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A posteriori error estimates for mixed finite element approximations of elliptic problems

Mats G. Larson* and Axel Målqvist†

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Abstract

We derive residual based a posteriori error estimates of the L^2-norm of the error in the flux for a general class of mixed methods for elliptic problem. The estimate is applicable to standard mixed methods such as the Raviart-Thomas-Nedelec and Taylor-Hood elements, as well as stabilized methods such as the Galerkin-Least squares method. The element residual in the estimate employs an elementwise computable postprocessed approximation of the pressure which gives optimal order.

1 Introduction

The Model Problem. We consider the mixed formulation of the Poisson equation with Neumann boundary conditions:

$$\begin{cases} \boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

Here Ω is a polygonal domain in $\mathbf{R}^n$ with boundary Γ. Assuming that $\int_\Omega f\,dx = 0$, we get a well posed problem with a solution $u \in H^1(\Omega)/\mathbf{R}$ and $\boldsymbol{\sigma} \in \boldsymbol{V} = \{\boldsymbol{v} \in H(\text{div};\Omega) : \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \Gamma\}$. See [9] for definitions of these function spaces.

Previous Work. Several works present a posteriori error estimates for mixed methods. In Carstensen [10] an error estimate in the $H(\text{div};\Omega)$ norm of the flux is presented. The $H(\text{div};\Omega)$ norm may be dominated by the div-part which is also directly computable. When it comes to error estimates in the of the flux in the L^2 norm of methods using richer

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spaces for the flux $\boldsymbol{\sigma}$ than the displacement u, such as Raviart-Thomas-Nedelec (RTN) elements, there are known difficulties. Braess and Verfürth presents a suboptimal estimate in [4]. The reason for the suboptimality is that the natural residual that arises from the first equation $\boldsymbol{\sigma} - \nabla u = 0$ in problem (1.1) may be large if the flux space is richer than the displacement space. In a recent paper Lovadina and Stenberg [13] derive an a posteriori error estimate of the L^2-norm of the flux for the RTN based methods which employs a particular postprocessed approximation of U. The proof is based on a posteriori error analysis of an equivalent method which involves the postprocessed approximation of U.

New Contributions. We derive a general a posteriori of the energy norm which is applicable to most mixed methods including the classical inf-sup stable elements, Raviart-Thomas elements, the BDM-elements and the Taylor-Hood. Our estimate is closely related to the estimate presented by Lovadina and Stenberg [13], however, our proof is more general and also reveals the fact that one can use any piecewise polynomial approximation of the pressure when computing the residual. By a small adjustment of the argument we finally, derive an estimate for the stabilized mixed method of Masud and Hughes [14]. The same technique applies to other stabilized schemes, for instance the Galerkin least squares method.

Outline. We start by presenting finite elements and the discrete version of equation (1.1) in Section 2 then we present the a posteriori error estimates in Section 3.

2 Weak Formulation and the Finite Element Method

Weak Formulation. We multiply the first equation in (1.1) by a test function $\boldsymbol{v} \in \boldsymbol{V}$ and integrate by parts. The second equation in (1.1) is multiplied by a test function $w \in W = L^2(\Omega)$. The weak form reads: find $\boldsymbol{\sigma} \in \boldsymbol{V}$ and $u \in W$ such that,

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{v}) + (u, \nabla \cdot \boldsymbol{v}) = 0 & \text{for all } \boldsymbol{v} \in \boldsymbol{V}, \\ (-\nabla \cdot \boldsymbol{\sigma}, w) = (f, w) & \text{for all } w \in W. \end{cases} \tag{2.1}$$

Our aim is to derive a posteriori error estimates of finite element approximations $\{\boldsymbol{\Sigma}, U\}$ of the exact solution $\{\boldsymbol{\sigma}, u\}$ in the energy norm $\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0$, here $\|\cdot\|_0$ denotes the $L^2(\Omega)$ norm.

