



Computability of the Lorenz system and the Solar system

Anders Logg

`logg@math.chalmers.se`

Chalmers Finite Element Center

Outline

- Multi-adaptive Galerkin
- Higher-order elements
- Solving the Lorenz system on $[0, T]$
- A few words about the Solar system

Ordinary Galerkin

Ordinary Galerkin cG(q) for $u' = f$:

$$\int_0^T (\dot{U}, v) dt = \int_0^T (f(U, \cdot), v) dt \quad \forall v \in W,$$

with $U \in V$, $U(0) = u_0$ and the trial and test spaces defined as

$$\begin{aligned} V &= \{v \in C^N([0, T]) : v_i|_{I_j} \in \mathcal{P}^q(I_j)\}, \\ W &= \{v : v_i|_{I_j} \in \mathcal{P}^{q-1}(I_j)\}. \end{aligned}$$

Multi-Adaptive Galerkin

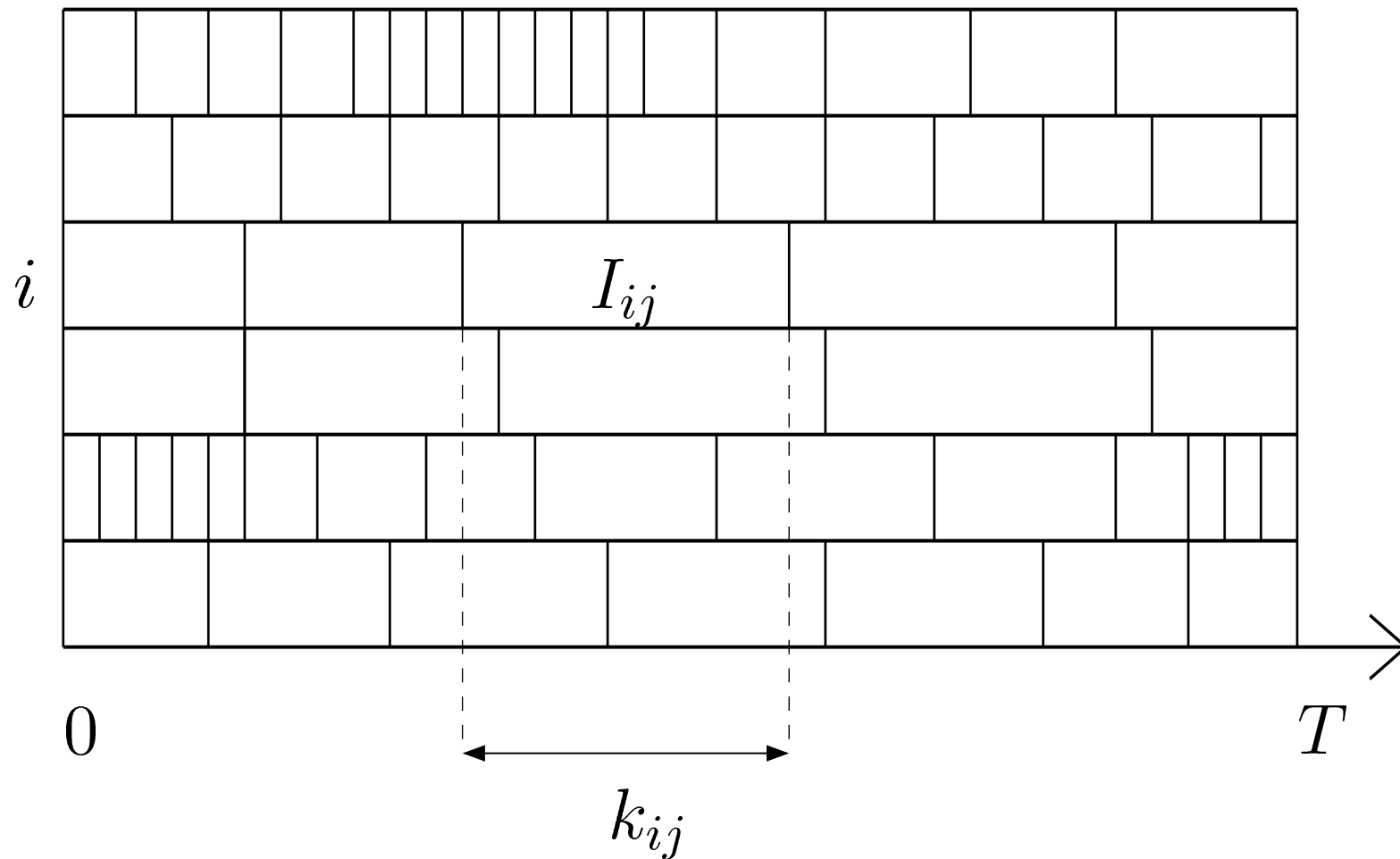
Multi-adaptive Galerkin mcG(q):

$$\int_0^T (\dot{U}, v) \, dt = \int_0^T (f(U, \cdot), v) \, dt \quad \forall v \in W,$$

with $U \in V$, $U(0) = u_0$ and the trial and test spaces now defined as

$$\begin{aligned} V &= \{v \in C^N([0, T]) : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})\}, \\ W &= \{v : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}-1}(I_{ij})\}. \end{aligned}$$

Individual Timesteps



The Discrete Equations

Making an ansatz for U_i on I_{ij} ,

$$U_i(t) = \sum_{n=0}^{q_{ij}} \xi_{ijn} \lambda_n^{[q_{ij}]}(\tau_{ij}(t)),$$

we end up with

$$\xi_{ijn} = \xi_0 + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t)) f_i(U, t) dt,$$

for certain weight functions $\{w_n^{[q]}\} \subset \mathcal{P}^{q-1}(0, 1)$.

The Discontinuous Method

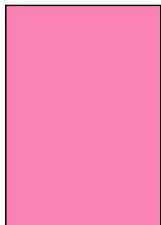
Similarly for the multi-adaptive version of the discontinuous method, mdG(q), we obtain

$$\xi_{ijm} = \xi_{ij0}^- + \int_{I_{ij}} w_m^{[q_{ij}]}(\tau_{ij}(t)) f_i(U, t) dt,$$

for certain weight functions $\{w_n^{[q]}\} \subset \mathcal{P}^q(0, 1)$.

Multi-adaptive Time-Stepping

Action: Redoing steps

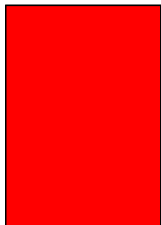


$t = 0$
 $n = 0$

last

Multi-adaptive Time-Stepping

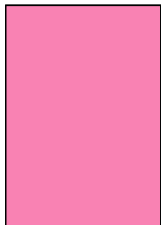
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

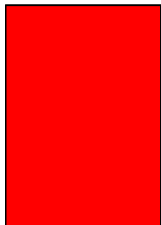
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

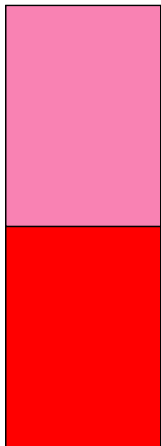
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

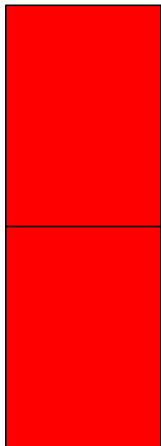
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

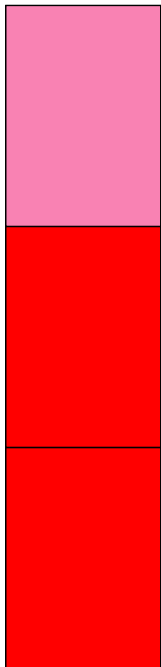
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

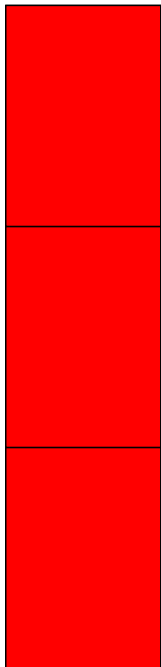
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

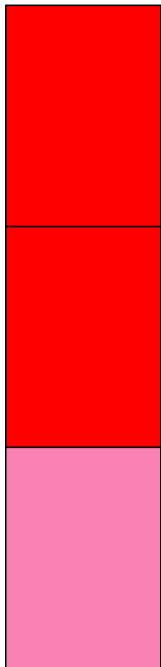
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

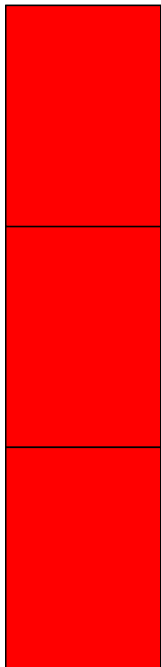
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

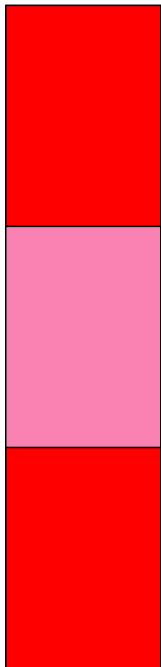
Action: Redoing steps



$t = 0.01$
 $n = 2$

Multi-adaptive Time-Stepping

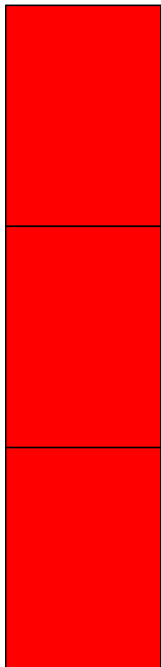
Action: Redoing steps



$t = 0.01$
 $n = 2$

Multi-adaptive Time-Stepping

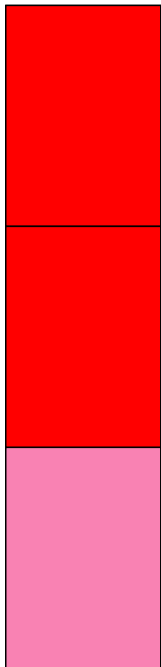
Action: Redoing steps



$t = 0.01$
 $n = 2$

Multi-adaptive Time-Stepping

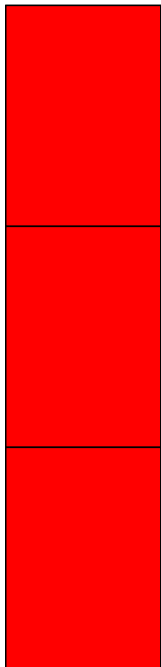
Action: Redoing steps



$t = 0.01$
 $n = 2$

Multi-adaptive Time-Stepping

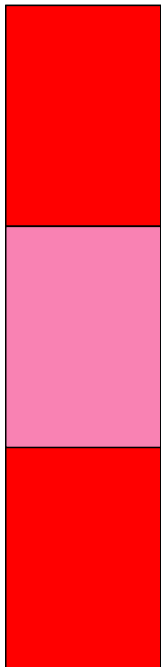
Action: Redoing steps



$t = 0.01$
 $n = 2.6667$

Multi-adaptive Time-Stepping

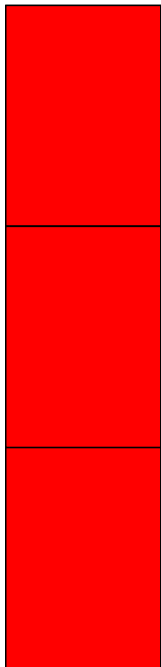
Action: Redoing steps



$t = 0.01$
 $n = 2.6667$

Multi-adaptive Time-Stepping

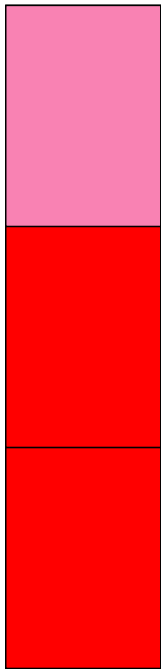
Action: Redoing steps



$t = 0.01$
 $n = 3$

Multi-adaptive Time-Stepping

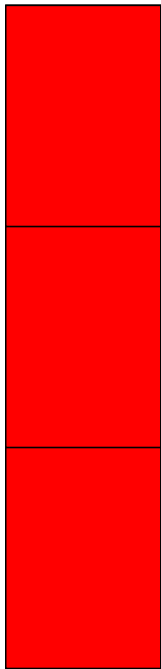
Action: Redoing steps



$t = 0.01$
 $n = 3$

Multi-adaptive Time-Stepping

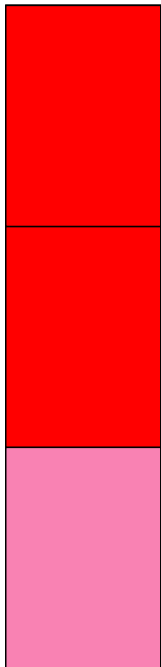
Action: Redoing steps



$t = 0.01$
 $n = 3$

Multi-adaptive Time-Stepping

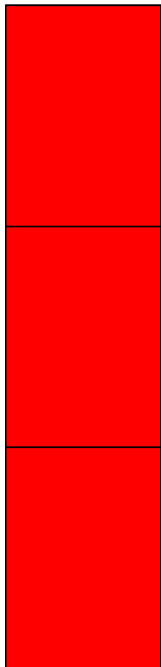
Action: Redoing steps



$t = 0.01$
 $n = 3$

Multi-adaptive Time-Stepping

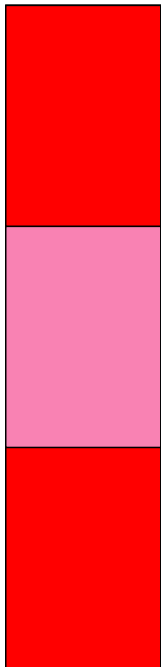
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

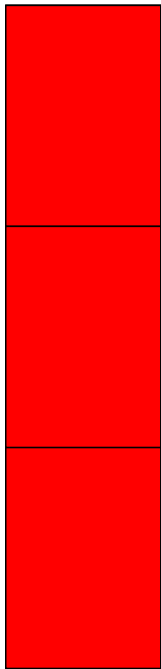
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

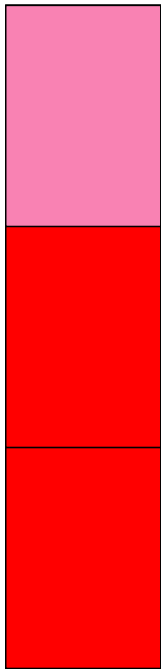
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

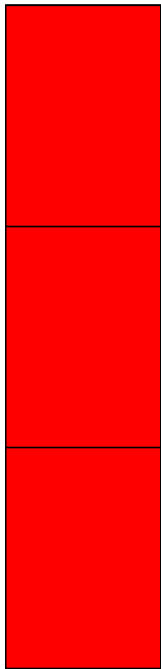
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

