assumptions (A1)–(A5) *and the assumptions of Lemma 5.1 hold, then there is a constant $C = C(q, c_k, \alpha)$, such that*

$$\|e(T)\|_{l_p} \leq C \bar{S}(T) \|k^{2q} \bar{\bar{u}}^{(2q)}\|_{L_\infty([0,T],l_1)}, \tag{5.7}$$

for $p = 2, \infty$, where the stability factor $\bar{S}(T)$ is given by $\bar{S}(T) = e^{C_q \|f\|_{[0,T]} T}$.

Corollary 5.2. (A priori error estimate for mdG(q)) *Let U be the* mdG(q) *solution of* (1.1). *Then, there is a constant $C = C(q) > 0$, such that*

$$\|e\|_{L_\infty([0,T],l_\infty)} \leq C S(T) \|k^{q+1} \bar{u}^{q+1}\|_{L_\infty([0,T],l_2)}, \tag{5.8}$$

where the stability factor $S(T) = \int_0^T \|J^\top(\pi_{\mathrm{dG}}^{[q]} u, u, \cdot)\Phi\|_{l_2}\, dt$ is taken as the maximum over $\psi = 0$ and $\|g\|_{L_1([0,T],l_\infty)} = 1$. Furthermore, if assumptions (A1)–(A5) *and the assumptions of Lemma 5.1 hold, then there is a constant $C = C(q, c_k, \alpha)$, such that*

$$\|e(T)\|_{l_p} \leq C \bar{S}(T) \|k^{2q+1} \bar{\bar{u}}^{(2q+1)}\|_{L_\infty([0,T],l_1)}, \tag{5.9}$$

for $p = 2, \infty$, where the stability factor $\bar{S}(T)$ is given by $\bar{S}(T) = e^{C_q \|f\|_{[0,T]} T}$.

5.2. **Linear problems.**

Theorem 5.3. (A priori error estimate for mcG(q)) *Let U be the* mcG(q) *solution of* (2.5), *and let Φ be the corresponding* mcG(q)* *solution of the dual problem* (2.6). *Then, there is a constant $C = C(q) > 0$, such that*

$$|L_{\psi,g}(\bar{e})| \leq C S(T) \|k^{q+1} \bar{u}^{q+1}\|_{L_\infty([0,T],l_2)}, \tag{5.10}$$

where $(k^{q+1}\bar{u}^{(q+1)})_i(t) = k_{ij}^{q_{ij}+1} \|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ for $t \in I_{ij}$, and where the stability factor $S(T)$ is given by $S(T) = \int_0^T \|A^\top \Phi\|_{l_2}\, dt$. Furthermore, if assumptions (B1)–(B4) *hold and $g = 0$ in (2.6), then there is a constant $C = C(q, c_k, \alpha) > 0$, such that*

$$|L_{\psi,g}(\bar{e})| \leq C \bar{S}(T) \|k^{2q} \bar{\bar{u}}^{(2q)}\|_{L_\infty([0,T],l_1)}, \tag{5.11}$$

where $(k^{2q}\bar{\bar{u}}^{(2q)})_i(t) = k_{ij}^{2q_{ij}} C_A^{q_{ij}-1} \|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ *for* $t \in I_{ij}$*, and where the stability factor* $\bar{S}(T)$ *is given by*

$$\bar{S}(T) = \int_0^T C_A \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\, dt = \sum_{n=1}^M K_n C_A \|\Phi\|_{L_\infty(\mathcal{T}_n,l_\infty)}.$$

Proof. By Corollary 3.1, we obtain

$$L_{\psi,g}(\bar{e}) = \int_0^T (A(u - \pi_{\mathrm{cG}}^{[q]} u), \Phi)\, dt = \int_0^T (u - \pi_{\mathrm{cG}}^{[q]}, A^\top \Phi)\, dt.$$

