Dynamic Subgrid Modeling for Time Dependent Convection-Diffusion-Reaction Equations with Fractal Solutions

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Plan for this talk

- DSM for a Fractal Model Problem
- Numerical Experiments
  - Reaction Dominated Problems
  - Convection Dominated Problems
- Future Work

Fractal Model Problem

Find $u : \Omega \times [0,T] \rightarrow \mathbb{R}^n$ such that

$$\dot{u} + Lu = \dot{u} - \epsilon\Delta u + \beta\cdot\nabla u = f(u),$$
$$u|_{\Gamma_D} = u_D, \quad \frac{\partial u}{\partial n}|_{\Gamma_N} = u_N, \quad \partial\Omega = \Gamma_D \cup \Gamma_N,$$
$$u(x,0) = u_0(x),$$

where ϵ is small, β and u_0 are fractal functions.

- Typically we cannot resolve all scales of u
- Want to approximate some average of u

The Running Average

- Let h denote the finest scale we are allowed to use (could be the computational scale).
- We define the running average u^h of u on h by

$$u^h(x,t) = \frac{1}{h^d} \int_{x_1-h/2}^{x_1+h/2} \dots \int_{x_d-h/2}^{x_d+h/2} u(y,t)\; dy_1 \dots dy_d$$

- We want to approximate u^h

The Simplified Problem

Find $u_h : \Omega \times [0, T] \to \mathbb{R}^n$ such that

$$\dot{u}_h + L_h u_h = \dot{u}_h + \beta^h \cdot \nabla u_h - \epsilon \Delta u_h = f(u_h),$$
$$u_h(x, 0) = u_0^h(x),$$

where L_h is a simplified operator on h.

- **Objective:**

  Minimize the error $u^h - u_h$ without refining h

The Averaged Problem

- The running average of the model problem $\Rightarrow$

$$\dot{u}^h + L_h u^h = f(u^h) + F_h(u),$$
$$u^h(x,0) = u_0^h(x),$$

$$F_h(u) = (f(u))^h - f(u^h) + L_h u^h - (Lu)^h$$

- $F_h(u)$ contains the information on how the unresolved scales influence u^h.

The Corrected Problem

- Approximate $F_h(u) \Rightarrow$ Corrected Problem

$$\dot{\tilde{u}}^h + (L_h + \tilde{L}_h)\tilde{u}^h = f(\tilde{u}_h) + \tilde{F}_h(\tilde{u}_h),$$
$$\tilde{u}_h(x,0) = u_0^h(x),$$
$$\tilde{F}_h(\tilde{u}_h) - \tilde{L}_h\tilde{u}_h \approx F_h(u).$$

- Free to choose Ansatz of the form
  - $\tilde{F}_h(\tilde{u}_h)$
  - $\tilde{L}_h\tilde{u}_h$
  - Or a combination of the above two

Analysis of $F_h(u)$

- If f is a second order reaction term $\Rightarrow$

$$(f(u))^h - f(u^h) = \sum (u_i u_j)^h - u_i^h u_j^h$$

- ϵ constant $\Rightarrow$

$$(Lu)^h - L_h u^h = (\beta \cdot \nabla u)^h - \beta^h \cdot \nabla u^h$$

- That is, $F_h(u) = \sum_k (v_k w_k)^h - v_k^h w_k^h$

P.w. Constant Approximation

- $\Omega = [0,1]^2$
- τ^h reg. quadratic mesh corresponding to h
- $[f]^h$ is the p.w. constant function on τ^h that equals f^h in the midpoints of the elements.

- We want to model $[v_k w_k]^h - [v_k]^h [w_k]^h$

Haar MRA in $L_2[0,1]$

Defined by the **wavelets**

$$\psi_{i,k}(x) = 2^{i/2}\psi(2^i x - k), \text{ where}$$

$$\psi(x) = \begin{cases} 1 & 0 < x < 1/2 \\ -1 & 1/2 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the **scale function** $\varphi = 1$ for $x \in [0,1]$.

Wavelets for Haar MRA in $L_2[0,1]^2$

-1	-1
$+1$	$+1$

$\Psi^H(x,y)$

$+1$	-1
$+1$	-1

$\Psi^V(x,y)$

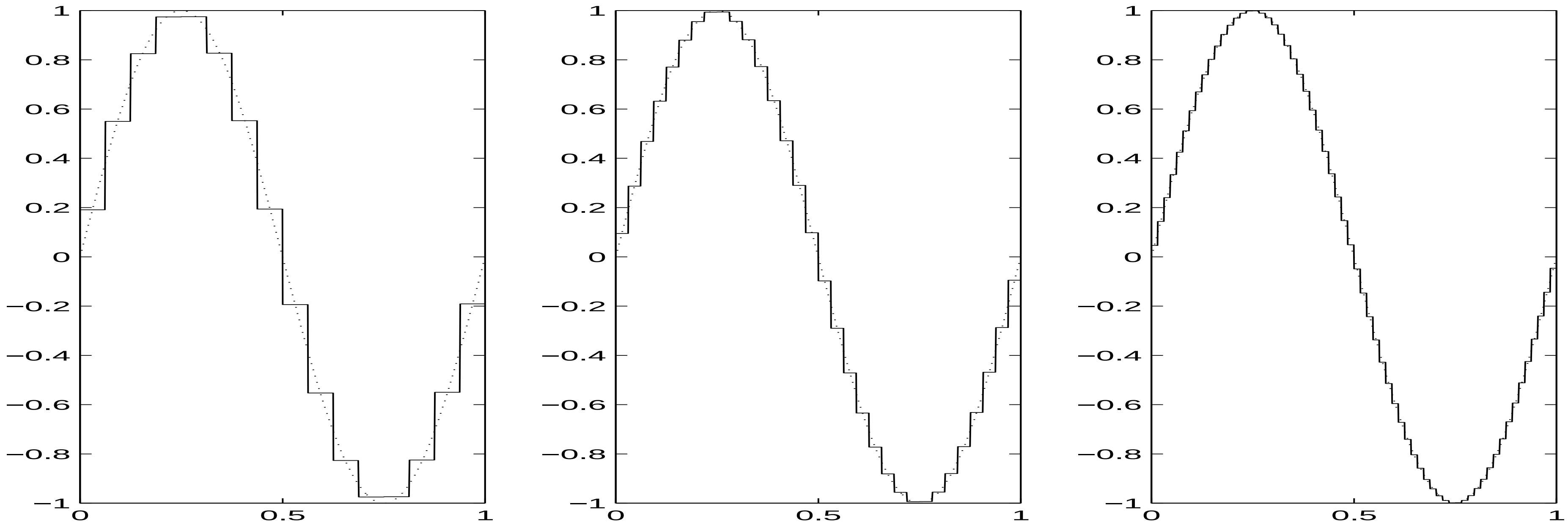
-1	$+1$
$+1$	-1

$\Psi^D(x,y)$

Haar Approximation ($h = 2^{-i}$)

- $f = f_\varphi \varphi + \sum_{j,k} (f^H_{j,k}\psi^H_{j,k} + f^V_{j,k}\psi^V_{j,k} + f^D_{j,k}\psi^D_{j,k})$
- $[f]^h = f_\varphi \varphi + \sum_{j<i,k} (f^H_{j,k}\psi^H_{j,k} + f^V_{j,k}\psi^V_{j,k} + f^D_{j,k}\psi^D_{j,k})$
- $[f]^h$ is the L_2-projection of f onto the space of p.w. constant functions on the scale h.

Haar Approximation



$[f]^{4h}$ $[f]^{2h}$ $[f]^{h}$

Haar Lemma

For a given $x \in \Omega = [0,1]^2$ ($h = 2^{-i}$):

$$[vw]^h - [v]^h[w]^h = \sum_{j \geq i} 2^{2j}(v_{j,l}^H w_{j,l}^H + v_{j,l}^V w_{j,l}^V + v_{j,l}^D w_{j,l}^D)$$

for $l : x \in \Omega_{i,l}$ (subdomain corresp. to el. in τ^h), $v_{j,l}^k, w_{j,l}^k$ are the Haar wavelet coefficients for v, w.

Scale Regularity in Turbulence

- Kolmogorov (1941): "$v(r+l) - v(r) \sim l^{1/3}$"
- Onsager (1949): "$v(r+l) - v(r) \sim l^{1/3} \Rightarrow$ v Hölder cont. with exponent $1/3$."
- Frisch (1995): "$v_j \sim C2^{-j\delta}$ stable solution for high Re"
- Scotti, Meneveau & Saddoughi (1995): "Experimental findings of fractal scaling of velocity signals in turbulent flow"

Ansatz

$$v_{j,k} = \alpha(x) 2^{-j(1/2+\delta(x))}, \;\; w_{j,k} = \beta(x) 2^{-j(1/2+\gamma(x))},$$

corresponding to a local fractal form

$$v_{j+1} = 2^{-\delta(x)} v_j, \quad w_{j+1} = 2^{-\gamma(x)} w_j.$$

Lemma gives that

$$[vw]^h - [v]^h [w]^h (x) \approx C(x) h^{\mu(x)},$$

where $C(x)$ and $\mu(x)$ are independent of h.

Subgrid Model $\tilde{F}_h(\tilde{u}_h) \approx F_h(u)$

$$\tilde{F}_h(\tilde{u}_h) = (\tilde{F}_h(\tilde{u}_h)_k), \quad \tilde{F}_h(\tilde{u}_h)_k = \sum_l E_h(\tilde{u}_h)_{k,l}$$

where

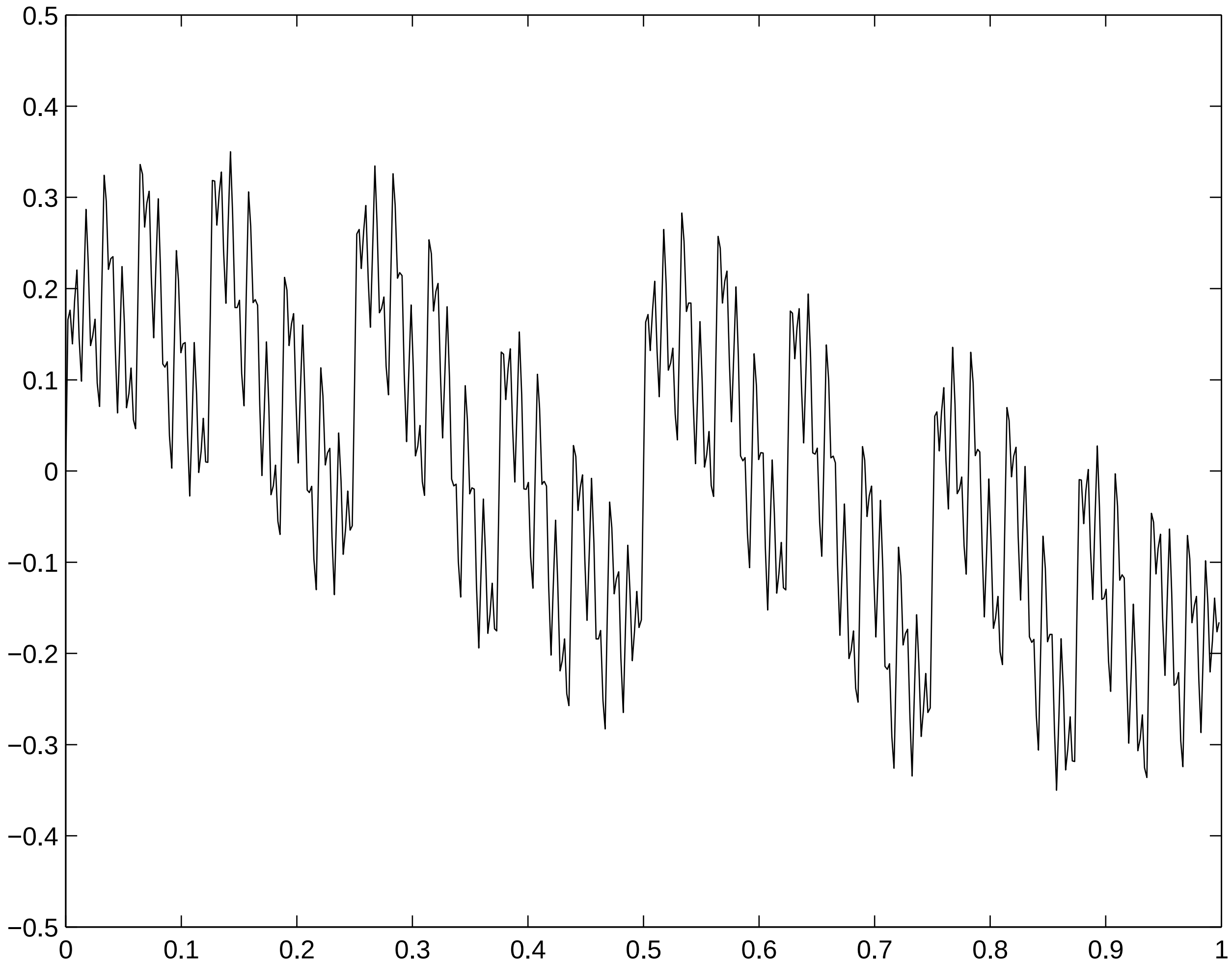
$$E_h(\tilde{u}_h)_{k,l}(x,t) = \frac{[E_{2h}]^{4h}(x,t)_k - [E_h]^{4h}(x,t)_k}{\frac{E_{4h}(x,t)_k - [E_{2h}]^{4h}(x,t)_k}{[E_{2h}]^{4h}(x,t)_k - [E_h]^{4h}(x,t)_k} - 1}$$

Weierstrass Functions

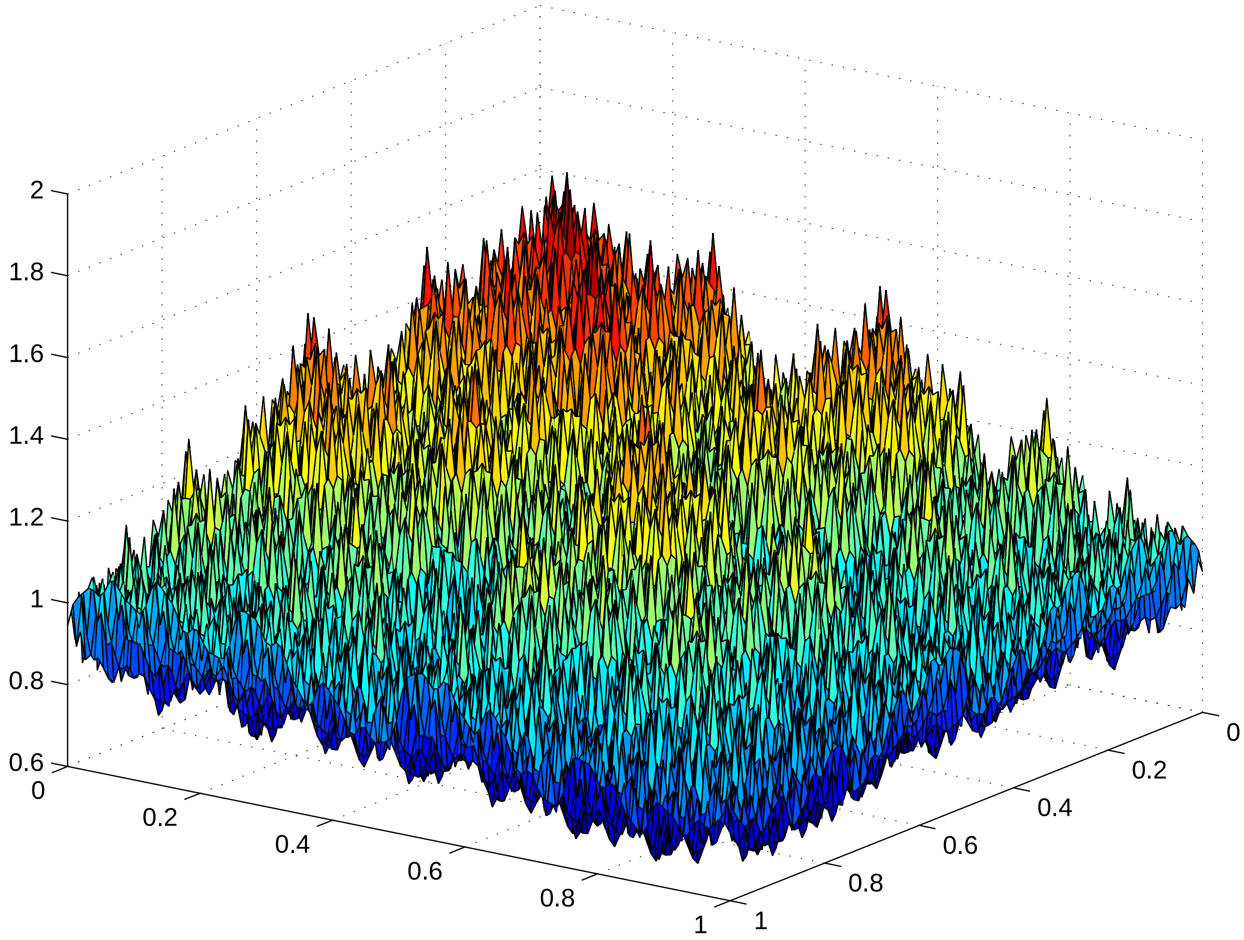
$$W_{\gamma,\delta}(x) = \gamma(x) \sum_{j=0}^{N} 2^{-j\delta(x)} \sin(2^j \cdot 2\pi x),$$

$$W_{2D} = W_{\gamma_1,\delta_1}(x_1) + W_{\gamma_2,\delta_2}(x_2) + W_{\gamma_3,\delta_3}(x_1) W_{\gamma_3,\delta_3}(x_2)$$

Weierstrass Function - 1D



Weierstrass Function - 2D



Reaction Dominated Problems

Find $u : [0,1]^2 \times [0,2] \to \mathbb{R}^n$ such that

$$\dot{u} - \epsilon\Delta u = f(u), \quad \Omega \times T = [0,1]^2 \times (0,2),$$
$$\frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad u(x,0) = u_0(x),$$

$\epsilon = 10^{-6}$, u_0 is fractal, $h = 2^{-5}$, ref. scale 2^{-9},

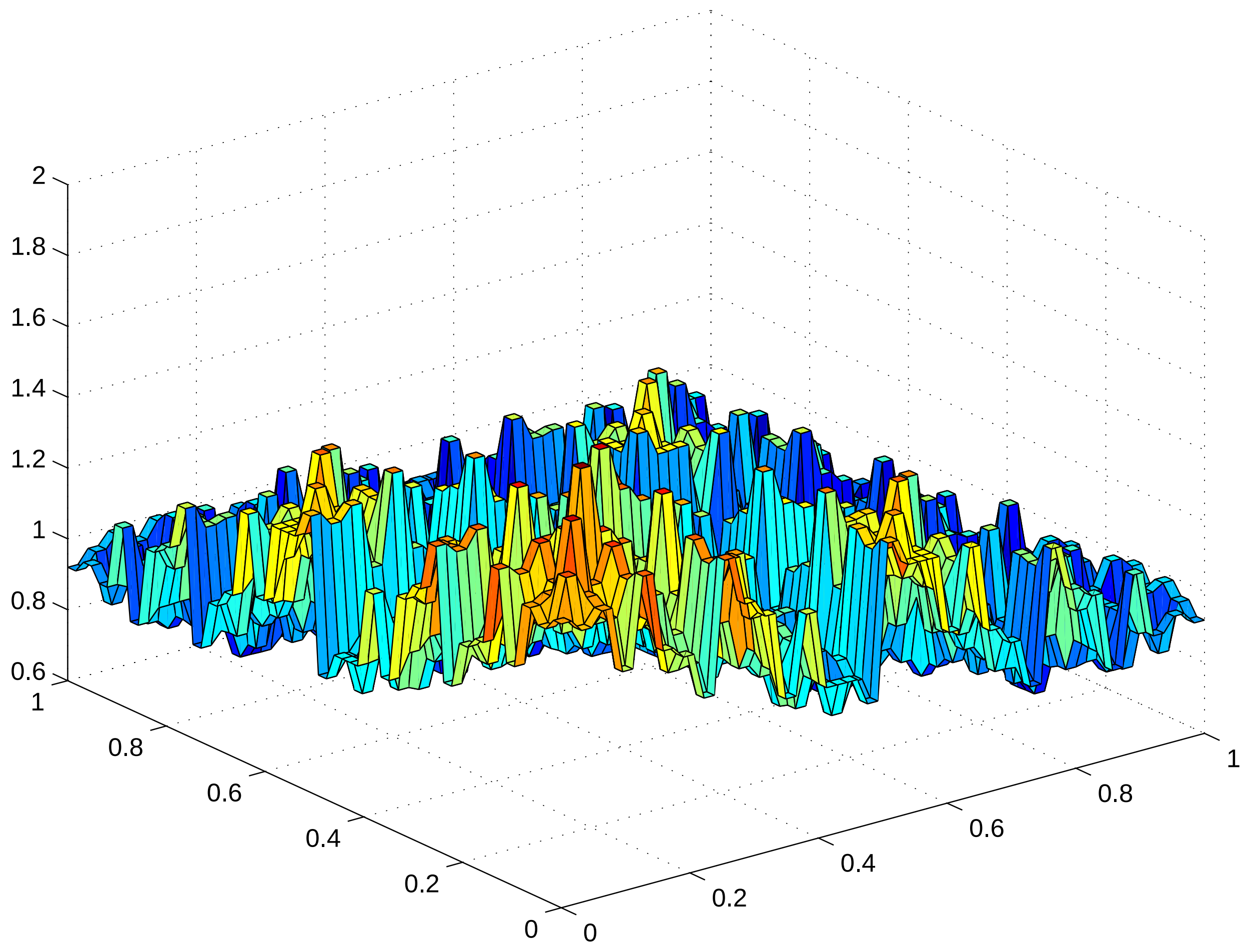
$$\dot{u}^h - \epsilon\Delta u^h = f(u^h) + F_h(u^h), \quad u^h(x,0) = u_0^h(x),$$
$$\dot{u}_h - \epsilon\Delta u_h = f(u_h), \quad u_h(x,0) = u_0^h(x),$$
$$\dot{\tilde{u}}_h - \epsilon\Delta\tilde{u}_h = f(\tilde{u}_h) + \tilde{F}_h(\tilde{u}_h), \quad \tilde{u}_h(x,0) = u_0^h(x).$$

$f(u) = u(1-u)$

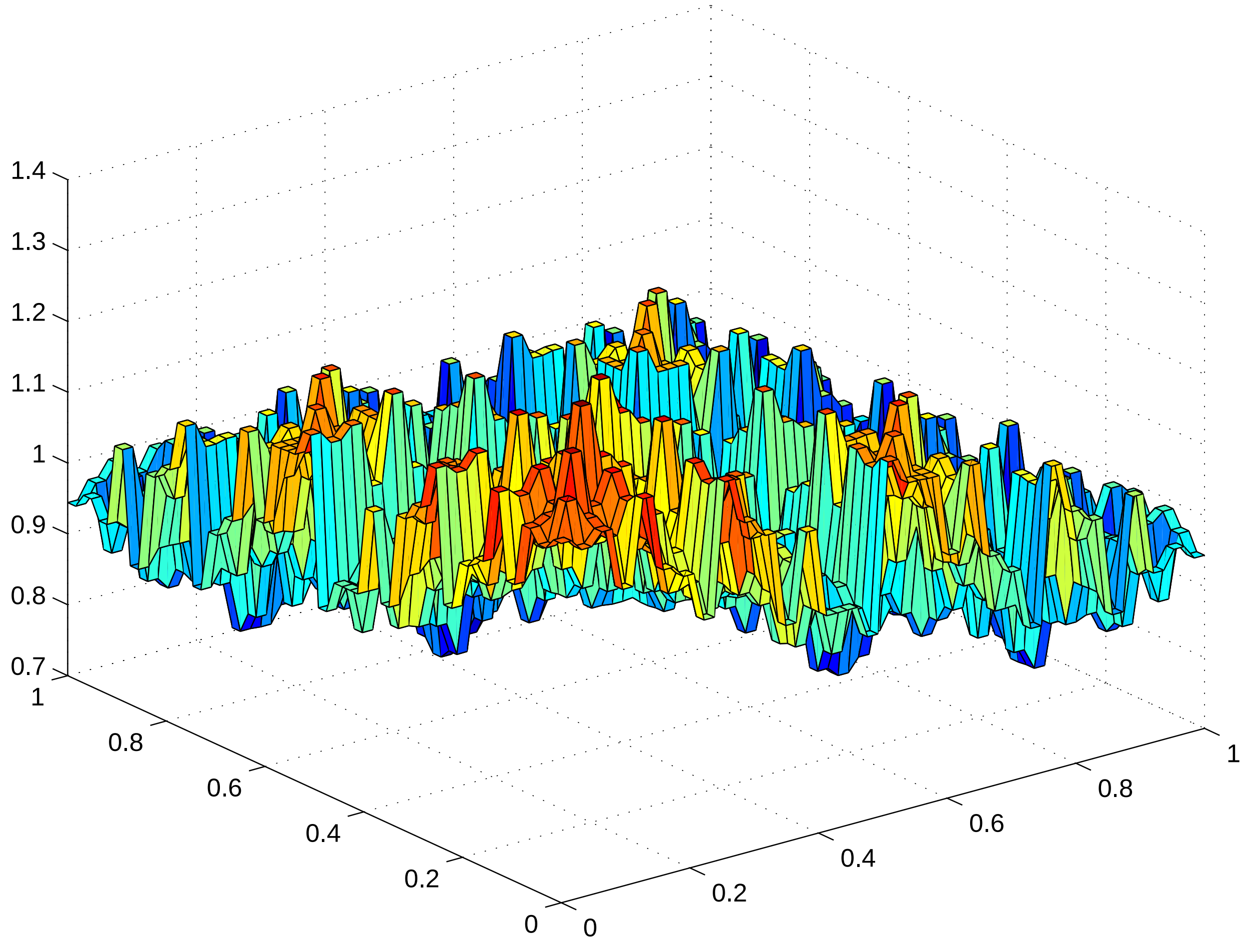
$$
\begin{aligned}
\dot{u} - \epsilon\Delta u &= u(1-u), & u(x,0) &= W_{2D} \\
\dot{u}^h - \epsilon\Delta u^h &= u^h(1-u^h) + F_h(u), & u^h(x,0) &= W_{2D}^h \\
\dot{u}_h - \epsilon\Delta u_h &= u_h(1-u_h), & u_h(x,0) &= W_{2D}^h \\
\dot{\tilde{u}}_h - \epsilon\Delta \tilde{u}_h &= \tilde{u}_h(1-\tilde{u}_h) + \tilde{F}_h(\tilde{u}_h, & \tilde{u}_h(x,0) &= W_{2D}^h
\end{aligned}
$$