The Mixed Finite Element Method. We let $\mathcal{K} = \{K\}$ be a partition of Ω into shape regular elements of diameter h_K and define the mesh function, $h(x) : \Omega \to \mathbf{R}^+$, by letting $h(x) = h_K$ for $x \in K$.

We seek an approximate solution in discrete spaces $\boldsymbol{V}_h \subset \boldsymbol{V}$ and $W_h \subset W$ defined on the partition $\mathcal{K}$. It is well known that for finite element methods based on the standard weak form (2.1) the discrete spaces must be chosen so that the inf-sup condition, see [9], is satisfied in order to guarantee a stable method. Only rather special constructions of the discrete spaces yield stable methods. In Section 3.3 we consider a stabilized mixed finite element method based on a modified weak formulation which can be based on standard continuous piecewise polynomials. We summarize some of the most well known choices of stable discrete spaces on triangles and tetrahedra for a given integer $k \geq 1$:

- *Raviart-Thomas-Nedelec* (RTN) elements, see [16, 15],
  $\boldsymbol{V}_h = \{\boldsymbol{v} \in H(\mathrm{div};\Omega) : \boldsymbol{v}|_K \in [P_{k-1}(K)]^n \oplus \boldsymbol{x}\tilde{P}_{k-1}(K) \text{ for all } K \in \mathcal{K}\}$,
  $W_h = \{w \in L^2(\Omega) : w|_K \in P_{k-1}(K) \text{ for all } K \in \mathcal{K}\}$.

- *Brezzi-Douglas-Marini* (BDM) elements, see [8, 7],
  $\boldsymbol{V}_h = \{\boldsymbol{v} \in H(\mathrm{div};\Omega) : \boldsymbol{v}|_K \in [P_k(K)]^n \text{ for all } K \in \mathcal{K}\}$,
  $W_h = \{w \in L^2(\Omega) : w|_K \in P_{k-1}(K) \text{ for all } K \in \mathcal{K}\}$.

- *Taylor-Hood* (TH), see [12],
  $\boldsymbol{V}_h = \{\boldsymbol{v} \in C(\Omega) : \boldsymbol{v}|_K \in [P_{k+1}(K)]^n \text{ for all } K \in \mathcal{K}\}$,
  $W_h = \{w \in C(\Omega) : w|_K \in P_k(K) \text{ for all } K \in \mathcal{K}\}$.

For a more complete account of inf-sup stable spaces we refer to Brezzi-Fortin, [9]. Here $C(\Omega)$ denotes the space of continuous functions on Ω, $P_k(K)$ the space of polynomials of degree k on element K, and $\tilde{P}_k(K)$ the set of homogeneous polynomials of degree k. The norms used in this paper are standard Sobolev norms following the notation, $\|\cdot\|_{s,\omega} = \|\cdot\|_{H^s(\omega)} = \|\cdot\|_{W_2^s(\omega)}$, see [1].

The mixed finite element method reads: find $\boldsymbol{\Sigma} \in \boldsymbol{V}_h$ and $U \in W_h$ such that:

$$\begin{cases} (\boldsymbol{\Sigma}, \mathbf{v}) + (U, \nabla\cdot\mathbf{v}) = 0 & \text{for all } \boldsymbol{v} \in \boldsymbol{V}_h, \\ (-\nabla\cdot\boldsymbol{\Sigma}, w) = (f, w) & \text{for all } w \in W_h. \end{cases} \tag{2.2}$$

3 A Posteriori Error Estimates

3.1 Estimate for Standard Mixed Methods

Here we present a general a posteriori estimate of the energy norm error $\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0$ involving a piecewise polynomial function Q, which may be obtained by postprocessing of U. The possibility to replace U by Q is important since it leads to a posteriori error estimates of optimal order. We are not interested in tracking the constants in the error estimates.