By Theorem 5.1 in [7], it now follows that $|L_{\psi,g}(\bar{e})| \leq C\|k^{q+1}\bar{u}^{q+1}\|_{L_\infty([0,T],l_2)} \int_0^T \|A^\top\Phi\|_{l_2}\, dt$, which proves (5.10). To prove (5.11), we note that by definition, $\pi_{\mathrm{cG}}^{[q_{ij}]} u_i - u_i$ is orthogonal to $\mathcal{P}^{q_{ij}-2}(I_{ij})$ for each local interval I_{ij}, and so

$$L_{\psi,g}(\bar{e}) \;=\; \sum_{i=1}^N \sum_{j=1}^{M_i} \int_{I_{ij}} (u_i - \pi_{\mathrm{cG}}^{[q_{ij}]} u_i)\varphi_i\, dt = \sum_{i=1}^N \sum_{j=1}^{M_i} \int_{I_{ij}} (u_i - \pi_{\mathrm{cG}}^{[q_{ij}]} u_i)(\varphi_i - \pi_{\mathrm{cG}}^{[q_{ij}-2]}\varphi_i)\, dt,$$

where $\varphi = A^\top \Phi$. By Theorem 5.1 in [7] and Lemma 4.2, it now follows that

$$\begin{aligned}
|L_{\psi,g}(\bar{e})| &\leq \int_0^T |(\pi_{\mathrm{cG}}^{[q]} u - u, \varphi - \pi_{\mathrm{cG}}^{[q-2]}\varphi)|\, dt \\
&= \int_0^T |(k^{q-1}C_A^{q-1}(\pi_{\mathrm{cG}}^{[q]} u - u), k^{-(q-1)}C_A^{-(q-1)}(\varphi - \pi_{\mathrm{cG}}^{[q-2]}\varphi))|\, dt \\
&\leq C\|k^{2q}\bar{\bar{u}}^{(2q)}\|_{L_\infty([0,T],l_1)} \int_0^T C_A\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\, dt = C\bar{S}(T)\|k^{2q}\bar{\bar{u}}^{(2q)}\|_{L_\infty([0,T],l_1)},
\end{aligned}$$

where the stability factor $\bar{S}(T)$ is given by

$$\bar{S}(T) = \int_0^T C_A \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\, dt = \sum_{n=1}^M K_n C_A \|\Phi\|_{L_\infty(\mathcal{T}_n,l_\infty)}.$$

□

Theorem 5.4. (A priori error estimate for mdG(q)) *Let* U *be the* mdG(q) *solution of* (2.5)*, and let* Φ *be the corresponding* mdG$(q)^*$ *solution of the dual problem* (2.6)*. Then, there is a constant* $C = C(q) > 0$*, such that*

$$|L_{\psi,g}(\bar{e})| \leq CS(T)\|k^{q+1}\bar{u}^{q+1}\|_{L_\infty([0,T],l_2)}, \tag{5.12}$$

where $(k^{q+1}\bar{u}^{(q+1)})_i(t) = k_{ij}^{q_{ij}+1}\|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ *for* $t \in I_{ij}$*, and where the stability factor* $S(T)$ *is given by* $S(T) = \int_0^T \|A^\top\Phi\|_{l_2}\, dt$*. Furthermore, if assumptions* (B1)–(B4) *hold and* $g = 0$ *in (2.6), then there is a constant* $C = C(q, c_k, \alpha) > 0$*, such that*

$$|L_{\psi,g}(\bar{e})| \leq C\bar{S}(T)\|k^{2q+1}\bar{\bar{u}}^{(2q+1)}\|_{L_\infty([0,T],l_1)}, \tag{5.13}$$

where $(k^{2q+1}\bar{\bar{u}}^{(2q+1)})_i(t) = k_{ij}^{2q_{ij}+1} C_A^{q_{ij}} \|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ *for* $t \in I_{ij}$*, and where the stability factor* $\bar{S}(T)$ *is given by*

$$\bar{S}(T) = \int_0^T C_A \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\, dt = \sum_{n=1}^{M} K_n C_A \|\Phi\|_{L_\infty(\mathcal{T}_n,l_\infty)}.$$

