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Adaptive Variational Multiscale Methods Based on A Posteriori Error Estimation

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Abstract

The variational multiscale method (VMM) provides a general framework for construction of multiscale finite element methods. In this paper we propose a method for parallel solution of the fine scale problem based on localized Dirichlet problems which are solved numerically. Next we present a posteriori error estimates for VMM which relates the error in linear functionals and the energy norm to the discretization errors, resolution and size of patches in the localized problems, in the fine scale approximation. Based on the a posteriori error estimates we propose an adaptive VMM with automatic tuning of the critical parameters. We primary study elliptic second order partial differential equations with highly oscillating coefficients or localized singularities.

1 Introduction

Many problems in science and engineering involve models of physical systems on many scales. For instance, models of materials with microstructure such as composites and flow in porous media. In such problems it is in general not feasible to seek for a numerical solution which resolves all scales. Instead we may seek to develop algorithms based on a suitable combination a global problem capturing the main features of the solution and localized problems which resolves the fine scales. Since the fine scale problems are localized the computation on the fine scales is parallel in nature.

Previous work. The Variational Multiscale Method (VMM) is a general framework for derivation of basic multiscale method in a variational context, see Hughes [8] and [10]. The basic idea is to decompose the solution into fine and coarse scale contributions, solve the

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fine scale equation in terms of the residual of the coarse scale solution, and finally eliminate the fine scale solution from the coarse scale equation. This procedure leads to a modified coarse scale equation where the modification accounts for the effect of fine scale behavior on the coarse scales. In practice it is necessary to approximate the fine scale equation to make the method realistic. In several works various ways of analytical modeling are investigated often based on bubbles or element Green's functions, see Oberai and Pinsky, [11] and Arbogast [1]. In [7] Hou and Wu present a different approach. Here the fine scale equations are solved numerically on a finer mesh. The fine scale equations are solved inside coarse elements and are thus totally decoupled.

New contributions. In this work we present a simple technique for numerical approximation of the fine scale equation in the variational multiscale method. The basic idea is to split the fine scale residual into localized contributions using a partition of unity and solving corresponding decoupled localized problems on patches with homogeneous Dirichlet boundary conditions. The fine scale solution is approximated by the sum $U_f = \sum_i U_{f,i}$ of the solutions $U_{f,i}$ to the localized problems. The accuracy of U_f depends on the fine scale mesh size h and the size of the patches. We note that the fine scale computation is naturally parallel.

To optimize performance we seek to construct an adaptive algorithm for automatic control of the coarse mesh size H, the fine mesh size h, and the size of patches. Our algorithm is based on the following a posteriori estimate of the error $e = u - U_c - U_f$ in the energy norm for the Poisson equation with variable coefficients a:

$$\begin{aligned} \|e\|_a \leq C \sum_{i \in \mathcal{C}} \|H\mathcal{R}(U_c)\|_{\omega_i} \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)} \\ + C \sum_{i \in \mathcal{F}} \left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i} \right) \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)}, \end{aligned} \tag{1.1}$$

where

$$(-\Sigma(U_{f,i}), v_f)_{\partial\omega_i} = (f + \nabla \cdot a\nabla U_c, \varphi_i v_f)_{\omega_i} - a(U_{f,i}, v_f)_{\omega_i}, \text{ for all } v_f \in V_f^h(\bar{\omega}_i), \tag{1.2}$$

$\mathcal{C}$ refers to nodes where no local problems have been solved, $\mathcal{F}$ to nodes where local problems are solved, U_c is the coarse scale solution, $U = U_c + U_f$, $\mathcal{R}(U)$ is a computable bound of the residual $f + \nabla \cdot a\nabla U$, $\mathcal{R}_i(U_{f,i})$ is a bound of the fine scale residual $\varphi_i(f + \nabla \cdot a\nabla U_c) + \nabla \cdot a\nabla U_{f,i}$, $\Sigma(U_{f,i})$ is related to the normal derivative of the fine scale solution $U_{f,i}$ and measures the effect of restriction to patches. If no fine scale equations are solved we obtain the first term in the estimate; the first part of the second sum measures the effect of restriction to patches; and finally the second part measures the influence of the fine scale mesh parameter h.

In addition to the energy norm error estimate we also derive error representation formulas for errors in linear functionals of the computed solution using duality techniques. The framework is fairly general and may be extended to other types of multiscale methods, for instance, based on localized Neumann problems.

Outline. In Section 2 we introduce the model problem and the variational multiscale formulation of this problem we also discuss the split of the coarse and fine scale spaces. In Section 3 we present a posteriori estimates of the error leading to Section 4 where we present an adaptive algorithm. In section Section 5 we present numerical results and finally Section 6 consists of concluding remarks and suggestions on future work.

2 The Variational Multiscale Method

2.1 Model Problem

We study the Poisson equation with a highly oscillating coefficient a and homogeneous Dirichlet boundary conditions: find $u \in H^1_0(\Omega)$ such that

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega, \tag{2.1}$$

where Ω is a polygonal domain in $\mathbf{R}^d$, $d = 1, 2$, or 3 with boundary Γ, $f \in H^{-1}(\Omega)$, and $a \in L^\infty(\Omega)$ such that $a(x) \geq \alpha_0 > 0$ for all $x \in \Omega$. The variational form of (2.1) reads: find $u \in \mathcal{V} = H^1_0(\Omega)$ such that

$$a(u, v) = (f, v) \quad \text{for all } v \in \mathcal{V}, \tag{2.2}$$

with the bilinear form

$$a(u, v) = (a\nabla u, \nabla v) \tag{2.3}$$

for all $u, v \in \mathcal{V}$.