Theorem 3.1 *It holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 \leq C \sum_{K\in\mathcal{K}} \left(h_K^2 \|f + \nabla\cdot\boldsymbol{\Sigma}\|_{0,K}^2 + \|\boldsymbol{\Sigma} - \nabla Q\|_{0,K}^2 + h_K^{-1} \|\, [Q]\, \|_{0,\partial K}^2 \right), \tag{3.1}$$

for arbitrary $Q \in \bigoplus_{K\in\mathcal{K}} P_l(K)$, with $l \geq 0$. The jump denoted $[\cdot]$ is the difference in function value over an face in the mesh.

Proof. Starting with the left hand side we have

$$
\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 &= (\boldsymbol{\sigma} - \boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma}) && (3.2)\\
&= (\boldsymbol{\sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma}) - (\boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma}) && (3.3)\\
&= -(u, \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma})) - (\boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma}) && (3.4)\\
&= -(u - Q, \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma})) + (Q, -\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma})) - (\boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma}) && (3.5)\\
&= (u - Q, f + \nabla \cdot \boldsymbol{\Sigma}) + \sum_{K \in \mathcal{K}} \Big((Q, -\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma}))_K - (\boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma})_K \Big) && (3.6)\\
&= I + II. && (3.7)
\end{aligned}
$$

We treat the two terms in equation (3.7) separately, beginning with I. From the second part of equation (2.2) we have the Galerkin orthogonality property $(f + \nabla \cdot \boldsymbol{\Sigma}, w) = 0$ for all $w \in W_h$. We let P_h denote the standard L^2-projection from W to W_h and proceed with the estimates as follows

$$
\begin{aligned}
I &\leq |(f + \nabla \cdot \boldsymbol{\Sigma}, u - Q)| && (3.8)\\
&\leq \|h(f + \nabla \cdot \boldsymbol{\Sigma})\|_0 \|h^{-1}(u - Q - P_h(u - Q))\|_0 && (3.9)\\
&\leq C\|h(f + \nabla \cdot \boldsymbol{\Sigma})\|_0 \|\nabla(u - Q)\|_0 && (3.10)\\
&= C\|h(f + \nabla \cdot \boldsymbol{\Sigma})\|_0 \|\boldsymbol{\sigma} - \boldsymbol{\Sigma} + \boldsymbol{\Sigma} - \nabla Q\|_0 && (3.11)\\
&\leq \frac{3C^2}{2}\|h(f + \nabla \cdot \boldsymbol{\Sigma})\|_0^2 + \frac{1}{4}\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 + \frac{1}{2}\|\boldsymbol{\Sigma} - \nabla Q\|_0^2. && (3.12)
\end{aligned}
$$

We now turn to the second term II in equation (3.7) and begin by integration by parts,

$$
\begin{aligned}
II &= \sum_{K \in \mathcal{K}} \Big((Q, -\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma}))_K - (\boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma})_K \Big) && (3.13)\\
&= \sum_{K \in \mathcal{K}} \Big((\nabla Q, \boldsymbol{\sigma} - \boldsymbol{\Sigma})_K - (Q, \boldsymbol{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma}))_{\partial K} - (\boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma})_K \Big) && (3.14)\\
&= (\nabla Q - \boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma}) - \sum_{K \in \mathcal{K}} (Q, \boldsymbol{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma}))_{\partial K} && (3.15)\\
&\leq \|\nabla Q - \boldsymbol{\Sigma}\|_0^2 + \frac{1}{4}\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 + \left| \sum_{K \in \mathcal{K}} (Q, \boldsymbol{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma}))_{\partial K} \right|. && (3.16)
\end{aligned}
$$