Proof. The estimate (5.12) is obtained in the same way as we obtained the estimate (5.10). To prove (5.13), we note that as in the proof of Theorem 5.1, we obtain $L_{\psi,g}(\bar{e}) = \int_0^T (u - \pi_{\mathrm{dG}}^{[q]} u, \varphi)\, dt$. By definition, $\pi_{\mathrm{dG}}^{[q_{ij}]} u_i - u_i$ is orthogonal to $\mathcal{P}^{q_{ij}-1}(I_{ij})$ for each local interval I_{ij}, and so

$$L_{\psi,g}(\bar{e}) = \sum_{i=1}^{N}\sum_{j=1}^{M_i} \int_{I_{ij}} (u_i - \pi_{\mathrm{dG}}^{[q_{ij}]} u_i)\varphi_i\, dt = \sum_{i=1}^{N}\sum_{j=1}^{M_i} \int_{I_{ij}} (u_i - \pi_{\mathrm{dG}}^{[q_{ij}]} u_i)(\varphi_i - \pi_{\mathrm{dG}}^{[q_{ij}-1]}\varphi_i)\, dt.$$

By Theorem 5.1 in [7] and Lemma 4.2, it now follows that

$$\begin{aligned} |L_{\psi,g}(\bar{e})| &\le \int_0^T |(\pi_{\mathrm{dG}}^{[q]} u - u, \varphi - \pi_{\mathrm{dG}}^{[q-1]}\varphi)|\, dt \\ &= \int_0^T |(k^q C_A^q(\pi_{\mathrm{dG}}^{[q]} u - u), k^{-q} C_A^{-q}(\varphi - \pi_{\mathrm{dG}}^{[q-1]}\varphi))|\, dt \\ &\le C \|k^{2q+1}\bar{\bar{u}}^{(2q+1)}\|_{L_\infty([0,T],l_1)} \int_0^T C_A \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\, dt. \end{aligned}$$

$\square$

We now use Lemma 5.1 to obtain a bound for the stability factor $\bar{S}(T)$. As for the non-linear problem, we note that for special choices of data ψ and g for the dual problem, we obtain error estimates in various norms, in particular the l_2-norm at final time.

Corollary 5.3. (A priori error estimate for mcG(q)) *Let* U *be the* mcG(q) *solution of* (2.5)*. Then, there is a constant* $C = C(q) > 0$*, such that*

$$\|e\|_{L_\infty([0,T],l_\infty)} \le C S(T) \|k^{q+1}\bar{u}^{q+1}\|_{L_\infty([0,T],l_2)}, \tag{5.14}$$

where the stability factor $S(T) = \int_0^T \|A^\top \Phi\|_{l_2}\, dt$ *is taken as the maximum over* $\psi = 0$ *and* $\|g\|_{L_1([0,T],l_\infty)} = 1$*. Furthermore, if assumptions* (B1)–(B4) *and the assumptions of Lemma 5.1 hold, then there is a constant* $C = C(q, c_k, \alpha)$*, such that*

$$\|e(T)\|_{l_p} \le C\bar{S}(T) \|k^{2q}\bar{\bar{u}}^{(2q)}\|_{L_\infty([0,T],l_1)}, \tag{5.15}$$

for $p = 2, \infty$*, where the stability factor* $\bar{S}(T)$ *is given by* $\bar{S}(T) = e^{C_q C_A T}$.

Corollary 5.4. (A priori error estimate for mdG(q)) *Let* U *be the* mdG(q) *solution of* (2.5)*. Then, there is a constant* $C = C(q) > 0$*, such that*

$$\|e\|_{L_\infty([0,T],l_\infty)} \le C S(T) \|k^{q+1}\bar{u}^{q+1}\|_{L_\infty([0,T],l_2)}, \tag{5.16}$$

where the stability factor $S(T) = \int_0^T \|A^\top \Phi\|_{l_2}\, dt$ *is taken as the maximum over* $\psi = 0$ *and* $\|g\|_{L_1([0,T],l_\infty)} = 1$. *Furthermore, if assumptions* (B1)–(B4) *and the assumptions of Lemma 5.1 hold, then there is a constant* $C = C(q, c_k, \alpha)$, *such that*

$$\|e(T)\|_{l_p} \leq C\bar{S}(T)\|k^{2q+1}\bar{\bar{u}}^{(2q+1)}\|_{L_\infty([0,T],l_1)}, \tag{5.17}$$

for $p = 2, \infty$, *where the stability factor* $\bar{S}(T)$ *is given by* $\bar{S}(T) = e^{C_q C_A T}$.