2.2 The Variational Multiscale Method

We employ the variational multiscale scale formulation, proposed by Hughes see [8, 10] for an overview, and introduce a coarse and a fine scale in the problem. We choose two spaces $\mathcal{V}_c \subset \mathcal{V}$ and $\mathcal{V}_f \subset \mathcal{V}$ such that

$$\mathcal{V} = \mathcal{V}_c \oplus \mathcal{V}_f. \tag{2.4}$$

Then we may pose (2.2) in the following way: find $u_c \in \mathcal{V}_c$ and $u_f \in \mathcal{V}_f$ such that

$$\begin{aligned} a(u_c, v_c) + a(u_f, v_c) &= (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(u_c, v_f) + a(u_f, v_f) &= (f, v_f) \quad \text{for all } v_f \in \mathcal{V}_f. \end{aligned} \tag{2.5}$$

Introducing the residual $R : \mathcal{V} \to \mathcal{V}'$ defined by

$$(R(v), w) = (f, w) - a(v, w) \quad \text{for all } w \in \mathcal{V}, \tag{2.6}$$

the fine scale equation takes the form: find $u_f \in \mathcal{V}_f$ such that

$$(f, v_f) - a(u_f, v_f) = (R(u_c), v_f) \quad \text{for all } v_f \in \mathcal{V}_f. \tag{2.7}$$

Thus the fine scale solution is driven by the residual of the coarse scale solution. Denoting the solution u_f to (2.7) by $u_f = TR(u_c)$ we get the modified coarse scale problem

$$a(u_c, v_c) + a(TR(u_c), v_c) = (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c. \tag{2.8}$$

Here the second term on the left hand side accounts for the effects of fine scales on the coarse scales.

2.3 A VMM Based on Localized Dirichlet Problems

We introduce a partition $\mathcal{K} = \{K\}$ of the domain Ω into shape regular elements K of diameter H_K and we let $\mathcal{N}$ be the set of nodes. Further we let $\mathcal{V}_c$ be the space of continuous piecewise polynomials of degree p defined on $\mathcal{K}$.

We shall now construct an algorithm which approximates the fine scale equation by a set of decoupled localized problems. We begin by writing $u_f = \sum_{i\in\mathcal{N}} u_{f,i}$ where

$$a(u_{f,i}, v_f) = (\varphi_i R(u_c), v_f) \quad \text{for all } v_f \in \mathcal{V}_f, \tag{2.9}$$

and $\{\varphi_i\}_{i\in\mathcal{N}}$ is the set of Lagrange basis functions in $\mathcal{V}_c$. Note that $\{\varphi_i\}_{i\in\mathcal{N}}$ is a partition of unity with support on the elements sharing the node i. We call the set of elements with one corner in node i a mesh star in node i and denote it S_1^i. Thus functions $u_{f,i}$ correspond to the fine scale effects created by the localized residuals $\varphi_i R(u_c)$. Introducing this expansion of u_f in the right hand side of the fine scale equation (2.5) and get: find $u_c \in \mathcal{V}_c$ and $u_f = \sum_{i\in\mathcal{N}} u_{f,i} \in \mathcal{V}_f$ such that

$$\begin{aligned} a(u_c, v_c) + a(u_f, v_c) &= (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(u_{f,i}, v_f) &= (\varphi_i R(u_c), v_f) \quad \text{for all } v_f \in \mathcal{V}_f \text{ and } i \in \mathcal{N}. \end{aligned} \tag{2.10}$$

We use this fact to construct a finite element method for solving (2.10) approximately in two steps.

- For each coarse node we define a patch ω_i such that $\text{supp}(\varphi_i) \subset \omega_i \subset \Omega$. We denote the boundary of ω_i by $\partial\omega_i$.
- On these patches we define piecewise polynomial spaces $\mathcal{V}_f^h(\omega_i)$ with respect to a fine mesh with mesh function $h = h(x)$ defined as a piecewise constant function on the fine mesh. Functions in $\mathcal{V}_f^h(\omega_i)$ are equal to zero on $\partial\omega_i$.

The resulting method reads: find $U_c \in \mathcal{V}_c$ and $U_f = \sum_i^n U_{f,i}$ where $U_{f,i} \in \mathcal{V}_f^h(\omega_i)$ such that

$$\begin{aligned} a(U_c, v_c) + a(U_f, v_c) &= (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(U_{f,i}, v_f) &= (\varphi_i R(U_c), v_f) \quad \text{for all } v_f \in \mathcal{V}_f^h(\omega_i) \text{ and } i \in \mathcal{N}. \end{aligned} \tag{2.11}$$

Since the functions in the local finite element spaces $\mathcal{V}_f^h(\omega_i)$ are equal to zero on $\partial\omega_i$, U_f and therefore U will be continuous.

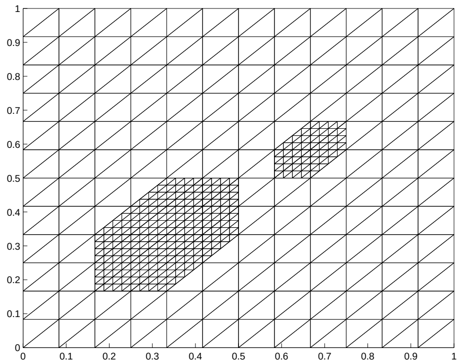

Figure 1: Two (left) and one (right) layer stars.