$$
F_h(u) = -(u^2)^h + (u^h)^2
$$

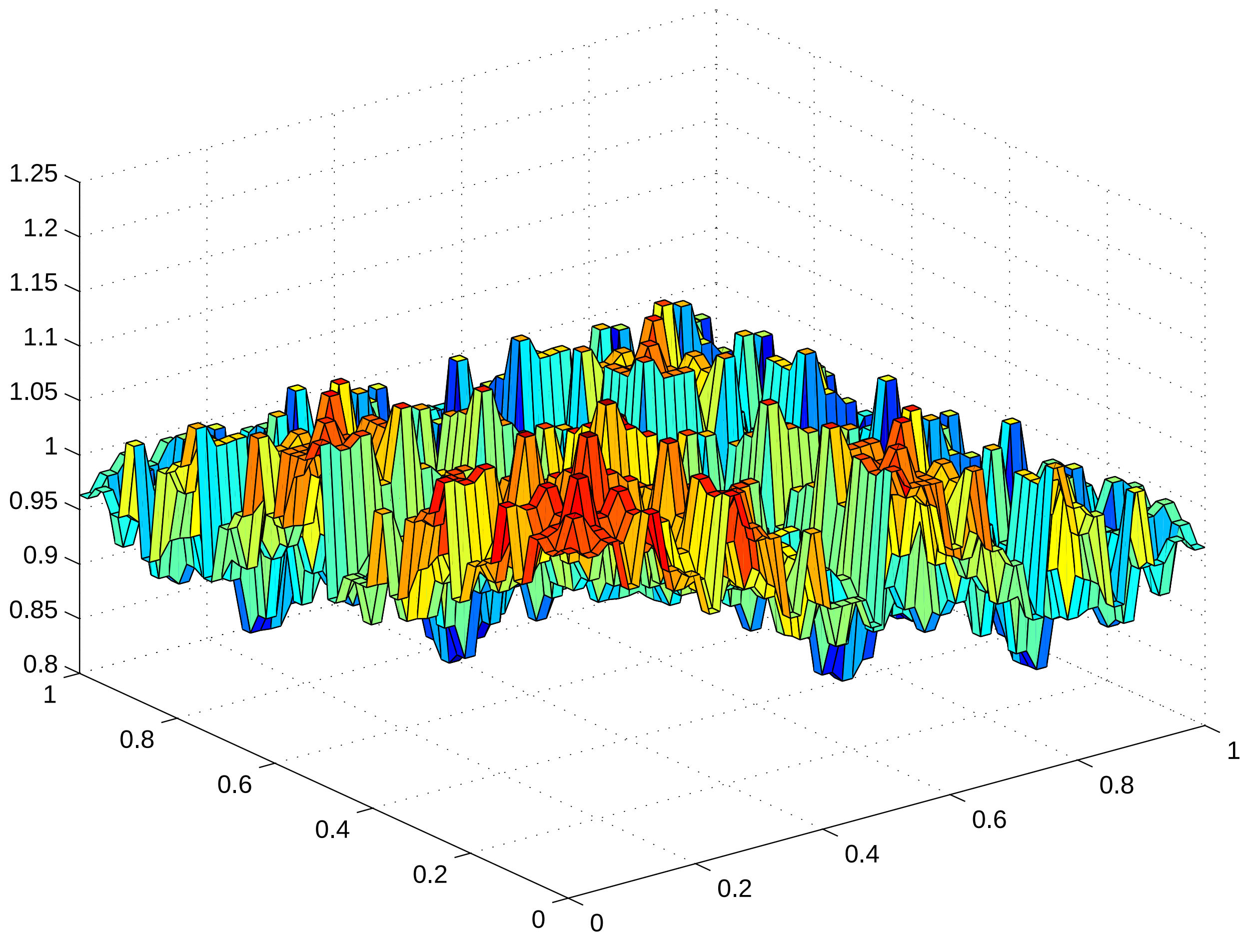
$u(t) \quad t = 0.0$



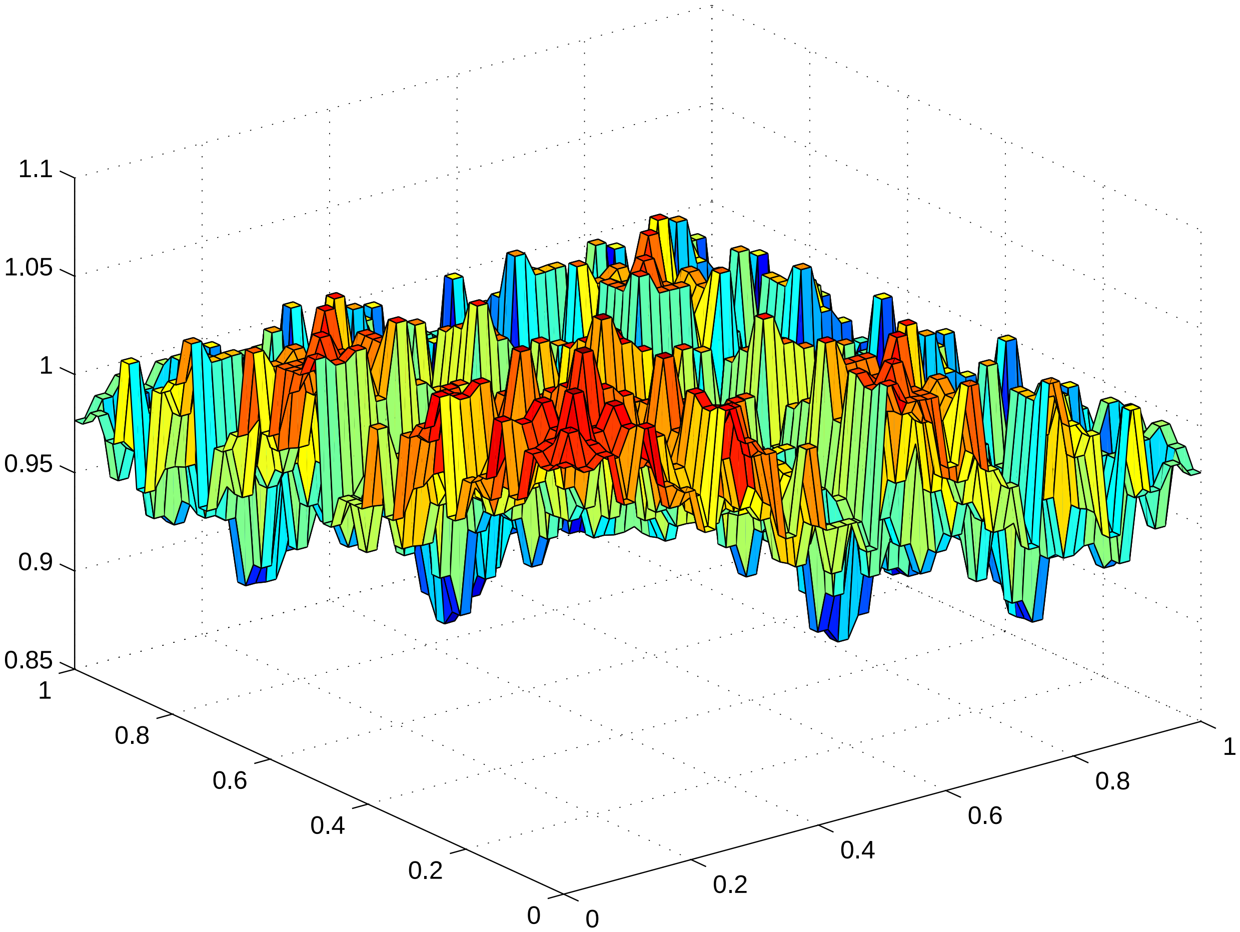
$u(t) \quad t = 0.5$



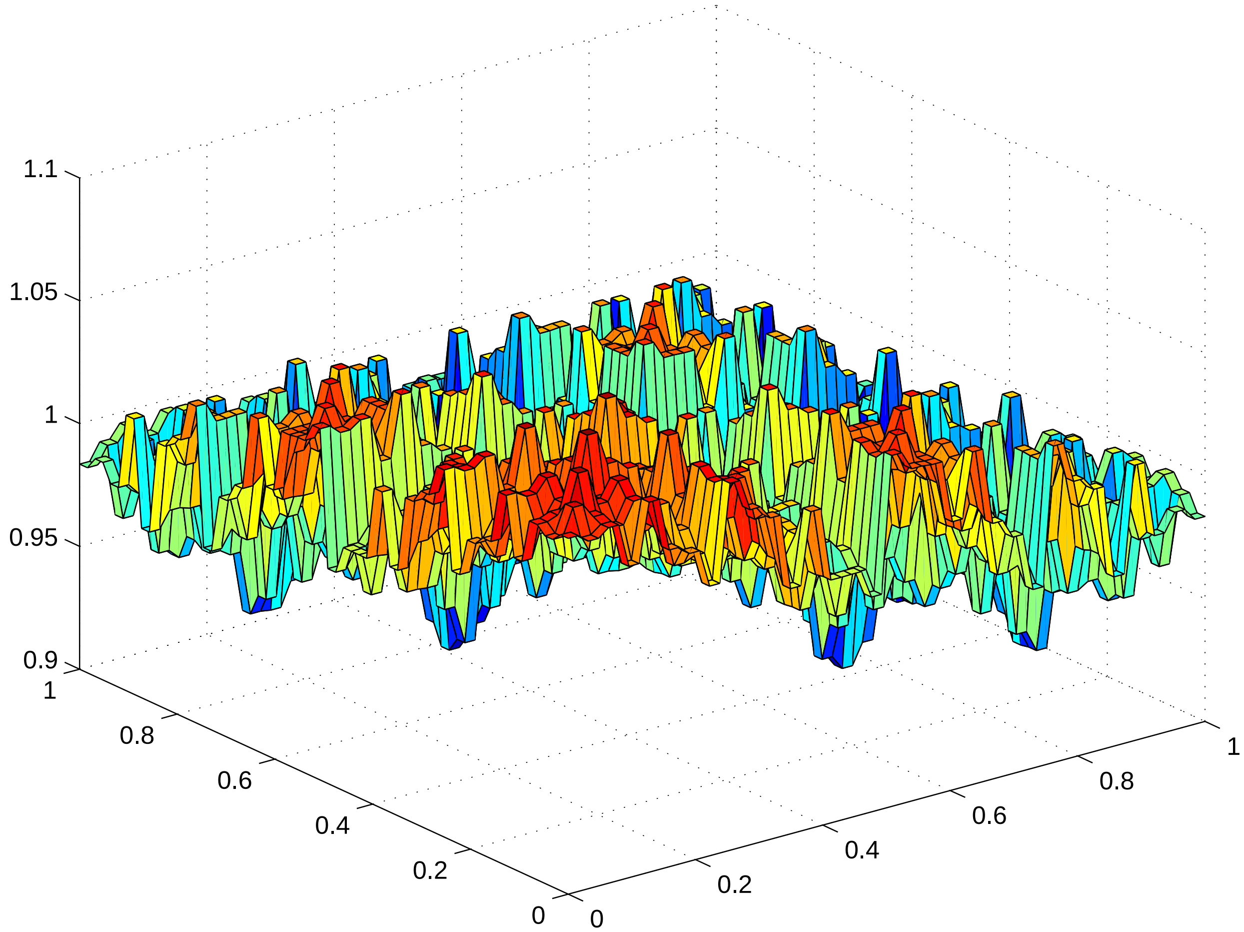
$u(t) \quad t = 1.0$



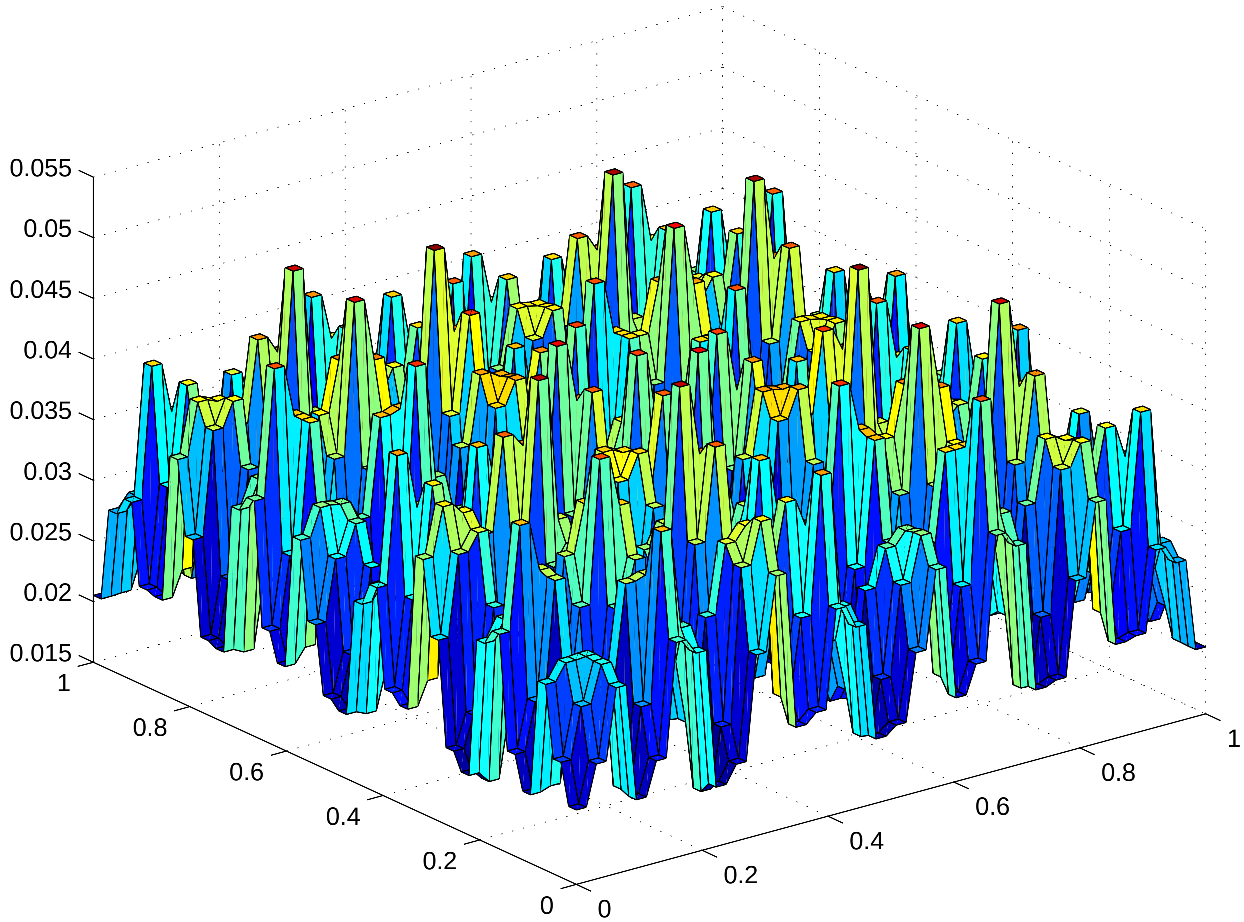
$u(t) \quad t = 1.5$



$u(t) \quad t = 2.0$



$F_h(u)(t) \quad t = 0.0$



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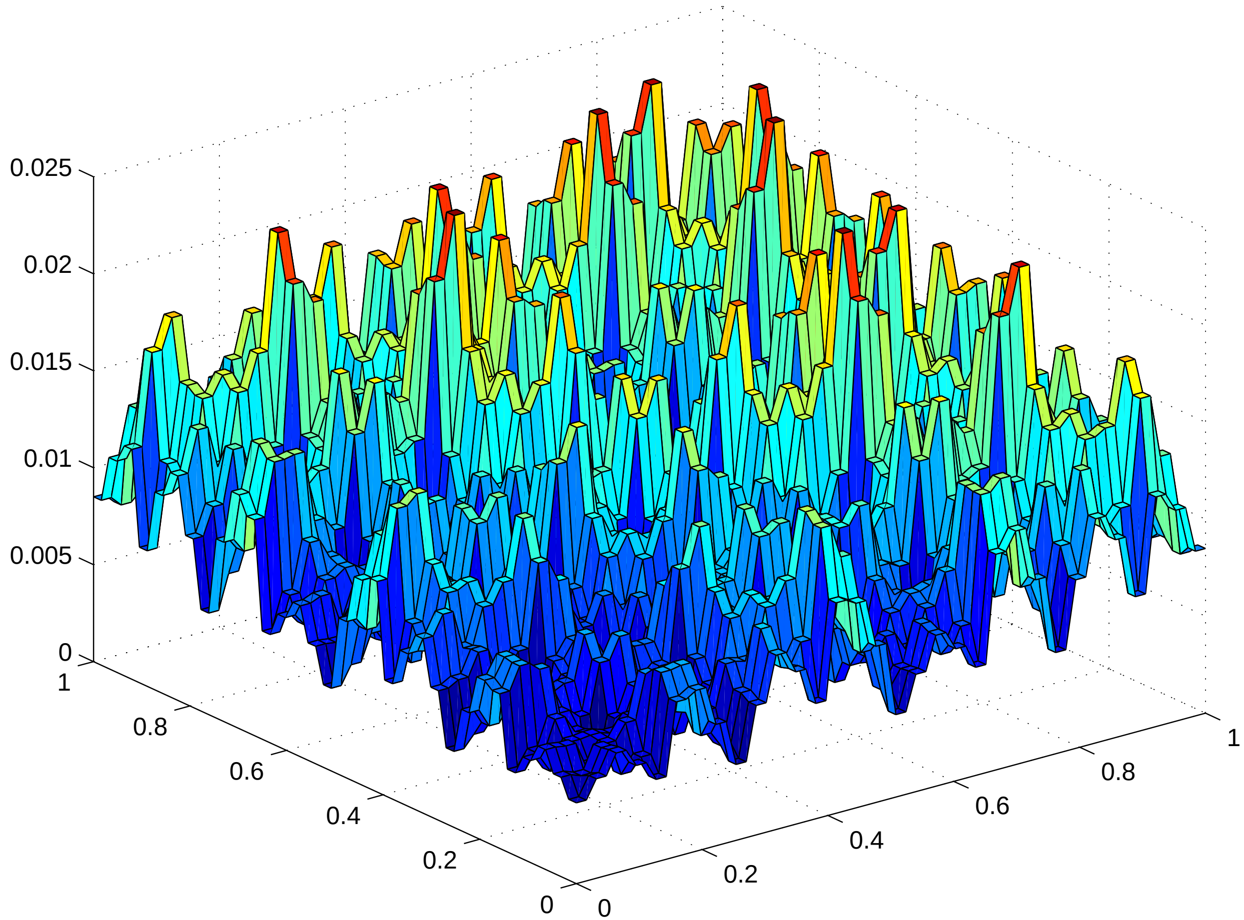
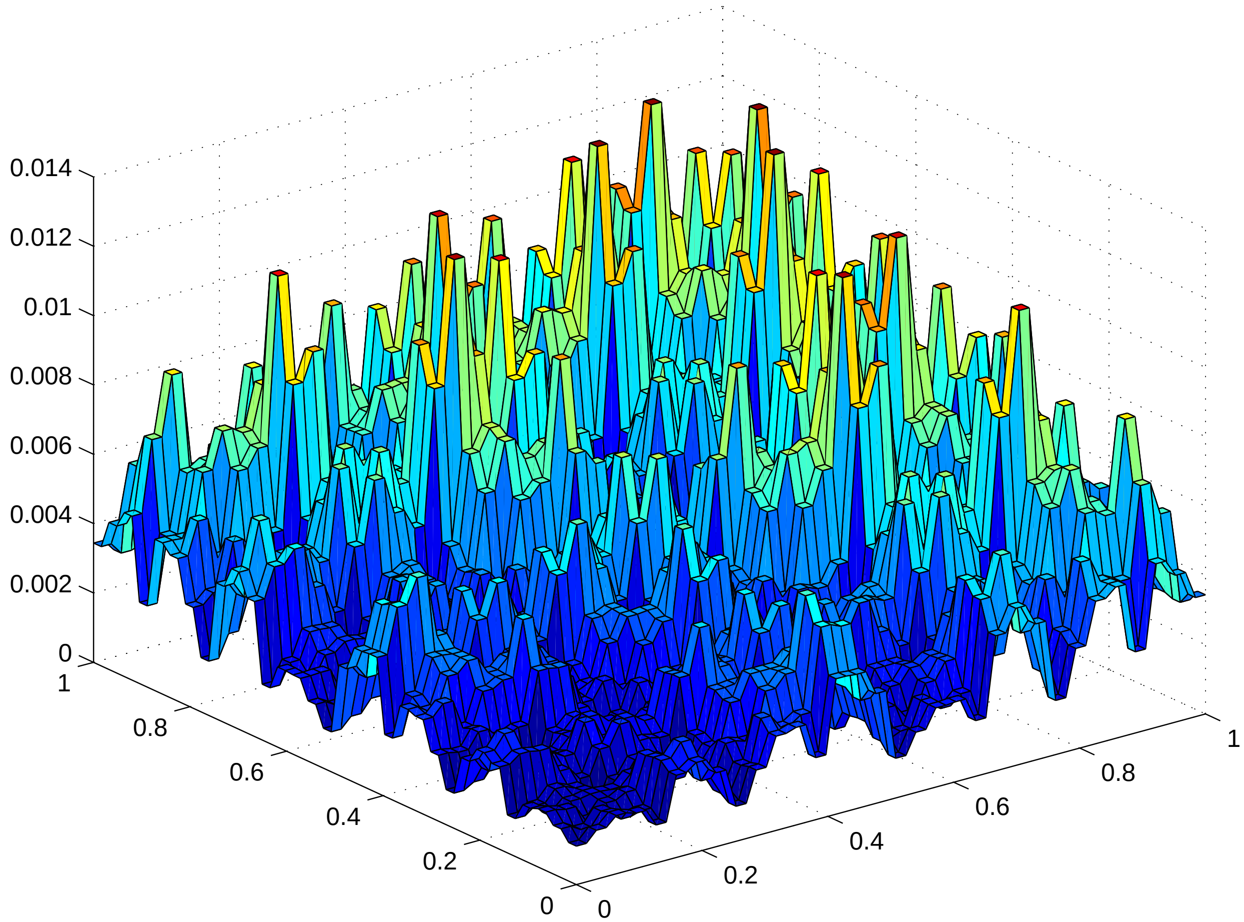
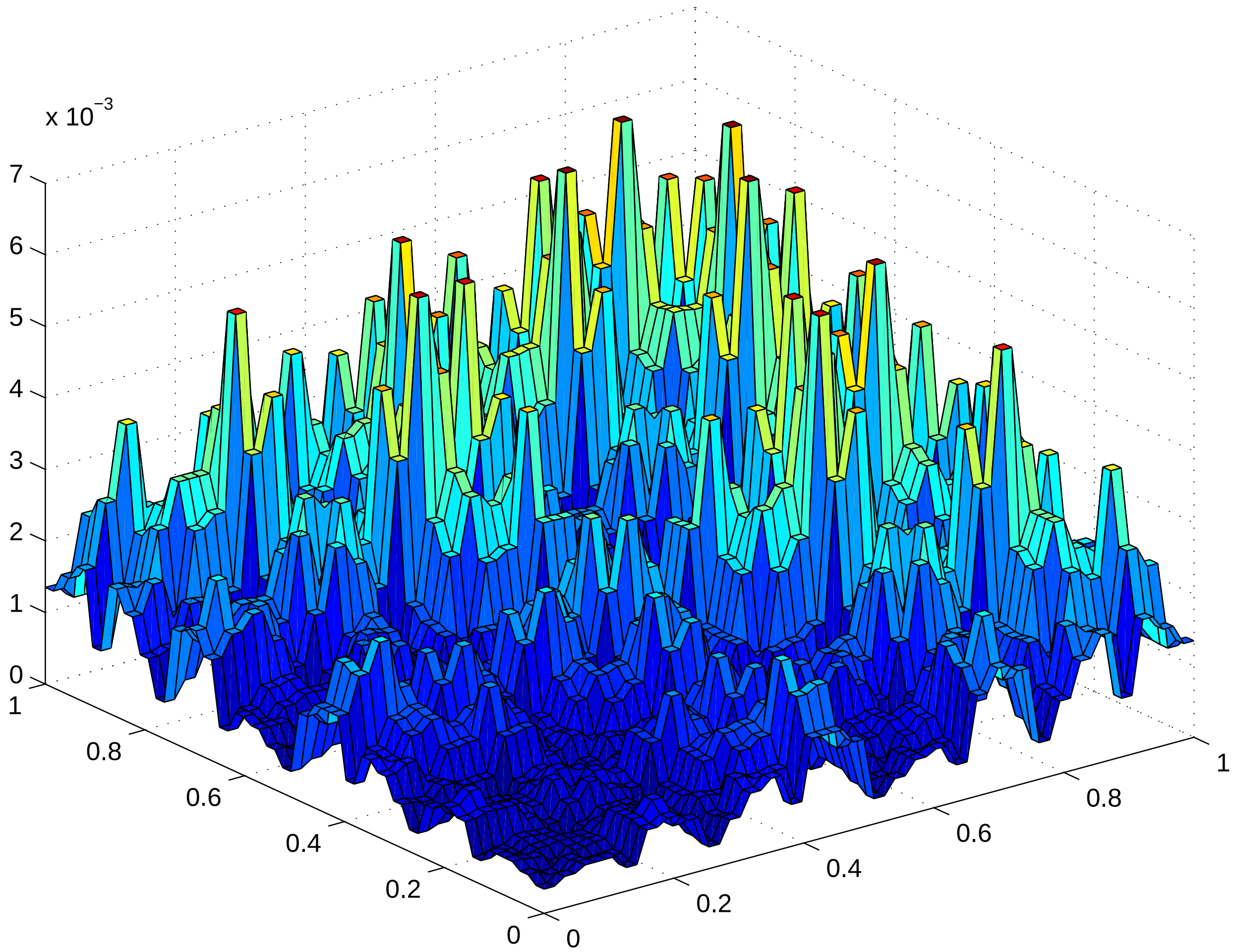