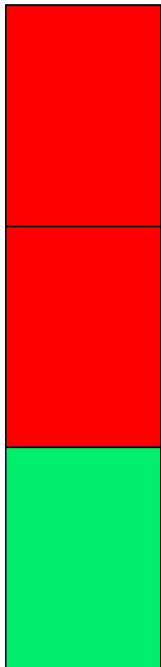
Action: Redoing steps



$t = 0.01$
 $n = 1$

Multi-adaptive Time-Stepping

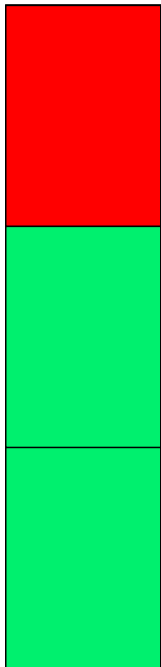
Action: Clearing steps



$t = 0.01$
 $n = 2$

Multi-adaptive Time-Stepping

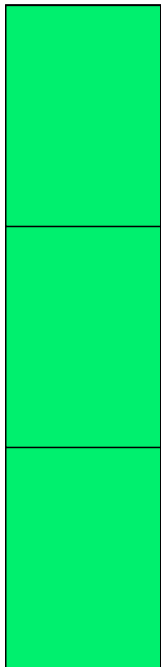
Action: Clearing steps



$t = 0.01$
 $n = 2$

Multi-adaptive Time-Stepping

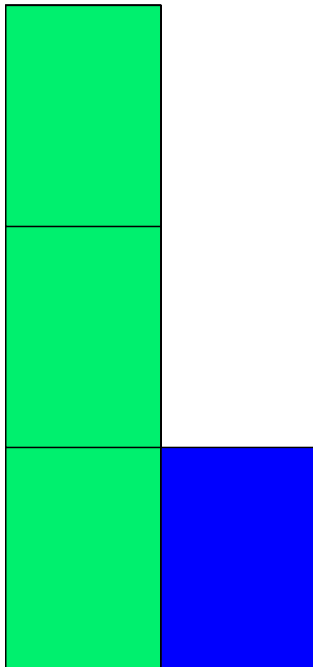
Action: Clearing steps



$t = 0.01$
 $n = 2$

Multi-adaptive Time-Stepping

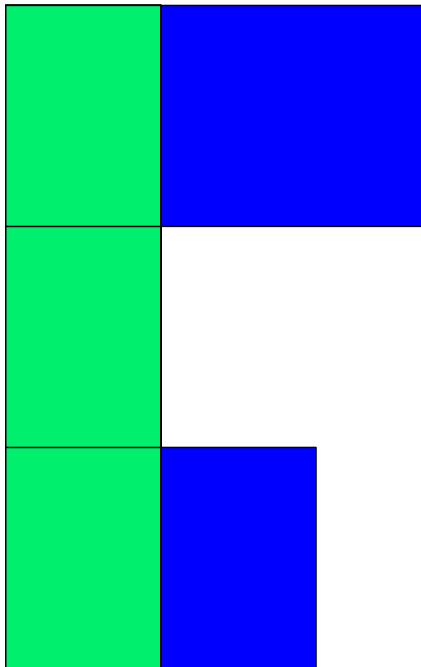
Action: Covering



$t = 0.02$
 $n = 2$

Multi-adaptive Time-Stepping

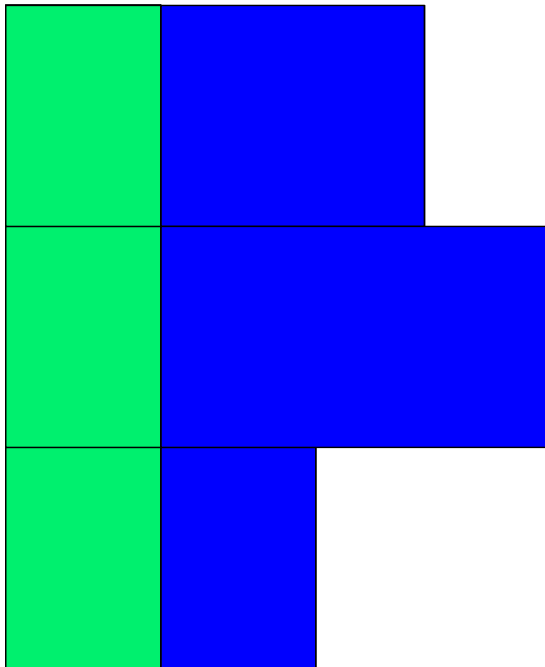
Action: Covering



$t = 0.027$
 $n = 2$

Multi-adaptive Time-Stepping

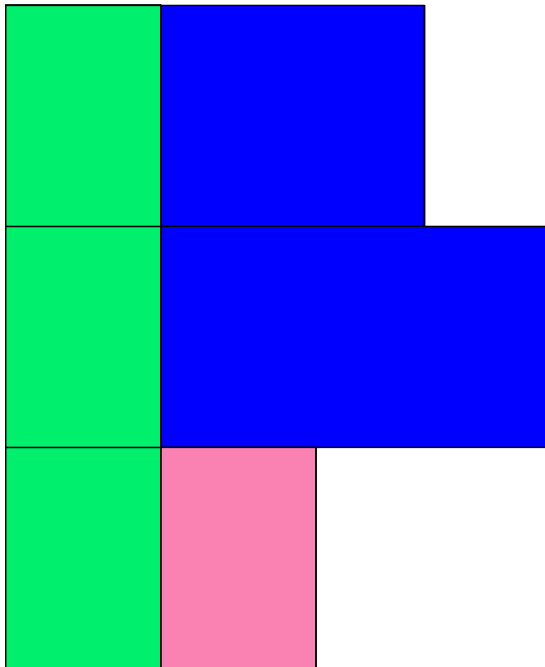
Action: Covering



$t = 0.035$
 $n = 2$

Multi-adaptive Time-Stepping

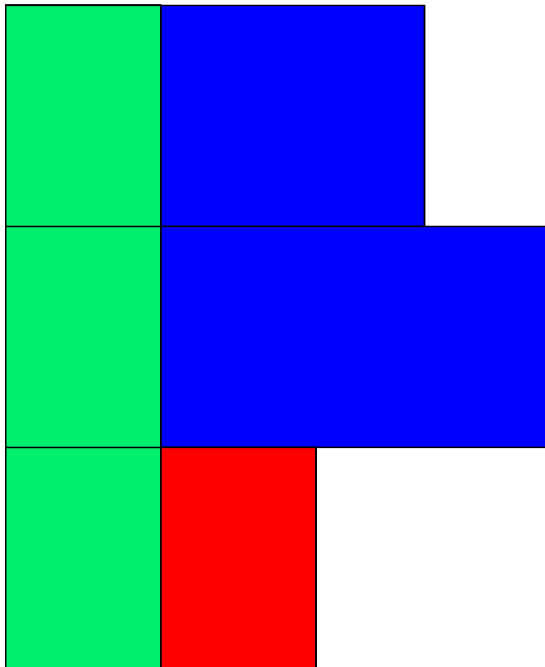
Action: Redoing steps



$t = 0.035$
 $n = 2$

Multi-adaptive Time-Stepping

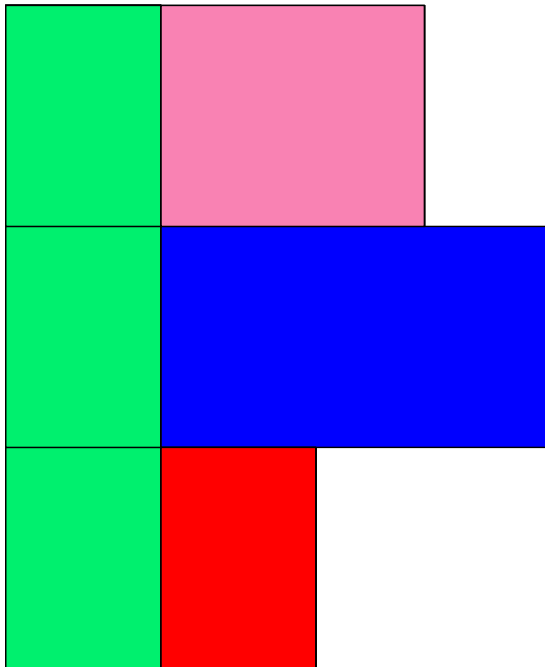
Action: Redoing steps



$t = 0.02$
 $n = 1$

Multi-adaptive Time-Stepping

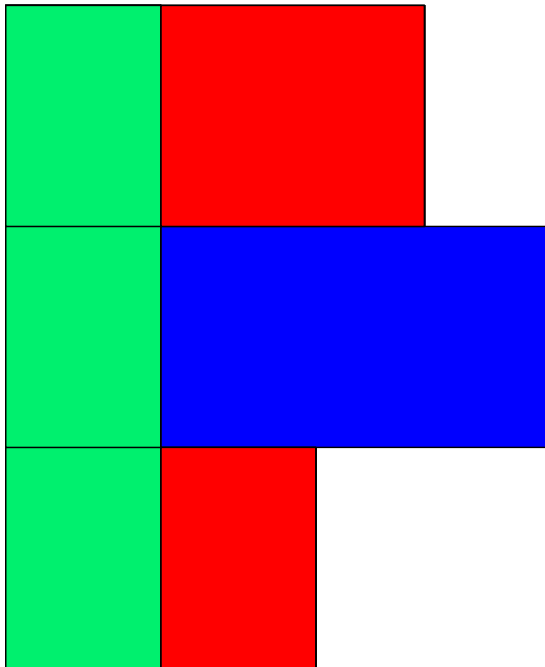
Action: Redoing steps



$t = 0.02$
 $n = 1$

Multi-adaptive Time-Stepping

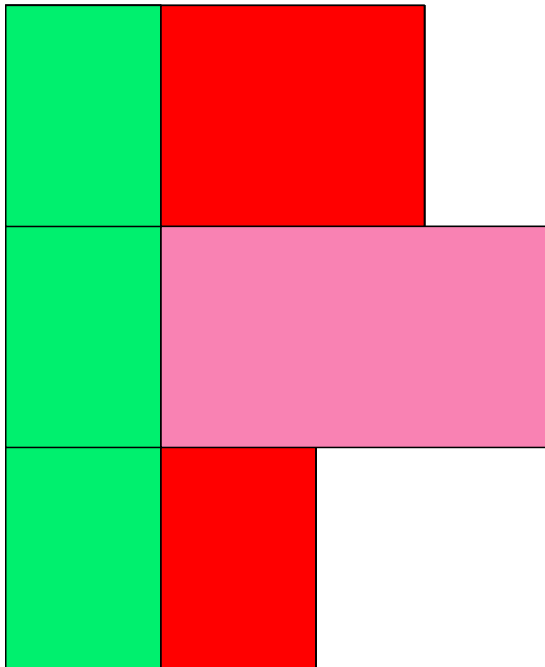
Action: Redoing steps