Remark 2.1 For problems with multiscale phenomena on a part of the domain it is not necessary to solve local problems for all coarse nodes. We let $\mathcal{C} \subset \mathcal{N}$ refer to nodes where no local problems are solved and $\mathcal{F} \subset \mathcal{N}$ refer to nodes where local problems are solved. Obviously $\mathcal{C} \cup \mathcal{F} = \mathcal{N}$. We let $U_{f,i} = 0$ for $i \in \mathcal{C}$.

Remark 2.2 The choice of the subdomains ω_i is crucial for the method. We introduce a notation for extended stars of many layers of coarse elements recursively in the following way. The extended mesh star $S_L^i = \cup_{j \in S_{L-1}^i} S_1^j$ for $L > 1$. We refer to L as layers, see Figure 1.

2.4 Subspaces

The choice of the fine scale space $\mathcal{V}_f$ can be done in different ways. In a paper by Aksoylu and Holst [4] three suggestions are made.

Hierarchical basis method. The first and perhaps easiest approach is to let $\mathcal{V}_f = \{v \in \mathcal{V} : v(x_j) = 0, j = \mathcal{N}\}$, where $\{x_i\}_{i \in \mathcal{N}}$ are the coarse mesh nodes. When $\mathcal{V}_f$ is discritized by the standard piecewise polynomials on the fine mesh this means that the fine scale base functions will have support on fine scale stars.

BPX preconditioner. The second approach is to let $\mathcal{V}_f$ be $L^2(\Omega)$ orthogonal to $\mathcal{V}_c$. In this case we will have global support for the fine scale base functions but for the discretized space we have rapid decay outside fine mesh stars.

Wavelet modified hierarchical basis method. The third choice is a mix of the other two. The fine scale space $\mathcal{V}_f$ is defined as an approximate $L^2(\Omega)$ orthogonal version of the

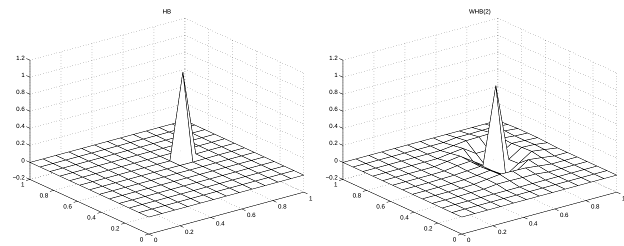

Figure 2: HB-function and WHB-function with two Jacobi iterations.

Hierarchical basis method. We let $Q_c^a v \in \mathcal{V}_c$ be an approximate solution (a small number of Jacobi iterations) to

$$(Q_c^a v, w) = (v, w), \quad \text{for all } w \in \mathcal{V}_c. \tag{2.12}$$

and define the Wavelet modified hierarchical basis function associated with the hierarchical basis function φ_{HB} to be,

$$\varphi_{WHB} = (I - Q_c^a)\varphi_{HB}, \tag{2.13}$$

see Figure 2.

For an extended description of these methods see [3, 4, 2]. In this paper we focus on the WHB method.

3 A Posteriori Error Estimates

3.1 The Dual Problem

To derive a posteriori error estimates of the error in a given linear functional (e, ψ) with $e = u - U$ and $\psi \in H^{-1}(\Omega)$ a given weight. We introduce the following dual problem: find $\phi \in \mathcal{V}$ such that

$$a(v, \phi) = (v, \psi) \quad \text{for all } v \in \mathcal{V}. \tag{3.1}$$

In the VMM setting this yields: find $\phi_c \in \mathcal{V}_c$ and $\phi_f \in \mathcal{V}_f$ such that

$$\begin{aligned} a(v_c, \phi_c) + a(v_c, \phi_f) &= (v_c, \psi), \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(v_f, \phi_f) + a(v_f, \phi_c) &= (v_f, \psi), \quad \text{for all } v_f \in \mathcal{V}_f. \end{aligned} \tag{3.2}$$

3.2 Error Representation Formula

We now derive an error representation formula involving both the coarse scale error $e_c = u_c - U_c$ and the fine scale error $e_f = \sum_{i \in \mathcal{N}} e_{f,i} := \sum_{i \in \mathcal{N}} (u_{f,i} - U_{f,i})$ that arises from using our finite element method (2.11).

We use the dual problem (3.2) to derive an a posteriori error estimate for a linear functional of the error $e = e_c + e_f$. If we subtract the coarse part of equation (2.11) from the coarse part of equation (2.10) we get the Galerkin orthogonality,

$$a(e_c, v_c) + a(e_f, v_c) = 0 \quad \text{for all } v_c \in \mathcal{V}_c. \tag{3.3}$$

The same argument on the fine scale equation gives for $i \in \mathcal{F}$,

$$a(e_{f,i}, v_f) = (f, \varphi_i v_f) - a(e_c, \varphi_i v_f), \quad \text{for all } v_f \in \mathcal{V}_f^h(\omega_i). \tag{3.4}$$

We are now ready to state the an error representation formula.

Theorem 3.1 *If $\psi \in H^{-1}(\Omega)$ then,*

$$(e, \psi) = \sum_{i \in \mathcal{C}} (\varphi_i R(U), \phi_f) + \sum_{i \in \mathcal{F}} \left((\varphi_i R(U_c), \phi_f - v_{f,i}^h)_{\omega_i} - a(U_{f,i}, \phi_f^h - v_{f,i}^h)_{\omega_i} \right) \tag{3.5}$$

for all $v_{f,i}^h \in \mathcal{V}_f^h(\omega_i)$ and $i \in \mathcal{F}$.