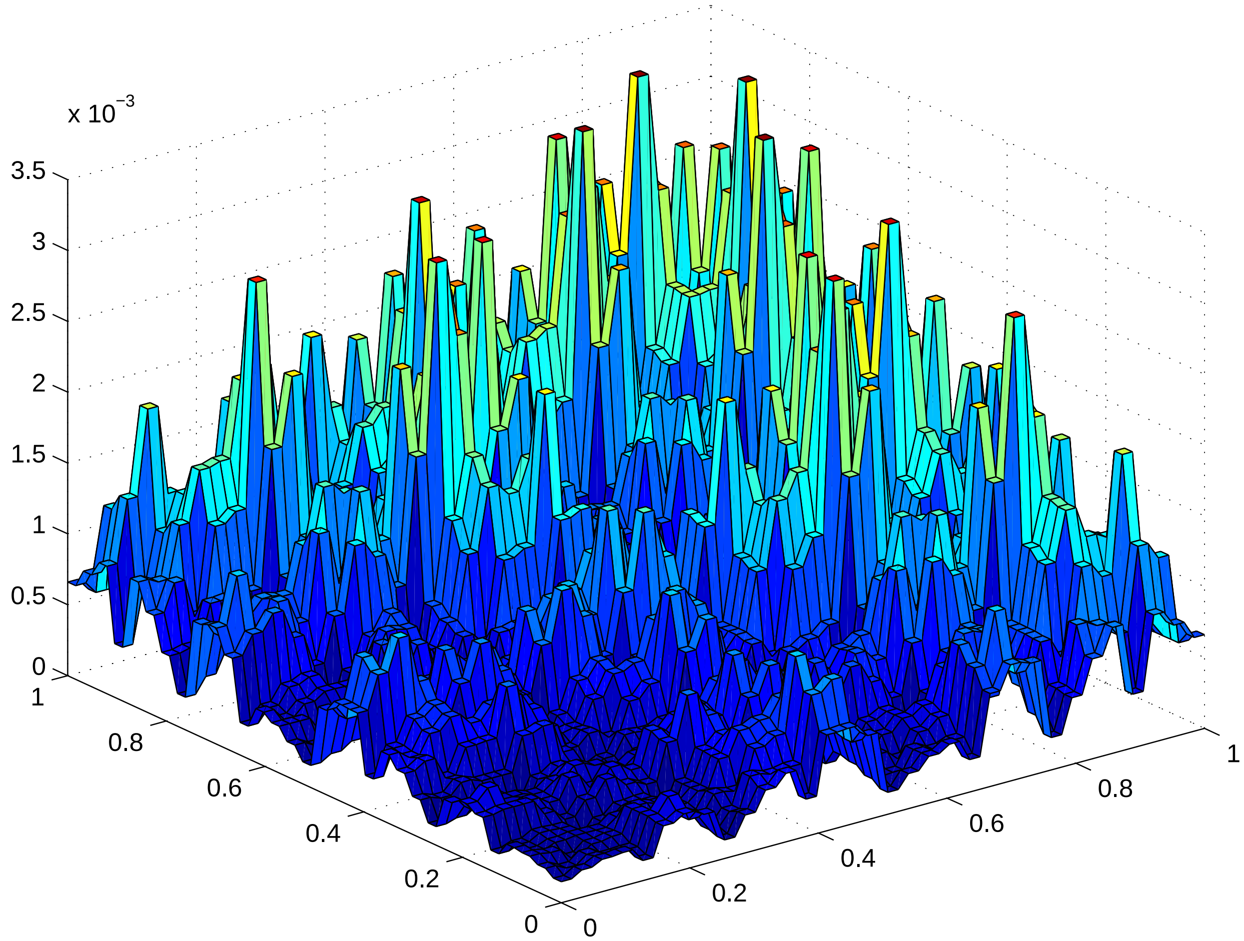
$F_h(u)(t) \quad t = 1.0$



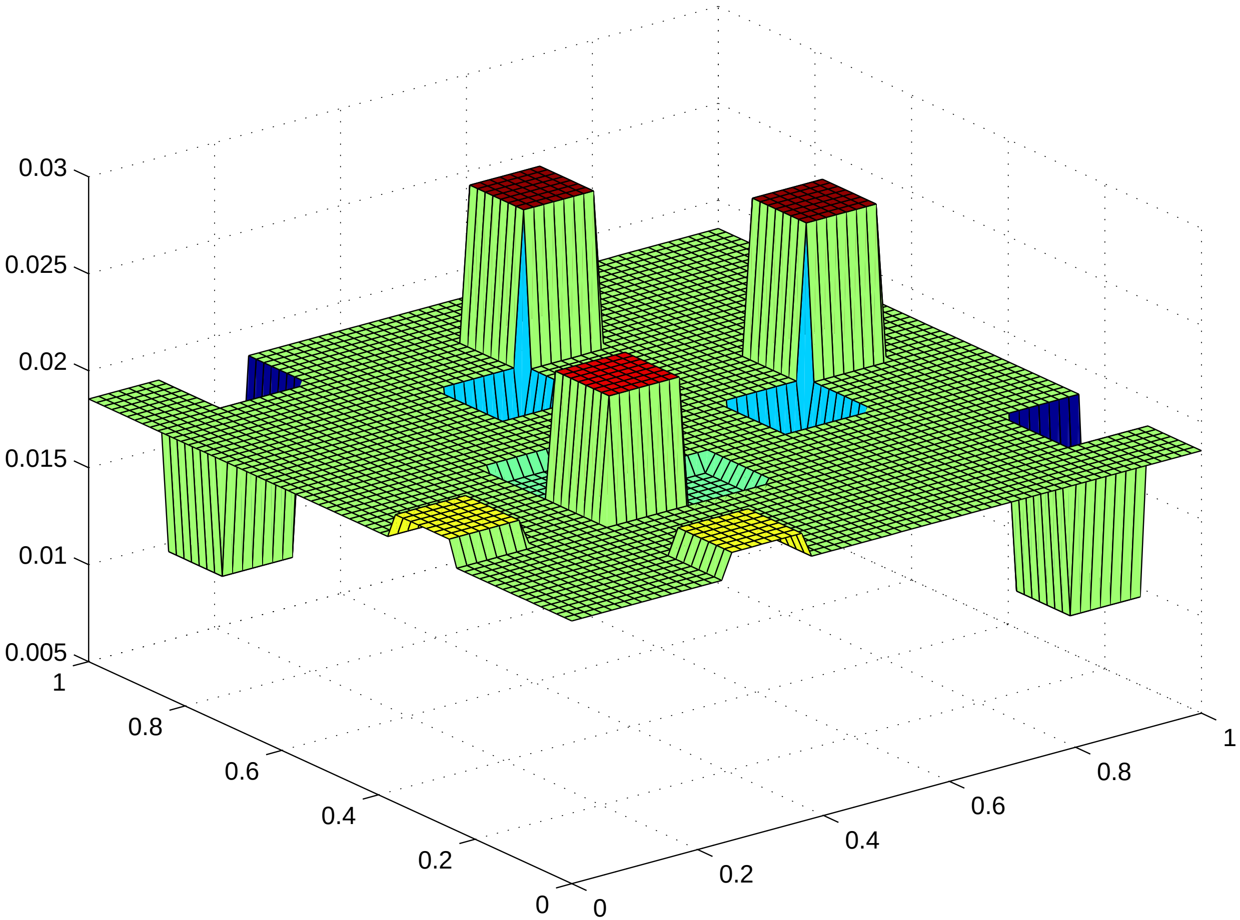
$F_h(u)(t) \quad t = 1.5$



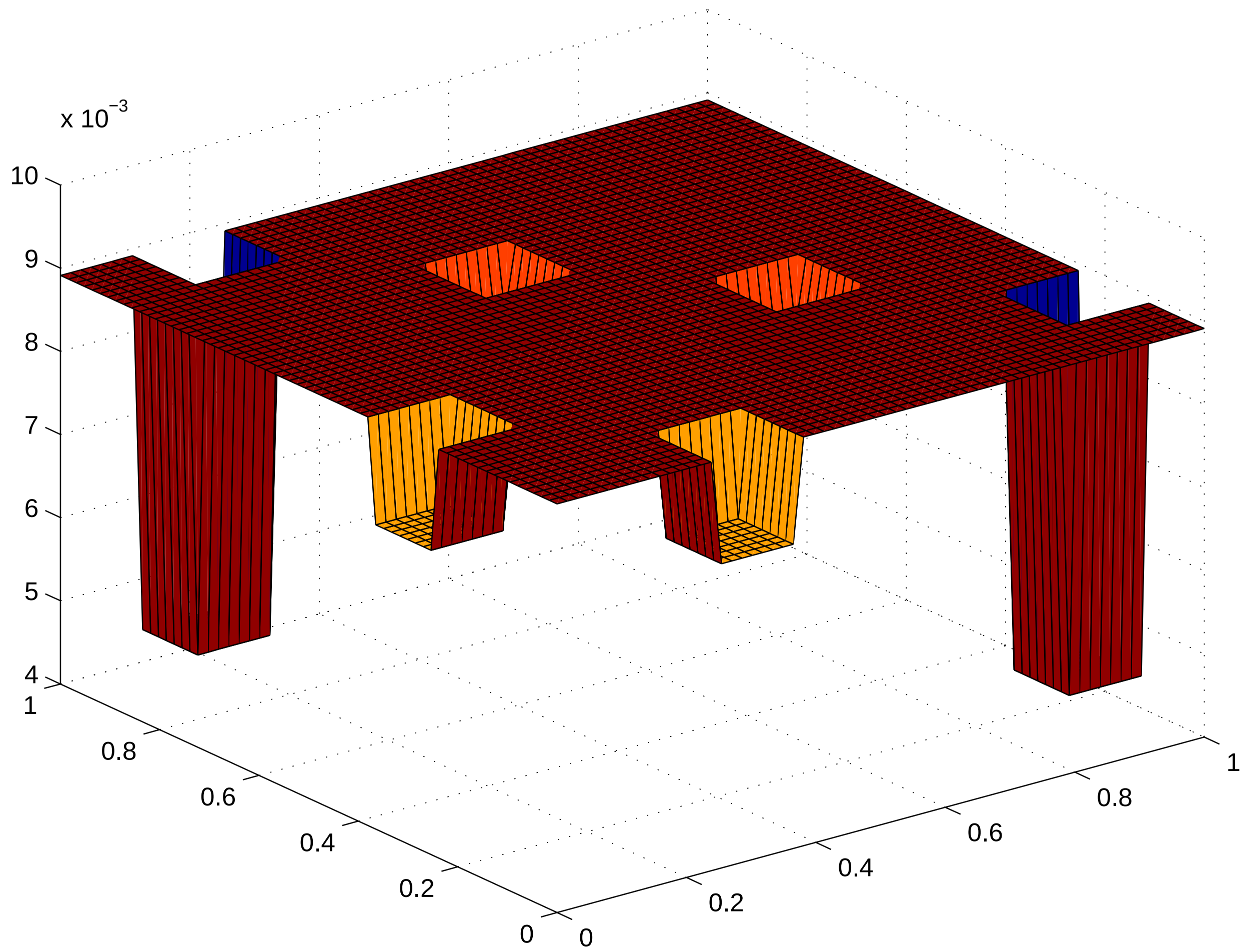
$F_h(u)(t) \quad t = 2.0$



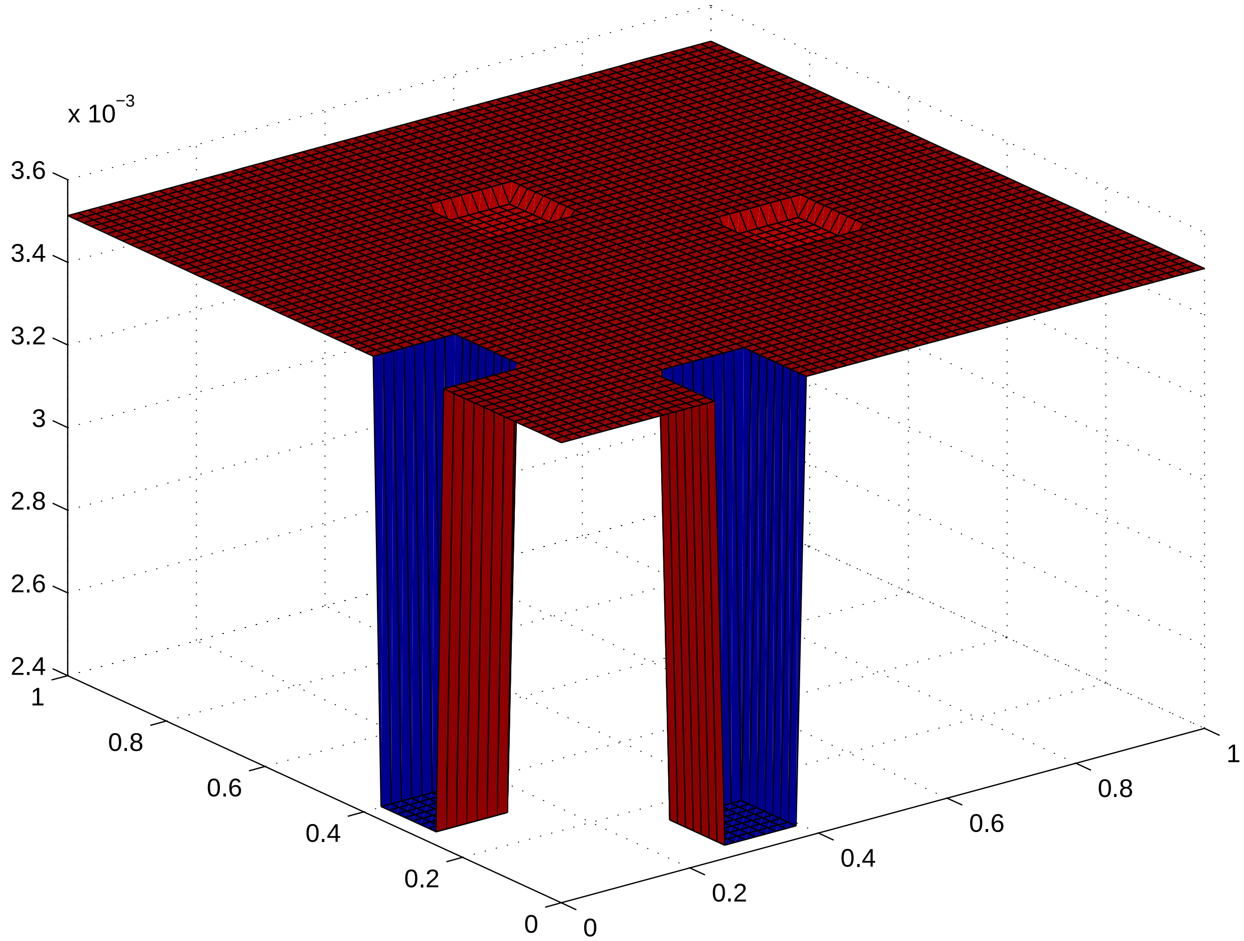
$\tilde{F}_h(\tilde{u}_h)(t) \quad t = 0.0$



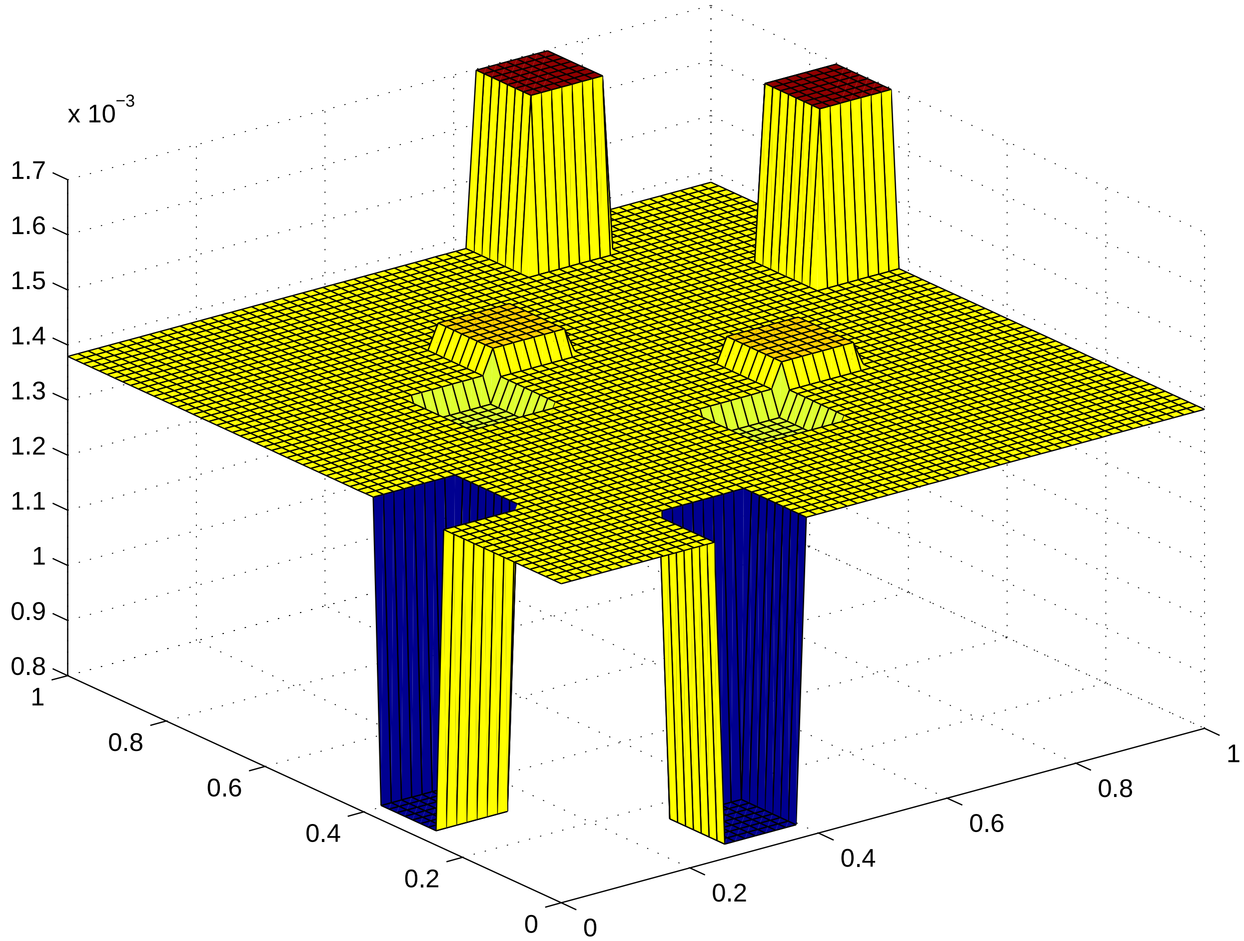
$\tilde{F}_h(\tilde{u}_h)(t) \quad t = 0.5$



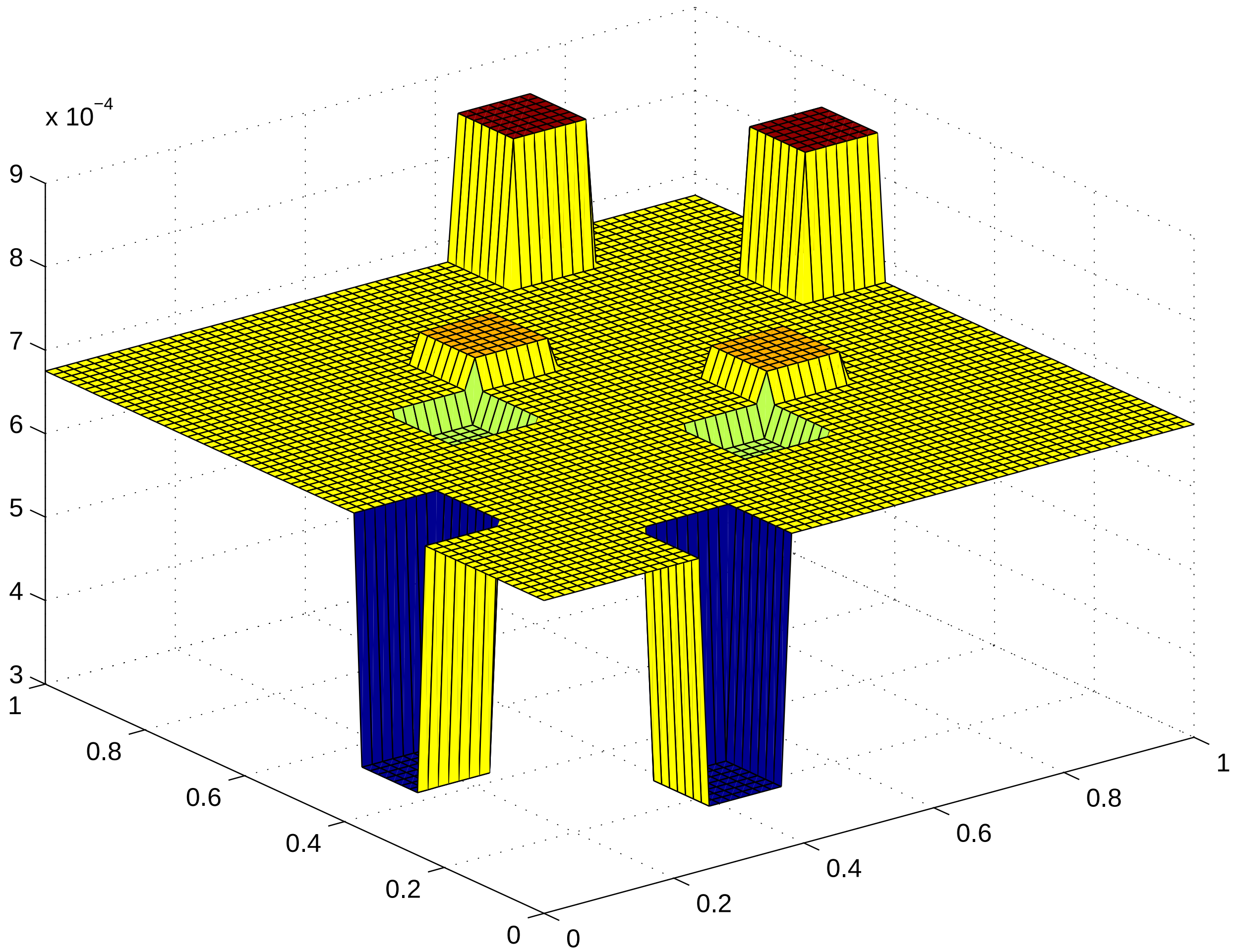
$\tilde{F}_h(\tilde{u}_h)(t) \quad t = 1.0$



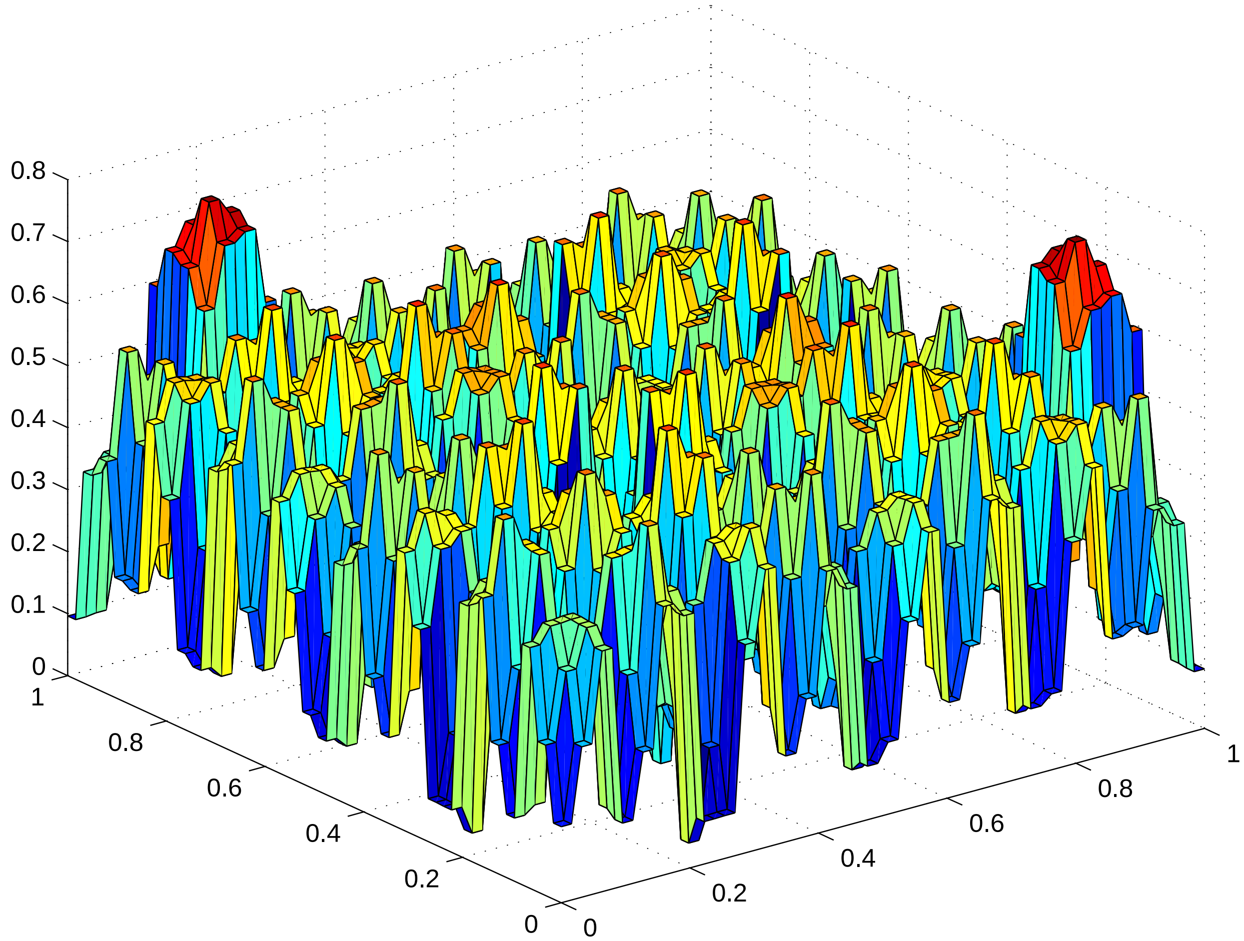
$\tilde{F}_h(\tilde{u}_h)(t) \quad t = 1.5$



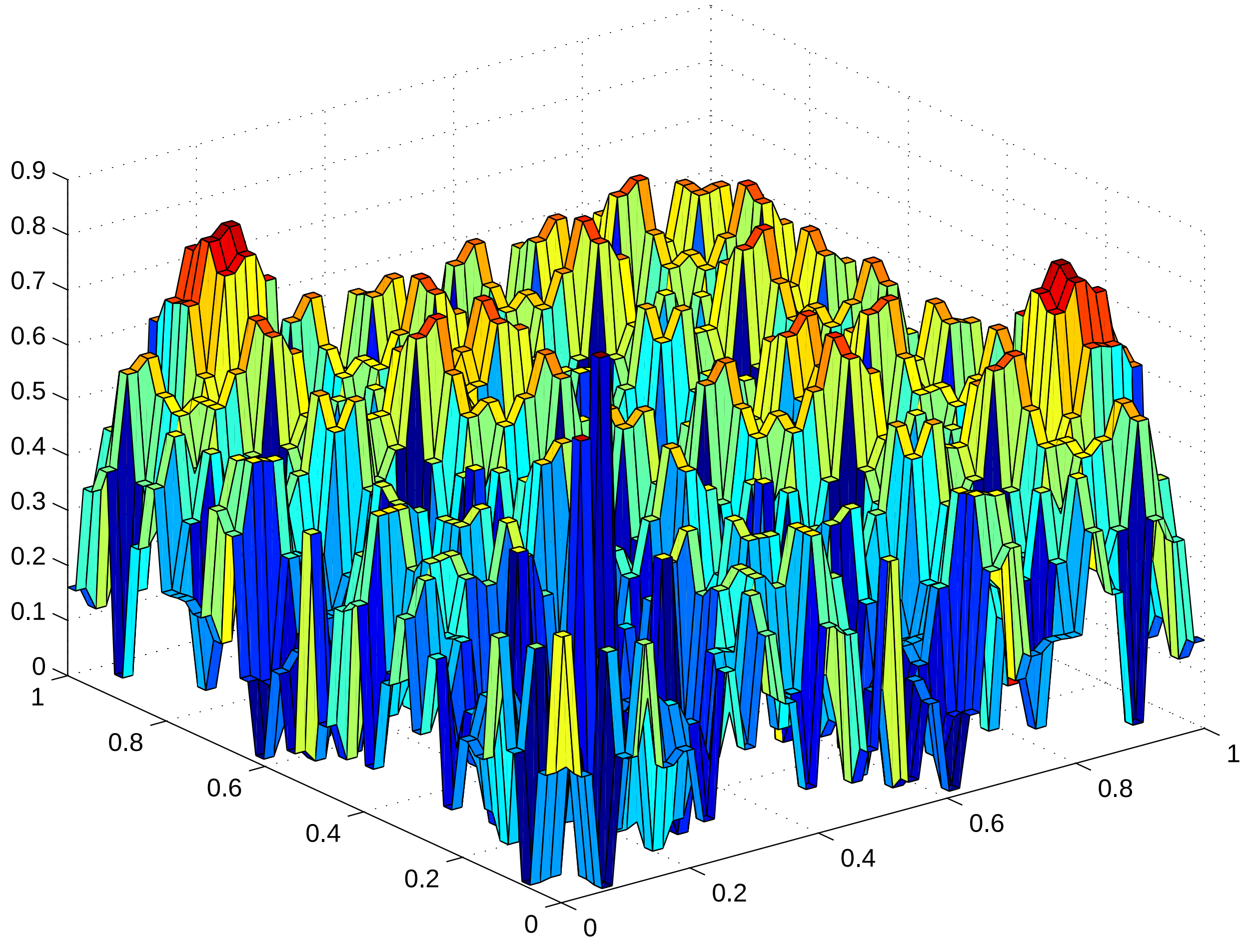
$\tilde{F}_h(\tilde{u}_h)(t) \quad t = 2.0$



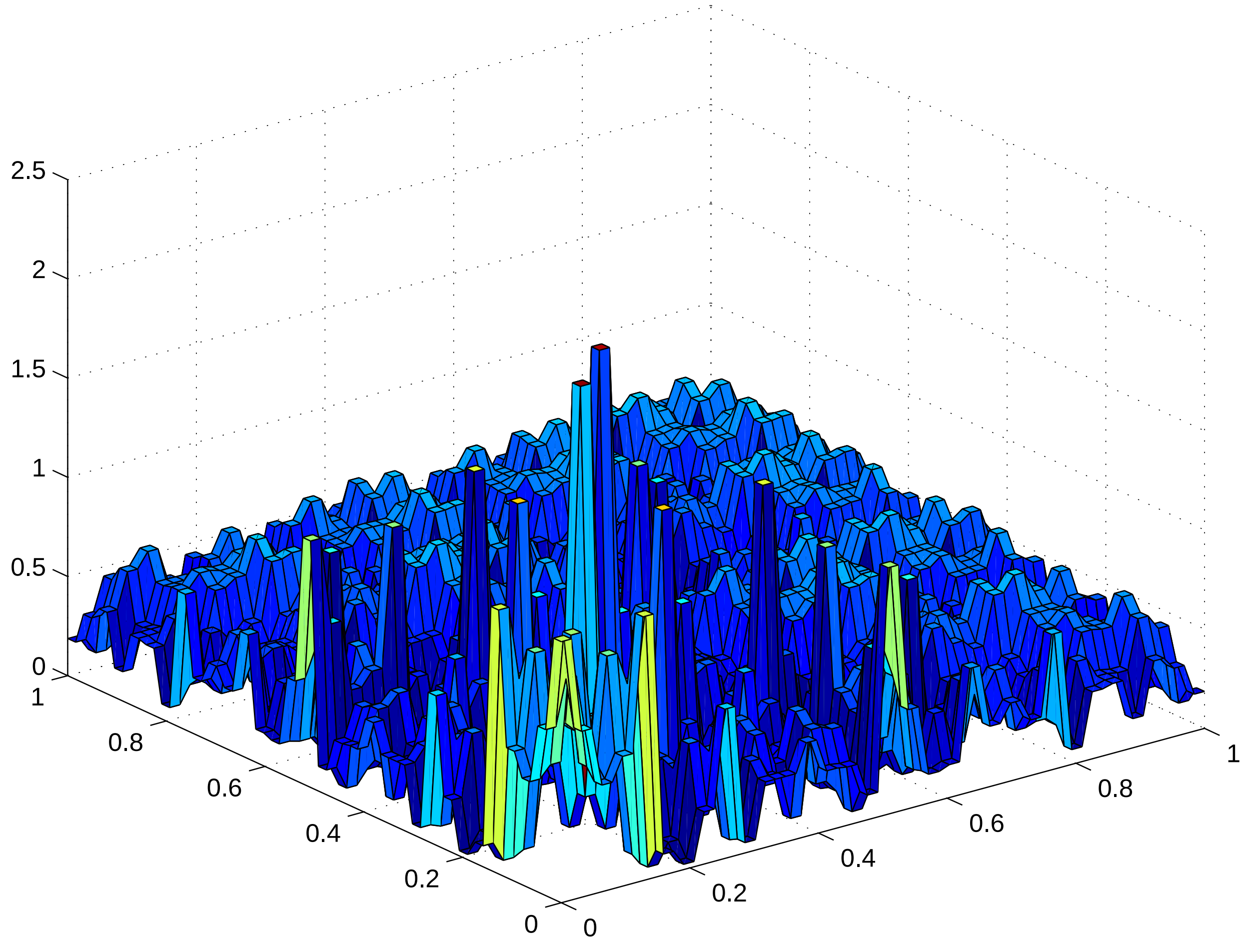
$|F_h(u) - \tilde{F}_h(\tilde{u}_h)|/|F_h(u)|(t) \quad t = 0.0$