Using that $\boldsymbol{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma})$ is continuous over element faces we can subtract an arbitrary function $v \in H^1(\Omega)$ in the term $\sum_{K \in \mathcal{K}} (Q, \boldsymbol{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma}))_{\partial K} = \sum_{K \in \mathcal{K}} (Q - v, \boldsymbol{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma}))_{\partial K}$. We then have the estimate

$$
II \leq \|\nabla Q - \boldsymbol{\Sigma}\|_0^2 + \frac{1}{4}\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 + \left| \inf_{v \in H^1(\Omega)} \sum_{K \in \mathcal{K}} (Q - v, \boldsymbol{n} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma}))_{\partial K} \right|. \qquad (3.17)
$$

We now use the Cauchy-Schwartz inequality followed by the trace inequality

$$\|\boldsymbol{n}\cdot\boldsymbol{w}\|_{-1/2,\partial K} \le C(\|\boldsymbol{w}\|_{0,K} + h_K\|\nabla\cdot\boldsymbol{w}\|_{0,K}), \tag{3.18}$$

see [11], to estimate the sum in equation (3.17) as follows

$$\left|\inf_{v\in H^1(\Omega)}\sum_{K\in\mathcal{K}}(Q-v,\boldsymbol{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\Sigma}))_{\partial K}\right| \tag{3.19}$$

$$\le \inf_{v\in H^1(\Omega)}\sum_{K\in\mathcal{K}}\|Q-v\|_{1/2,\partial K}\|\boldsymbol{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\Sigma})\|_{-1/2,\partial K} \tag{3.20}$$

$$\le \inf_{v\in H^1(\Omega)}\left(\sum_{K\in\mathcal{K}}\|Q-v\|^2_{1/2,\partial K}\right)^{1/2}\left(\sum_{K\in\mathcal{K}}\|\boldsymbol{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\Sigma})\|^2_{-1/2,\partial K}\right)^{1/2} \tag{3.21}$$

$$\le C\inf_{v\in H^1(\Omega)}\left(\sum_{K\in\mathcal{K}}\|Q-v\|^2_{1/2,\partial K}\right)^{1/2}\left(\sum_{K\in\mathcal{K}}\left(\|\boldsymbol{\sigma}-\boldsymbol{\Sigma}\|^2_{0,K} + h_K\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\Sigma})\|^2_{0,K}\right)\right)^{1/2} \tag{3.22}$$

$$\le C\inf_{v\in H^1(\Omega)}\left(\sum_{K\in\mathcal{K}}\|Q-v\|^2_{1/2,\partial K}\right)^{1/2}\left(\|\boldsymbol{\sigma}-\boldsymbol{\Sigma}\|^2_0 + \|h(f+\nabla\cdot\boldsymbol{\Sigma})\|^2_0\right)^{1/2} \tag{3.23}$$

$$\le \frac{3C^2}{2}\inf_{v\in H^1(\Omega)}\sum_{K\in\mathcal{K}}\|Q-v\|^2_{1/2,\partial K} + \frac{1}{4}\|\boldsymbol{\sigma}-\boldsymbol{\Sigma}\|^2_0 + \frac{1}{2}\|h(f+\nabla\cdot\boldsymbol{\Sigma})\|^2_0. \tag{3.24}$$

Together equation (3.17) and equation (3.19-3.23) gives a bound for the second term, II, in equation (3.7),

$$II \le \|\nabla Q-\boldsymbol{\Sigma}\|^2_0 + \frac{1}{2}\|\boldsymbol{\sigma}-\boldsymbol{\Sigma}\|^2_0 + \frac{1}{2}\|h(f+\nabla\cdot\boldsymbol{\Sigma})\|^2_0 + \frac{3C^2}{2}\inf_{v\in H^1(\Omega)}\sum_{K\in\mathcal{K}}\|Q-v\|^2_{1/2,\partial K}. \tag{3.25}$$

We combine equation (3.8) and equation (3.25) to get,

$$I+II \le C\|\nabla Q-\boldsymbol{\Sigma}\|^2_0 + \frac{3}{4}\|\boldsymbol{\sigma}-\boldsymbol{\Sigma}\|^2_0 + C\|h(f+\nabla\cdot\boldsymbol{\Sigma})\|^2_0 + C\inf_{v\in H^1(\Omega)}\sum_{K\in\mathcal{K}}\|Q-v\|^2_{1/2,\partial K}. \tag{3.26}$$