Proof. Starting from the definition of the dual problem and letting $v = e = u - U_c - U_f$ we get

$$\begin{aligned}
(e, \psi) &= a(e, \phi) && (3.6)\\
&= a(e, \phi_f) && (3.7)\\
&= a(u - U_c, \phi_f) - a(U_f, \phi_f) && (3.8)\\
&= (R(U_c), \phi_f) - a(U_f, \phi_f) && (3.9)\\
&= (R(U_c), \phi_f) - \sum_{i \in \mathcal{F}} a(U_{f,i}, \phi_f) && (3.10)\\
&= \sum_{i \in \mathcal{C}} (\varphi_i R(U_c), \phi_f) && (3.11)\\
&\quad + \sum_{i \in \mathcal{F}} (\varphi_i R(U_c), \phi_f) - a(U_{f,i}, \phi_f). && (3.12)
\end{aligned}$$

Since equation (2.11) holds we can subtract functions $v_{f,i}^h \in \mathcal{V}_f^h(\omega_i)$ where $i \in \mathcal{F}$ from equation (3.12). We end up with

$$(e, \psi) = \sum_{i \in \mathcal{C}} (\varphi_i R(U_c), \phi_f) + \sum_{i \in \mathcal{F}} (\varphi_i R(U_c), \phi_f - v_{f,i}^h) - a(U_{f,i}, \phi_f - v_{f,i}^h), \tag{3.13}$$

which proves the theorem. □

For example we can choose $v_f^h = \pi_h \phi_f$, where $\pi_h \phi_f$ is the Scott-Zhang interpolant of ϕ_f onto $\mathcal{V}_f^h(\omega_i)$.

Remark 3.1 In practice the dual problem has to be solved numerically and the solution has to be in a finer space then the primal solution. To achieve this we can increase the number of layers when solving the dual problem.

3.3 Energy Norm Estimate

Next we introduce a notation for a bound of the residual. Let $\mathcal{R}(U)$ be a bound of the residual defined in the following way, see [6]:

$$\mathcal{R}(U) = |f + \nabla \cdot a\nabla U| + \frac{1}{2}\max_{\partial K\setminus\Gamma} h_K^{-1}|[a\partial_n U]| \quad \text{on } K \in \mathcal{K}, \tag{3.14}$$

where $\mathcal{K}$ is the set of elements in the mesh and $[\cdot]$ is the difference in function value over the current interior edge. We note that $|(R(U), v)| \leq \|h^s\mathcal{R}(U)\|\|h^{-s}v\|$ for $s \in \mathbf{R}$. We define $\mathcal{R}_i(U_{f,i})$ in the same way as $\mathcal{R}(U)$ on the local mesh but with U replaced by $U_{f,i}$ and f replaced by $\varphi_i\mathcal{R}(U_c)$.

We also define a new space on the patches. Let $V_f^h(\bar{\omega}_i)$ be the space of piecewise polynomials of degree p on ω_i. This space is identic to $V_f^h(\omega_i)$ with the difference that $V_f^h(\bar{\omega}_i)$ is not necessarily zero on the boundary $\partial\omega_i$. This means that $V_f^h(\omega_i) \subset V_f^h(\bar{\omega}_i)$.

We now state the following estimate for the error in the energy norm, $\|e\|_a = a(e,e)^{1/2}$.

Theorem 3.2 *It holds,*

$$\begin{aligned}\|e\|_a \leq C\sum_{i\in\mathcal{C}} \|H\mathcal{R}(U_c)\|_{\omega_i}\|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)} \\ + C\sum_{i\in\mathcal{F}}\left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}\right)\|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)},\end{aligned} \tag{3.15}$$

where

$$(-\Sigma(U_{f,i}), v_f)_{\partial\omega_i} = (\varphi_i R(U_c), v_f)_{\omega_i} - a(U_{f,i}, v_f)_{\omega_i}, \quad \textit{for all } v_f \in V_f^h(\bar{\omega}_i). \tag{3.16}$$

Proof. We start with similar arguments as in the proof of Theorem 3.1. We use the error equation (3.3) with v_c as the Scott-Zhang interpolant $\pi_c e$ onto the coarse space $\mathcal{V}_c$, see [5], to get,

$$
\begin{aligned}
\|e\|_a^2 &= a(e,e) && (3.17)\\
&= a(e, e-\pi_c e) && (3.18)\\
&= a(u-U_c, e-\pi_c e) - a(U_f, e-\pi_c e) && (3.19)\\
&= (R(U_c), e-\pi_c e) - a(U_f, e-\pi_c e) && (3.20)\\
&= \sum_{i\in\mathcal{C}} (\varphi_i R(U_c), e-\pi_c e) && (3.21)\\
&\quad + \sum_{i\in\mathcal{F}} (\varphi_i R(U_c), e-\pi_c e) - a(U_{f,i}, e-\pi_c e)\\
&= \sum_{i\in\mathcal{C}} (\varphi_i R(U_c), e-\pi_c e) && (3.22)\\
&\quad + \sum_{i\in\mathcal{F}} (\varphi_i R(U_c), \pi_{f,i}(e-\pi_c e)) - a(U_{f,i}, \pi_{f,i}(e-\pi_c e))\\
&\quad + \sum_{i\in\mathcal{F}} (\varphi_i R(U_c), e-\pi_c e-\pi_{f,i}(e-\pi_c e))\\
&\quad - \sum_{i\in\mathcal{F}} a(U_{f,i}, e-\pi_c e-\pi_{f,i}(e-\pi_c e))\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} && (3.23)
\end{aligned}
$$

where $\pi_{f,i}$ is the Scott-Zhang interpolant onto $\mathcal{V}_f(\omega_i)$. We start by estimating the first term of equation (3.23), I. From interpolation theory [5] we have,

$$
\begin{aligned}
\sum_{i\in\mathcal{C}} (\varphi_i R(U_c), e-\pi_c e) &\le \sum_{i\in\mathcal{C}} \|\varphi_i R(U_c)\|_{\omega_i} \|e-\pi_c e\|_{\omega_i} && (3.24)\\
&\le C \sum_{i\in\mathcal{C}} \|H\mathcal{R}(U_c)\|_{\omega_i} \|\nabla e\|_{\omega_i}. && (3.25)
\end{aligned}
$$