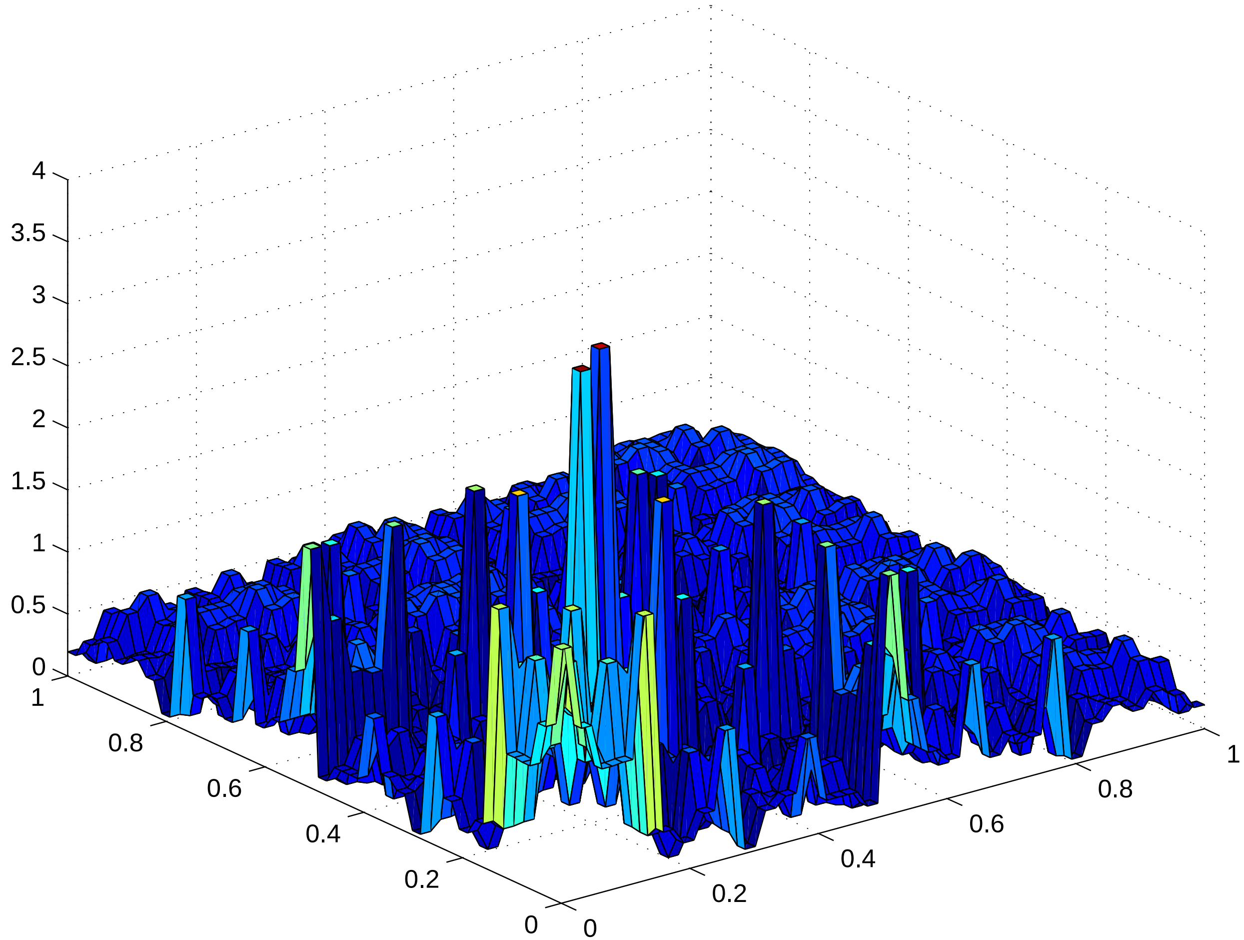
$|F_h(u) - \tilde{F}_h(\tilde{u}_h)|/|F_h(u)|(t) \quad t = 0.5$



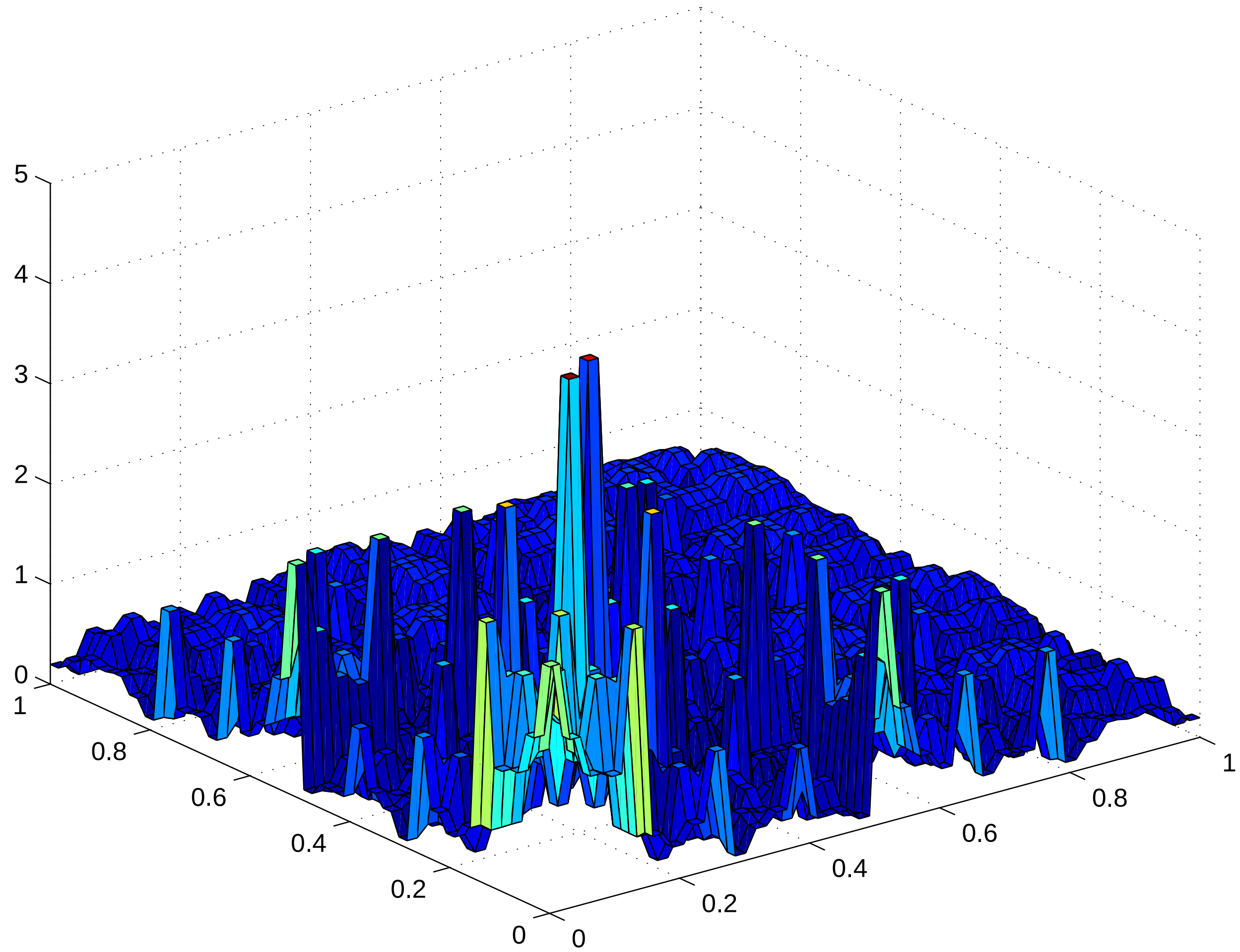
$|F_h(u) - \tilde{F}_h(\tilde{u}_h)|/|F_h(u)|(t) \quad t = 1.0$



$|F_h(u) - \tilde{F}_h(\tilde{u}_h)|/|F_h(u)|(t) \quad t = 1.5$



$|F_h(u) - \tilde{F}_h(\tilde{u}_h)|/|F_h(u)|(t) \quad t = 2.0$

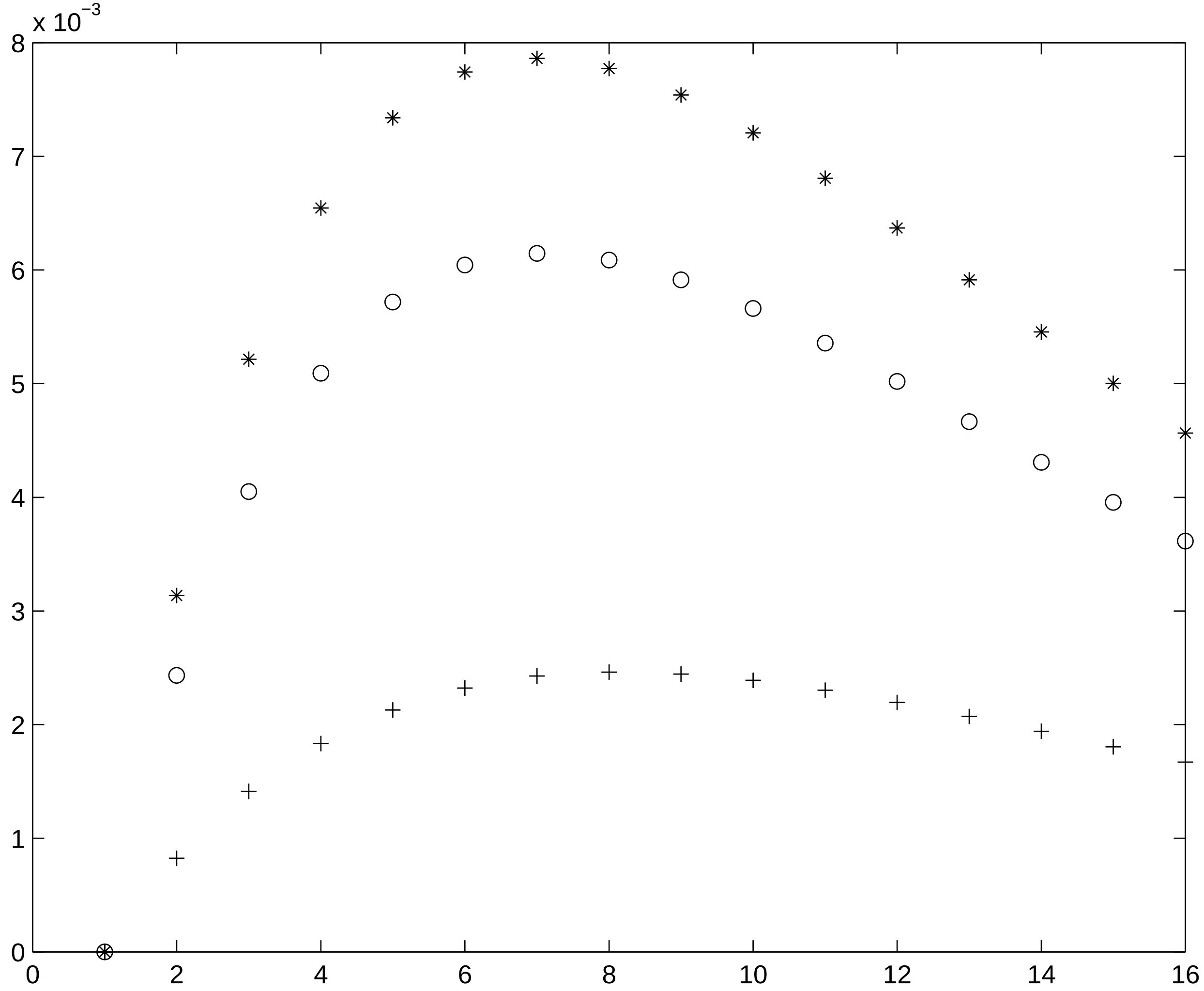


Error with and without subgrid model

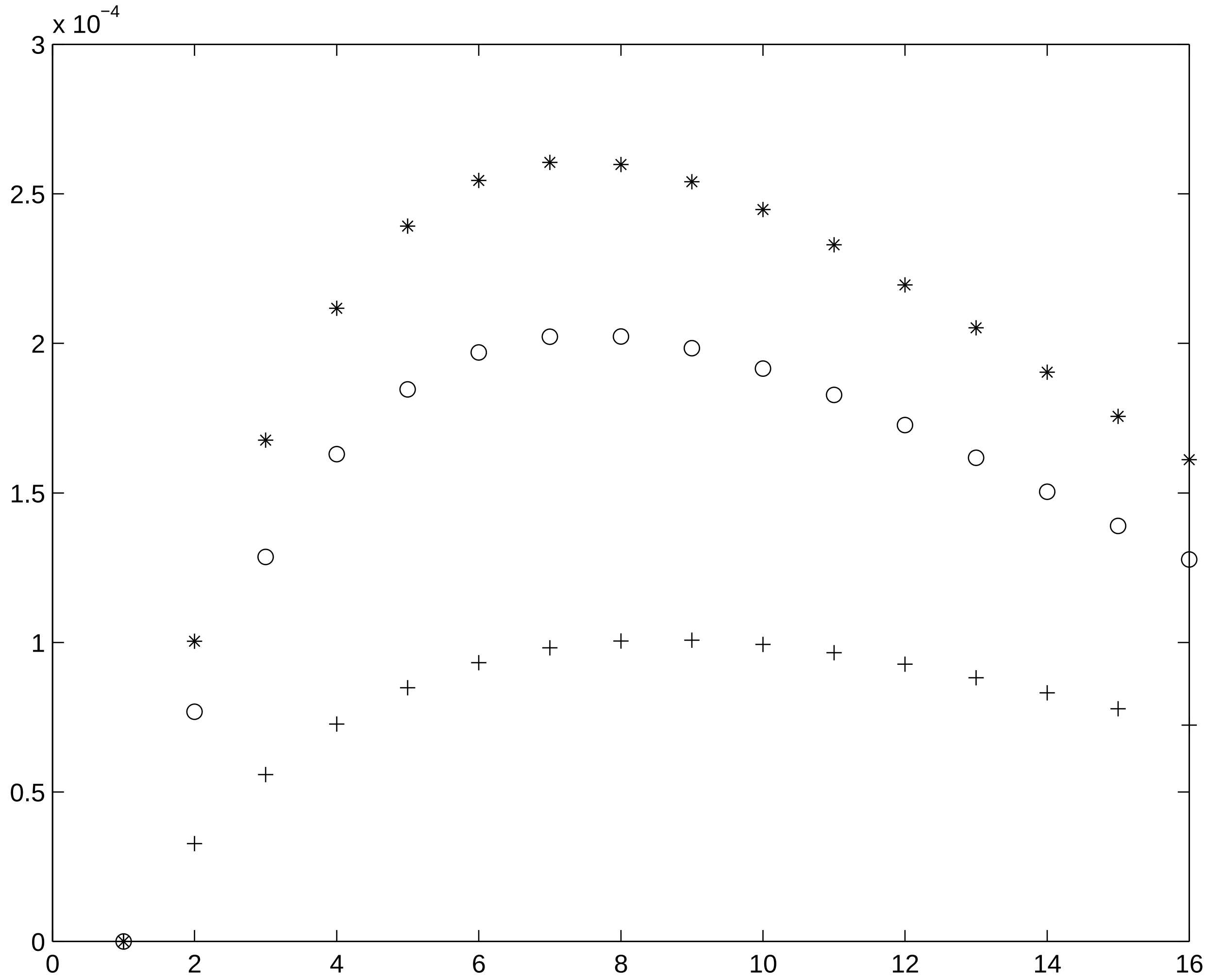
$$\|u-u_h\|_1/\|u-\tilde{u}_h\|_1 = 3.1$$
$$\|u-u_h\|_2/\|u-\tilde{u}_h\|_2 = 2.5$$
$$\|u-u_h\|_\infty/\|u-\tilde{u}_h\|_\infty = 1.6$$

$$\|u-u_{h/2}\|_1/\|u-\tilde{u}_h\|_1 = 2.5$$
$$\|u-u_{h/2}\|_2/\|u-\tilde{u}_h\|_2 = 2.0$$
$$\|u-u_{h/2}\|_\infty/\|u-\tilde{u}_h\|_\infty = 1.1$$

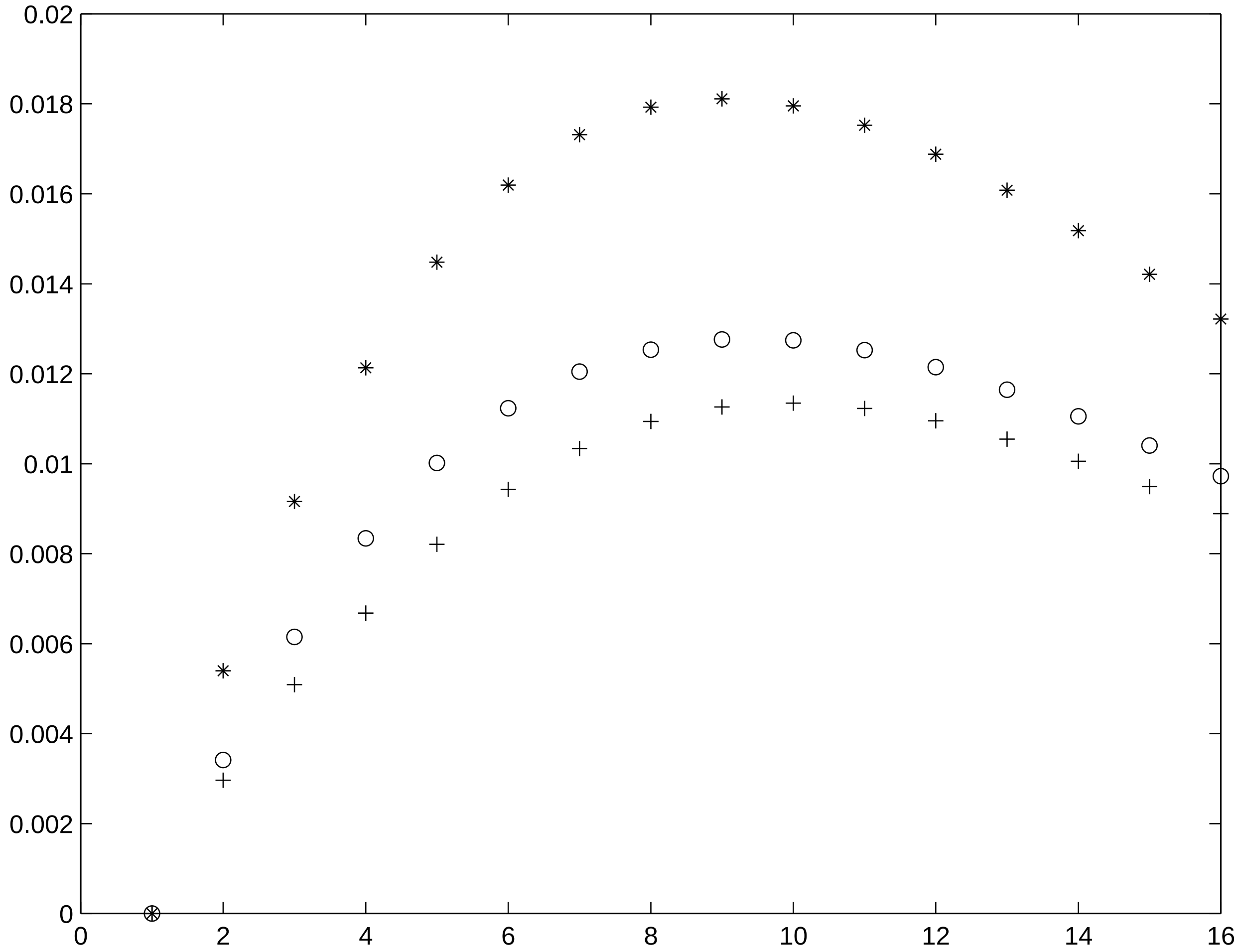
* - $\|[e_h]^h\|_1$, o - $\|[e_{h/2}]^h\|_1$, + - $\|[\tilde{e}_h]^h\|_1$



$*$ - $\|[e_h]^h\|_2$, o - $\|[e_{h/2}]^h\|_2$, + - $\|[\tilde{e}_h]^h\|_2$



* - $\|[e_h]^h\|_\infty$, o - $\|[e_{h/2}]^h\|_\infty$, + - $\|[\tilde{e}_h]^h\|_\infty$

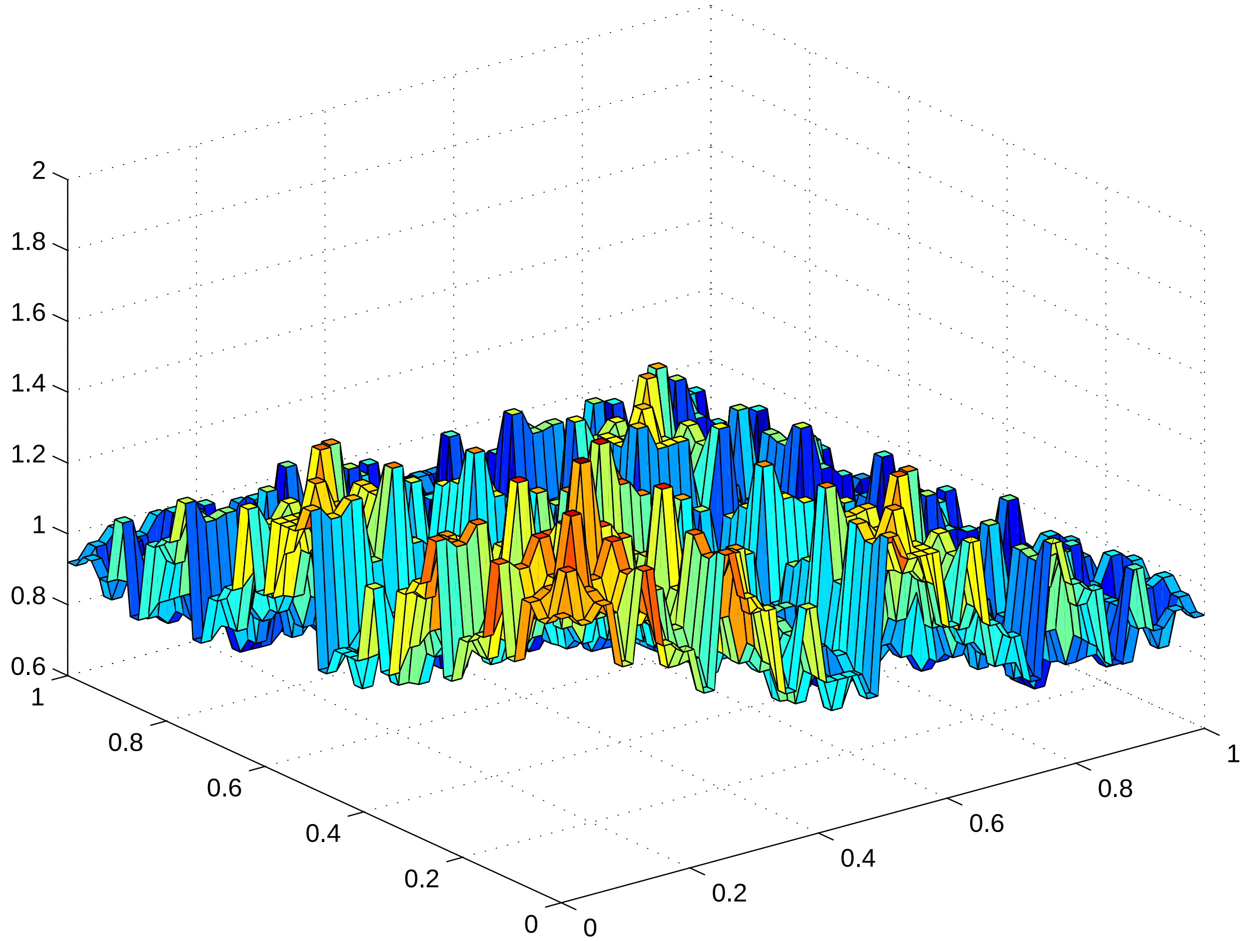


$f(u) = (u_1(1-u_2), u_2(u_1-1))$

$$
\begin{aligned}
\dot{u}_1 - \epsilon\Delta u_1 &= u_1(1-u_2),\\
\dot{u}_2 - \epsilon\Delta u_2 &= u_2(u_1-1),\\
u(x,0) &= (W_{2D},1)
\end{aligned}
$$

$$
\begin{aligned}
F_h(u)_1 &= -(u_1u_2)^h + u_1^h u_2^h,\\
F_h(u)_2 &= (u_2u_1)^h - u_2^h u_1^h
\end{aligned}
$$

$u_1(t) \quad t = 0.0$



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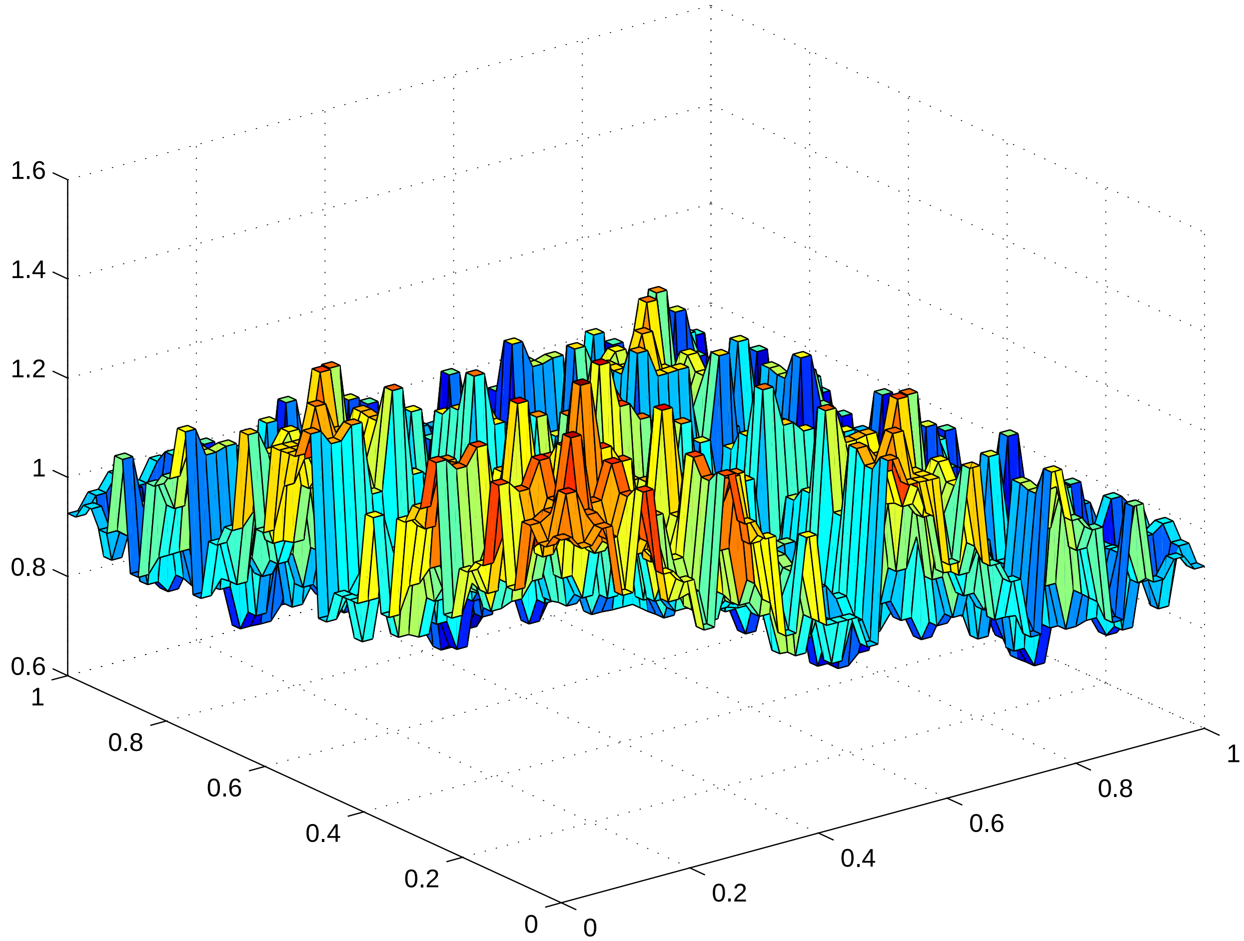
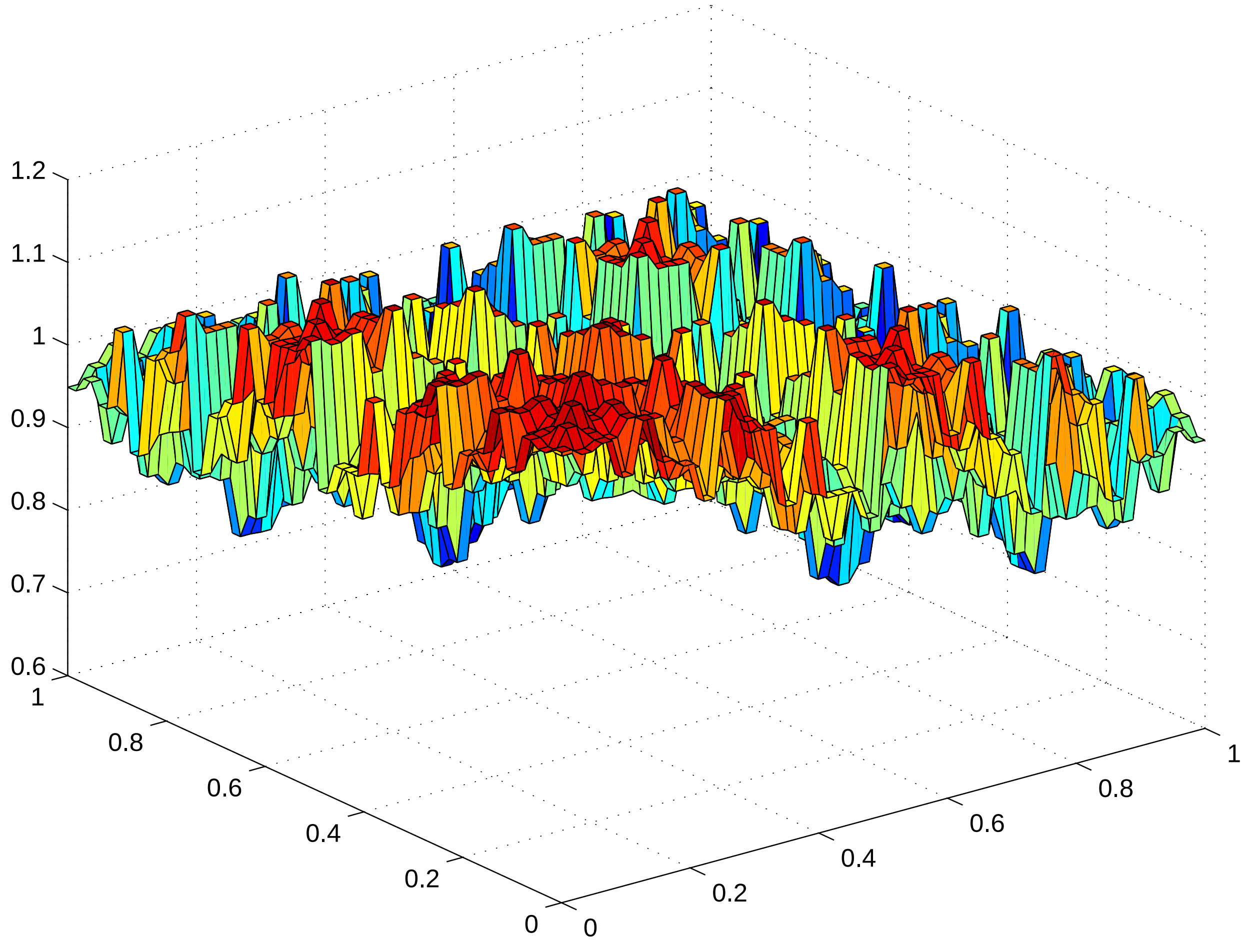
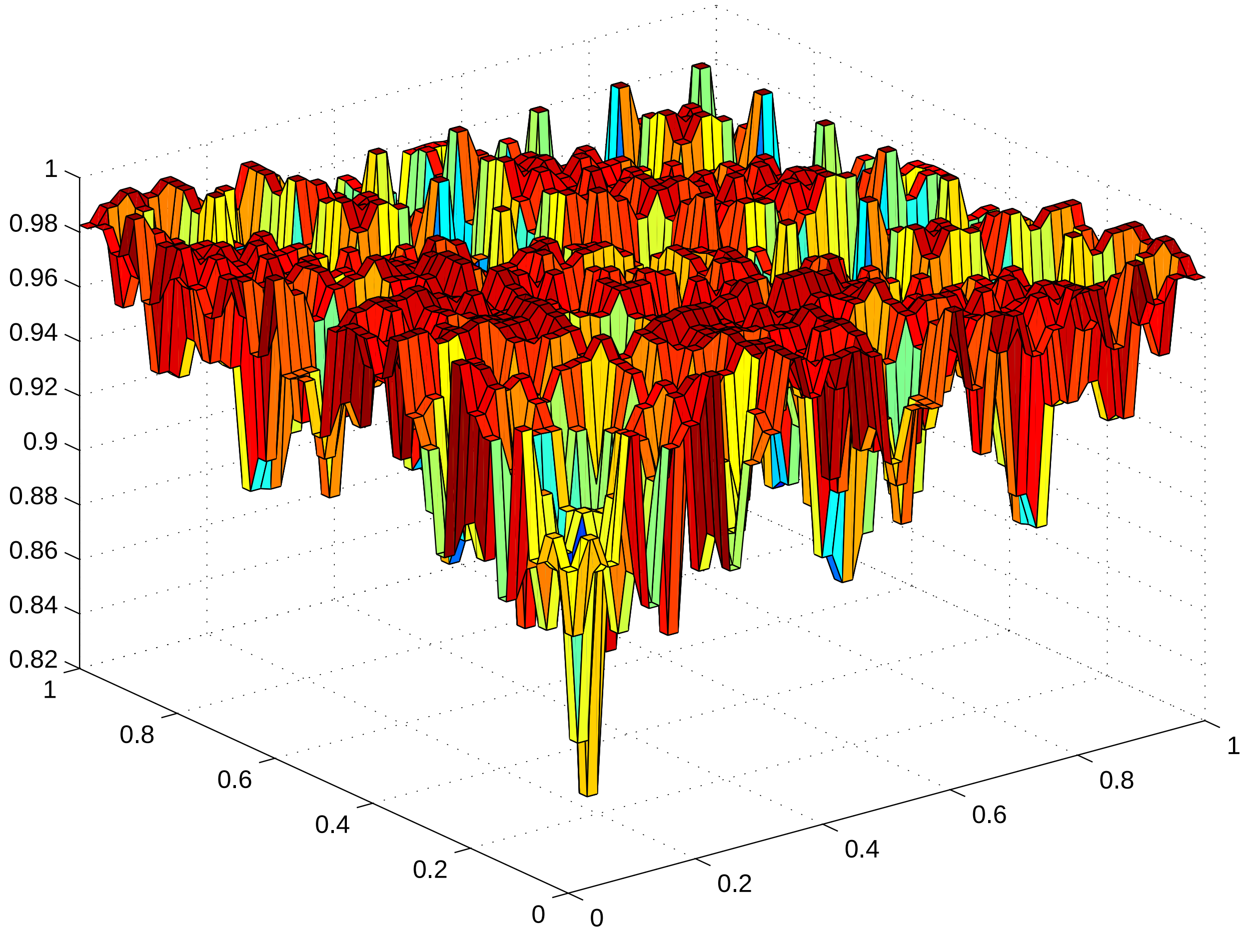