5.3. **Parabolic problems.** We consider the linear parabolic model problem,

$$\begin{aligned} \dot{u}(t) + A(t)u(t) &= 0, \quad t \in (0, T], \\ u(0) &= u_0, \end{aligned} \tag{5.18}$$

with A a positive semidefinite and symmetric $N \times N$ matrix for all $t \in (0, T]$, and prove an a priori error estimate for the mdG(q) method. The estimate is based on the error representation for the mdG(q) method presented in Section 3 and the strong stability estimate derived in [8]. To prove the a priori error estimate, we need to make the following assumptions. We first assume that q is constant within each time slab, that is, for each pair of intervals I_{ij} and I_{mn} within a given time slab, we have as before

$$q_{ij} = q_{mn} = \bar{q}. \tag{C1}$$

Furthermore, we assume that A is invertible on $(0, T)$ for $q \geq 2$ and refer to this as assumption (C2).

Additional assumptions, (C3)–(C6), are needed for the strong stability estimate of [8]. We first assume that there is a constant $\gamma \geq 1$, such that

$$(T - T_n)\int_{T_{n-1}}^{T_n} (Av, Av)\, dt \leq \gamma \int_{T_{n-1}}^{T_n} (Av, \pi(\bar{t}Av))\, dt, \quad n = 1, \ldots, M-1, \tag{C3}$$

for all trial functions v, that is, all v discontinuous and piecewise polynomial with $v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})$, where $\bar{t} = \bar{t}(t)$ is the piecewise constant right-continuous function defined by $\bar{t}(t) = \min_{ij}\{T - t_{i,j-1} : t \geq t_{i,j-1}\}$. If Av is not close to being orthogonal to the trial space, then γ is of moderate size. We also assume that there is a constant $\sigma > 1$, such that

$$\sigma K_n \leq (T - T_n), \quad n = 1, \ldots, M-1. \tag{C4}$$

This condition corresponds to the condition $\sigma K_n \leq T_{n-1}$ used in the strong stability estimate for the discrete dual problem in [8]. We further assume that all components use the same time step on the last time slab $\mathcal{T}_M$,

$$k_{ij} = K_M \quad \forall\, I_{ij} \in \mathcal{T}_M. \tag{C5}$$

Finally, we assume that A is constant and refer to this as assumption (C6).

Theorem 5.5. (A priori error estimate for parabolic problems) *Let* U *be the* mdG(q) *solution of* (5.18), *and assume that* (C1) *and* (C2) *hold. Then, there is a constant* $C = C(q)$, *such that*

$$\|e(T)\|_{l_2} \leq CS(T)\max_{[0,T]}\|k^{2q+1}A^q\bar{u}^{(q+1)}\|_{l_2} + \mathcal{E}, \tag{5.23}$$

where $(k^{2q+1}A^q\bar{u}^{(q+1)})_i(t) = k_{ij}^{2q_{ij}+1}\|(A^{q_{ij}}u)_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$ *for* $t \in I_{ij}$, $\mathcal{E} = 0$ *for* $q = 0$, *and* $\mathcal{E} = \int_0^T (\pi_{\mathrm{dG}}^{[q]} A^q u - A^q \pi_{\mathrm{dG}}^{[q]} u, A^{1-q}\Phi)\, dt$ *for* $q > 0$. *The stability factor* $S(T)$ *is given by*

$$S(T) = \int_0^T \|k^{-q}(A^{1-q}\Phi - \pi_{\mathrm{dG}}^{[q-1]} A^{1-q}\Phi)\|_{l_2}\, dt. \tag{5.24}$$

For $q = 0, 1$, *we obtain the following analytical bound for* $S(T)$, *using assumptions* (C3)–(C6),

$$S(T) \leq C\left(\log \frac{T}{K_M} + 1\right)^{1/2}, \tag{5.25}$$

where $C = C(q, \gamma, \sigma) > 0$.