Next we turn our attention to the second term of equation (3.23), II. We introduce $\Sigma(U_{f,i})$ which the piecewise polynomial defined on $\partial\omega_i$ that uniquely solves,

$$(-\Sigma(U_{f,i}), v_f)_{\partial\omega_i} = (R(U_c), \varphi_i v_f)_{\omega_i} - a(U_{f,i}, v_f)_{\omega_i}, \quad \text{for all } v_f \in V_f^h(\bar{\omega}_i). \tag{3.26}$$

With this definition we get the following estimate for the second term,

$$
\begin{aligned}
\mathrm{II} &= \sum_{i\in\mathcal{F}} (-\Sigma(U_{f,i}), \pi_{f,i}(e-\pi_c e))_{\partial\omega_i} && (3.27)\\
&\le \sum_{i\in\mathcal{F}} \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} \|\frac{1}{\sqrt{H}}\pi_{f,i}(e-\pi_c e)\|_{\partial\omega_i}. && (3.28)
\end{aligned}
$$

We use the the following trace inequality from [5],

$$\|\pi_{f,i}(e-\pi_c e)\|_{\partial\omega_i}^2 \le C\left(\frac{1}{H}\|\pi_{f,i}(e-\pi_c e)\|_{\omega_i}^2 + H\|\nabla\pi_{f,i}(e-\pi_c e)\|_{\omega_i}^2\right). \tag{3.29}$$

Next we use that the Scott-Zhang interpolant is both L^2 and H^1 stable from [5] to get,

$$\|\pi_{f,i}(e-\pi_c e)\|^2_{\partial\omega_i} \leq C\left(\frac{1}{H}\|e-\pi_c e\|^2_{\omega_i} + H\|\nabla(e-\pi_c e)\|^2_{\omega_i}\right) \tag{3.30}$$

$$\leq CH\|\nabla e\|^2_{\omega_i}. \tag{3.31}$$

We conclude

$$\mathrm{II} \leq C\sum_{i\in\mathcal{F}} \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}\|\nabla e\|_{\omega_i}. \tag{3.32}$$

We now take on the third term in equation (3.23), $\sum_{i\in\mathcal{F}}(\varphi_i R(U_c), e-\pi_c e-\pi_{f,i}(e-\pi_c e)) - a(U_{f,i}, e-\pi_c e-\pi_{f,i}(e-\pi_c e))$,

$$\mathrm{III} \leq C\sum_{i\in\mathcal{F}} \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}\|\nabla(e-\pi_c e)\|_{\omega_i} \tag{3.33}$$

$$\leq C\sum_{i\in\mathcal{F}} \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}\|\nabla e\|_{\omega_i}. \tag{3.34}$$

We need to do the following simple observation,

$$\|\nabla e\|_{\omega_i} \leq \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)}\|\sqrt{a}\nabla e\|_{\omega_i}, \tag{3.35}$$

by Hölder's inequality. We go back to equation (3.17) and use the estimates of the three terms together with equation (3.35)

$$\|e\|_a^2 \leq \sum_{i\in\mathcal{C}}(\varphi_i R(U_c), e-\pi_c e) \tag{3.36}$$
$$+\sum_{i\in\mathcal{F}}(\varphi_i R(U_c), \pi_{f,i}(e-\pi_c e)) - a(U_{f,i}, \pi_{f,i}(e-\pi_c e))$$
$$+\sum_{i\in\mathcal{F}}(\varphi_i R(U_c), e-\pi_c e - \pi_{f,i}(e-\pi_c e))$$
$$-\sum_{i\in\mathcal{F}} a(U_{f,i}, e-\pi_c e-\pi_{f,i}(e-\pi_c e))$$
$$\leq C\sum_{i\in\mathcal{C}}\|H\mathcal{R}(U_c)\|_{\omega_i}\|\nabla e\|_{\omega_i} \tag{3.37}$$
$$+C\sum_{i\in\mathcal{F}}\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}\|\nabla e\|_{\omega_i}$$
$$+C\sum_{i\in\mathcal{F}}\|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}\|\nabla e\|_{\omega_i}$$
$$\leq C\left(\sum_{i\in\mathcal{C}}\|H\mathcal{R}(U_c)\|_{\omega_i}\|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)}\right)\|e\|_a \tag{3.38}$$
$$+C\left(\sum_{i\in\mathcal{F}}\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}\|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)}\right)\|e\|_a$$
$$+C\left(\sum_{i\in\mathcal{F}}\|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}\|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)}\right)\|e\|_a$$