To estimate the last term on the right hand side in inequality equation (3.26) we employ the technique of Lemma 4 in [3]. For completeness we include the details of the proof. We let $\mathcal{N}$ be the set of nodes in the mesh, $\{\phi\}_{i\in\mathcal{N}}$ piecewise linear base functions, $\omega_i = \text{supp}(\phi_i)$, CP_i continuous piecewise polynomials on ω_i , and $CP = \oplus_{i\in\mathcal{N}}\phi_i CP_i \in H^1(\Omega)$.

Using that $CP \subset H^1(\Omega)$ followed by the inverse inequality $\|Q-v\|^2_{1/2,\partial K} \le Ch_K^{-1}\|Q-v\|^2_{0,\partial K}$, which holds since both v and Q are piecewise polynomials, we get

$$\inf_{v\in H^1(\Omega)}\sum_{K\in\mathcal{K}}\|Q-v\|^2_{1/2,\partial K} \le \inf_{v\in CP}\sum_{K\in\mathcal{K}}\|Q-v\|^2_{1/2,\partial K} \le C\inf_{v\in CP}\sum_{K\in\mathcal{K}}h_K^{-1}\|Q-v\|^2_{0,\partial K}. \tag{3.27}$$

We write $v = \sum_{i \in \mathcal{N}} \phi_i v_i \in CP$ and proceed with the estimate as follows

$$\|Q - v\|_{0,\partial K}^2 = \sum_{i \in \mathcal{N}} (Q - v, \phi_i (v_i - Q))_{\partial K}, \tag{3.28}$$

$$\leq \sum_{i \in \mathcal{N}} \|\phi_i^{1/2} (Q - v)\|_{0,\partial K} \|\phi_i^{1/2} (v_i - Q)\|_{0,\partial K}, \tag{3.29}$$

$$\leq \|Q - v\|_{0,\partial K} \left(\sum_{i \in \mathcal{N}} \|\phi_i^{1/2} (v_i - Q)\|_{0,\partial K}^2 \right)^{1/2}, \tag{3.30}$$

where we used that $\{\phi_i\}_{i \in \mathcal{N}}$ is a partition of unity. We have

$$\inf_{v \in H^1(\Omega)} \sum_{K \in \mathcal{K}} \|Q - v\|_{1/2,\partial K}^2 \leq \inf_{v \in CP} \sum_{i \in \mathcal{N}} \sum_{K \in \mathcal{K}} h_K^{-1} \|\phi_i^{1/2} (v_i - Q)\|_{0,\partial K}^2. \tag{3.31}$$

Further the following bound holds,

$$\inf_{v_i \in CP_i} \sum_{K \in \mathcal{K}} h_K^{-1} \|\phi_i^{1/2} (v_i - Q)\|_{0,\partial K}^2 \leq C \sum_{K \in \mathcal{K}} h_K^{-1} \|\phi_i^{1/2} \, [Q] \, \|_{0,\partial K}^2, \tag{3.32}$$

since the right hand side of equation (3.32) is zero when Q in continuous on ω_i so we may choose $v_i = Q|_{\omega_i}$ which means that the left hand side is also zero. The estimate follows from finite dimensionality and scaling. We end up with,

$$\inf_{v \in H^1(\Omega)} \sum_{K \in \mathcal{K}} \|Q - v\|_{1/2,\partial K}^2 \leq C \sum_{K \in \mathcal{K}} \sum_{i \in \mathcal{N}} h_K^{-1} \|\phi_i^{1/2} \, [Q] \, \|_{0,\partial K}^2 = C \sum_{K \in \mathcal{K}} h_K^{-1} \| \, [Q] \, \|_{0,\partial K}^2, \tag{3.33}$$

again after using that $\{\phi_i\}_{i \in \mathcal{N}}$ is a partition of unity.