$t = 0.027$
 $n = 1$

Multi-adaptive Time-Stepping

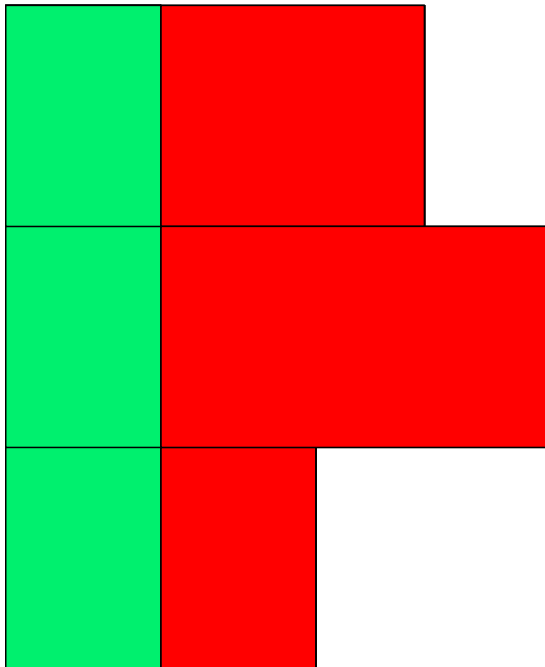
Action: Redoing steps



$t = 0.027$
 $n = 1$

Multi-adaptive Time-Stepping

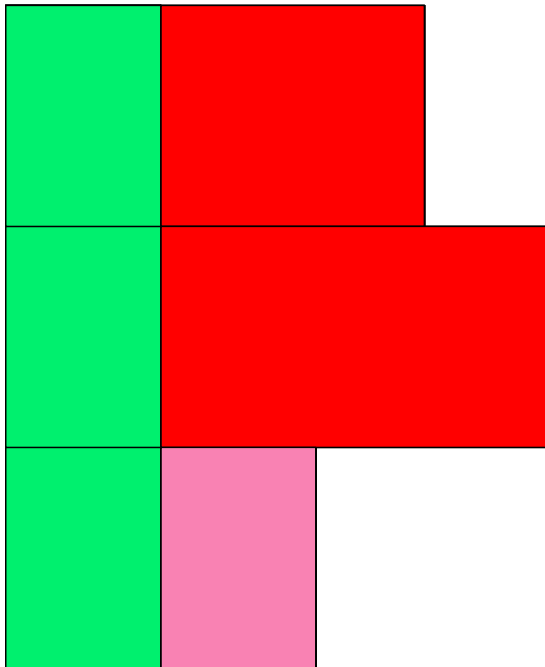
Action: Redoing steps



$t = 0.035$
 $n = 1$

Multi-adaptive Time-Stepping

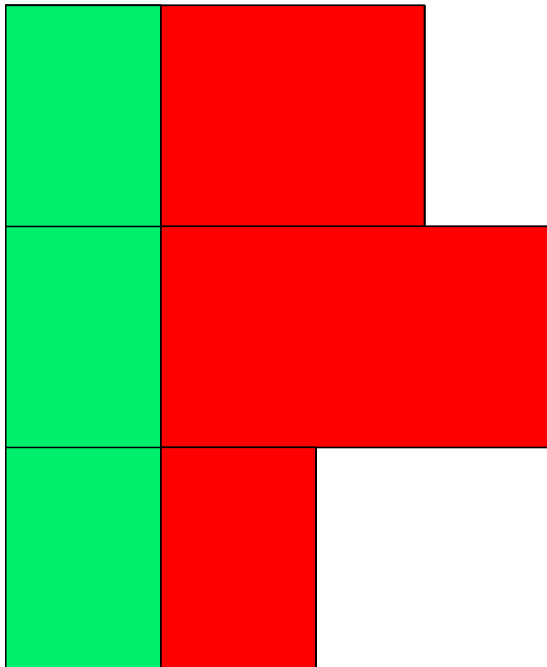
Action: Redoing steps



$t = 0.035$
 $n = 1$

Multi-adaptive Time-Stepping

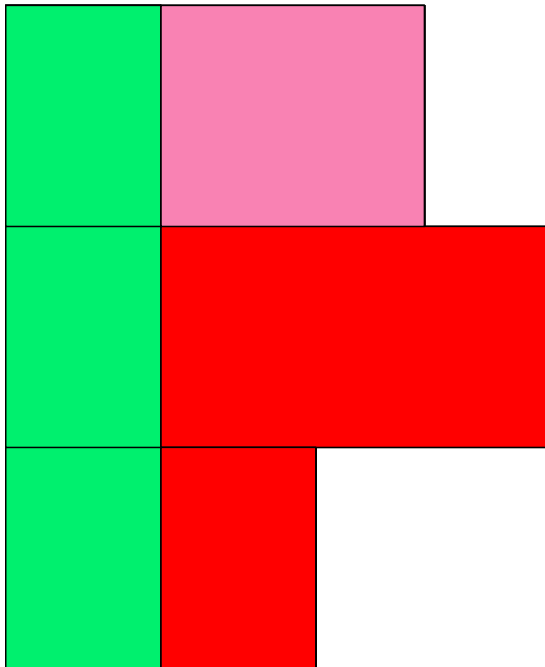
Action: Redoing steps



$t = 0.02$
 $n = 2$

Multi-adaptive Time-Stepping

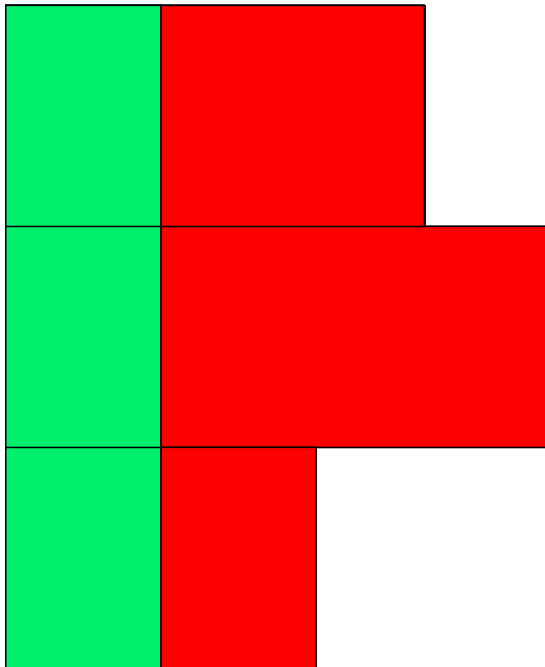
Action: Redoing steps



$t = 0.02$
 $n = 2$

Multi-adaptive Time-Stepping

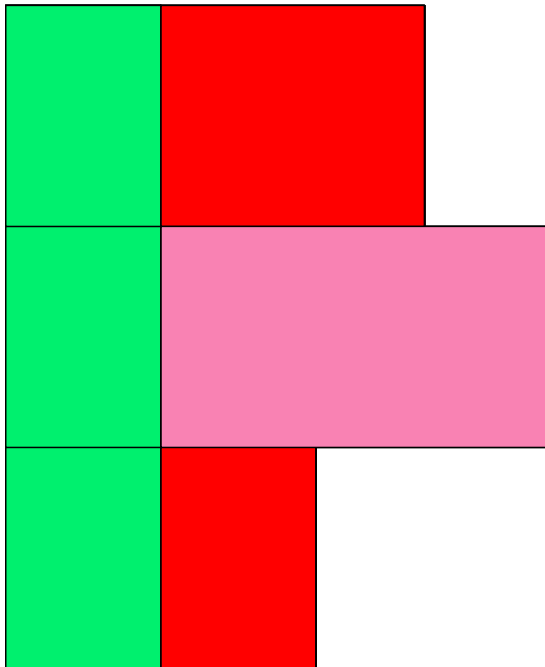
Action: Redoing steps



$t = 0.027$
 $n = 2$

Multi-adaptive Time-Stepping

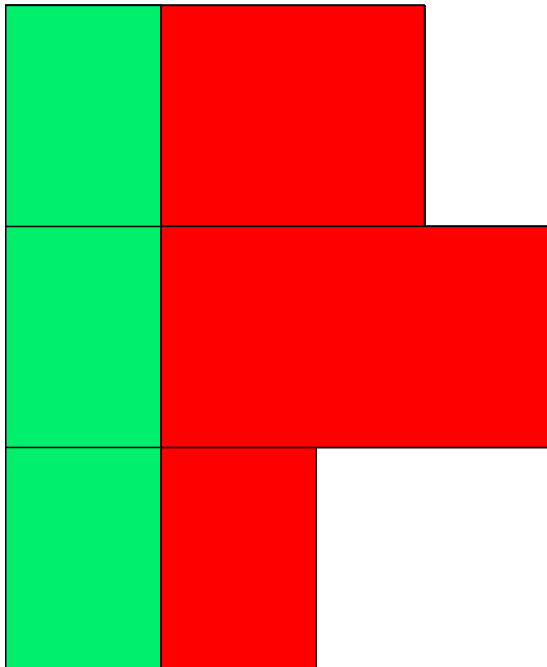
Action: Redoing steps



$t = 0.027$
 $n = 2$

Multi-adaptive Time-Stepping

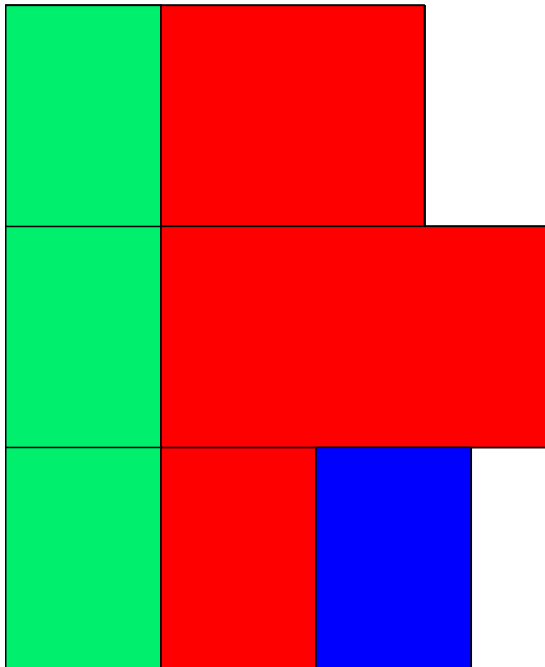
Action: Redoing steps



$t = 0.035$
 $n = 2.6667$

Multi-adaptive Time-Stepping

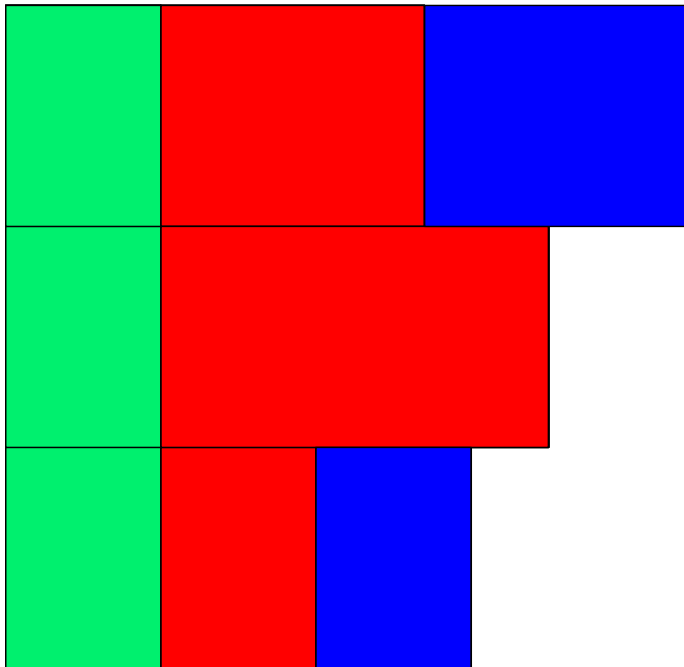
Action: Covering



$t = 0.03$
 $n = 3$

Multi-adaptive Time-Stepping

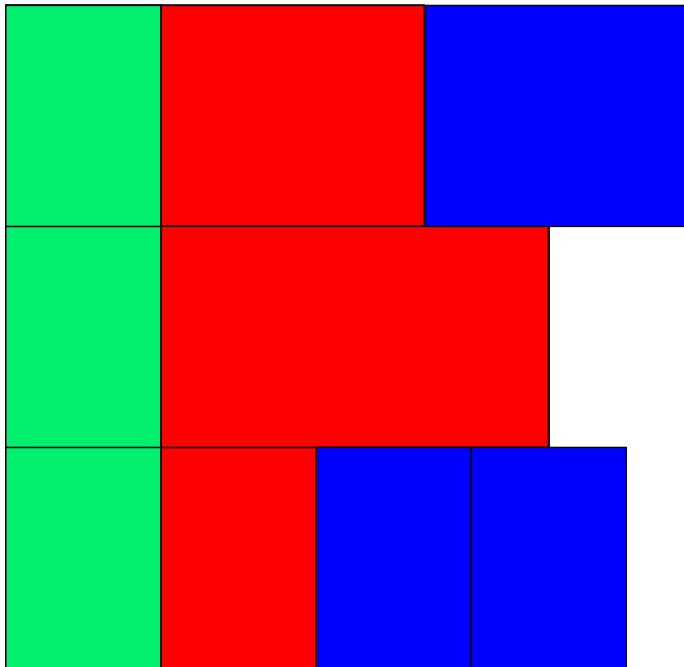
Action: Covering



$t = 0.044$
 $n = 3$

Multi-adaptive Time-Stepping