Proof. With $\psi = e(T)/\|e(T)\|_{l_2}$ and $g = 0$, it follows by Corollary 3.2 that

$$\begin{aligned}
\|e(T)\|_{l_2} &= \int_0^T (u - \pi_{\mathrm{dG}}^{[q]} u, A\Phi)\, dt = \int_0^T (A^q(u - \pi_{\mathrm{dG}}^{[q]} u), A^{1-q}\Phi)\, dt \\
&= \int_0^T (A^q u - \pi_{\mathrm{dG}}^{[q]} A^q u + \pi_{\mathrm{dG}}^{[q]} A^q u - A^q \pi_{\mathrm{dG}}^{[q]} u, A^{1-q}\Phi)\, dt \\
&= \int_0^T (A^q u - \pi_{\mathrm{dG}}^{[q]} A^q u, A^{1-q}\Phi)\, dt + \int_0^T (\pi_{\mathrm{dG}}^{[q]} A^q u - A^q \pi_{\mathrm{dG}}^{[q]} u, A^{1-q}\Phi)\, dt \\
&= \int_0^T (A^q u - \pi_{\mathrm{dG}}^{[q]} A^q u, A^{1-q}\Phi - \pi_{\mathrm{dG}}^{[q-1]} A^{1-q}\Phi)\, dt + \mathcal{E},
\end{aligned}$$

where we have assumed that A is invertible for $q \geq 2$. With $S(T) = \int_0^T \|k^{-q}(A^{1-q}\Phi - \pi_{\mathrm{dG}}^{[q-1]} A^{1-q}\Phi)\|_{l_2}\, dt$, we thus obtain

$$\begin{aligned}
\|e(T)\|_{l_2} &\leq S(T) \max_{[0,T]} \|k^q(A^q u - \pi_{\mathrm{dG}}^{[q]} A^q u)\|_{l_2} + \mathcal{E} \\
&= S(T) \max_{[0,T]} \left(\sum_{i=1}^N [k_i^{q_i}((A^q u)_i - \pi_{\mathrm{dG}}^{[q_i]}(A^q u)_i)]^2\right)^{1/2} + \mathcal{E} \\
&\leq C S(T) \max_{t\in[0,T]} \left(\sum_{i=1}^N [k_i^{2q_i(t)+1}(t)\|(A^q u)_i^{(q_i(t)+1)}\|_{L_\infty(I_{ij}(t))}]^2\right)^{1/2} + \mathcal{E} \\
&= C S(T) \max_{[0,T]} \|k^{2q+1} A^q \bar{u}^{(q+1)}\|_{l_2} + \mathcal{E}.
\end{aligned}$$

Note that we use an interpolation estimate for $A^q u$ which is straightforward since the exact solution u is smooth. We conclude by estimating the stability factor $S(T)$ for $q = 0, 1$, using the strong stability estimate for the discrete dual solution Φ. For $q = 0$, it follows directly by Theorem 4.3 in [8], that

$$S(T) = \int_0^T \|A\Phi\|_{l_2}\, dt \leq C\left(\log \frac{T}{K_M} + 1\right)^{1/2},$$

and for $q = 1$, we obtain

$$S(T) = \int_0^T \|k^{-1}(\Phi - \pi_{\mathrm{dG}}^{[0]}\Phi)\|_{l_2}\, dt \leq C \int_0^T \|\dot{\Phi}\|_{l_2}\, dt,$$

using an interpolation estimate in combination with an inverse estimate, and so the estimate $S(T) \leq C \left(\log \frac{T}{K_M} + 1\right)^{1/2}$ follows again by Theorem 4.3 in [8].

□

The stability factor $S(T)$ that appears in the a priori error estimate is obtained from the discrete solution Φ of the dual problem (2.6), and can thus be computed exactly by solving the discrete dual problem. Allowing numerical computation of the stability factor, the additional assumptions (C3)–(C6) needed to obtain the analytical bound for $S(T)$ are no longer needed. Numerical computation of the stability factor also directly reveals whether the problem is parabolic or not; if the stability factor is of unit size and does not grow, then the problem is parabolic by definition, see [2].

6. Numerical examples

We conclude by demonstrating the convergence of the multi-adaptive methods in the case of a simple test problem. We also present some results in support of assumption (C3).