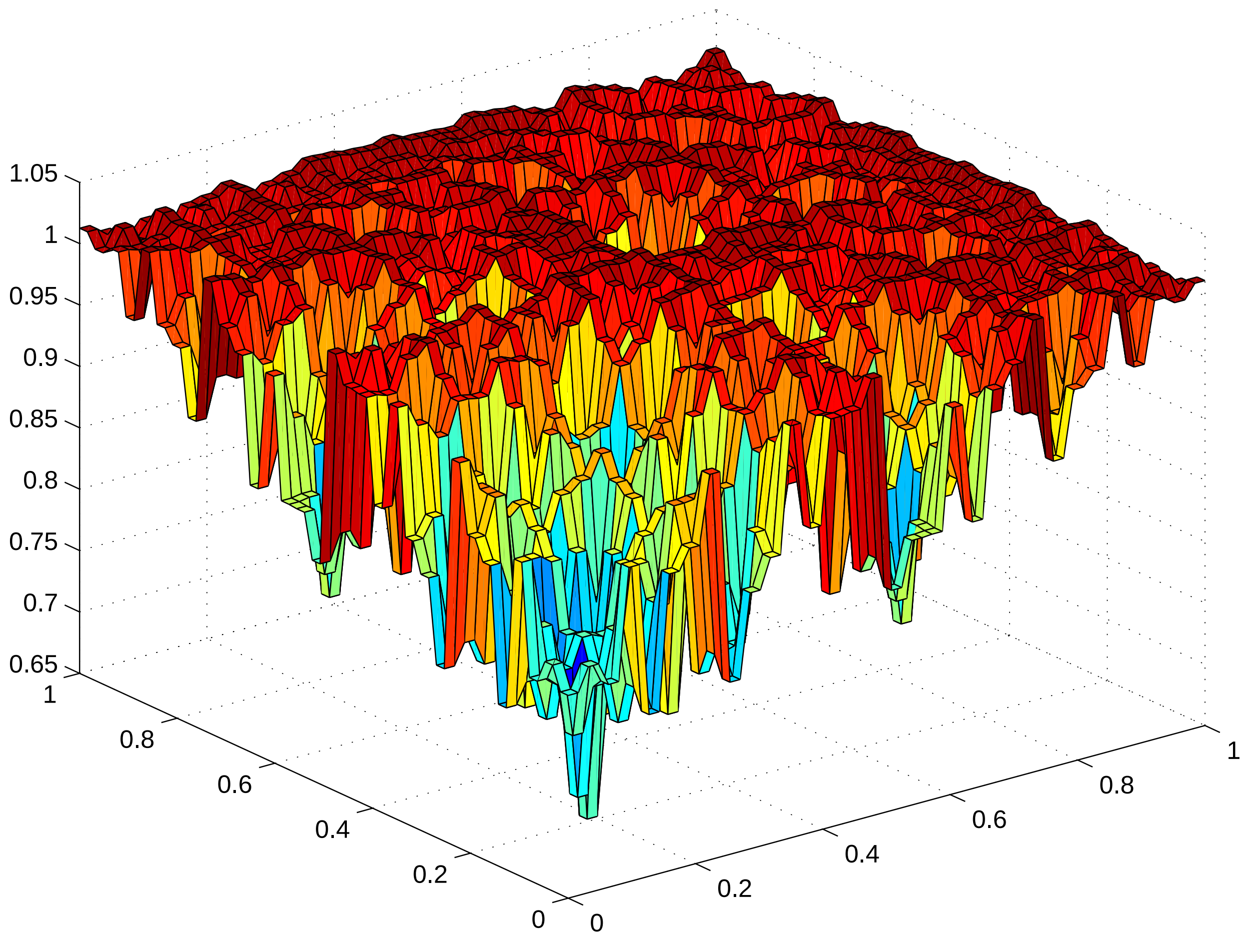
$u_1(t) \quad t = 1.0$



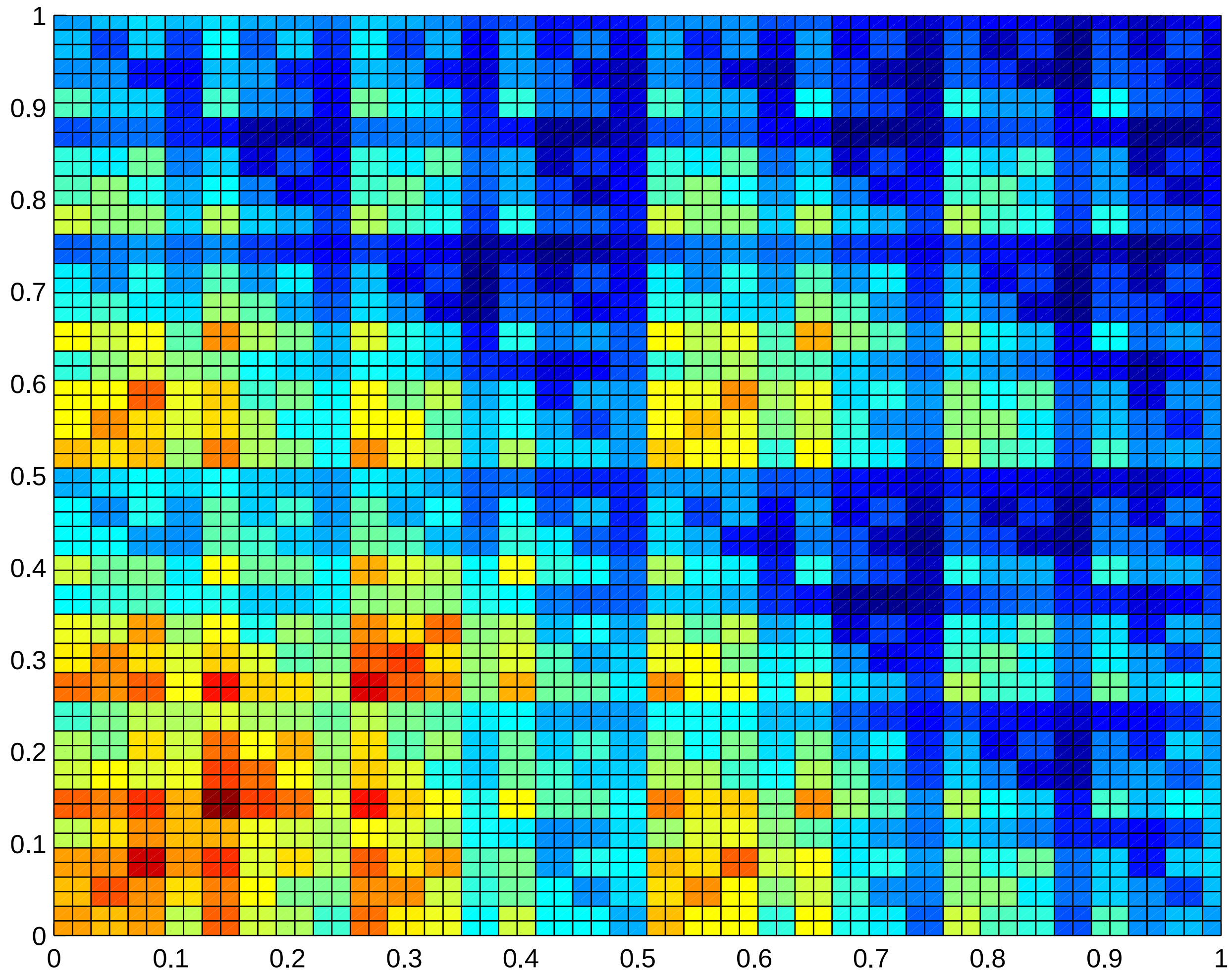
$u_1(t) \quad t = 1.5$



$u_1(t) \quad t = 2.0$



$u_1(t) \quad t = 0.0$



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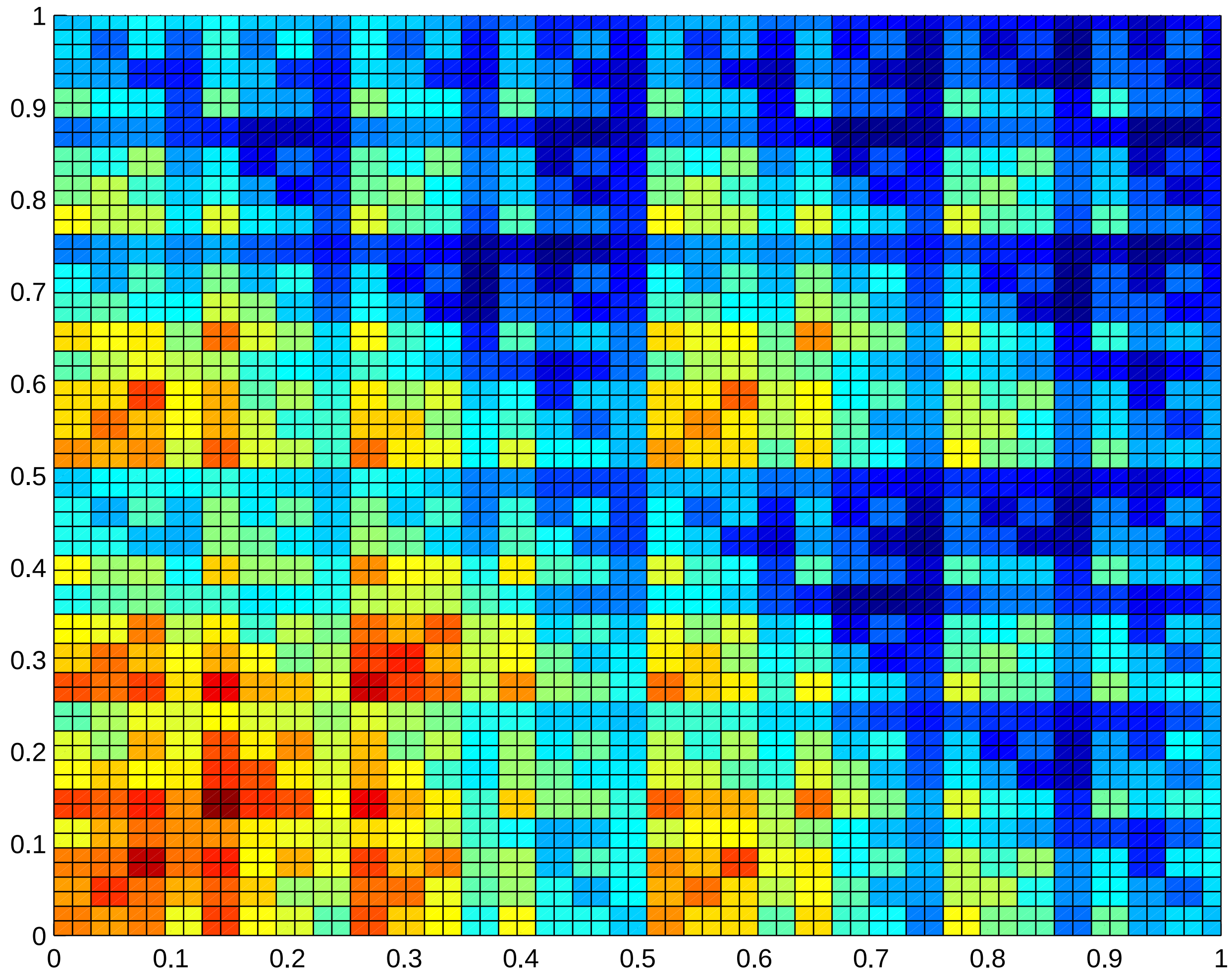
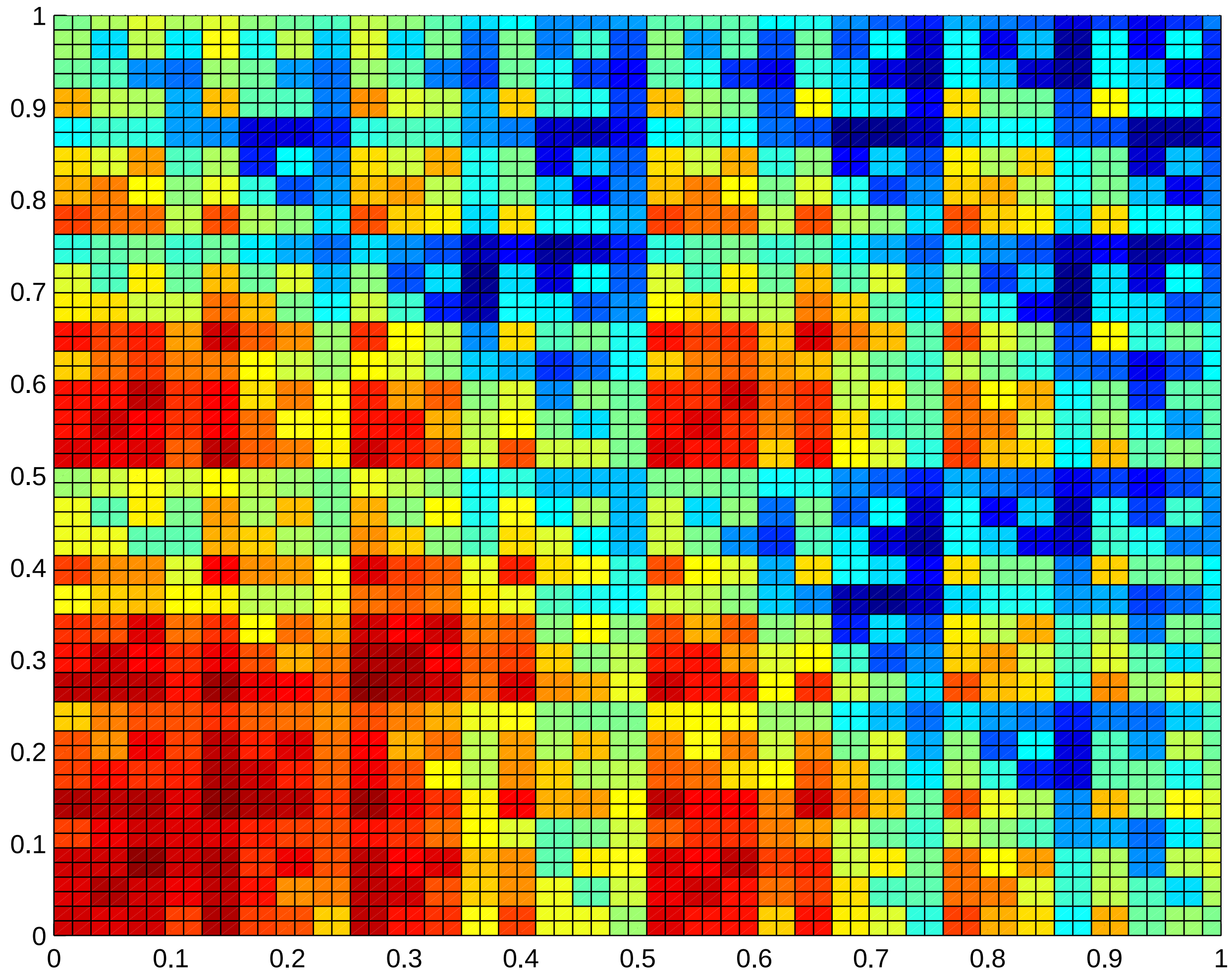
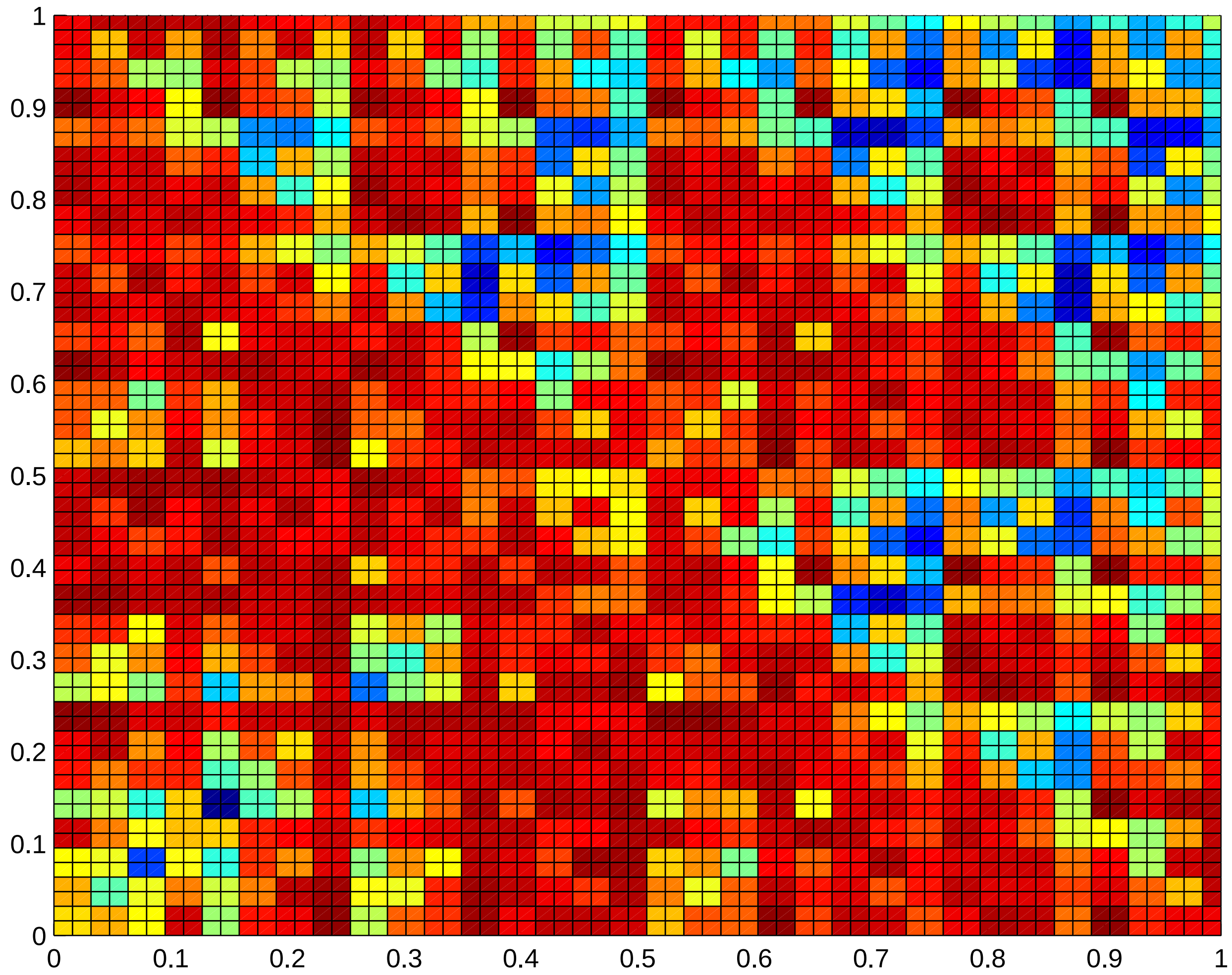