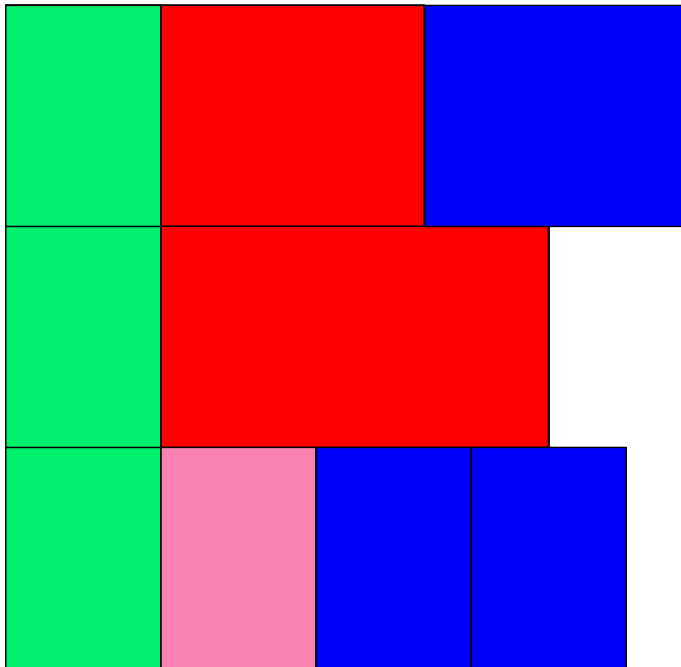
Action: Covering



$t = 0.04$
 $n = 3$

Multi-adaptive Time-Stepping

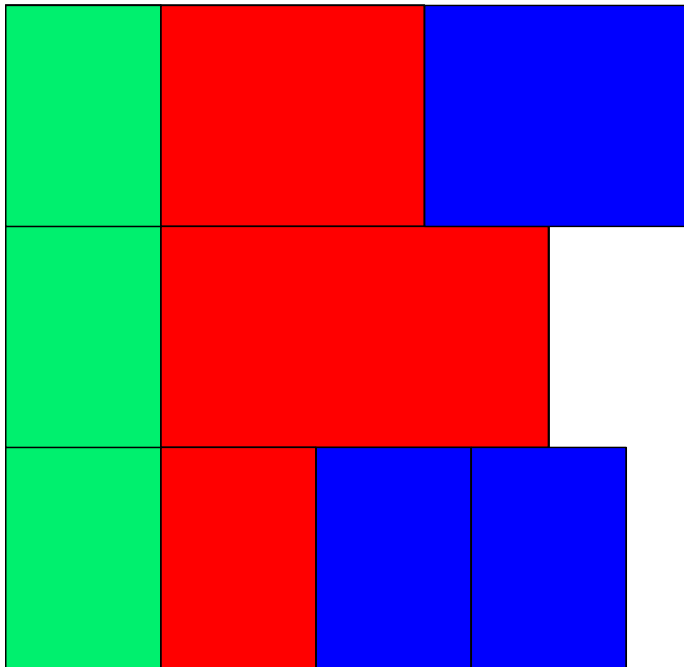
Action: Redoing steps



$t = 0.04$
 $n = 3$

Multi-adaptive Time-Stepping

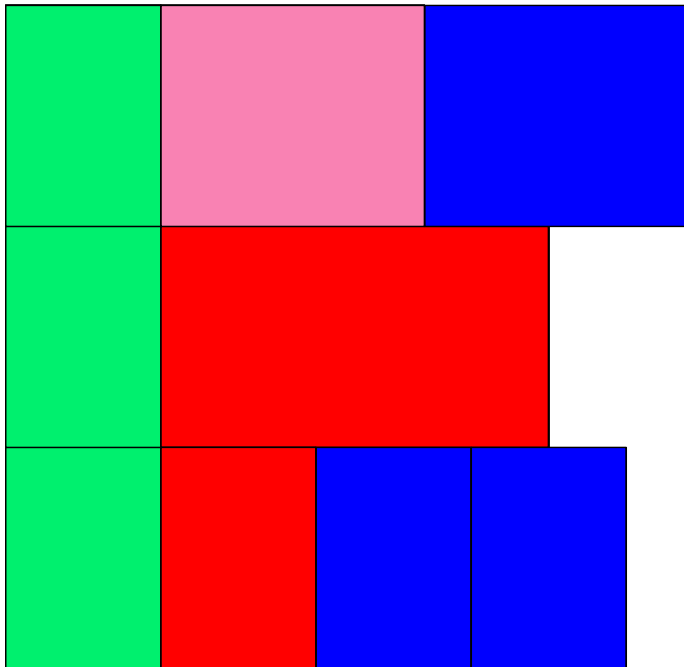
Action: Redoing steps



$t = 0.02$
 $n = 1$

Multi-adaptive Time-Stepping

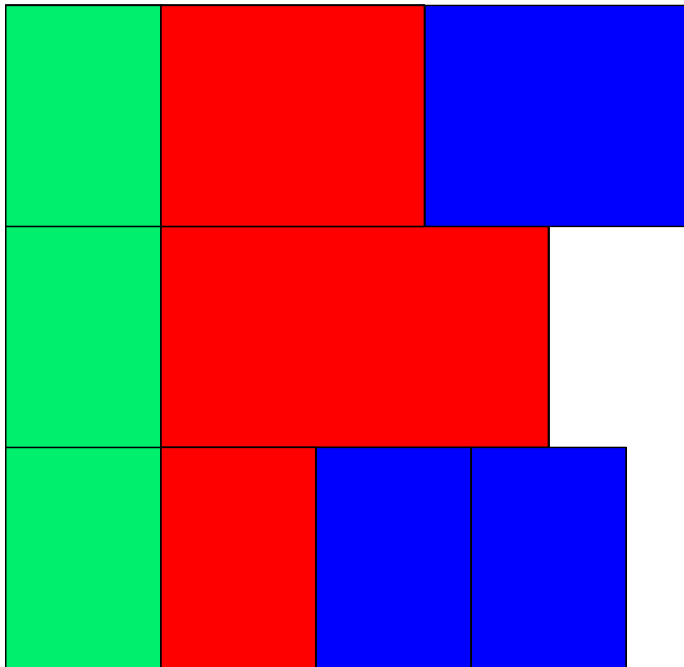
Action: Redoing steps



$t = 0.02$
 $n = 1$

Multi-adaptive Time-Stepping

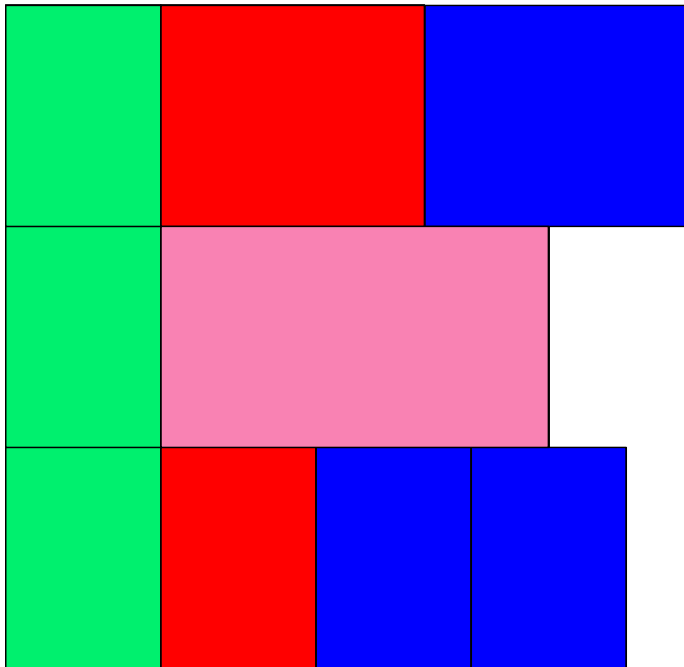
Action: Redoing steps



$t = 0.027$
 $n = 1$

Multi-adaptive Time-Stepping

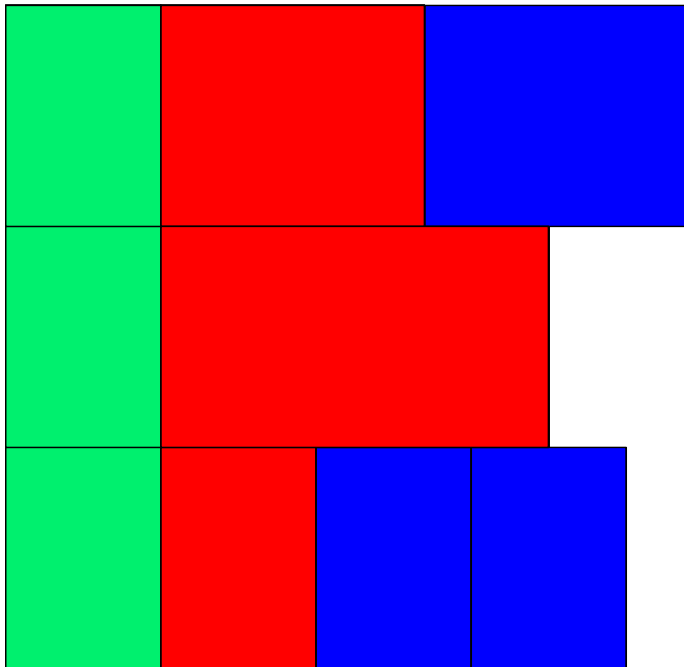
Action: Redoing steps



$t = 0.027$
 $n = 1$

Multi-adaptive Time-Stepping

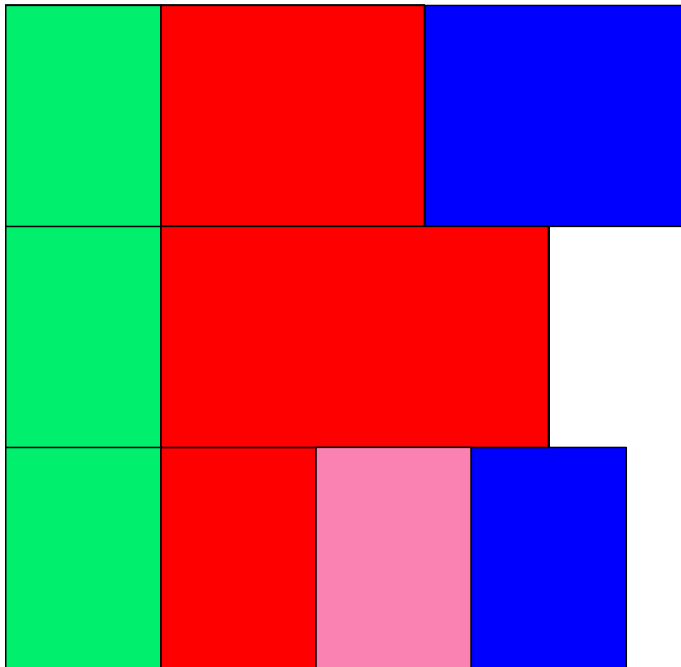
Action: Redoing steps



$t = 0.035$
 $n = 1$

Multi-adaptive Time-Stepping

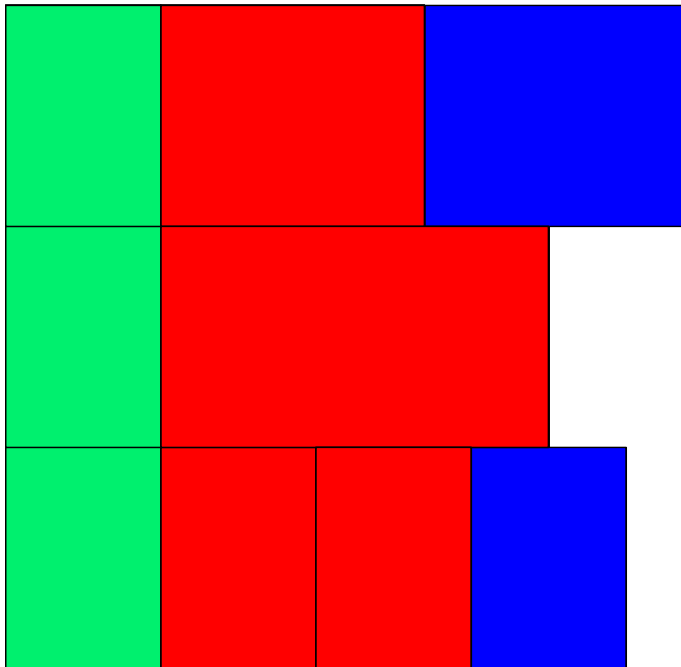
Action: Redoing steps



$t = 0.035$
 $n = 1$

Multi-adaptive Time-Stepping

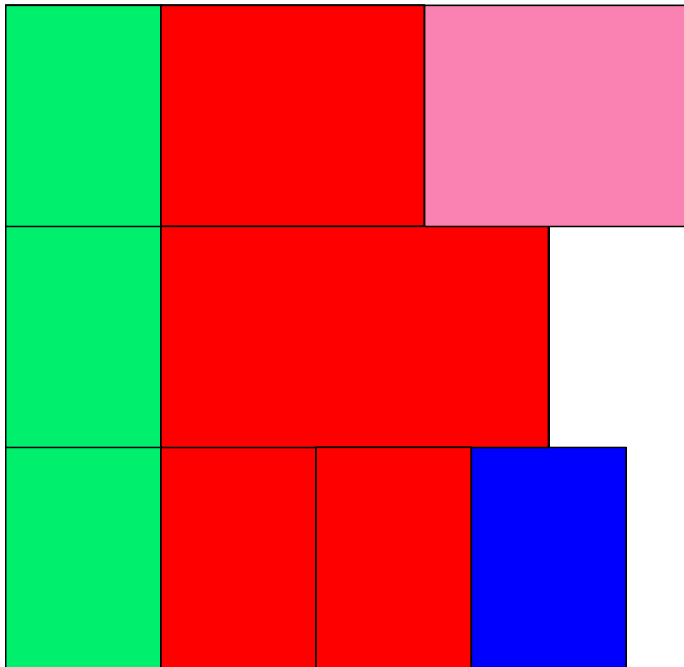
Action: Redoing steps



$t = 0.03$
 $n = 1$

Multi-adaptive Time-Stepping

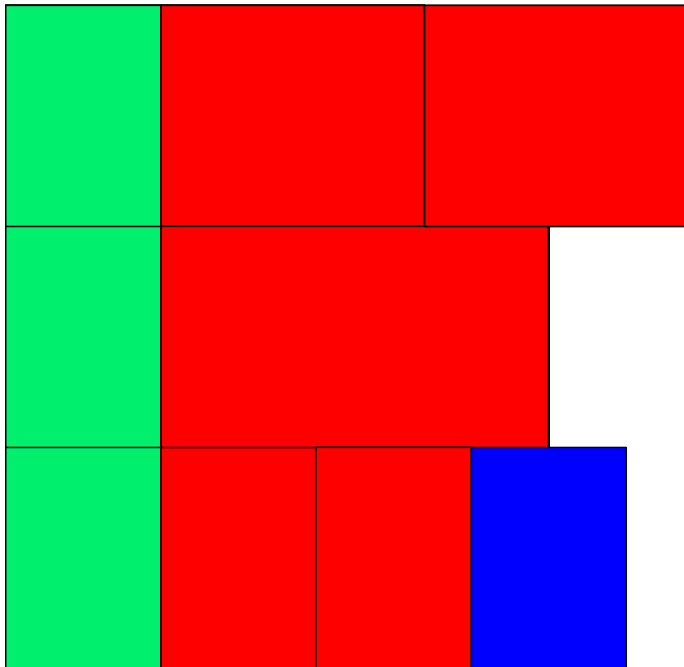
Action: Redoing steps



$t = 0.03$
 $n = 1$

Multi-adaptive Time-Stepping

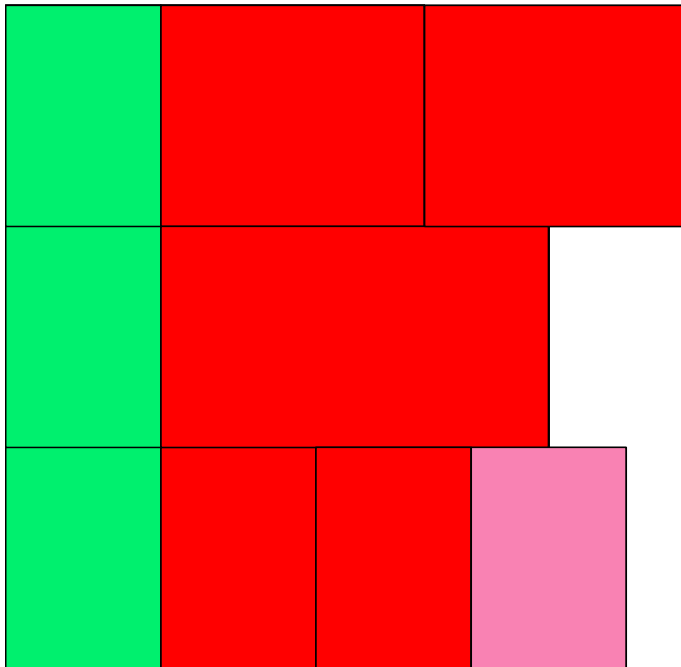
Action: Redoing steps



$t = 0.044$