6.1. **Convergence.** Consider the problem

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= -u_1, \\ \dot{u}_3 &= -u_2 + 2u_4, \\ \dot{u}_4 &= u_1 - 2u_3, \\ \dot{u}_5 &= -u_2 - 2u_4 + 4u_6, \\ \dot{u}_6 &= u_1 + 2u_3 - 4u_5, \end{aligned} \tag{6.1}$$

on $[0, 1]$ with initial condition $u(0) = (0, 1, 0, 2, 0, 3)$. The solution is given by $u(t) = (\sin t, \cos t, \sin t + \sin 2t, \cos t + \cos 2t, \sin t + \sin 2t + \sin 4t, \cos t + \cos 2t + \cos 4t)$. For given $k_0 > 0$, we take $k_i(t) = k_0$ for $i = 1, 2$, $k_i(t) = k_0/2$ for $i = 3, 4$, and $k_i(t) = k_0/4$ for $i = 5, 6$, and study the convergence of the error $\|e(T)\|_{l_2}$ with decreasing k_0. From the results presented in Figure 3, Table 1, and Table 2, it is clear that the predicted order of convergence is obtained, both for mcG(q) and mdG(q).

mcG(q)	1	2	3	4	5
p	1.99	3.96	5.92	7.82	9.67
$2q$	2	4	6	8	10

Table 1. Order of convergence p for mcG(q).

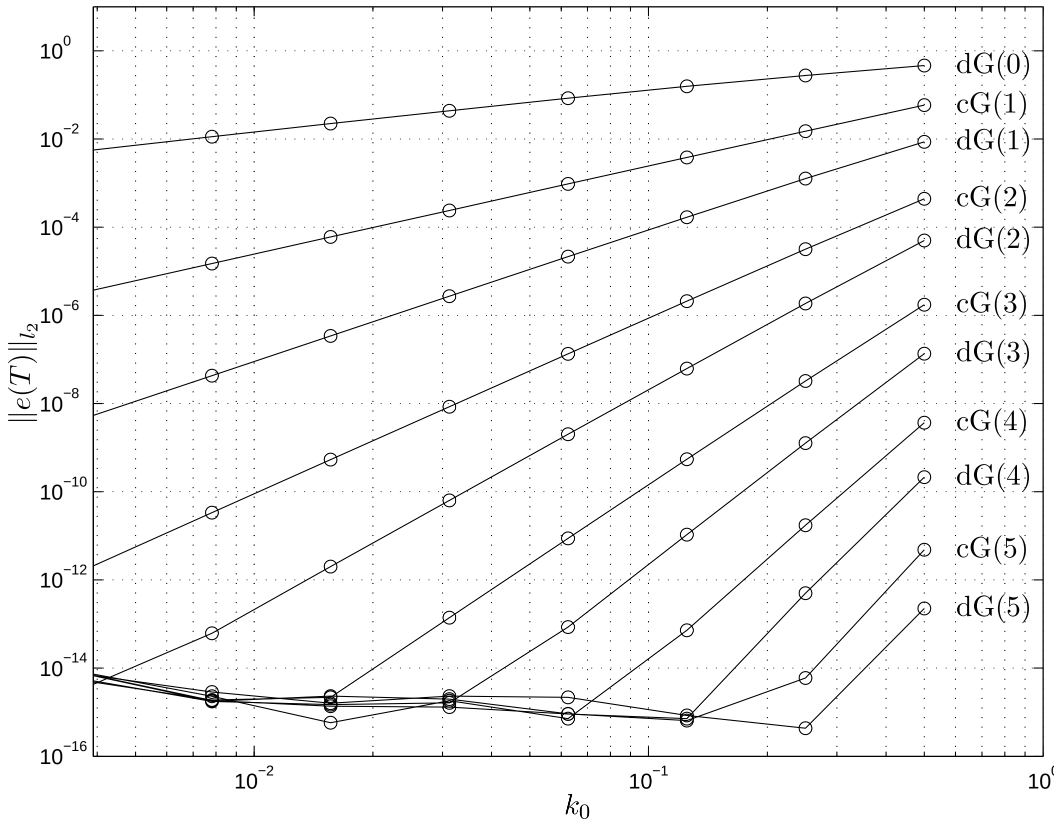

FIGURE 3. Convergence of the error at final time for the solution of the test problem (6.1) with mcG(q) and mdG(q), $q \leq 5$.

mdG(q)	0	1	2	3	4	5
p	0.92	2.96	4.94	6.87	9.10	–
$2q+1$	1	3	5	7	9	11

TABLE 2. Order of convergence p for mdG(q).