Combining equation (3.25) and (3.33) we get,

$$I + II \leq C \|\nabla Q - \boldsymbol{\Sigma}\|_0^2 + \frac{3}{4} \|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 + C \|h(f + \nabla \cdot \boldsymbol{\Sigma})\|_0^2 + C \sum_{K \in \mathcal{K}} h_K^{-1} \| \, [Q] \, \|_{0,\partial K}^2. \tag{3.34}$$

Since $I + II = \|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2$ from equation (3.2-3.7) we just need to subtract $3/4 \|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2$ from both sides of equation (3.34) to prove the theorem. □

3.2 Estimate Based on Postprocessing

We now turn to the question of how to choose Q in Theorem 3.1. We know that choosing $Q = U$ results in a suboptimal estimate of the energy norm error, [4]. A natural idea is to choose Q to be a postprocessed version of U. There have been several works [8, 5, 17, 13] following Arnold and Brezzi [2], published in the mid eighties, on postprocessing methods where information from the calculated flux $\boldsymbol{\Sigma}$ is used to compute an improved approximation of u.

We focus on the method proposed in Lovadina and Stenberg in [13] and show that Theorem 3.1 directly gives the estimate presented in [13]. We denote the postprocessed version of U by U^*. To define U^* we introduce some notations. For all $K \in \mathcal{K}$ we let $P_{h,K} : L^2(\Omega) \to W_{h,K}$ be the L^2 projection onto $W^*_{h,K}$, where $W^*_{h,K}$ is defined as $W^*_{h,K} = P_k(K)$ for RTN elements, $W^*_{h,K} = P_{k+1}(K)$ for BDM elements, and $W^*_{h,K} = P_{k+2}(K)$ for TH elements.

Definition 3.1 *(Postprocessing method) Find* $U^* = \bigoplus_{K\in\mathcal{K}} U^*_K \in \bigoplus_{K\in\mathcal{K}} W^*_{h,K}$ *such that*

$$P_{h,K} U^*_K = U_K, \tag{3.35}$$

and

$$(\nabla U^*, \nabla v)_K = (\boldsymbol{\Sigma}, \nabla v)_K \quad \text{for all } v \in (I - P_{h,K}) W^*_{h,K}. \tag{3.36}$$

Proposition 3.1 *It holds,*

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 \le C \sum_{K\in\mathcal{K}} \left(h_K^2 \|f + \nabla \cdot \boldsymbol{\Sigma}\|_{0,K}^2 + \|\boldsymbol{\Sigma} - \nabla U^*\|_{0,K}^2 + h_K^{-1} \| \left[U^* \right] \|_{0,\partial K}^2 \right), \tag{3.37}$$

where U^* is taken from Definition 3.1.

Proof. The proof follows directly from Theorem 3.1 with $Q = U^*$. □

3.3 Estimate for Stabilized Methods

Here we extend our estimate to stabilized mixed methods, in particular, we consider the recent method presented in Masud and Hughes [14]. Stabilized methods are based on a modified weak formulation which yields a stable method for standard continuous piecewise polynomial, of equal or different order, approximation of the pressure and flux.

The stabilized method of Masud and Hughes reads: find $\boldsymbol{\Sigma} \in \boldsymbol{V}_h$ and $U \in W_h$ such that,

$$(-\nabla \cdot \boldsymbol{\Sigma}, w) + (\boldsymbol{\Sigma}, \boldsymbol{v}) + (U, \nabla \cdot \boldsymbol{v}) - \frac{1}{2}(\boldsymbol{\Sigma} - \nabla U, \boldsymbol{v} + \nabla w) = (f, w), \tag{3.38}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_h$ and $w \in W_h$. Applying the same ideas as in Theorem 3.1 to this stabilized method we obtain the following a posteriori error estimate. The argument may be modified to cover other stabilized methods such as the Galerkin least squares method.