Finally we get

$$\|e\|_a \leq C\sum_{i\in\mathcal{C}}\|H\mathcal{R}(U_c)\|_{\omega_i}\|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)} \tag{3.39}$$
$$+C\sum_{i\in\mathcal{F}}\left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}+\|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}\right)\|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)},$$

which proves the theorem. □

Remark 3.2 We need to motivate the definition of $\Sigma(U_{f,i})$:

$$(-\Sigma(U_{f,i}), v_f)_{\partial\omega_i} = (\varphi_i R(U_c), v_f)_{\omega_i} - (a\nabla U_{f,i}, \nabla v_f)_{\omega_i}, \quad \text{for all } v_f \in V_f^h(\bar{\omega}_i), \tag{3.40}$$

in equation (3.16). The function $\Sigma(U_{f,i})$ is a piecewise polynomial defined on the boundary of patch ω_i. Remember that

$$(\varphi_i R(U_c), v_f)_{\omega_i} - (a\nabla U_{f,i}, \nabla v_f)_{\omega_i} = 0, \quad \text{for all } v_f \in V_f^h(\bar{\omega}_i), \tag{3.41}$$

This means that have the same number of unknowns and equations and in practice calculating $\Sigma(U_{f,i})$ will come down to solving a linear system with a mass matrix defined on the boundary of the patch. There is a close connection between $\Sigma(U_{f,i})$ and $n \cdot a\nabla U_{f,i}$ in fact $\Sigma(U_{f,i})$ is the $L^2(\partial\omega_i)$ projection of $n \cdot a\nabla U_{f,i}$. This is further discussed in [9].

3.4 Application to A Posteriori Error Estimates for the Standard Galerkin Method

We use the variational mutiscale method on a dual problem to estimate the error of the standard Galerkin solution on the coarse mesh: find $U \in \mathcal{V}_c$ such that

$$a(U, v) = (f, v), \quad \text{for all } v \in \mathcal{V}_c. \tag{3.42}$$

The corresponding discrete variational multiscale method for the dual reads: find $\Phi_c \in \mathcal{V}_c$ and $\Phi_f = \sum_{i\in\mathcal{N}} \Phi_{f,i}$ where $\Phi_{f,i} \in \mathcal{V}_f^h(\omega_i)$ such that

$$\begin{aligned} a(v_c, \Phi_c) + a(v_c, \Phi_f) = (v_c, \psi) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(v_f, \Phi_{f,i}) = (\varphi_i v_f, \psi) - a(\varphi_i v_f, \Phi_c) \quad \text{for all } v_f \in \mathcal{V}_f^h(\omega_i). \end{aligned} \tag{3.43}$$

Since we have $a(u, v) = (f, v)$ for all $v \in \mathcal{V}_c$ we can subtract equation (3.42) from this equation to get the Galerkin orthogonality,

$$a(u - U, v) = 0, \quad \text{for all } v \in \mathcal{V}_c. \tag{3.44}$$

We formulate an error representation formula for the standard Galerkin method in the following proposition.

Proposition 3.1 *It holds*

$$(u - U, \psi) = \sum_{i\in\mathcal{N}} (R(U), \Phi_{f,i}) + (R(U), \phi_f - \Phi_f). \tag{3.45}$$

Proof. Together equation (3.44) and equation (3.2) gives

$$(u - U, \psi) = a(u, \phi_c + \phi_f) - a(U, \phi_c + \phi_f) \tag{3.46}$$
$$= (f, \phi_c + \phi_f) - a(U, \phi_c + \phi_f) \tag{3.47}$$
$$= (R(U), \phi_f) \tag{3.48}$$

Finally we add and subtract the Φ_f term. □

If we can get a bound of $\phi_f - \Phi_f$ in terms of the fine mesh parameter h and the size of the subdomains ω_i, the computable terms $(R(U), \Phi_{f,i})$ will serve as local error estimators that points out elements where the fine scale influence is significant. This is done in the following theorem

Theorem 3.3 *It holds,*

$$
\begin{aligned}
|(R(U), \phi_f - \Phi_f)| \leq{}& C_a \|H\mathcal{R}(U)\| \sum_{i\in\mathcal{N}} \|\sqrt{H}\Sigma(\Phi_{f,i})\|_{\partial\omega_i} \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)} \\
&+ C_a \|H\mathcal{R}(U)\| \sum_{i\in\mathcal{N}} \|h\mathcal{R}_i(\Phi_{f,i})\|_{\omega_i} \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)},
\end{aligned} \tag{3.49}
$$

where

$$
(\Sigma(\Phi_{f,i}), v_f)_{\partial\omega_i} = a(\Phi_{f,i}, v_f)_{\omega_i} - (\psi + \nabla \cdot a\nabla\Phi_c, v_f)_{\omega_i}, \text{ for all } v_f \in V_f^h(\bar{\omega}_i), \tag{3.50}
$$

and $\mathcal{R}_i(\Phi_{f,i})$ is defined in analogy with with the earlier definition for $\mathcal{R}_i(U_{f,i})$.

Proof. We start with the rest term of equation (3.45),

$$
\begin{aligned}
|(R(U), \phi_f - \Phi_f)| &= |a(e, \phi_f - \Phi_f)| && (3.51)\\
&\leq \|e\|_a \, \|\phi_f - \Phi_f\|_a && (3.52)\\
&\leq \|e\|_a \, \|\phi - (\Phi_c + \Phi_f)\|_a. && (3.53)
\end{aligned}
$$