 $n = 1$

Multi-adaptive Time-Stepping

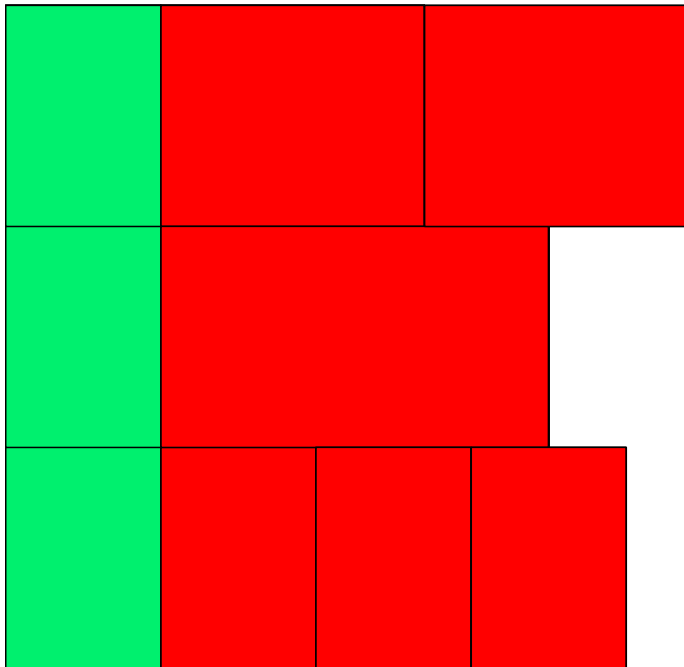
Action: Redoing steps



$t = 0.044$
 $n = 1$

Multi-adaptive Time-Stepping

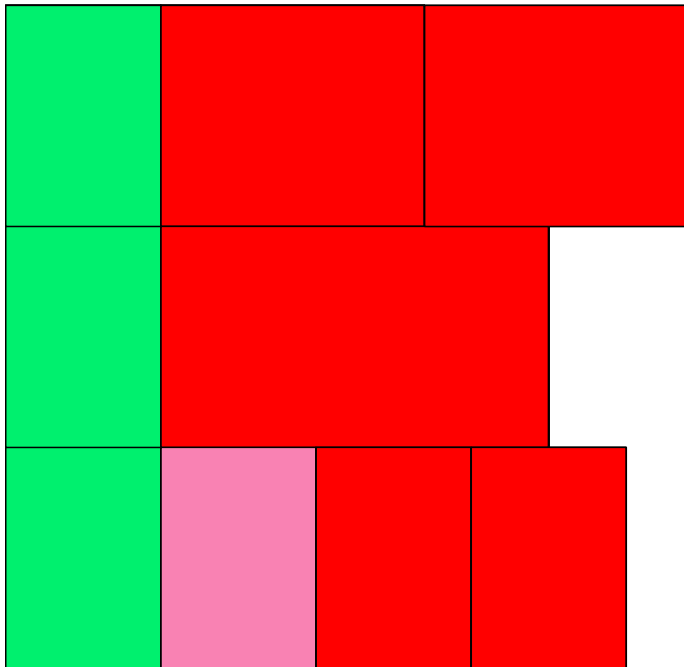
Action: Redoing steps



$t = 0.04$
 $n = 1$

Multi-adaptive Time-Stepping

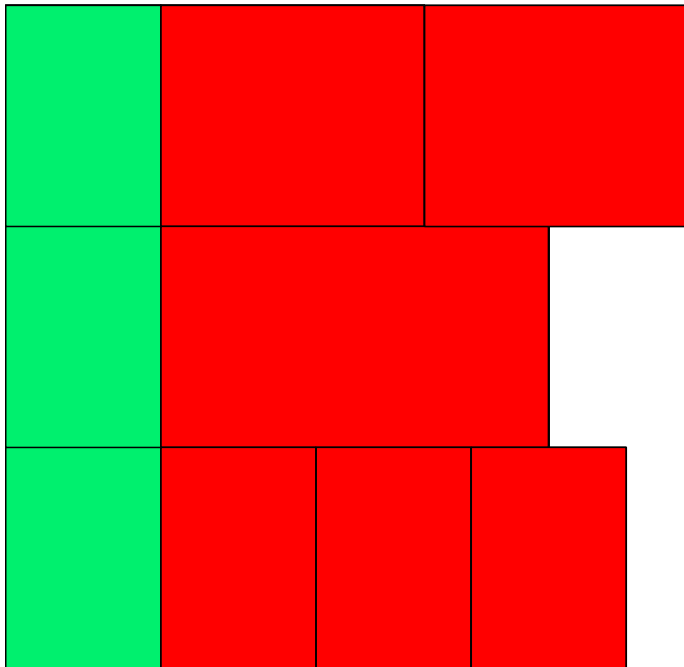
Action: Redoing steps



$t = 0.04$
 $n = 1$

Multi-adaptive Time-Stepping

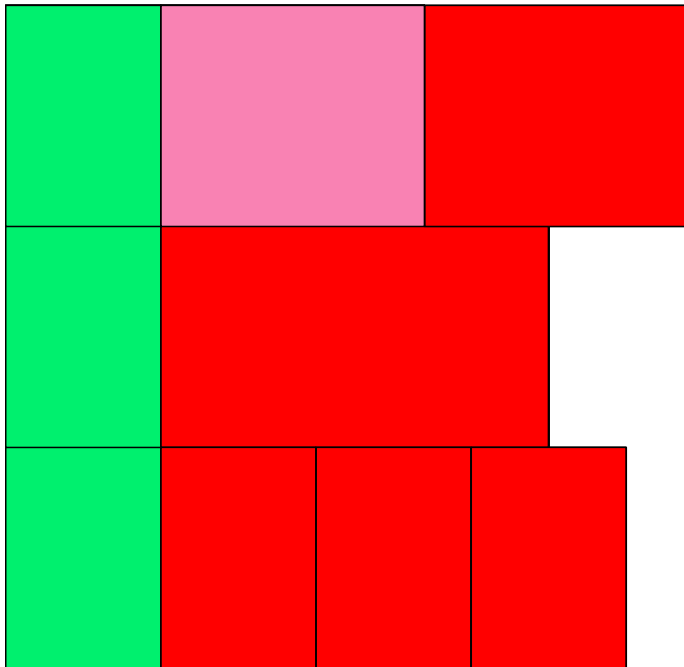
Action: Redoing steps



$t = 0.02$
 $n = 2$

Multi-adaptive Time-Stepping

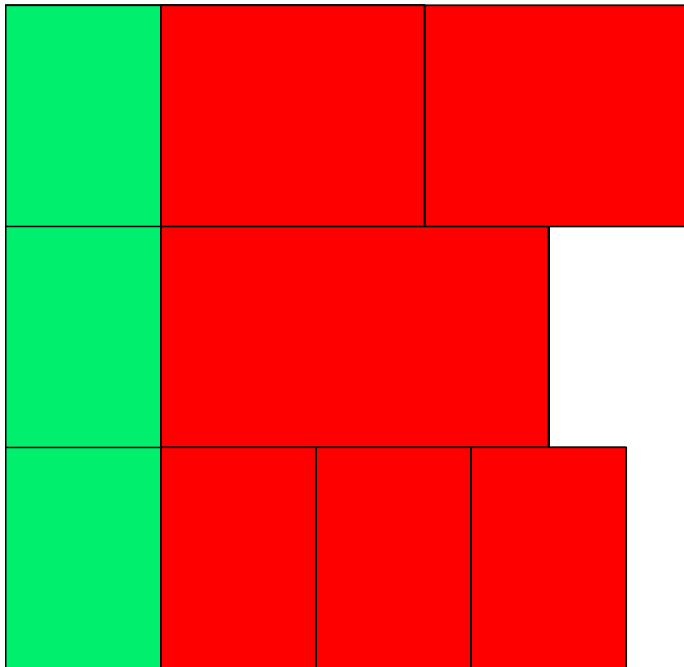
Action: Redoing steps



$t = 0.02$
 $n = 2$

Multi-adaptive Time-Stepping

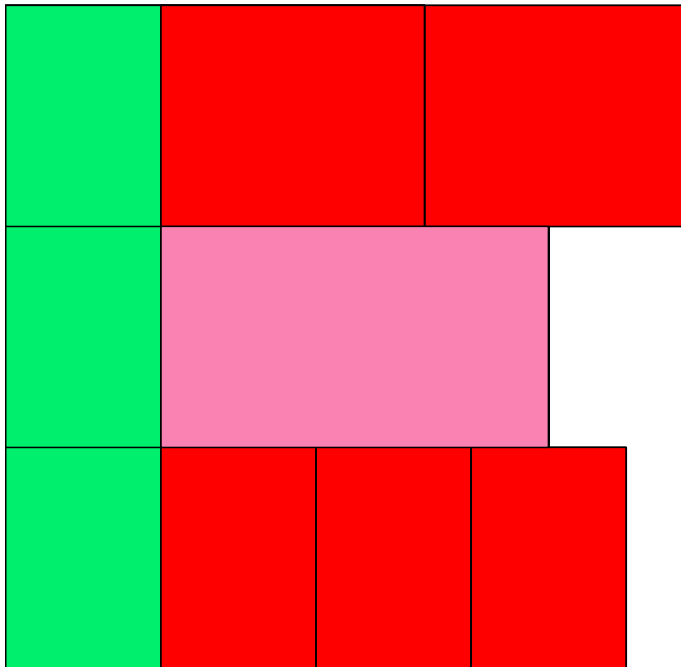
Action: Redoing steps



$t = 0.027$
 $n = 2$

Multi-adaptive Time-Stepping

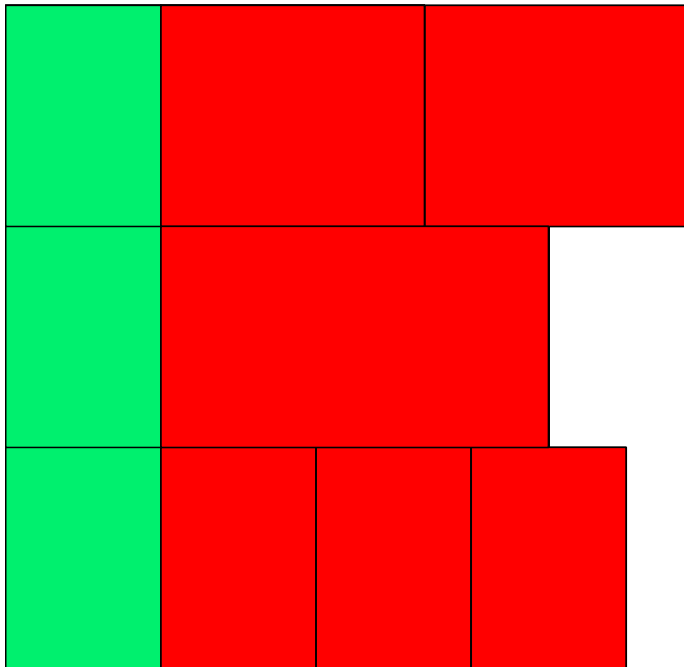
Action: Redoing steps



$t = 0.027$
 $n = 2$

Multi-adaptive Time-Stepping

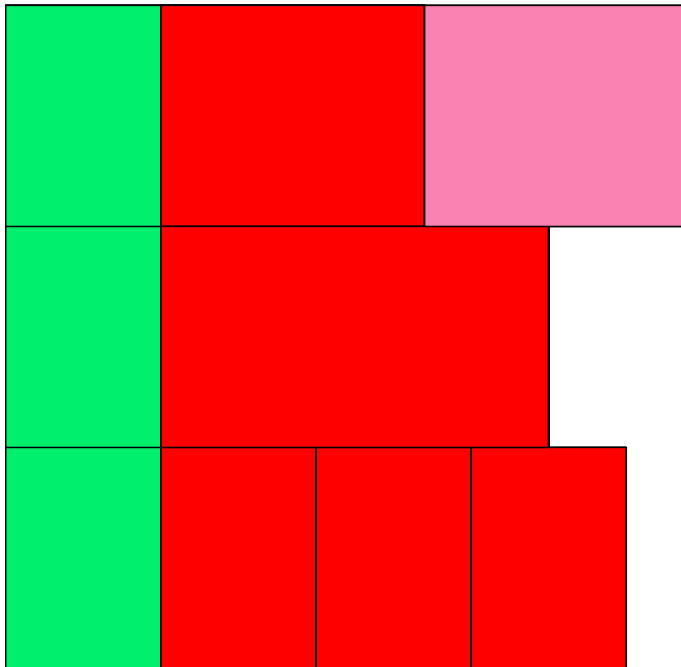
Action: Redoing steps



$t = 0.035$
 $n = 2$

Multi-adaptive Time-Stepping

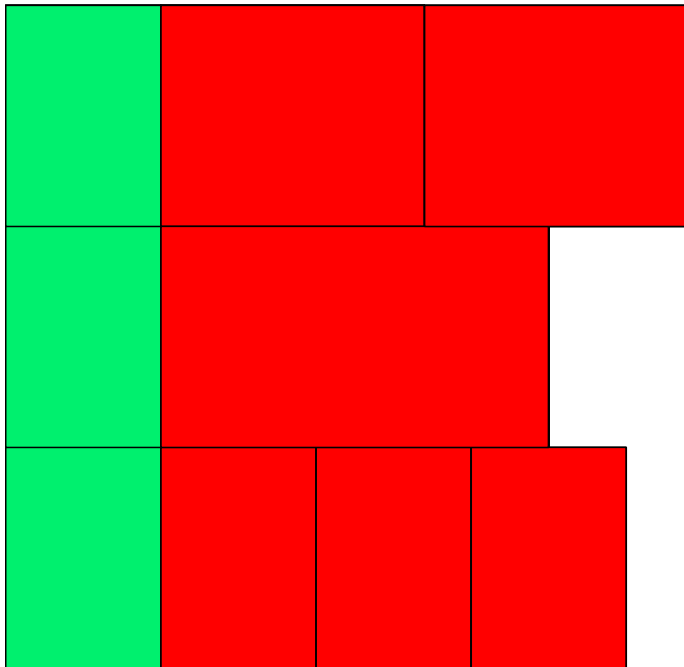
Action: Redoing steps



$t = 0.035$
 $n = 2$

Multi-adaptive Time-Stepping

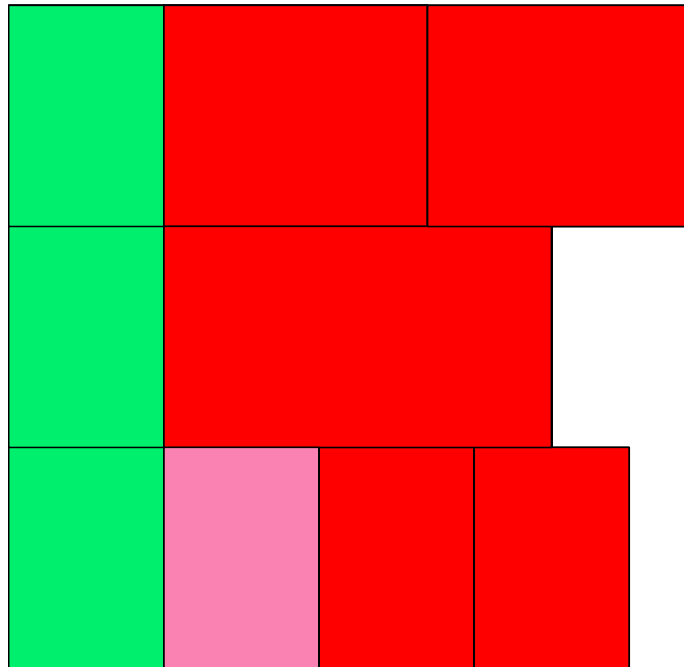
Action: Redoing steps



$t = 0.044$
 $n = 2$

Multi-adaptive Time-Stepping