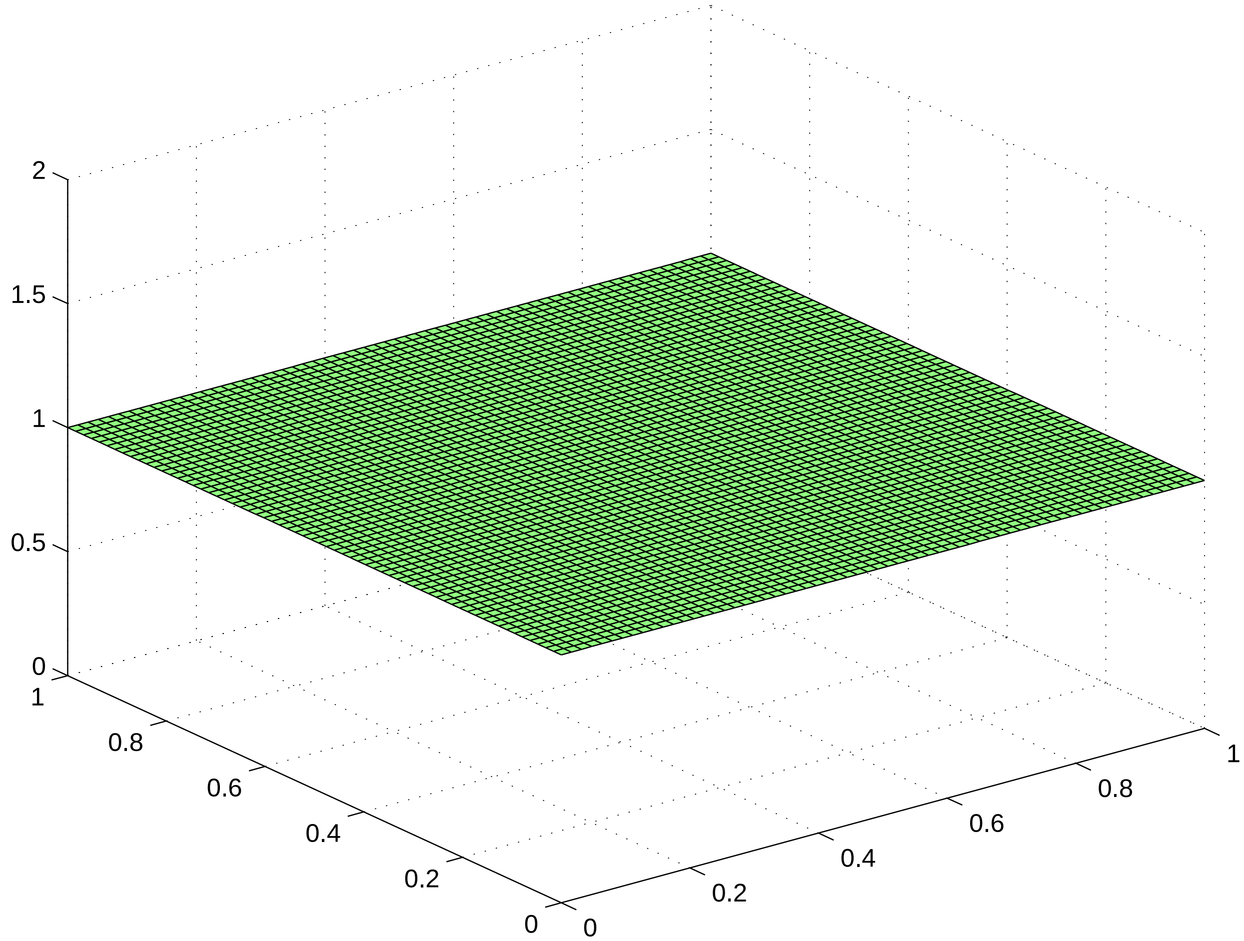
$u_1(t) \quad t = 1.0$



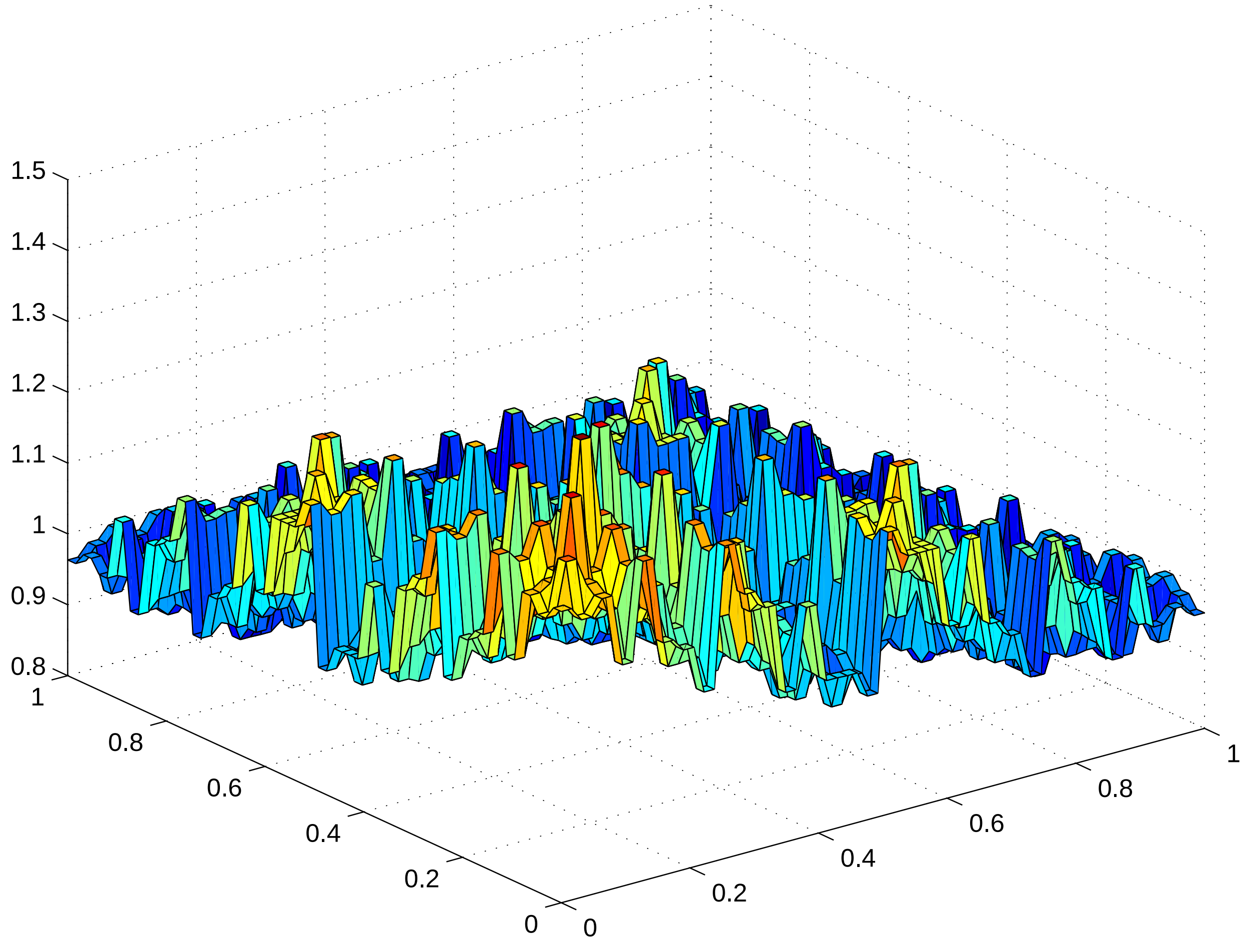
$u_1(t) \quad t = 1.5$



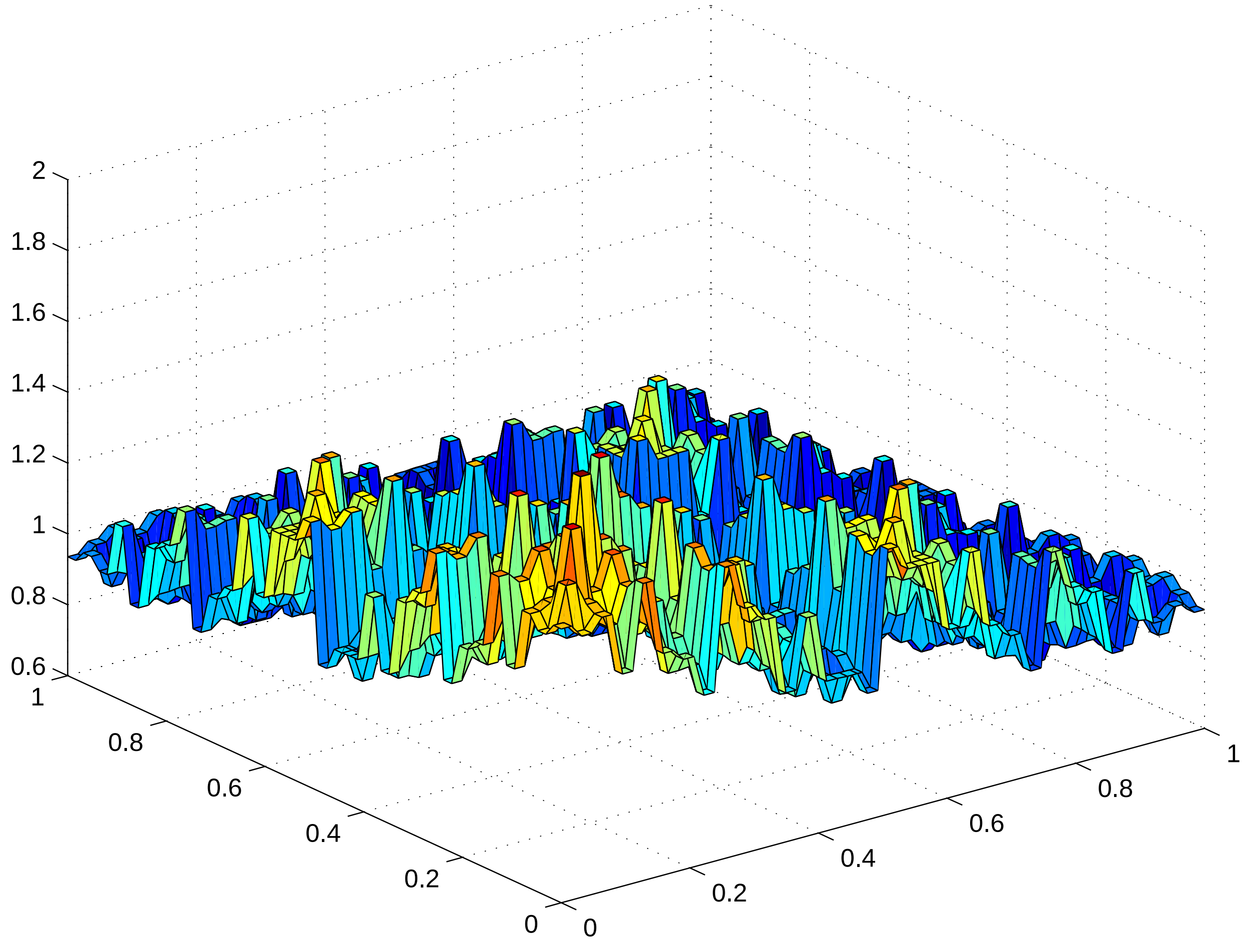
$u_2(t) \quad t = 0.0$



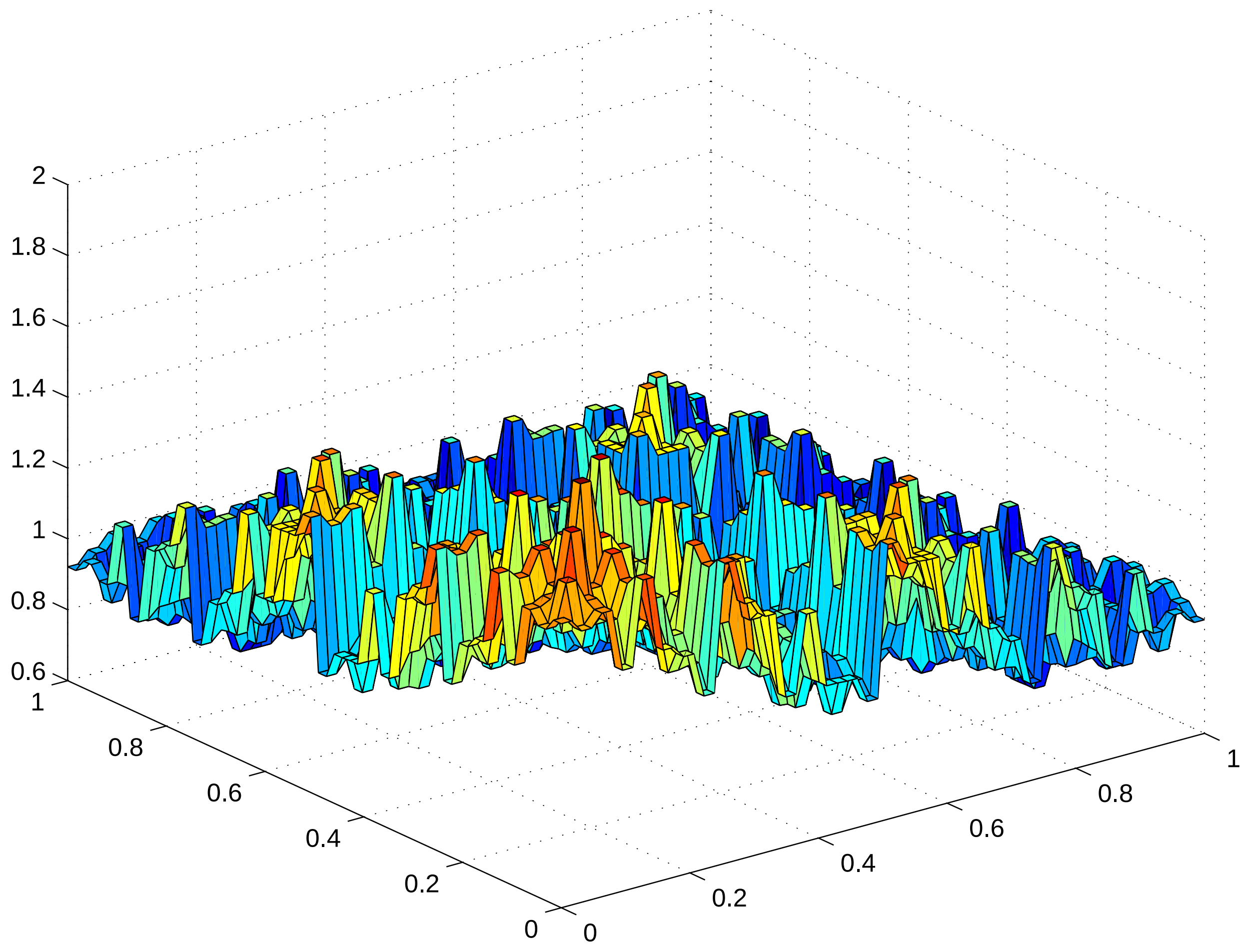
$u_2(t) \quad t = 0.5$



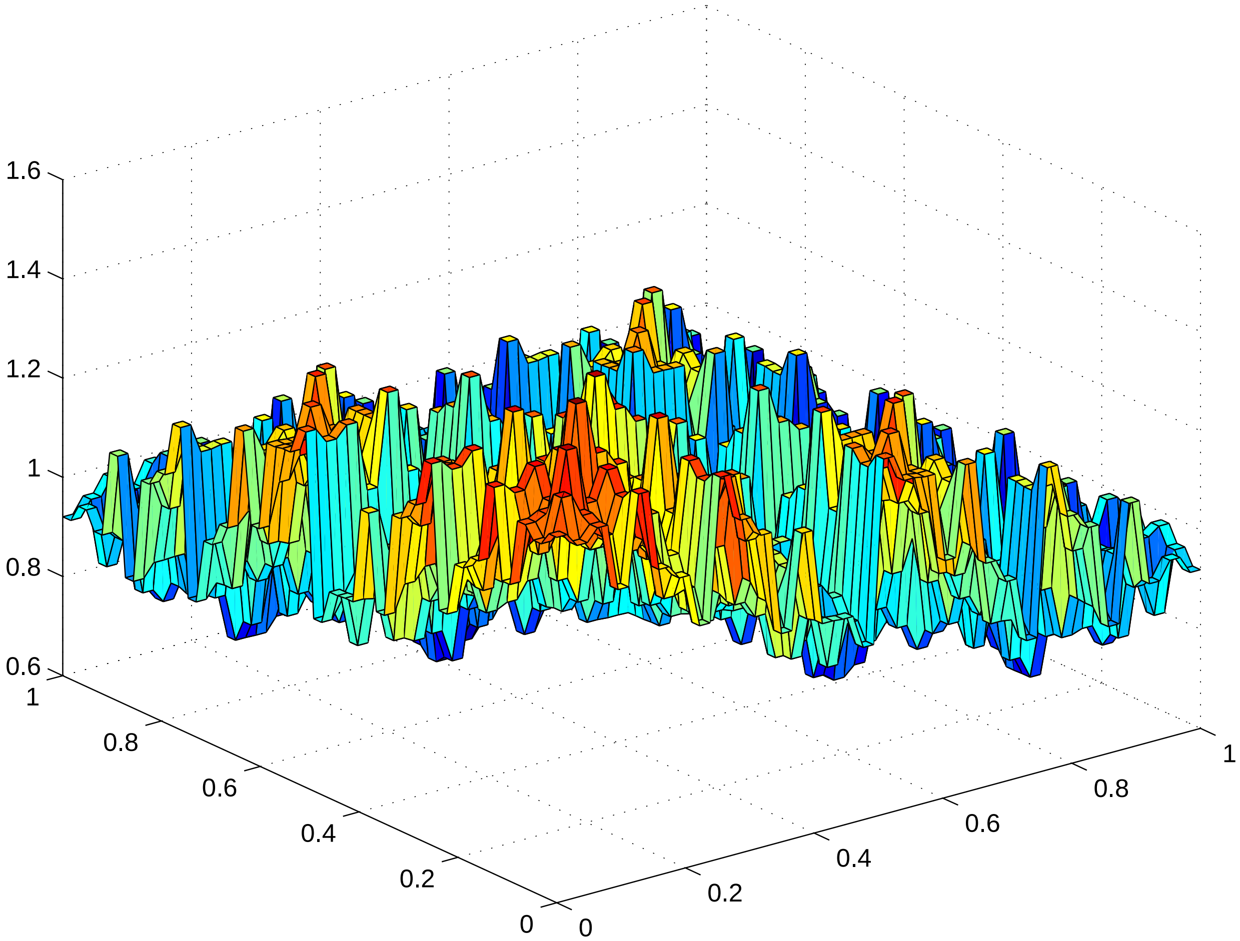
$u_2(t) \quad t = 1.0$



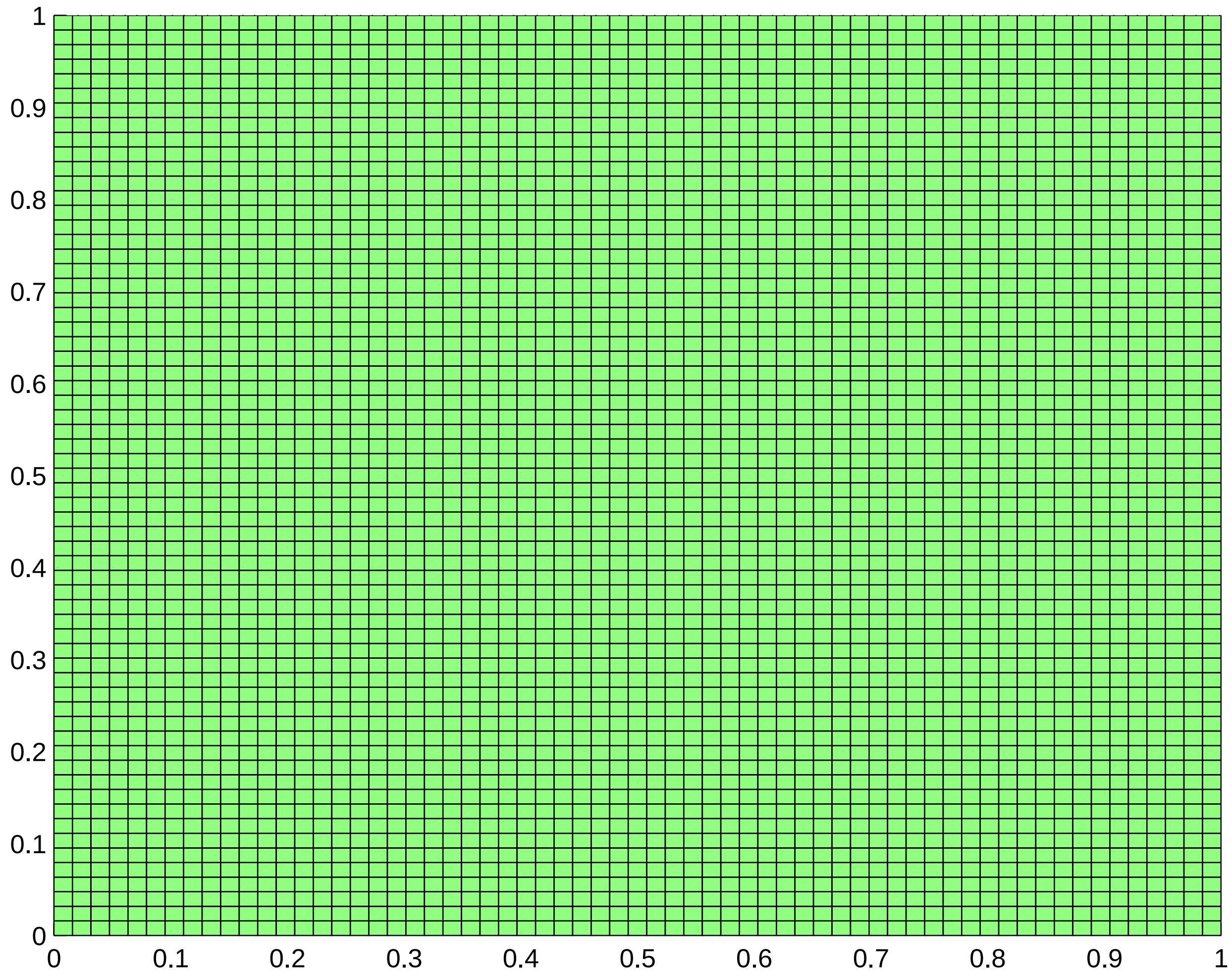
$u_2(t) \quad t = 1.5$



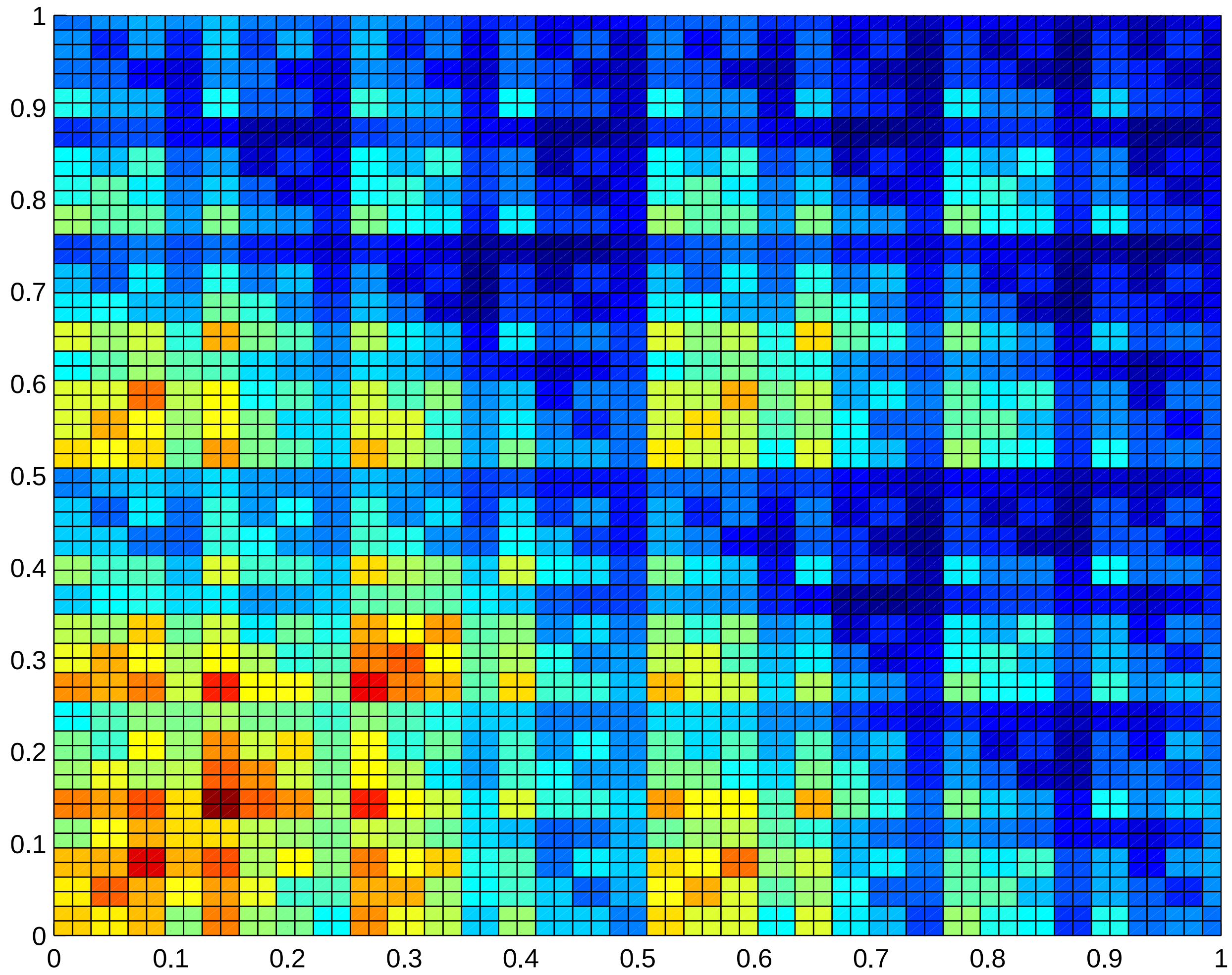
$u_2(t) \quad t = 2.0$



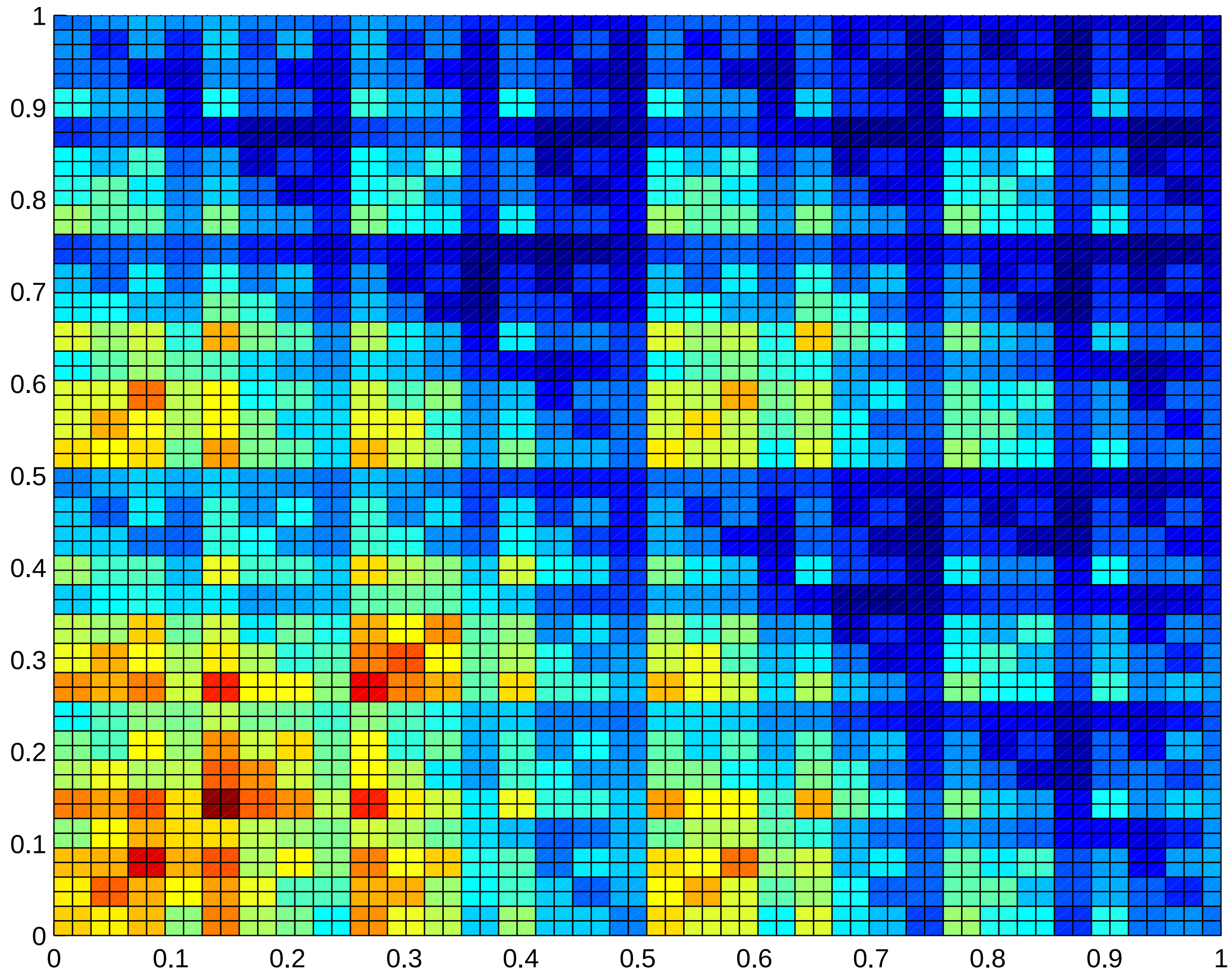
$u_2(t) \quad t = 0.0$



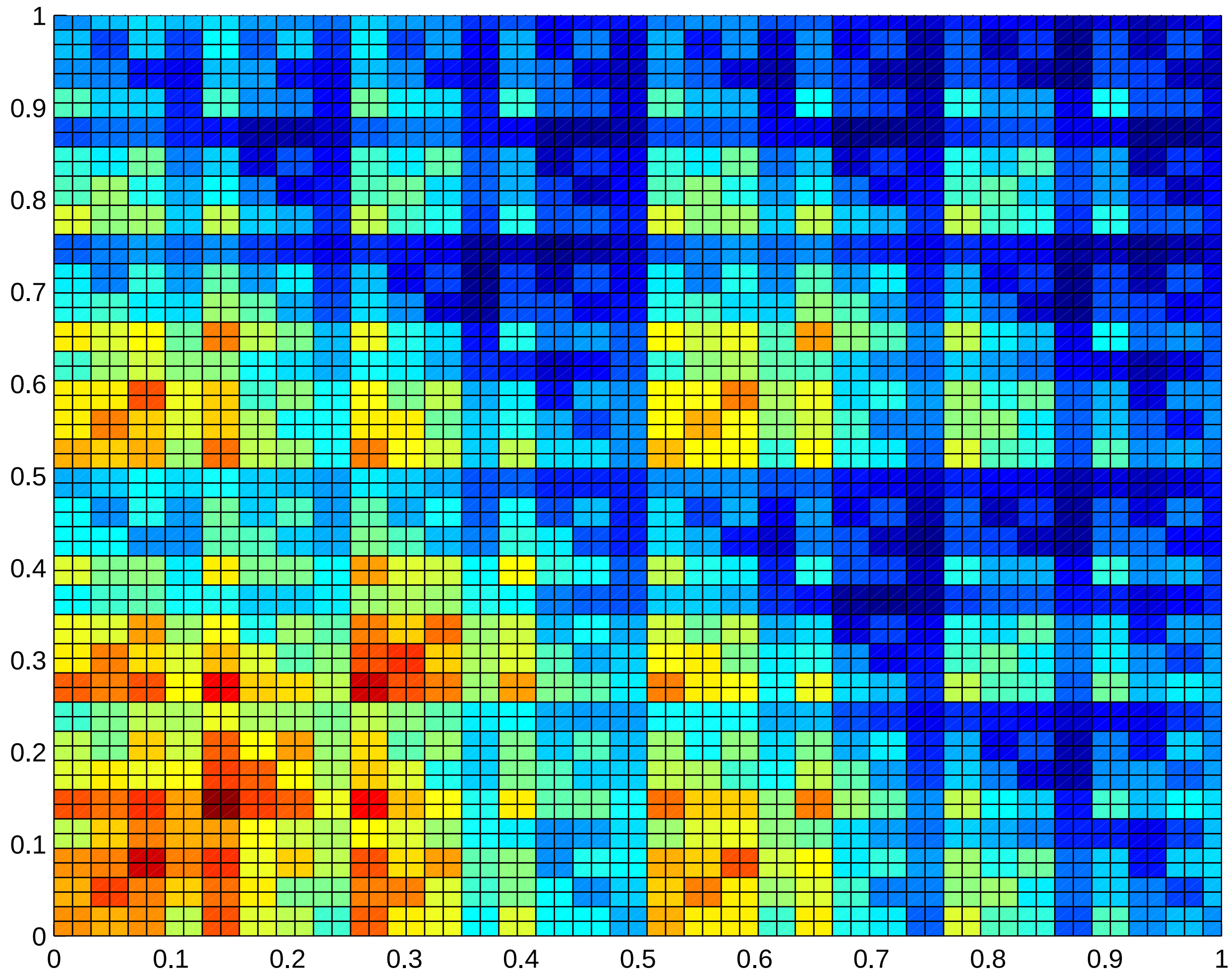
$u_2(t) \quad t = 0.5$



$u_2(t) \quad t = 1.0$



$u_2(t) \quad t = 1.5$

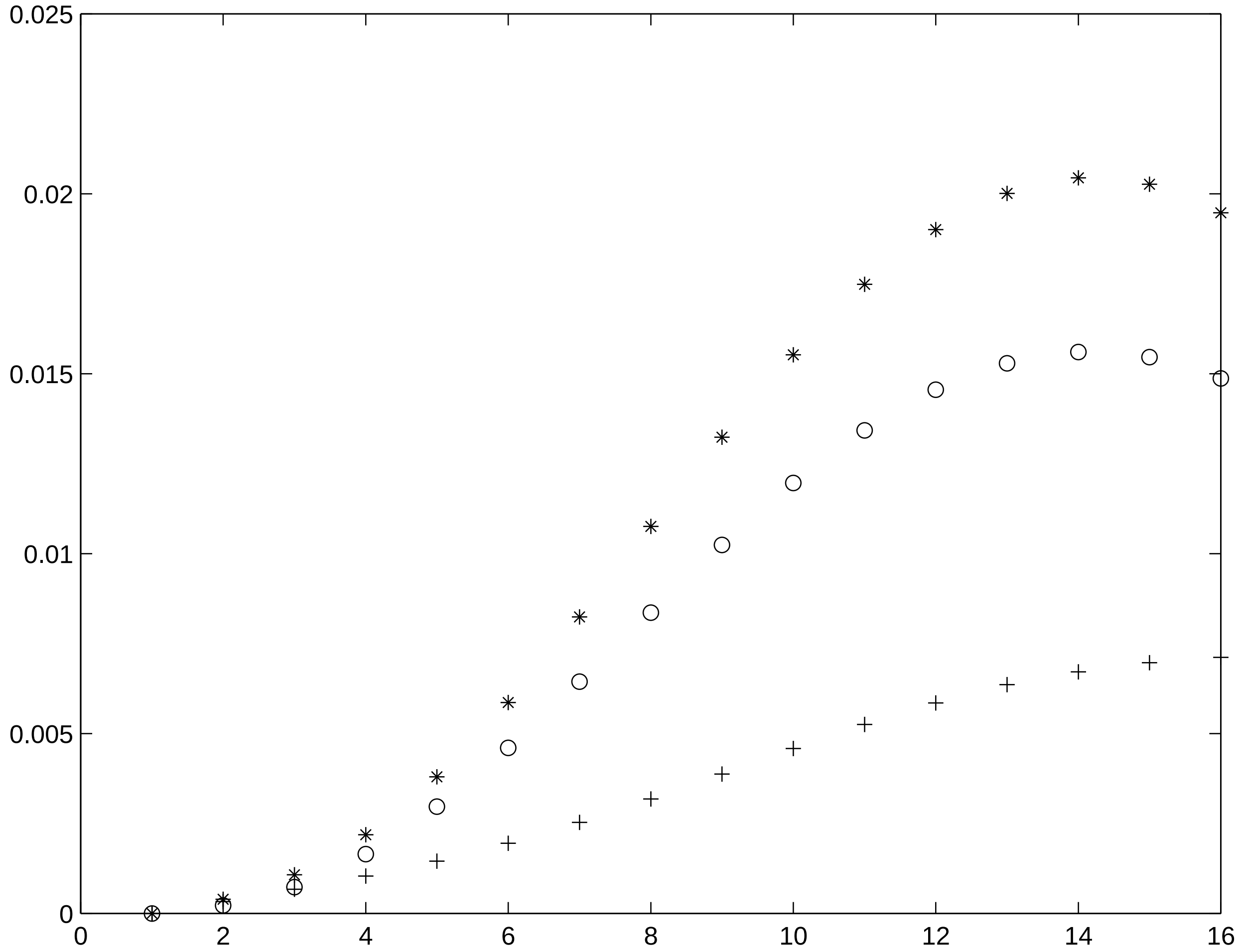


Error with and without subgrid model

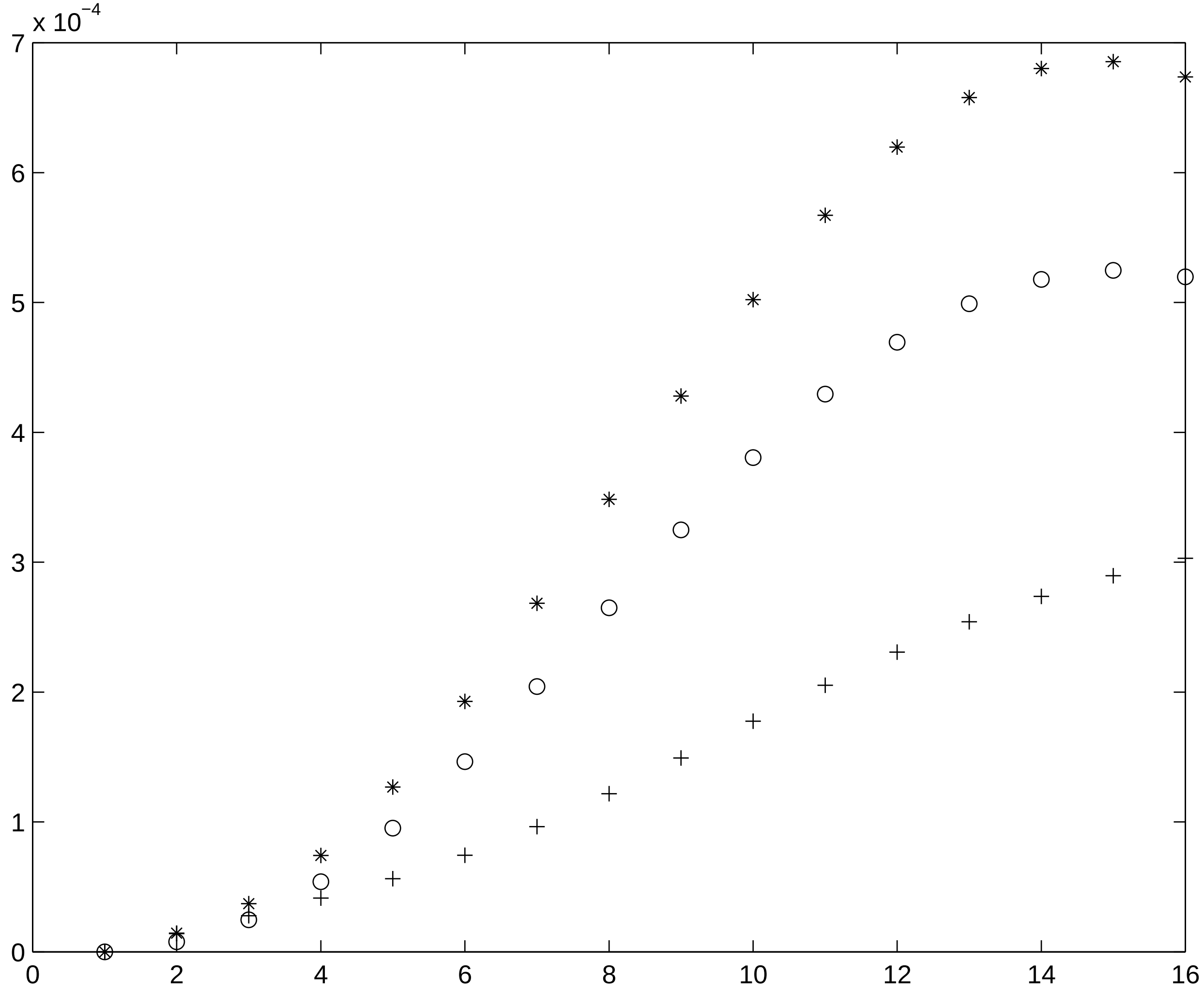
$$\begin{aligned}
\|u-u_h\|_1/\|u-\tilde{u}_h\|_1 &= (3.0, 2.1)\\
\|u-u_h\|_2/\|u-\tilde{u}_h\|_2 &= (2.7, 1.8)\\
\|u-u_h\|_\infty/\|u-\tilde{u}_h\|_\infty &= (2.4, 1.2)
\end{aligned}$$

$$\begin{aligned}
\|u-u_{h/2}\|_1/\|u-\tilde{u}_h\|_1 &= (1.9, 1.4)\\