From standard a posteriori theory we know that $\|e\|_a \leq C_a \|HR(U)\|$, for some constant C_a depending on a, and from Theorem 3.2 with $f = \psi$, $u = \phi$, $U_c = \Phi_c$, $U_f = \Phi_f$, $\mathcal{C} = \emptyset$, and $U_{f,i} = \Phi_{f,i}$ we have,

$$
\begin{aligned}
\|\phi - (\Phi_c + \Phi_f)\|_a \leq{}& C \sum_{i\in\mathcal{N}} \|\sqrt{H}\Sigma(\Phi_{f,i})\|_{\partial\omega_i} \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)} \\
&+ C \sum_{i\in\mathcal{N}} \|h\mathcal{R}_i(\Phi_{f,i})\|_{\omega_i} \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)},
\end{aligned} \tag{3.54}
$$

with $\Sigma(\Phi_{f,i})$ defined as in equation (3.50). The theorem follows immediately by combining equation (3.53) and equation (3.54). □

4 Adaptive Algorithm

We use the energy norm estimate in Theorem 3.2 to construct an adaptive algorithm. We remember the result,

$$
\begin{aligned}
\|e\|_a \leq{}& C \sum_{i\in\mathcal{C}} \|H\mathcal{R}(U_c)\|_{\omega_i} \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)} \\
&+ C \sum_{i\in\mathcal{F}} \left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i} \right) \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)},
\end{aligned} \tag{4.1}
$$

These contributions to the error can easily be understood. The first term is the standard a posteriori estimate for a Galerkin solution on the coarse mesh i.e. this is what we get if we do not solve any local problems. The first part of the second sum represents the error arising from the fact that we solve the local problems on patches ω_i instead of the whole domain. Remember that $\Sigma(U_{f,i})$ is closely related to the normal derivative of the fine scale solution on the boundary of the patches. Finally, the second part of the second sum represents the fine scale resolution. The two contributions to the second sum clearly points out the parameters of interest when using our method. The first one is the patch size, increasing patch size will decrease $\|\sqrt{H}\Sigma_i\|_{\partial\omega_i}$, the second one is the fine scale mesh size h.

From equation (4.1) we now state the following adaptive algorithm:

Adaptive Algorithm.

- Start with no nodes in $\mathcal{F}$.
- Calculate a solution U on the coarse mesh.
- Calculate the residuals for each coarse node, $R_i = \|H\mathcal{R}(U_c)\|_{\omega_i}$.
- Calculate the contributions from the first term of the local problems, $S_i = \|\sqrt{H}\Sigma_i\|_{\partial\omega_i}$.
- Calculate the contributions from the second term of the local problems, $W_i = \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}$.
- For large values in R_i add i to $\mathcal{F}$, for large values in S_i or W_i either increase the number of layers or decrease the fine scale mesh size h for local problem i. Return to 2 or stop if the desired tolerance is reached.

5 Numerical Examples

We solve two dimensional model problems with linear base functions defined on a uniform triangular mesh.

Example 1. In the first example we let $a = 1$, $f = 1$, and Ω be the unit square with a slit, see Figure 5. The solution u is forced to be zero on the boundary including the slit. We solve the problem by using the adaptive algorithm above with a refinement level of 10 % each iteration. Figure 5 shown the adaptive choice of refinement level k, where $h = H \cdot 2^{-k}$, and number of layers L for the local problems after one and two iterations. We plot the difference between our solution and a reference solution in Figure 5. We see that the Galerkin solution has a large error in the singularity and that we can take care of this singularity by solving local problems chosen in an adaptive fashion.

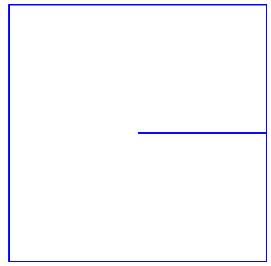

Figure 3: Unit square with a slit between $(0.5, 0.5)$ and $(1, 0.5)$.

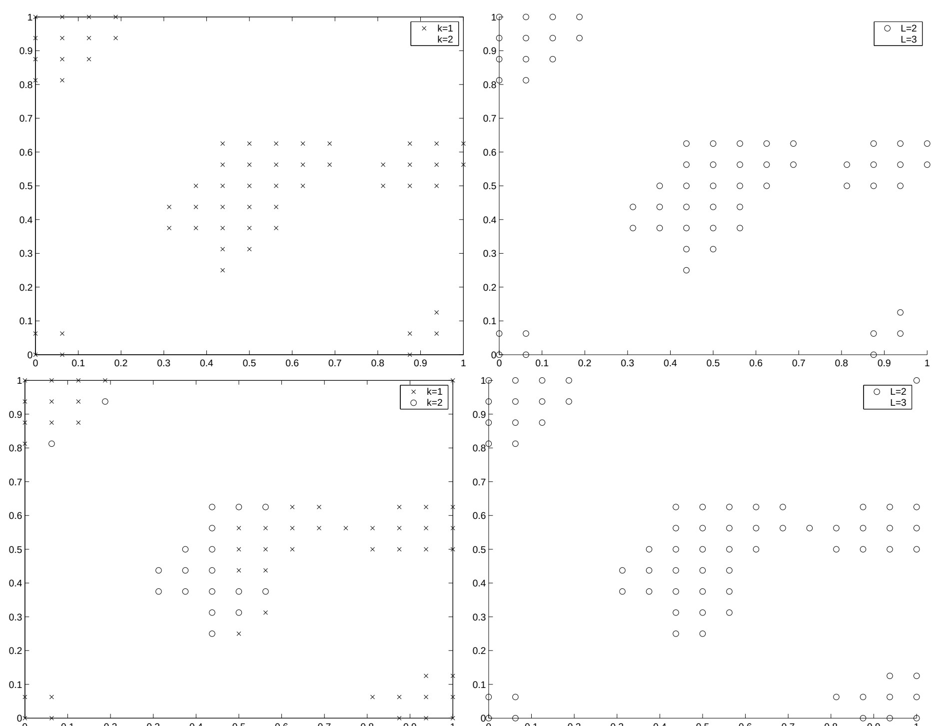

Figure 4: Refinement level, $h = H \cdot 2^{-k}$, and number of layers L for each coarse node. The upper pictures are after one iteration in the adaptive algorithm and the lower pictures are after two iterations.