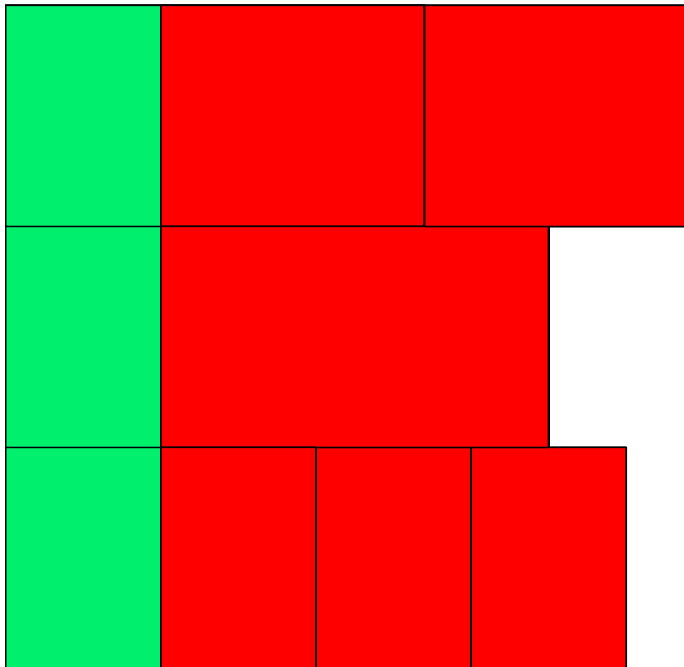
Action: Redoing steps



$t = 0.044$
 $n = 2$

Multi-adaptive Time-Stepping

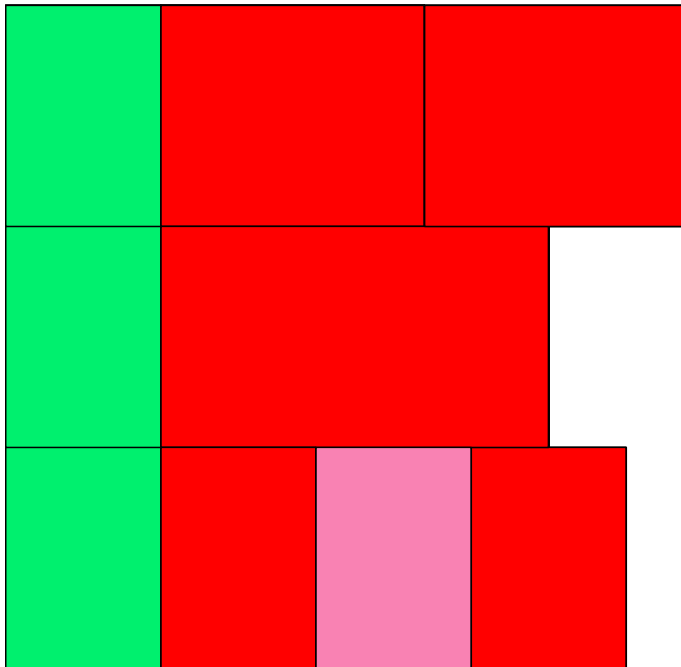
Action: Redoing steps



$t = 0.02$
 $n = 2.6667$

Multi-adaptive Time-Stepping

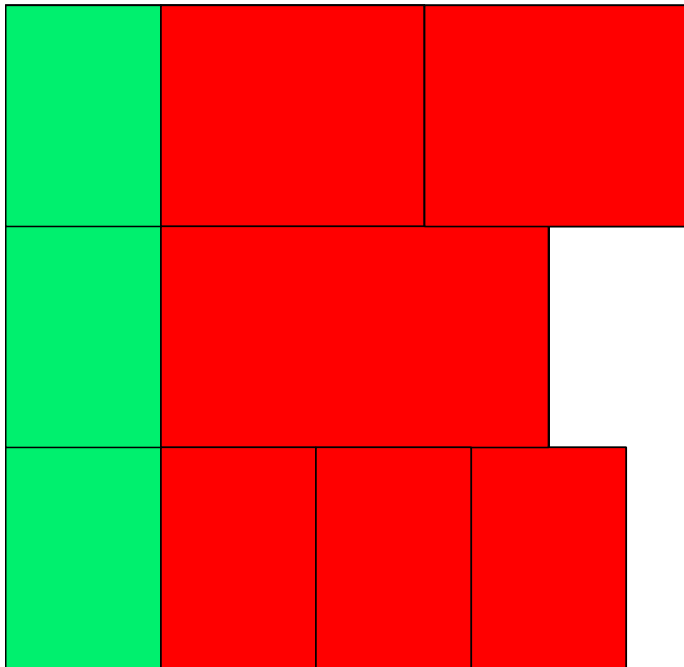
Action: Redoing steps



$t = 0.02$
 $n = 2.6667$

Multi-adaptive Time-Stepping

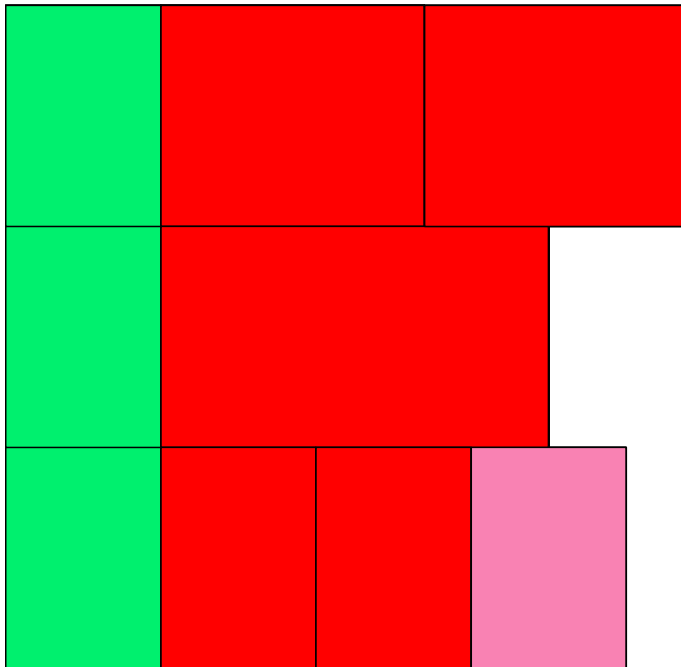
Action: Redoing steps



$t = 0.03$
 $n = 2.8333$

Multi-adaptive Time-Stepping

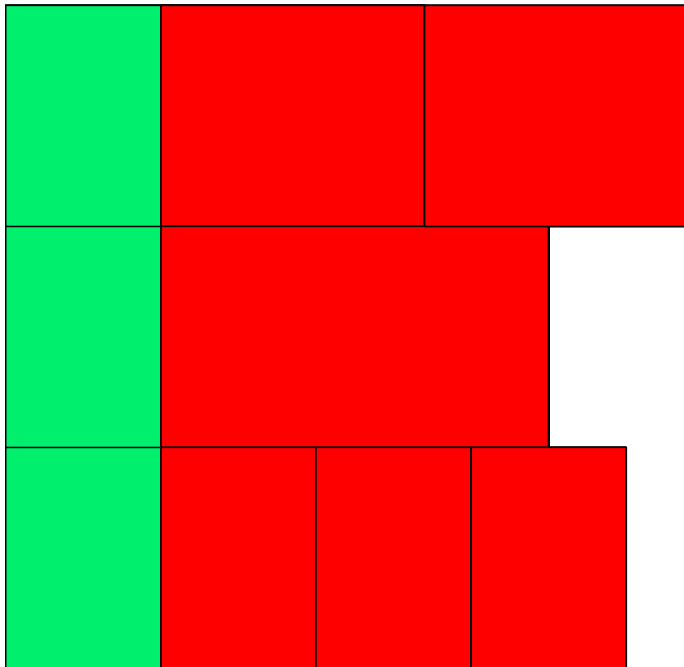
Action: Redoing steps



$t = 0.03$
 $n = 2.8333$

Multi-adaptive Time-Stepping

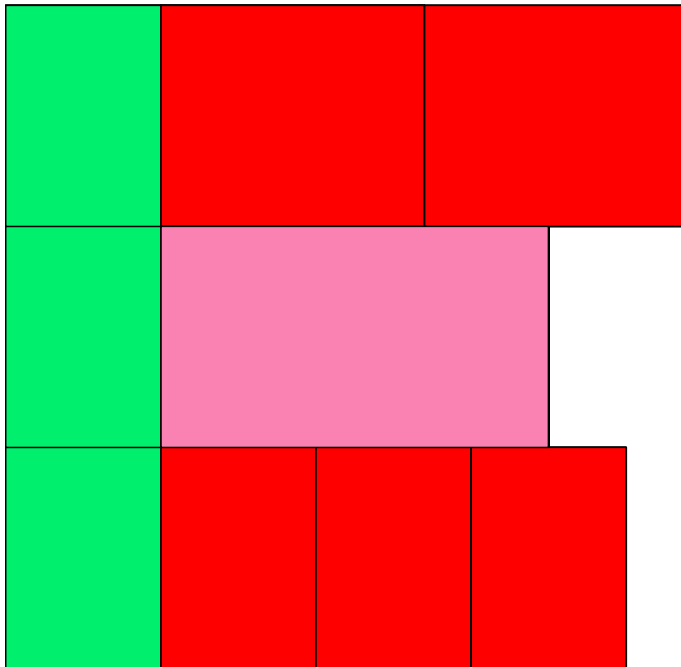
Action: Redoing steps



$t = 0.04$
 $n = 2.8333$

Multi-adaptive Time-Stepping

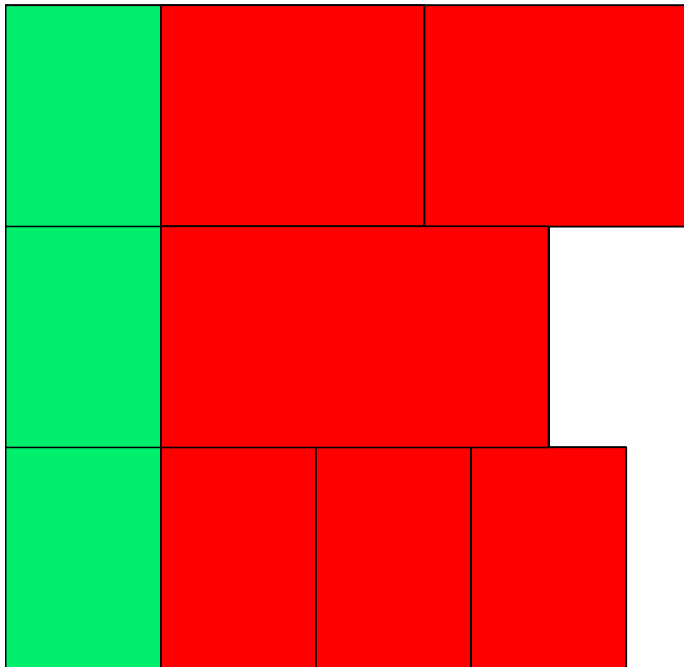
Action: Redoing steps



$t = 0.04$
 $n = 2.8333$

Multi-adaptive Time-Stepping

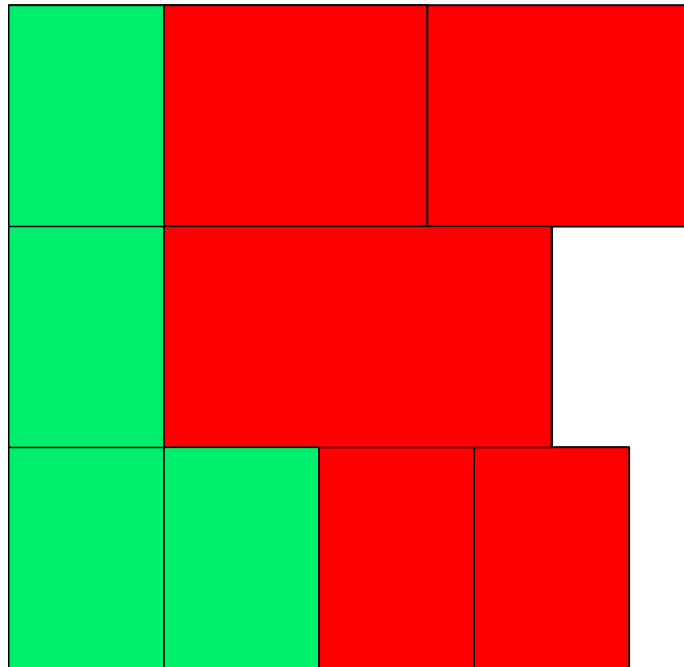
Action: Redoing steps



$t = 0.035$
 $n = 3.1667$

Multi-adaptive Time-Stepping

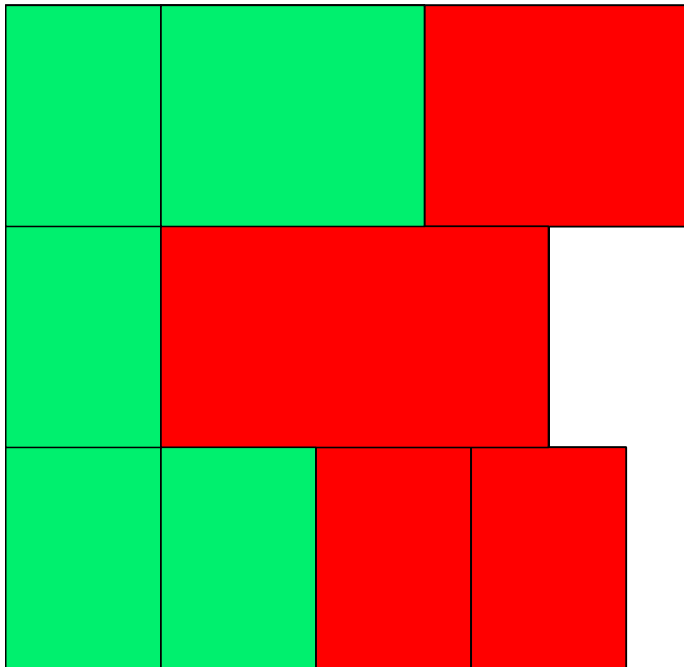
Action: Clearing steps



$t = 0.02$
 $n = 3.3333$

Multi-adaptive Time-Stepping

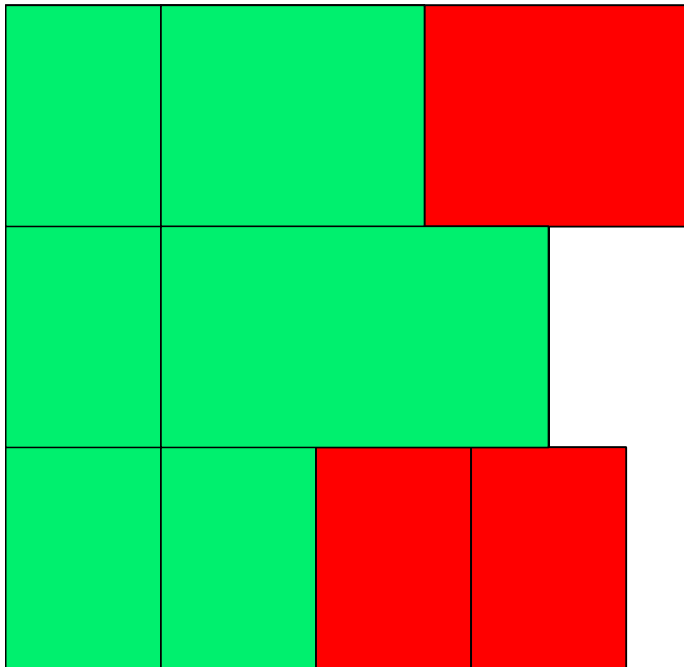
Action: Clearing steps



$t = 0.027$
 $n = 3.3333$

Multi-adaptive Time-Stepping