Proposition 3.2 *For the approximate solution of equation (3.38) using continuous piecewise polynomials it holds,*

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 \le C \sum_{K\in\mathcal{K}} \left(h_K^2 \|f + \nabla \cdot \boldsymbol{\Sigma}\|_{0,K}^2 + \|\boldsymbol{\Sigma} - \nabla U\|_{0,K}^2 \right). \tag{3.39}$$

Proof. Using the same arguments as in equations (3.2-3.7) in the proof of Theorem 3.1, we obtain the following error representation formula,

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 = (p - Q, f + \nabla \cdot \boldsymbol{\Sigma}) + (Q, -\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma})) - (\boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma}). \tag{3.40}$$

Next, setting $\boldsymbol{v} = 0$ in (3.38) we have the Galerkin orthogonality property

$$(f + \nabla \cdot \boldsymbol{\Sigma}, w) = -\frac{1}{2}(\boldsymbol{\Sigma} - \nabla U, \nabla w), \tag{3.41}$$

for all $w \in W_h$. Subtracting the Scott-Zhang interpolant [6], $\pi_h(p - Q)$, of $p - Q$, using (3.41) followed by an interpolation estimate we get

$$\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 &= (p - Q - \pi_h(p - Q), f + \nabla \cdot \boldsymbol{\Sigma}) - \frac{1}{2}(\nabla \pi_h(p - Q), \boldsymbol{\Sigma} - \nabla U) && (3.42)\\
&\quad + (Q, -\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\Sigma})) - (\boldsymbol{\Sigma}, \boldsymbol{\sigma} - \boldsymbol{\Sigma}) && (3.43)\\
&\leq C \sum_{K \in \mathcal{K}} \left(h_K^2 \|f + \nabla \cdot \boldsymbol{\Sigma}\|_{0,K}^2 + h_K^{-1} \| \, [Q] \, \|_{0,\partial K}^2 \right) + C\|\boldsymbol{\Sigma} - \nabla Q\|_0^2 && (3.44)\\
&\quad + \frac{1}{2}\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 + \|\boldsymbol{\Sigma} - \nabla U\|_0 \|\nabla \pi_h(p - Q)\|_0. && (3.45)
\end{aligned}$$

To get this estimate we also use arguments that are identical with the ones in the proof of Theorem 3.1. We choose $Q = U$. Since U is continuous the jump terms will vanish. We also use the stability of the interpolant π_h in $H^1(\Omega)$,

$$\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 &\leq C \sum_{K \in \mathcal{K}} h_K^2 \|f + \nabla \cdot \boldsymbol{\Sigma}\|_{0,K}^2 + C\|\boldsymbol{\Sigma} - \nabla U\|_0^2 && (3.46)\\
&\quad + \frac{1}{2}\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 + C\|\boldsymbol{\Sigma} - \nabla U\|_0 \|\boldsymbol{\sigma} - \nabla U\|_0 && (3.47)\\
&\leq C \sum_{K \in \mathcal{K}} h_K^2 \|f + \nabla \cdot \boldsymbol{\Sigma}\|_{0,K}^2 + C\|\boldsymbol{\Sigma} - \nabla U\|_0^2 && (3.48)\\
&\quad + \frac{1}{2}\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 + \frac{1}{4}\|\boldsymbol{\sigma} - \nabla U\|_0^2 && (3.49)
\end{aligned}$$

But since $\|\boldsymbol{\sigma} - \nabla U\|_0 \leq \|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0 + \|\boldsymbol{\Sigma} - \nabla U\|_0$ we have,

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2 \leq C \sum_{K \in \mathcal{K}} h_K^2 \|f + \nabla \cdot \boldsymbol{\Sigma}\|_{0,K}^2 + C\|\boldsymbol{\Sigma} - \nabla U\|_0^2 + \frac{3}{4}\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2, \tag{3.50}$$

so the proposition follows immediately after subtracting $3/4\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_0^2$ from both sides. $\square$

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