6.2. **Numerical evidence for assumption (C3).** The strong stability estimate Theorem 4.3 in [8], which is used in the proof of Theorem 5.5, relies on assumption (C3), which for the dual problem (with time reversed) can be stated in the form

$$T_{n-1} \int_{T_{n-1}}^{T_n} (Av, Av)\, dt \leq \gamma \int_{T_{n-1}}^{T_n} (Av, \pi(\bar{t}Av))\, dt, \quad n = 2, \ldots, M, \tag{6.2}$$

where $\bar{t} = \bar{t}(t)$ is the piecewise constant left-continuous function defined by $\bar{t}(t) = \min_{ij}\{t_{ij} : t \leq t_{ij}\}$. As mentioned, this may fail to hold if Av is close to orthogonal to the trial space. On the other hand, if every pair of components which are coupled through A use approximately the same step size, then $\pi(Av) \approx Av$ and (6.2) holds. We illustrate this in the case of the mdG(0) method, where interpolation is given by taking the right end-point value within each local interval, for A given by

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

We take $k_i(t) = 1/i$ for $i = 1, \ldots, 10$ on $[0, 1]$, and randomize the piecewise constant function v on this partition. Examining the quotient

$$\gamma = \max_v \frac{C \int_0^1 (Av, Av)\, dt}{\int_0^1 (Av, \pi(\bar{t}Av))\, dt},$$

with $\bar{t}(t) = \min_{ij}\{C + t_{ij} : t \leq t_{ij}\}$ for C large, we find $\gamma \lessapprox 1.5$. Here, C corresponds to T_{n-1} in (6.2). In Figure 4, we present an example in which $C = 100$ and $\gamma = 1.05$.

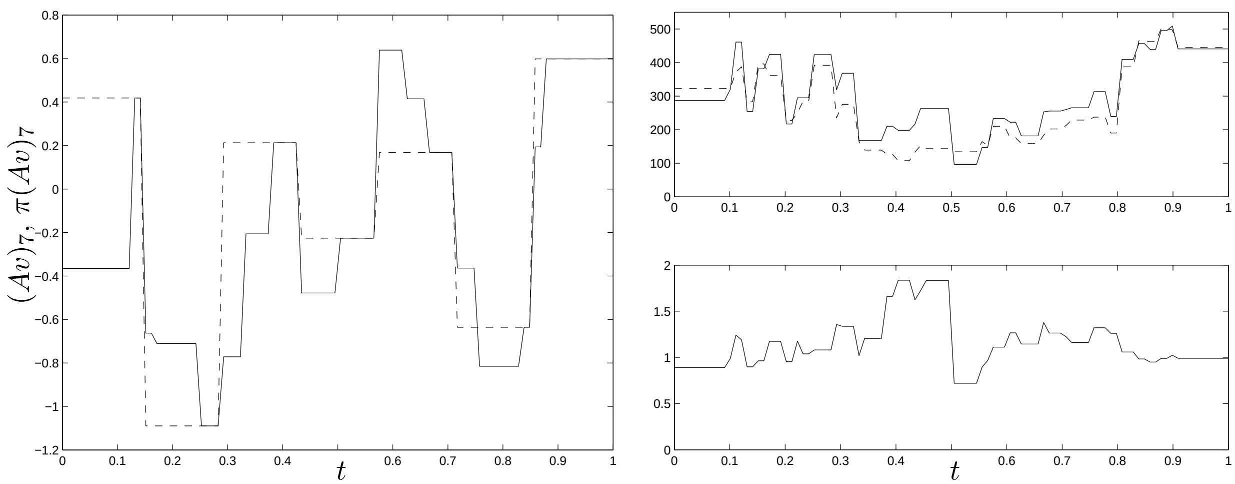

FIGURE 4. Left, we plot a component of the function Av (solid) and its interpolant $\pi(Av)$ (dashed) for the partition discussed in the text. Above right, we plot $C(Av, Av)$ (solid) and $(Av, \pi(\bar{t}Av))$ (dashed), and below right, we plot the corresponding quotient $C(Av, Av)/(Av, \pi(\bar{t}Av))$.

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