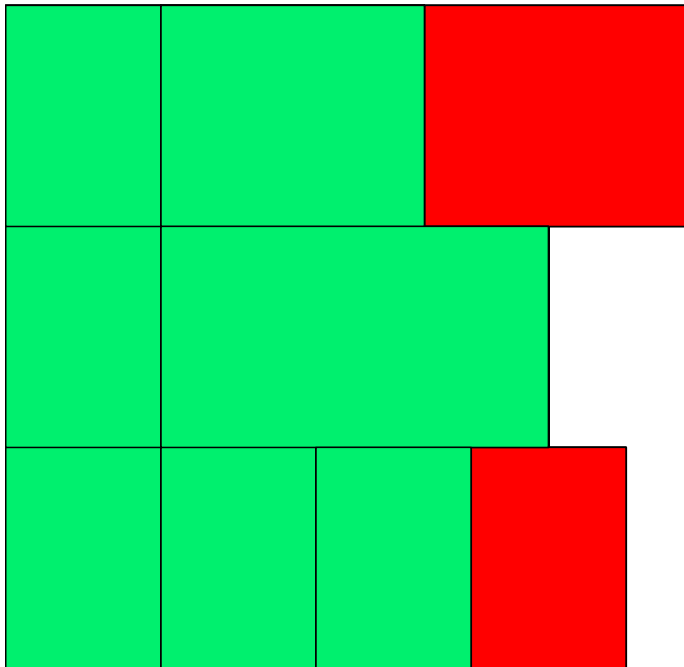
Action: Clearing steps



$t = 0.035$
 $n = 3.3333$

Multi-adaptive Time-Stepping

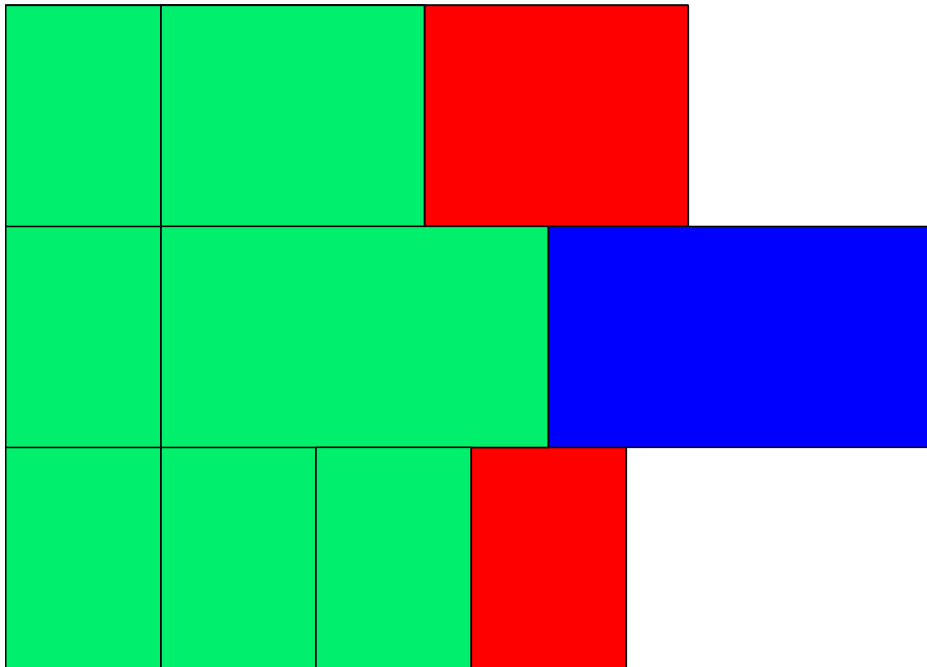
Action: Clearing steps



$t = 0.03$
 $n = 3.3333$

Multi-adaptive Time-Stepping

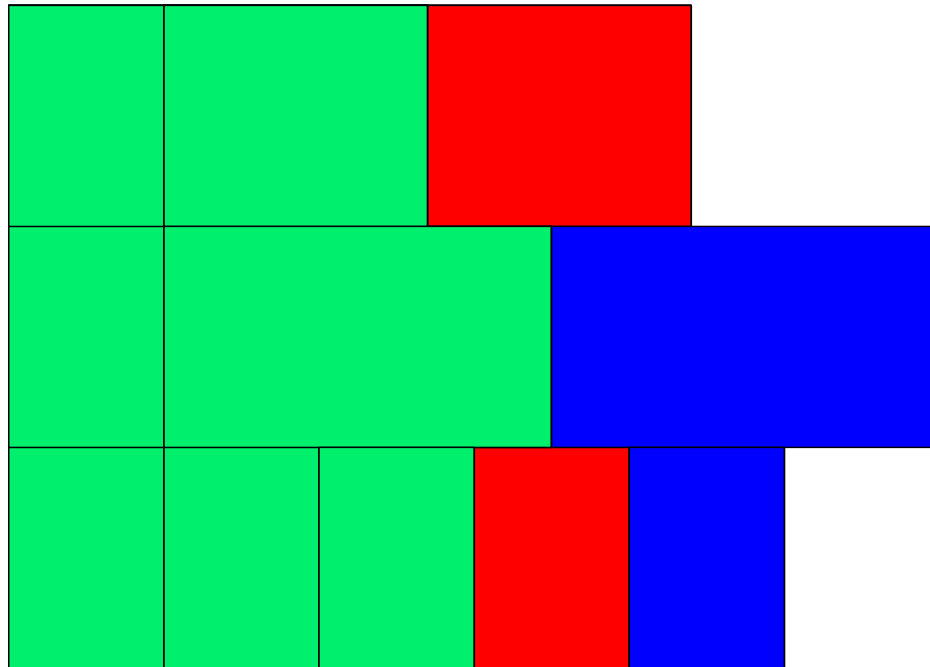
Action: Covering



$t = 0.06$
 $n = 3.3333$

Multi-adaptive Time-Stepping

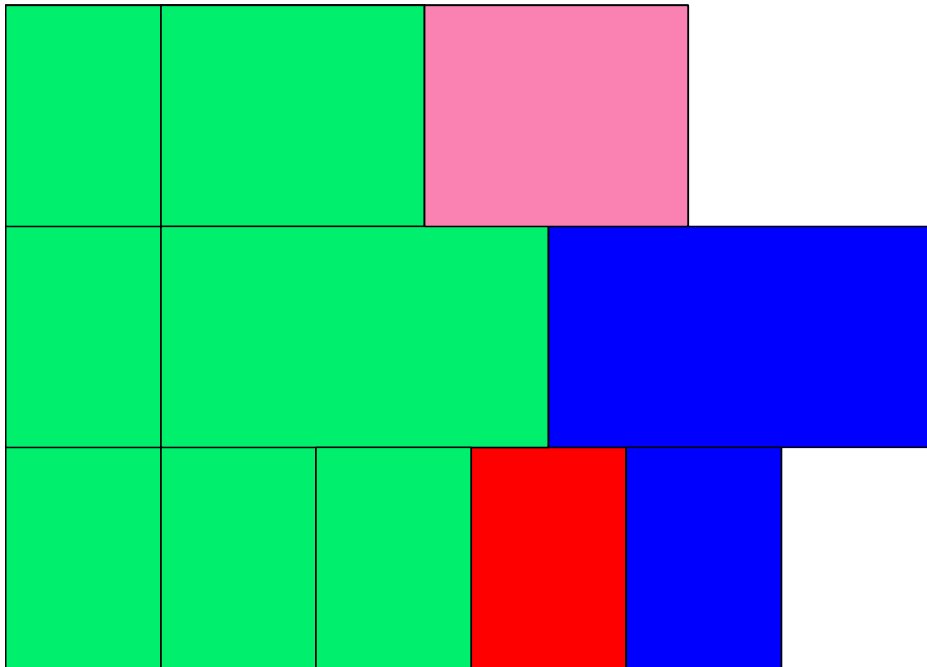
Action: Covering



$t = 0.05$
 $n = 3.3333$

Multi-adaptive Time-Stepping

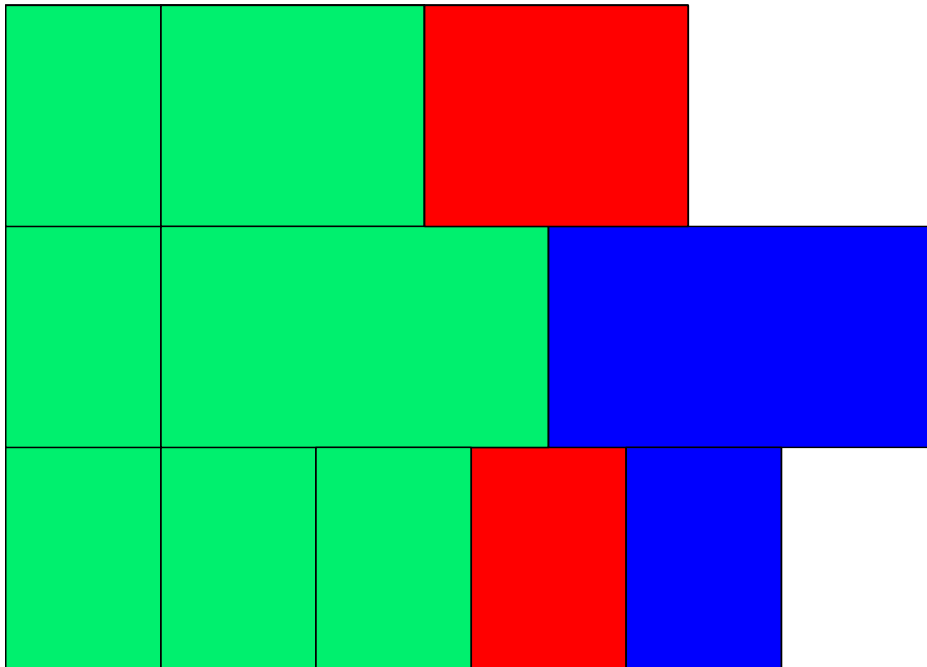
Action: Redoing steps



$t = 0.05$
 $n = 3.3333$

Multi-adaptive Time-Stepping

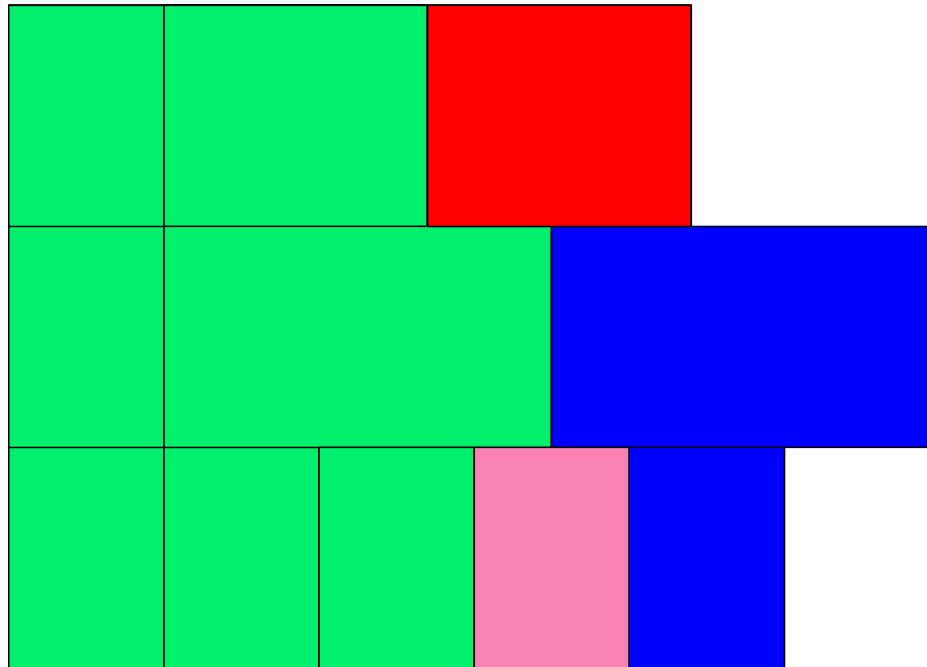
Action: Redoing steps



$t = 0.044$
 $n = 1$

Multi-adaptive Time-Stepping

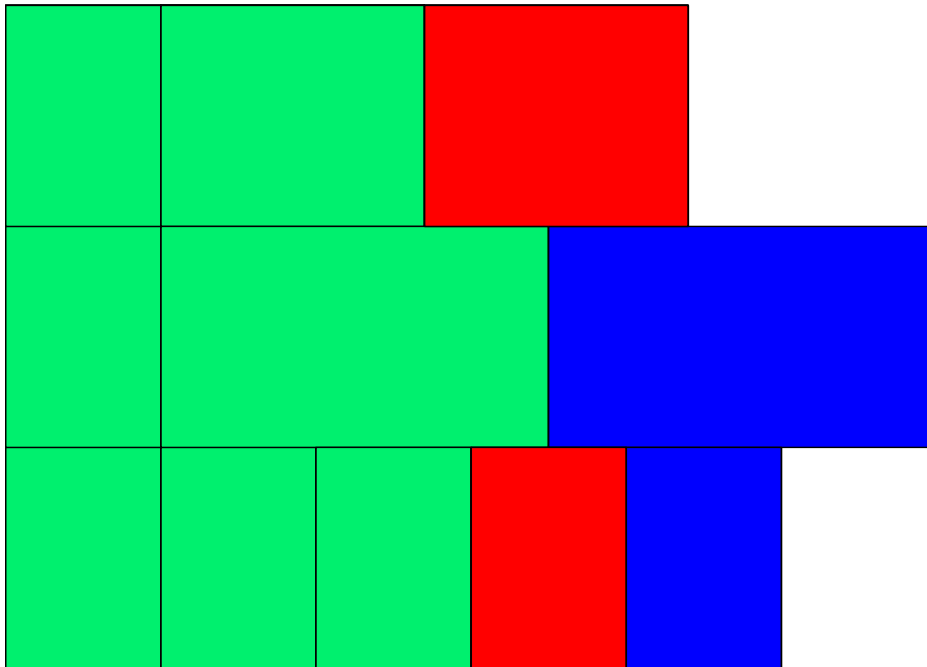
Action: Redoing steps



$t = 0.044$
 $n = 1$

Multi-adaptive Time-Stepping

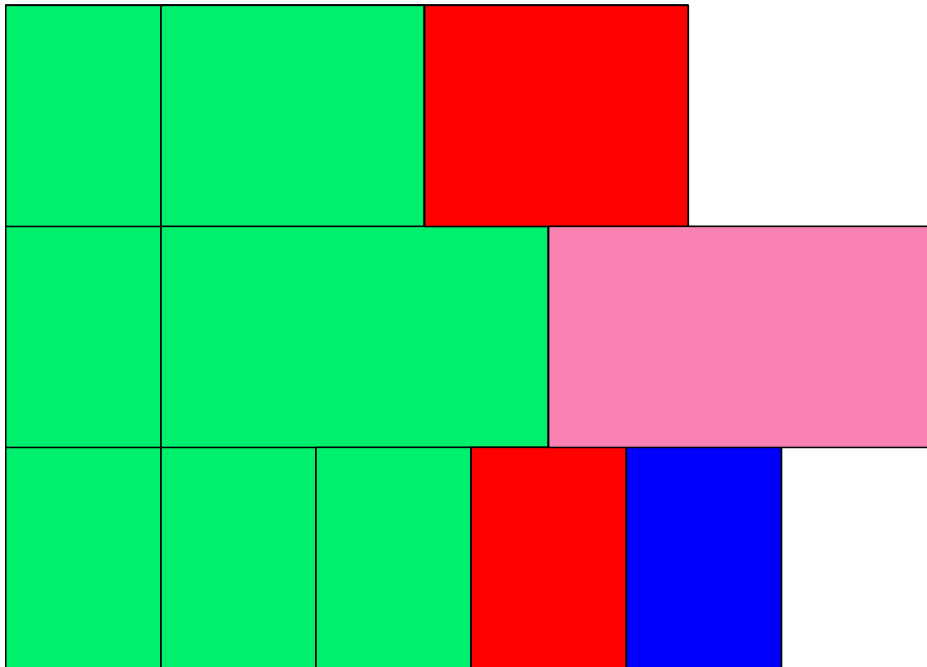
Action: Redoing steps



$t = 0.04$
 $n = 1$

Multi-adaptive Time-Stepping

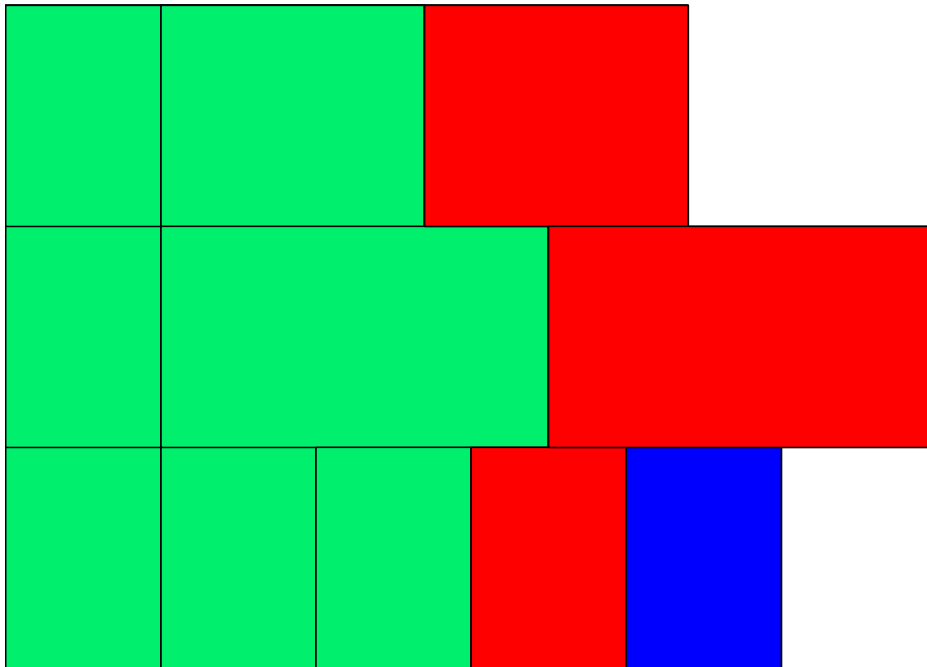
Action: Redoing steps



$t = 0.04$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.06$
 $n = 1$

Multi-adaptive Time-Stepping

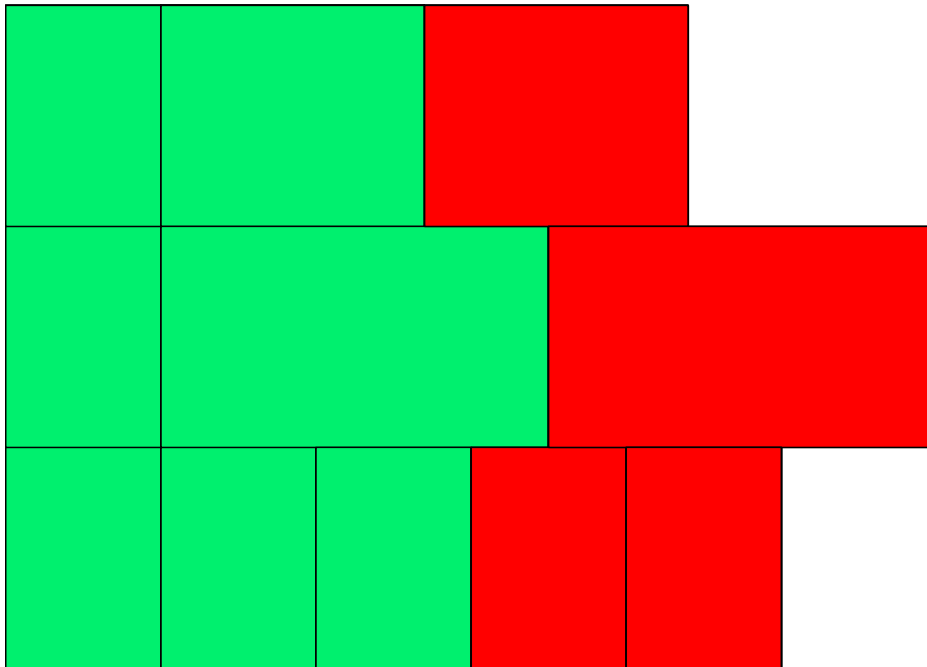
Action: Redoing steps



$t = 0.06$
 $n = 1$

Multi-adaptive Time-Stepping