\|u-u_{h/2}\|_2/\|u-\tilde{u}_h\|_2 &= (1.7, 1.2)\\
\|u-u_{h/2}\|_\infty/\|u-\tilde{u}_h\|_\infty &= (1.1, 1.0)
\end{aligned}$$

* - $\|[e_h^1]^h\|_1$, o - $\|[e_{h/2}^1]^h\|_1$, + - $\|[\tilde{e}_h^1]^h\|_1$



$*$ - $\|[e_h^1]^h\|_2$, o - $\|[e_{h/2}^1]^h\|_2$, + - $\|[\tilde{e}_h^1]^h\|_2$



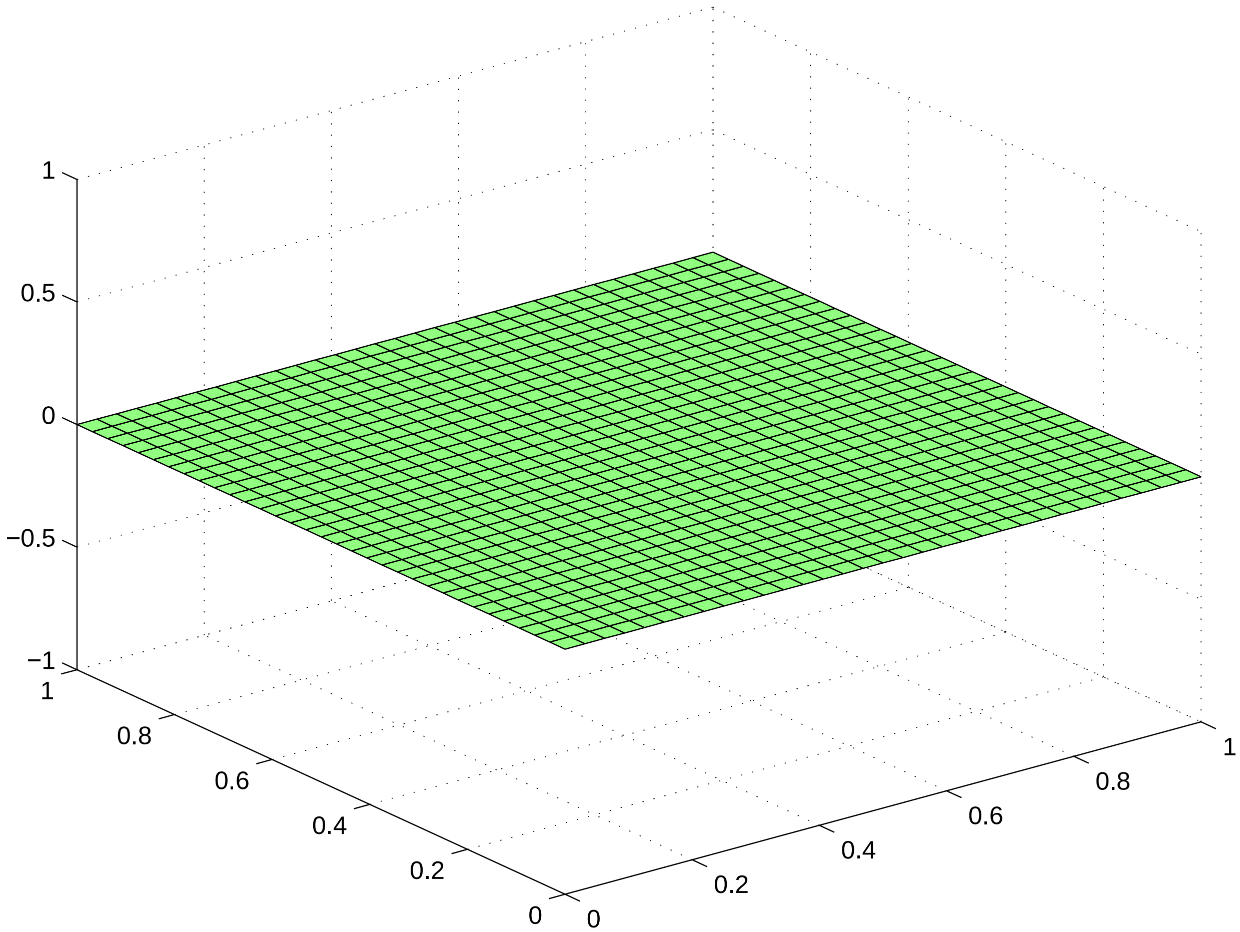
Convection Dominated Problems

$$\dot{u} - \epsilon \Delta u + \beta \cdot \nabla u = 1, \quad \Omega \times T = [0,1]^2 \times (0,2),$$
$$\frac{\partial u}{\partial n}|_{x_1=1, x_2=1} = 0, \quad u|_{x_1=0, x_2=0}, \quad u(x,0) = 0,$$

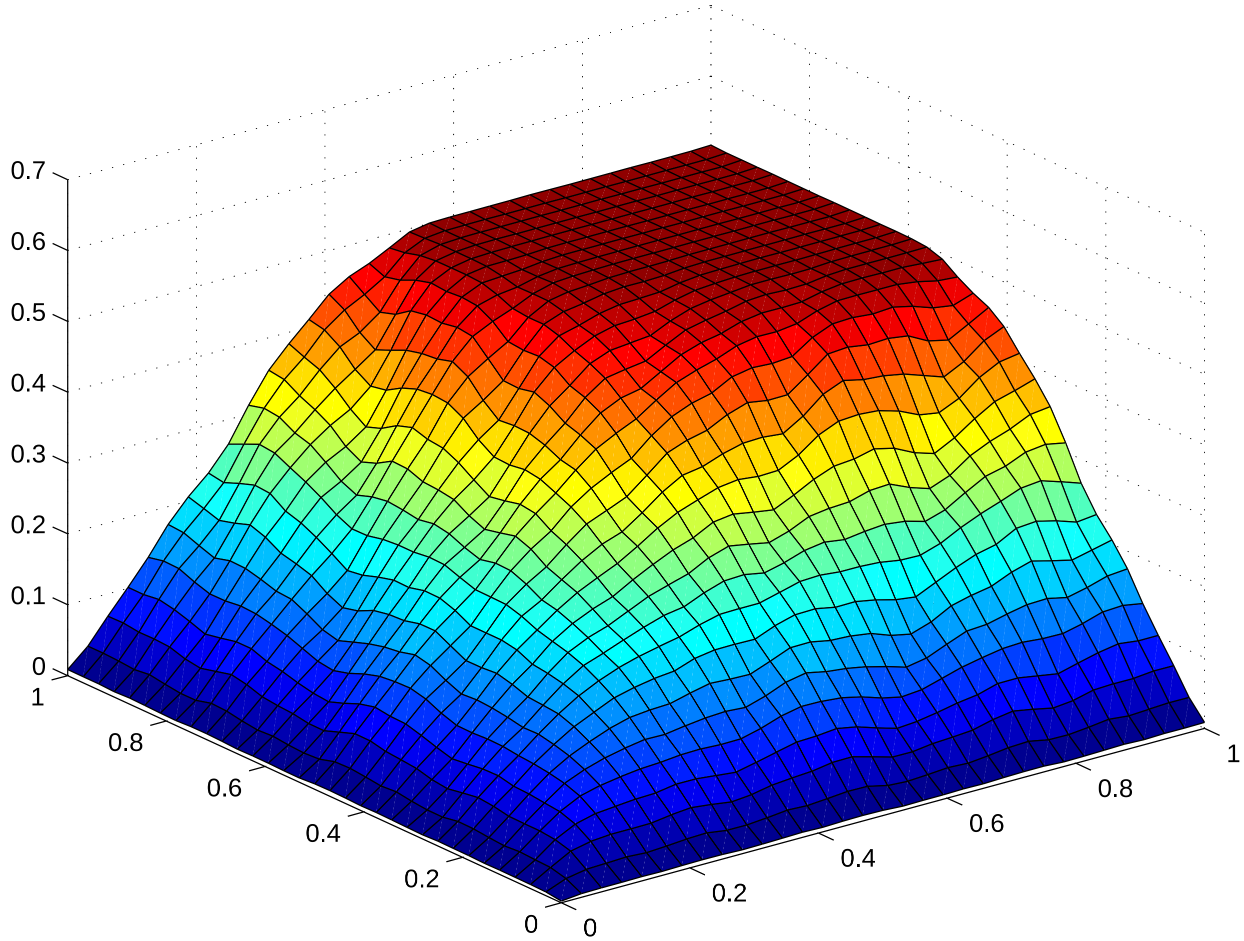
$\epsilon = 10^{-6}$, $\beta = W_{2D}$, $h = 2^{-5}$, ref. scale is 2^{-7},

$$\dot{u}^h - \epsilon \Delta u^h + \beta^h \cdot \nabla u^h = 1 + F_h(u^h), \quad u^h(x,0) = 0,$$
$$\dot{u}_h - \epsilon \Delta u_h + \beta^h \cdot \nabla u_h = 1, \quad u_h(x,0) = 0,$$
$$\dot{\tilde{u}}_h - \epsilon \Delta \tilde{u}_h + \beta^h \cdot \nabla \tilde{u}^h = 1 + \tilde{F}_h(\tilde{u}_h), \quad \tilde{u}_h(x,0) = 0.$$