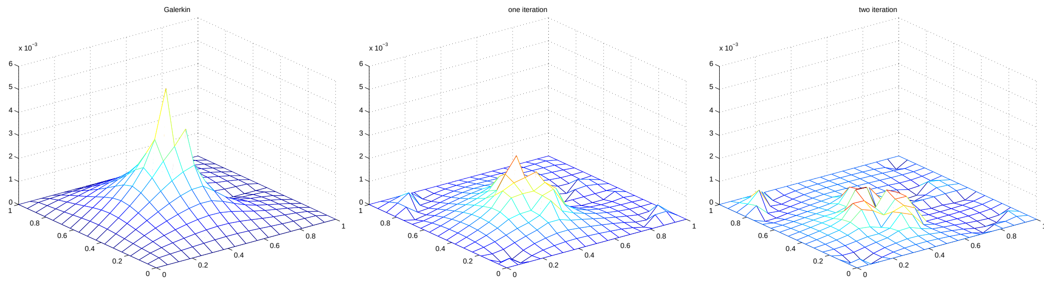

Figure 5: The error in the Galerkin solution (left), after one step in the adaptive algorithm (middle), and after two steps (right).

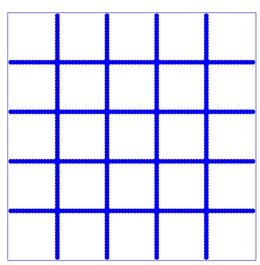

Figure 6: The coefficient is discontinuous with the values $a = 1$ on the white squares and $a = 0.05$ on the lattice.

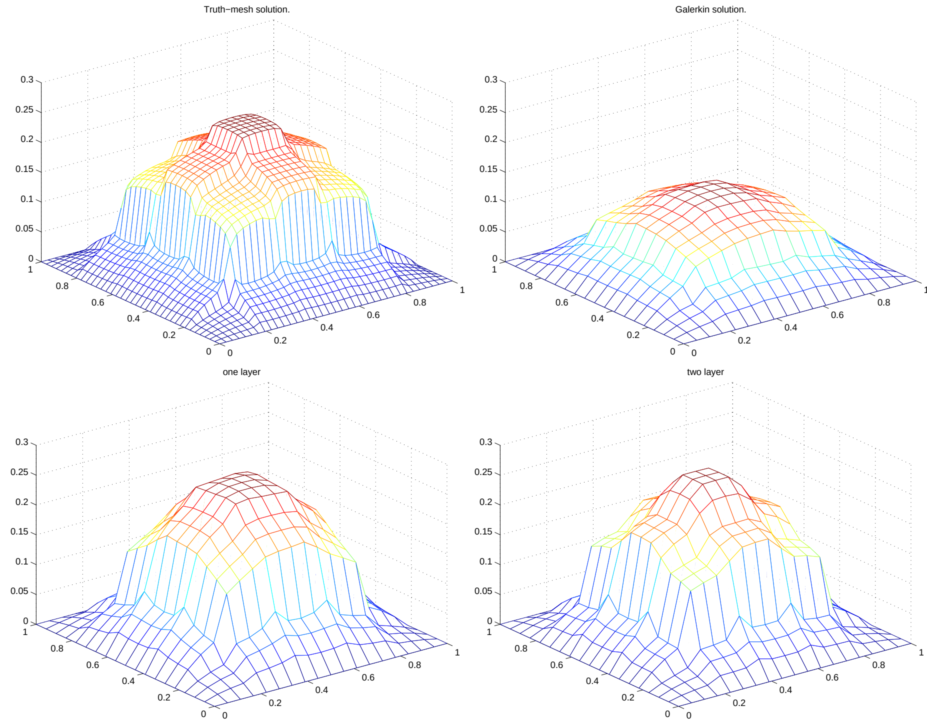

Figure 7: Reference solution (upper left), standard Galerkin on coarse mesh (upper right), solution with local problems using one layer stars (lower left), and finally local problems using two layer stars (lower right).

Example 2. In this example we use a simple geometry, the unit square, but we let the coefficient a oscillate rapidly according to Figure 5. We calculate a reference solution on the fine space and compare it to the standard Galerkin on the coarse mesh with and without solving local problems. We see that standard Galerkin on a coarse mesh performs badly for this problem, Figure 5. Solving local problems using one layer stars give the solution the correct magnitude and if we use two layers we see that the fine scale features of the solution starts to fall into place. In this example no adaptivity was used. Local problems was solved for all coarse nodes.

6 Conclusions and Future Work

We have presented a method for parallel solution of the fine scale equations in the variational multiscale method based on solution of localized Dirichlet problems on patches and developed an a posteriori error analysis for the method. Based on the estimates we design a basic adaptive algorithm for automatic tuning of the critical parameters: resolution and size of patches in the fine scale problems. It is also possible to decide wether a fine scale computation is necessary or not and thus the proposed scheme may be combined with a standard adaptive algorithm on the coarse scales. The method is thus very general in nature and may be applied to any problem where adaptivity is needed.

In this paper we have focused on two scales in two spatial dimensions. A natural extension would be to solve three dimensional problems with multiple scales. It is also natural to extend this theory to other equations modeling for instance flow and materials.

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