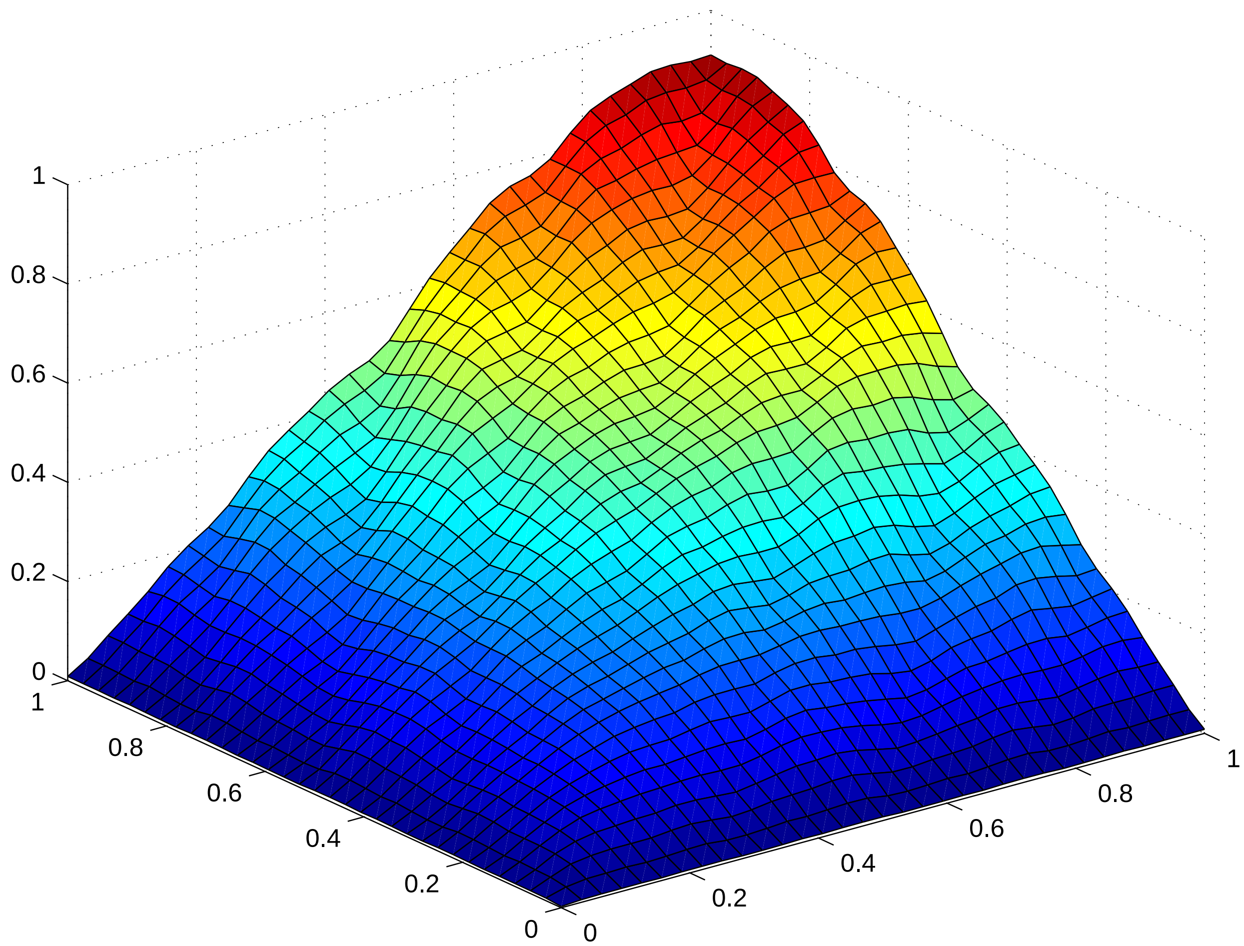
$u(t) \quad t = 0.0$



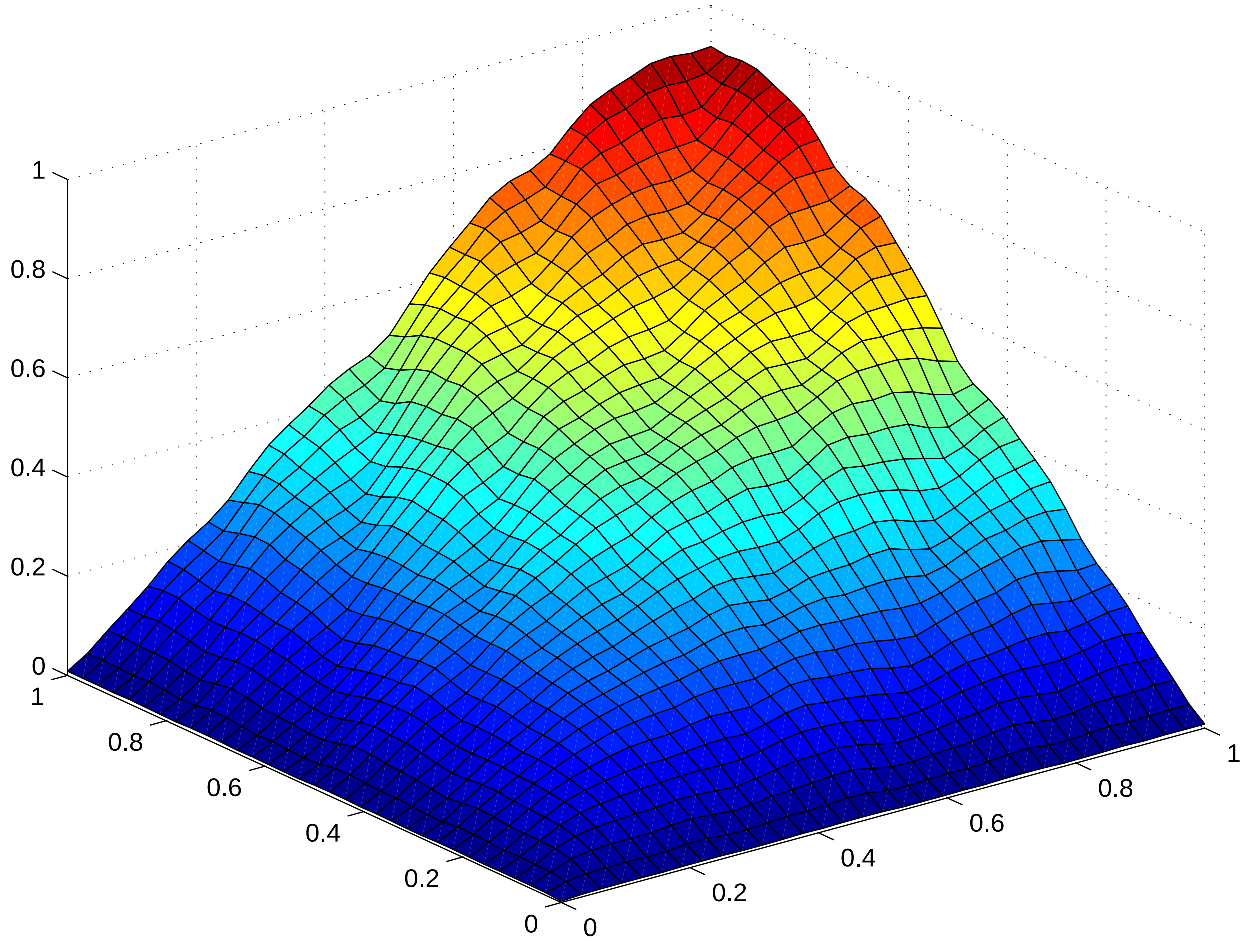
$u(t) \quad t = 0.5$



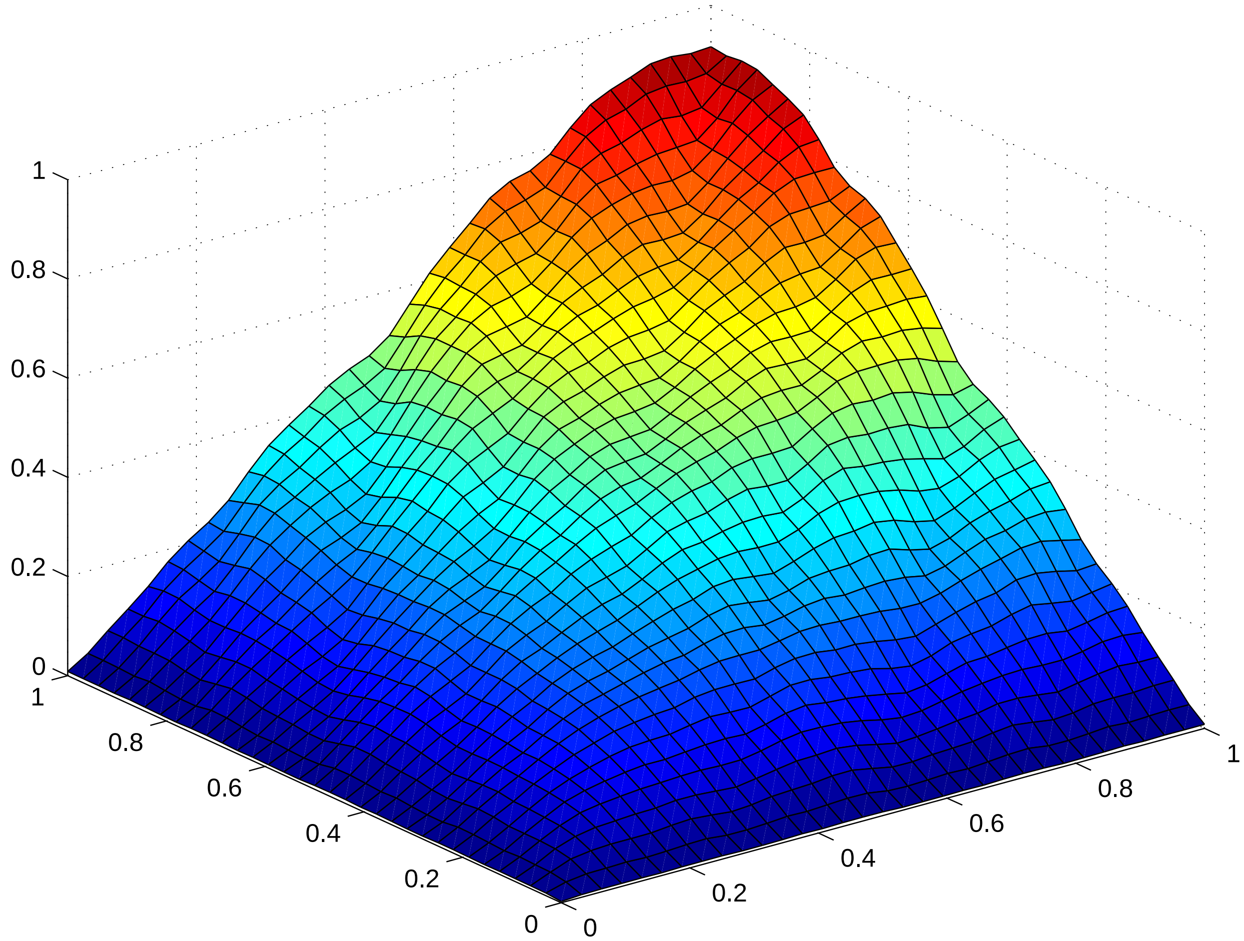
$u(t) \quad t = 1.0$



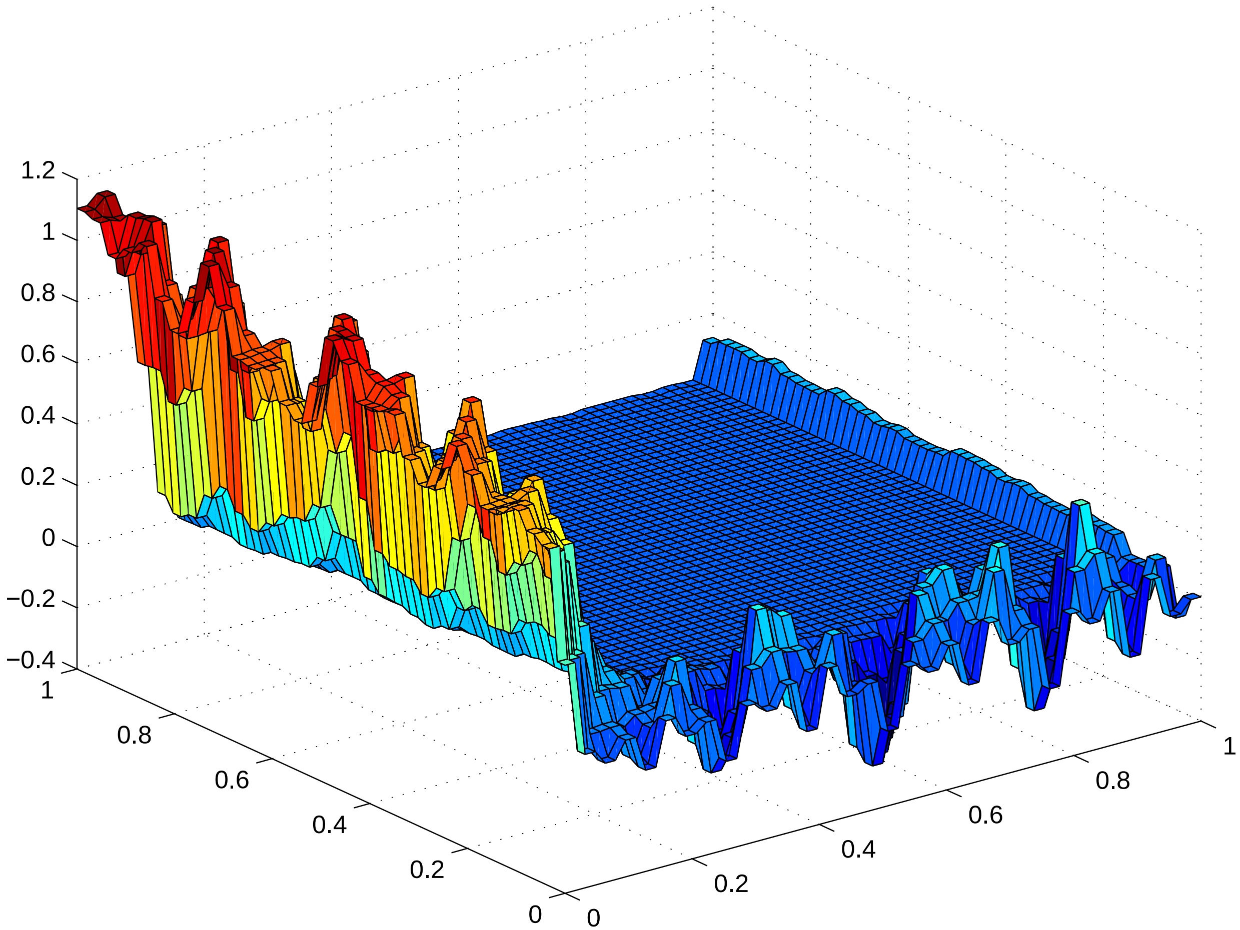
$u(t) \quad t = 1.5$



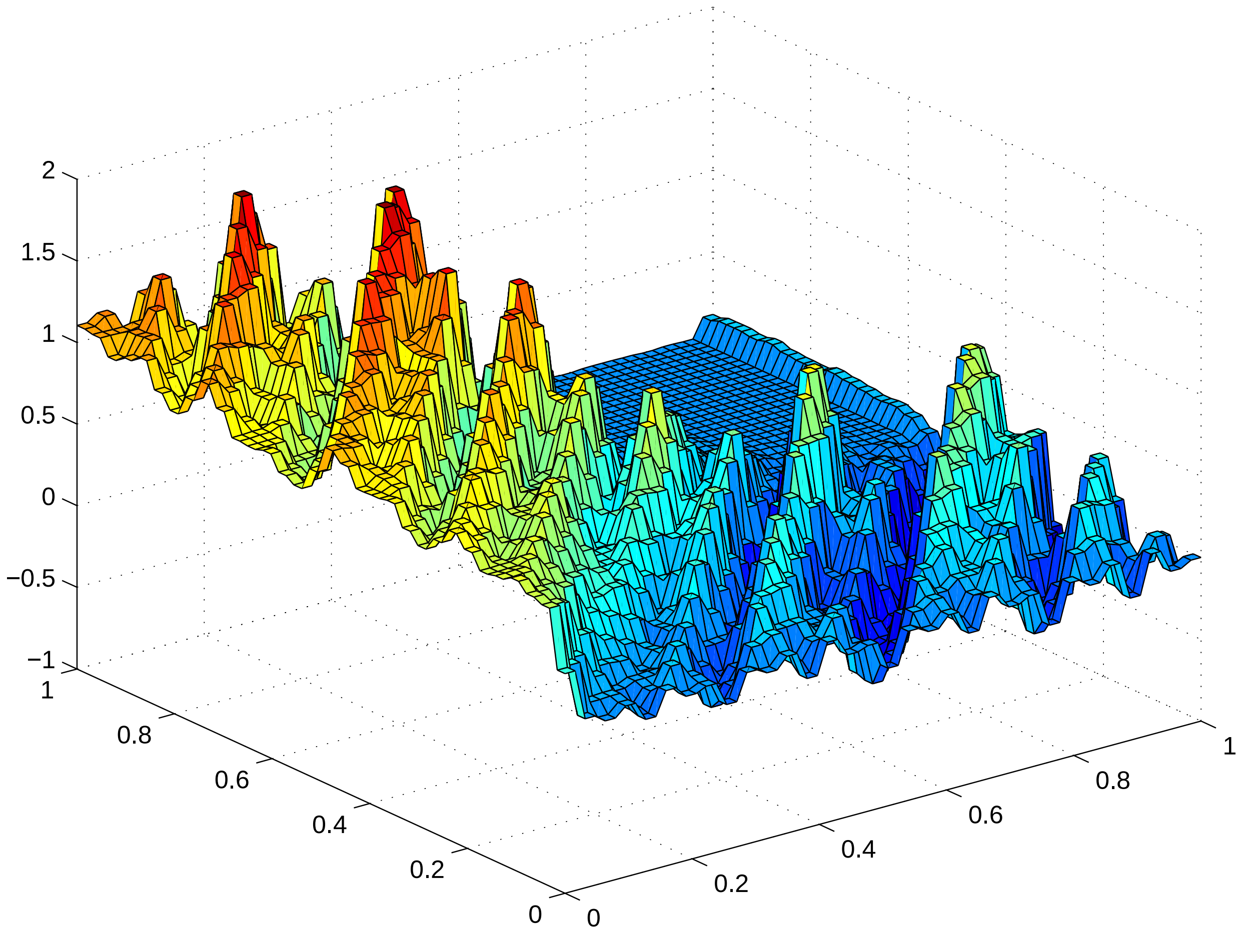
$u(t) \quad t = 2.0$



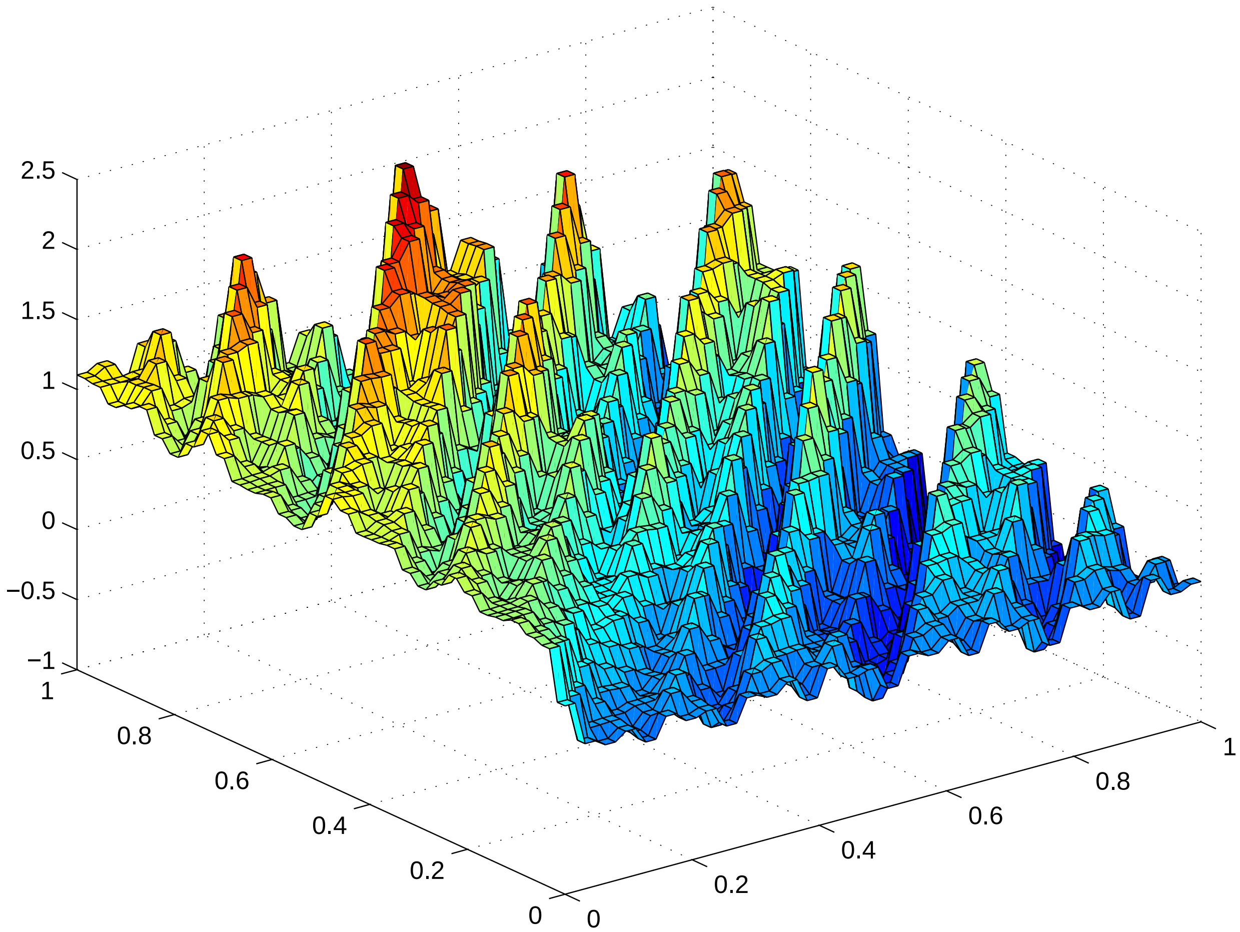
$\partial u/\partial x_1 \quad t = 0.0$



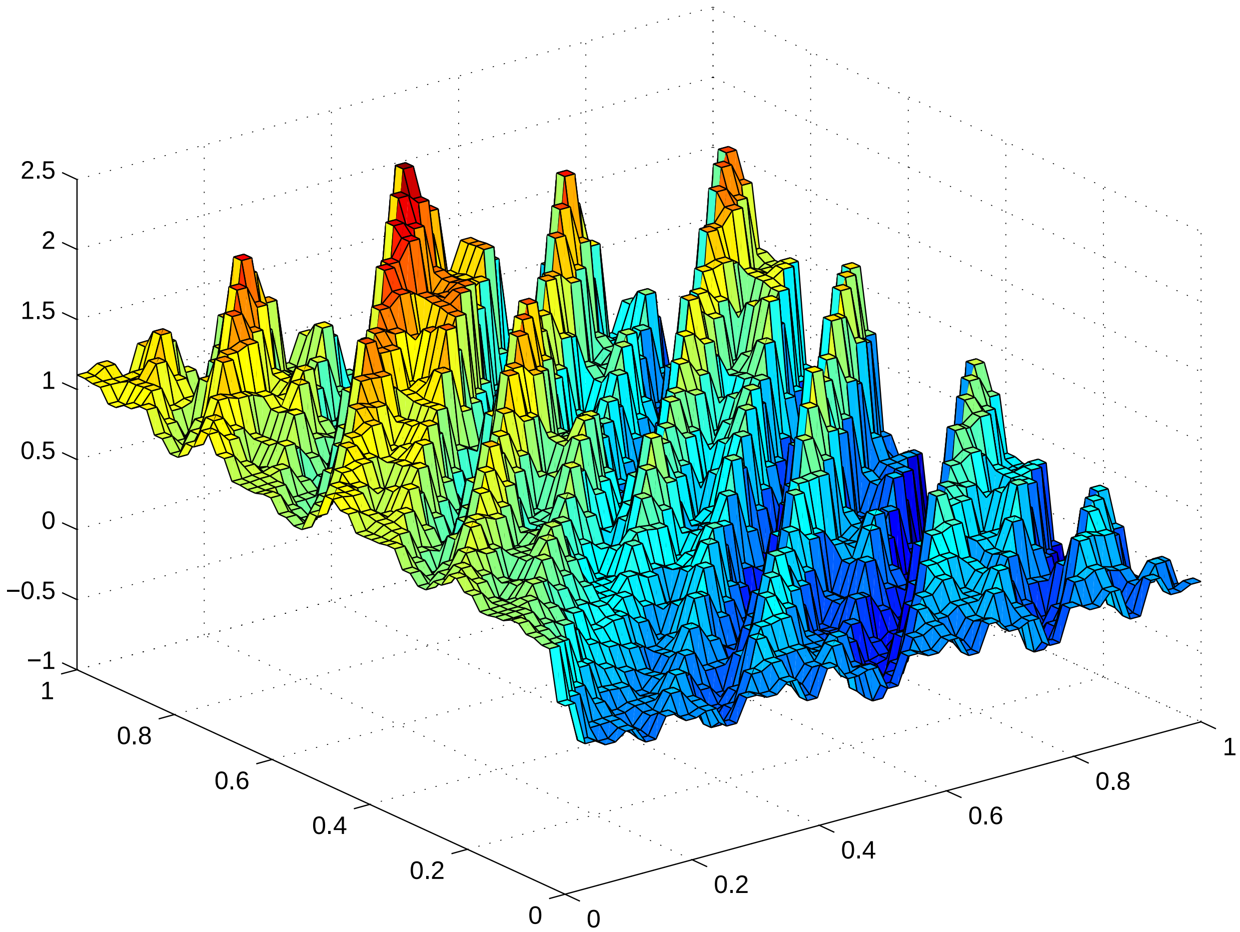
$\partial u/\partial x_1 \quad t = 0.5$



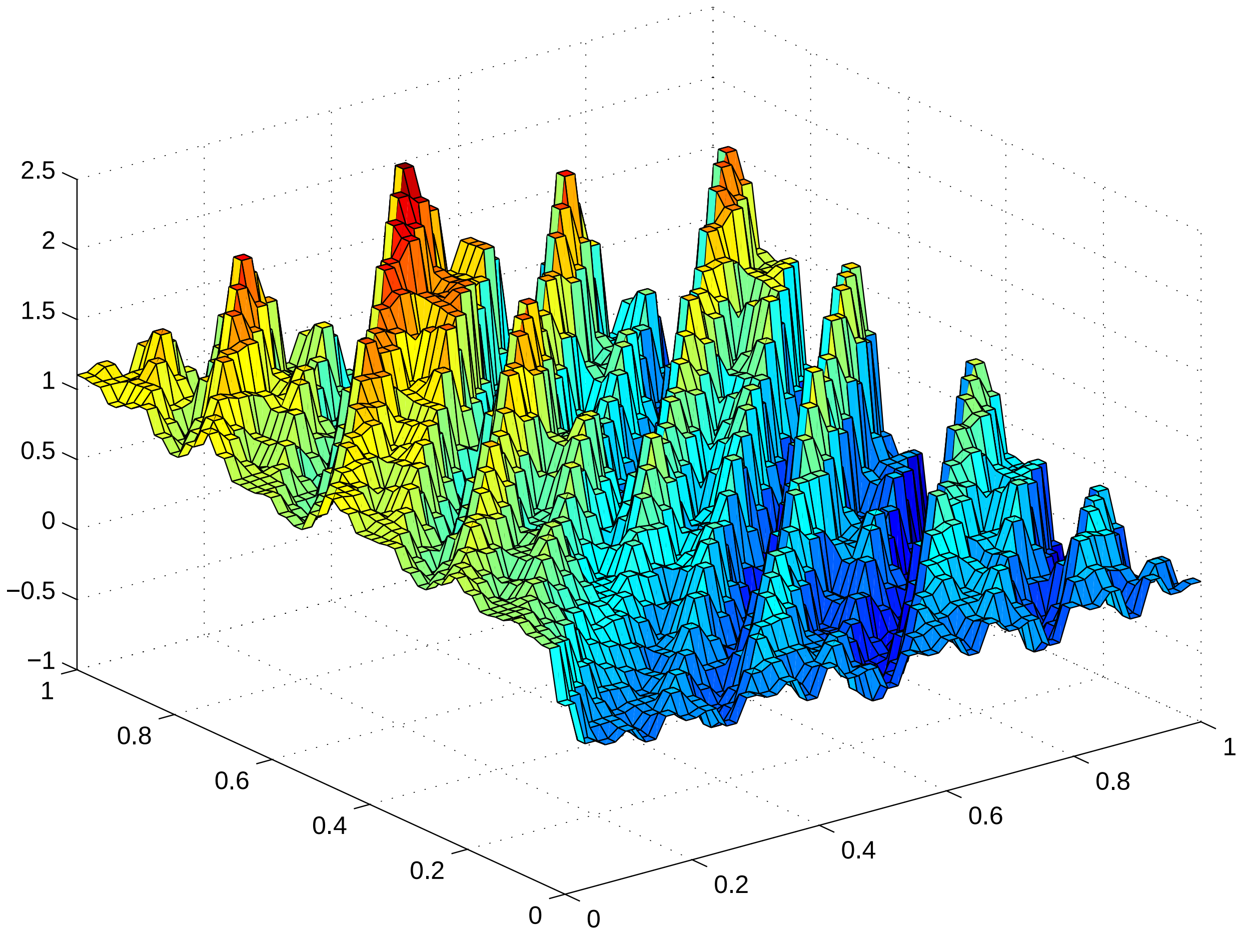
$\partial u/\partial x_1 \quad t = 1.0$



$\partial u / \partial x_1 \quad t = 1.5$



$\partial u/\partial x_1 \quad t = 2.0$

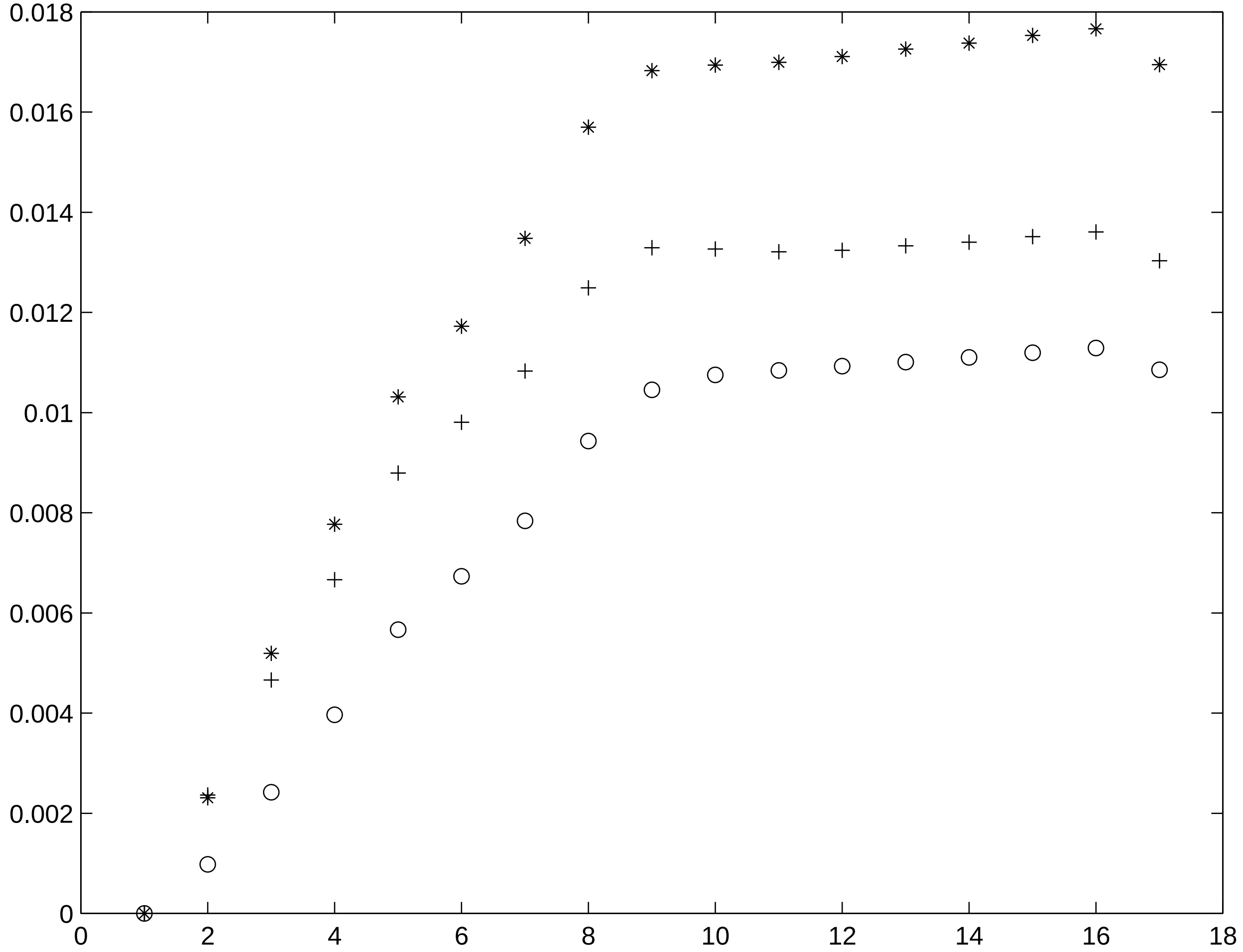


Error with and without subgrid model

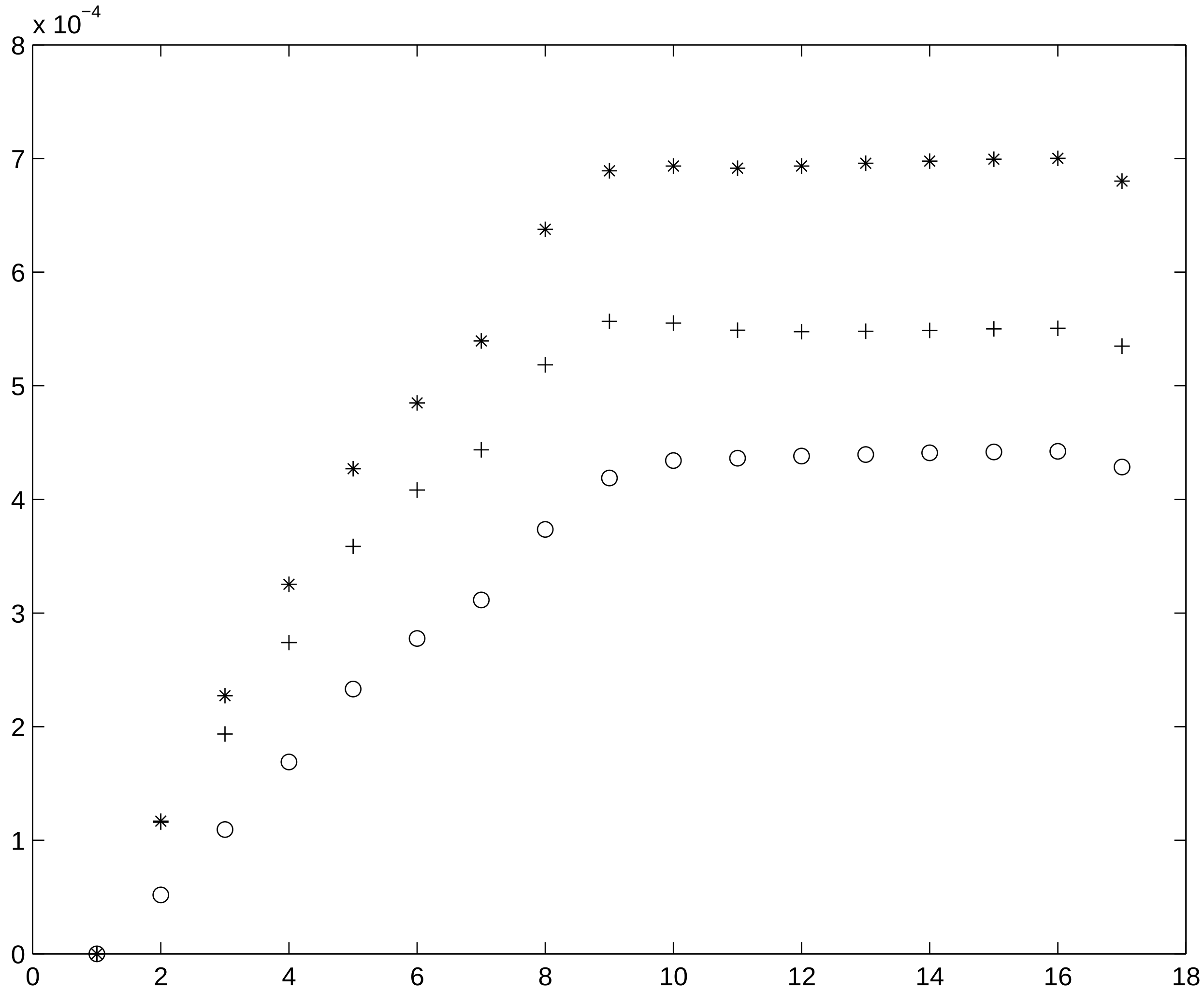
$$\begin{aligned}
\|u - u_h\|_1 / \|u - \tilde{u}_h\|_1 &= 1.3 \\
\|u - u_h\|_2 / \|u - \tilde{u}_h\|_2 &= 1.2 \\
\|u - u_h\|_\infty / \|u - \tilde{u}_h\|_\infty &= 1.1
\end{aligned}$$

$$\begin{aligned}
\|u - u_{h/2}\|_1 / \|u - \tilde{u}_h\|_1 &= 0.77 \\
\|u - u_{h/2}\|_2 / \|u - \tilde{u}_h\|_2 &= 0.77 \\
\|u - u_{h/2}\|_\infty / \|u - \tilde{u}_h\|_\infty &= 0.60
\end{aligned}$$

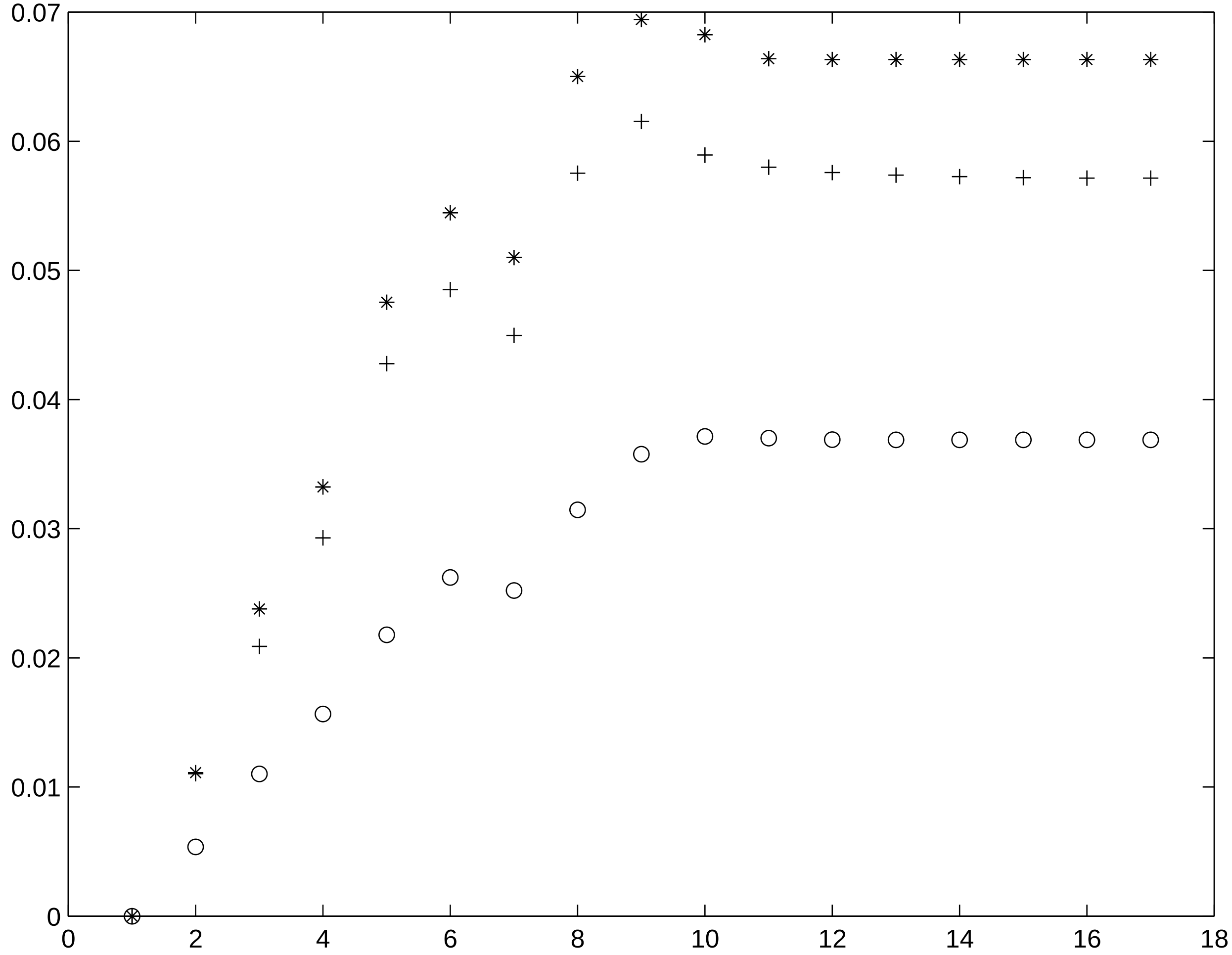
* - $\|[e_h]^h\|_1$, o - $\|[e_{h/2}]^h\|_1$, + - $\|[\tilde{e}_h]^h\|_1$



* - $\|[e_h]^h\|_2$, o - $\|[e_{h/2}]^h\|_2$, + - $\|[\tilde{e}_h]^h\|_2$



* - $\|[e_h]^h\|_\infty$, o - $\|[e_{h/2}]^h\|_\infty$, + - $\|[\tilde{e}_h]^h\|_\infty$



Future Work

- LES - Navier Stokes Equations
- Model influence of details of $F_h(u)$
- Adaptivity

$$\int (R_{num} + R_{mod})\varphi, \qquad R_{mod} = F_h(u) - \tilde{F}_h(\tilde{u}_h),$$

φ dual solution linearized at u^h