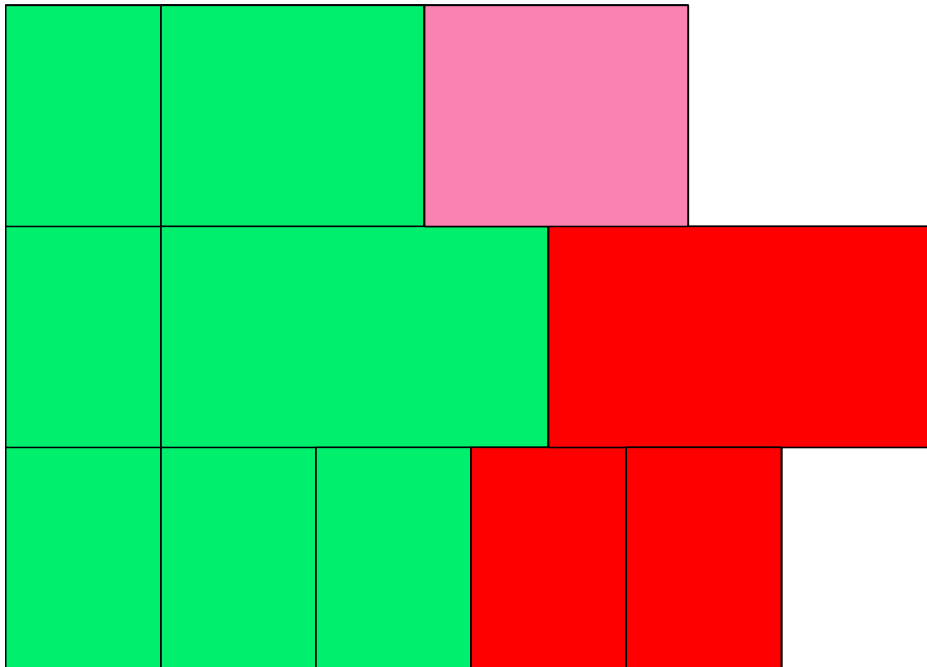
Action: Redoing steps



$t = 0.05$
 $n = 1$

Multi-adaptive Time-Stepping

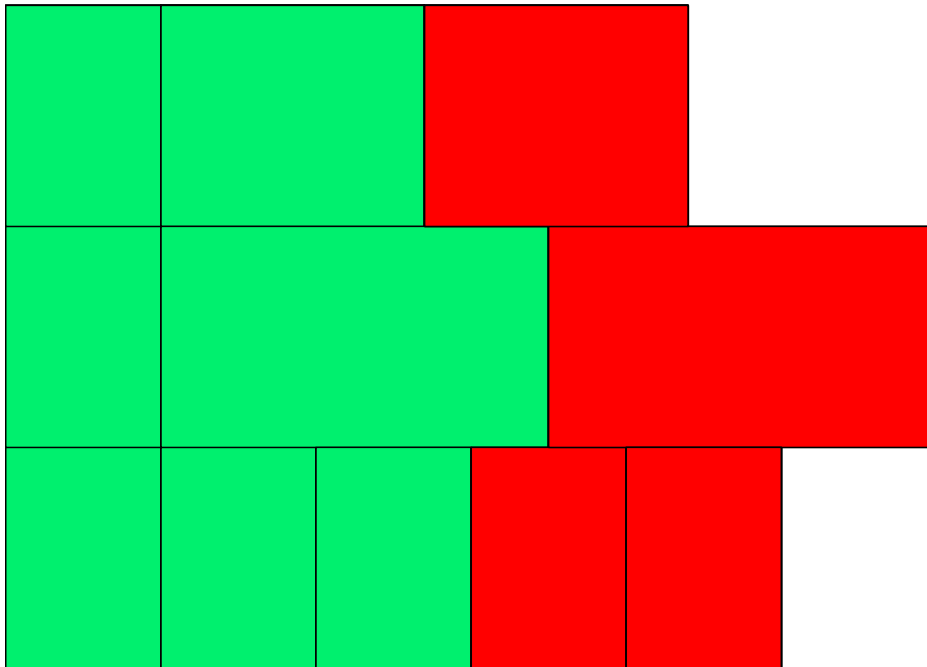
Action: Redoing steps



$t = 0.05$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.044$
 $n = 2$

Multi-adaptive Time-Stepping

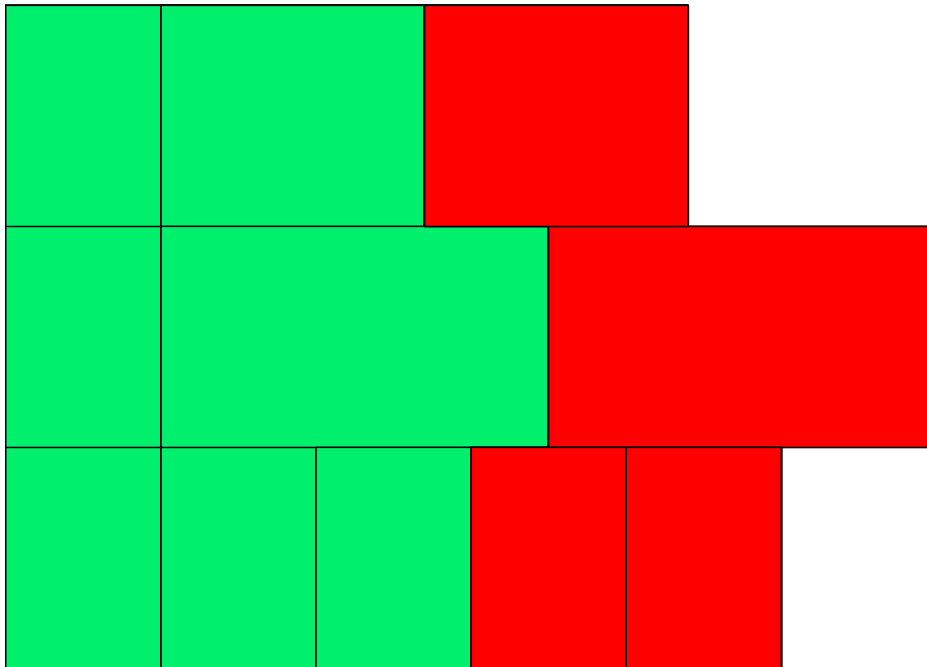
Action: Redoing steps



$t = 0.044$
 $n = 2$

Multi-adaptive Time-Stepping

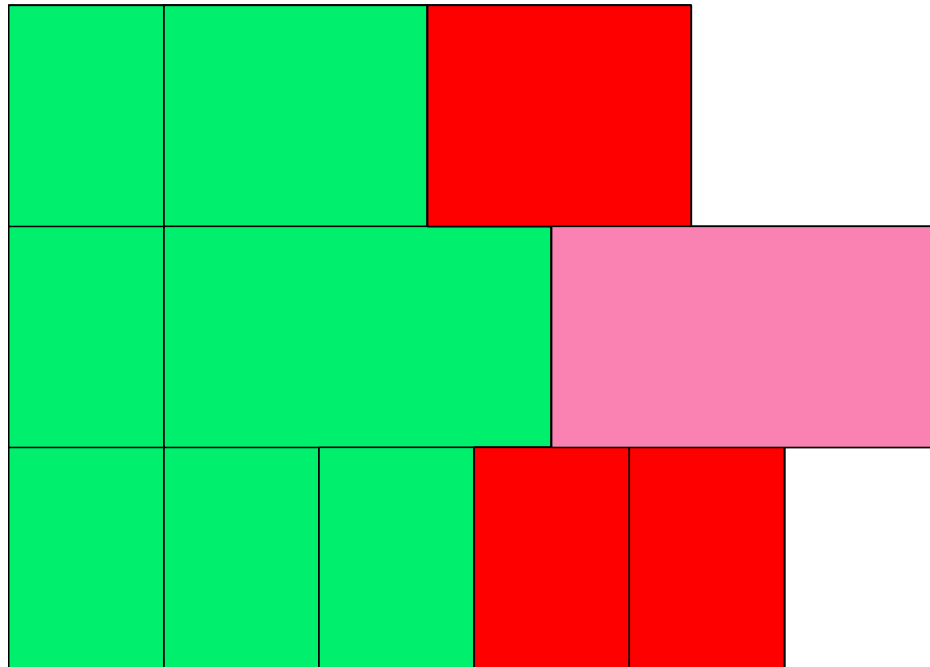
Action: Redoing steps



$t = 0.04$
 $n = 2$

Multi-adaptive Time-Stepping

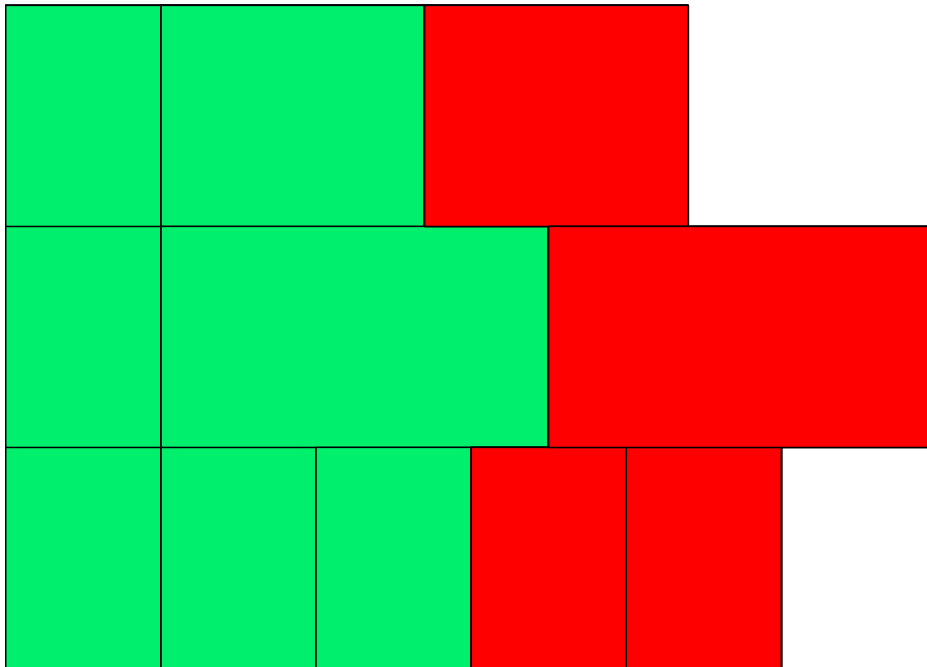
Action: Redoing steps



$t = 0.04$
 $n = 2$

Multi-adaptive Time-Stepping

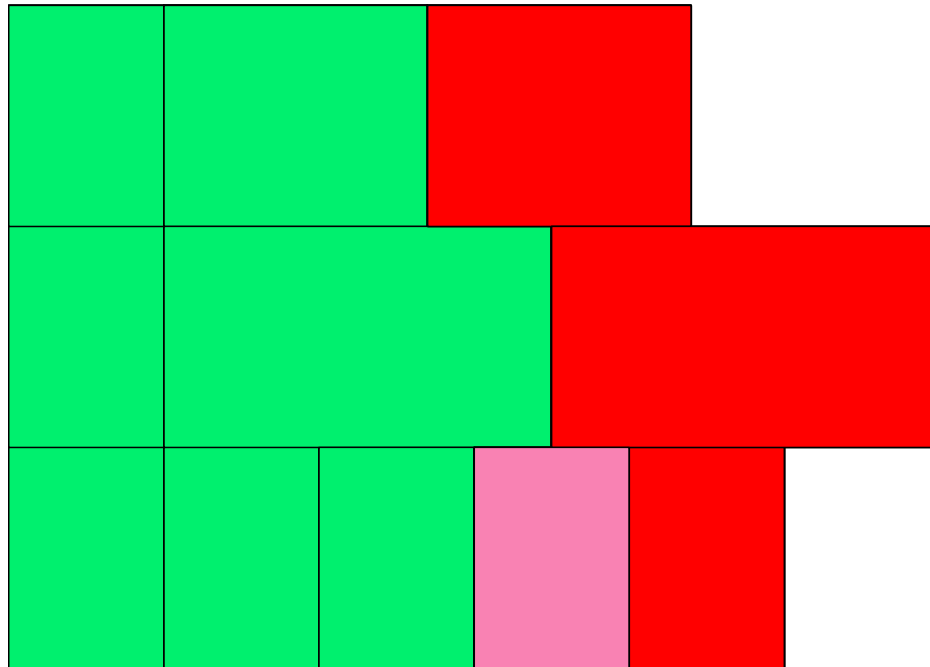
Action: Redoing steps



$t = 0.06$
 $n = 2$

Multi-adaptive Time-Stepping

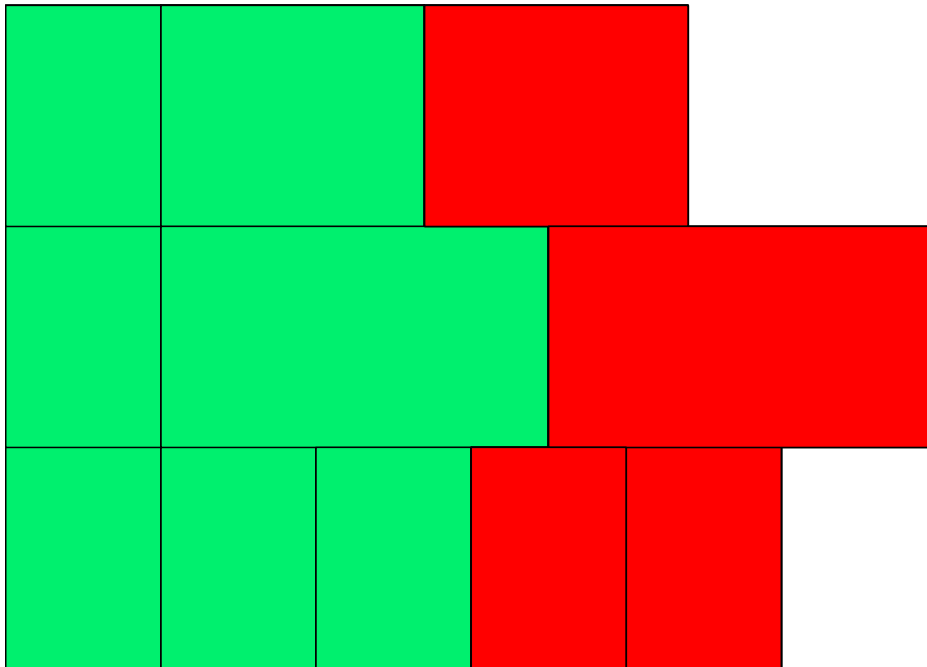
Action: Redoing steps



$t = 0.06$
 $n = 2$

Multi-adaptive Time-Stepping

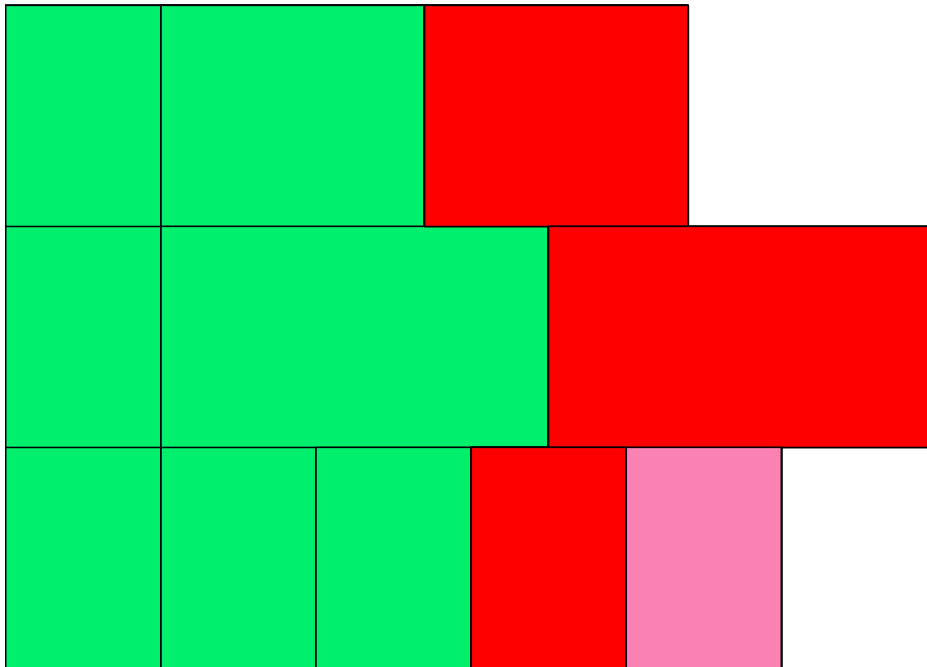
Action: Redoing steps



$t = 0.04$
 $n = 2.75$

Multi-adaptive Time-Stepping

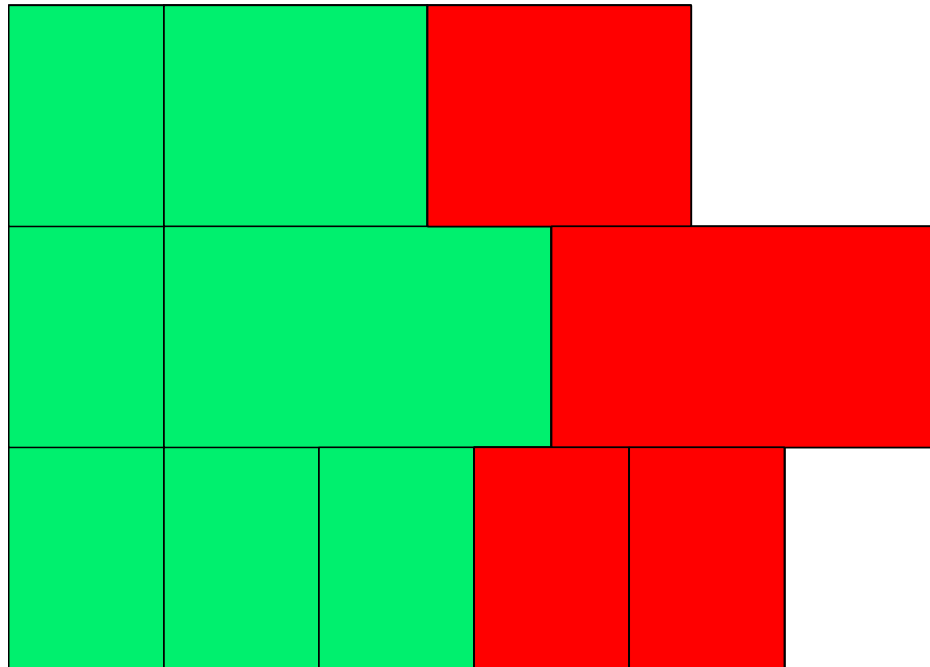
Action: Redoing steps



$t = 0.04$
 $n = 2.75$

Multi-adaptive Time-Stepping

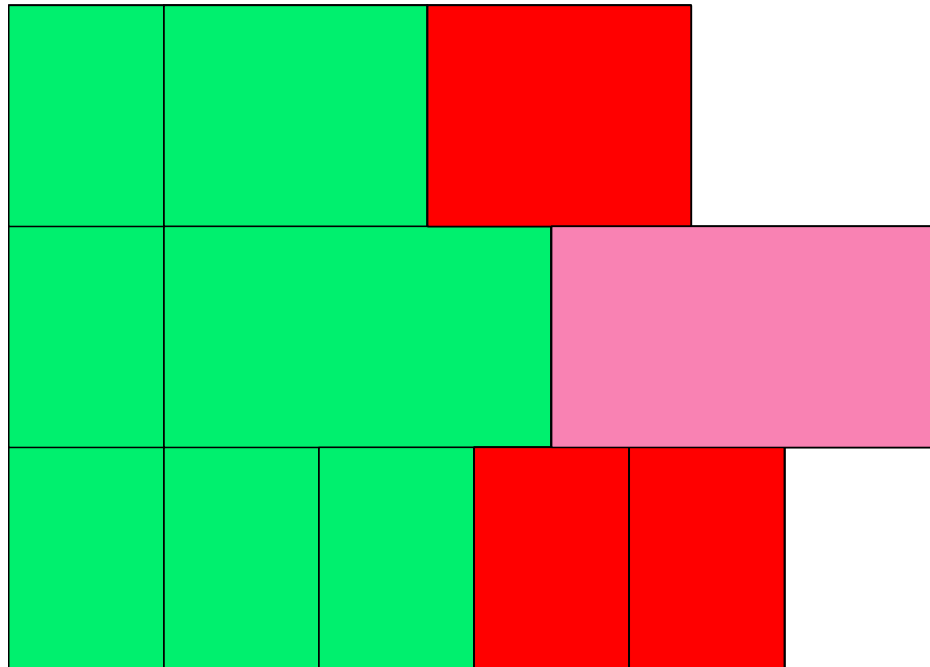
Action: Redoing steps



$t = 0.05$
 $n = 3$

Multi-adaptive Time-Stepping

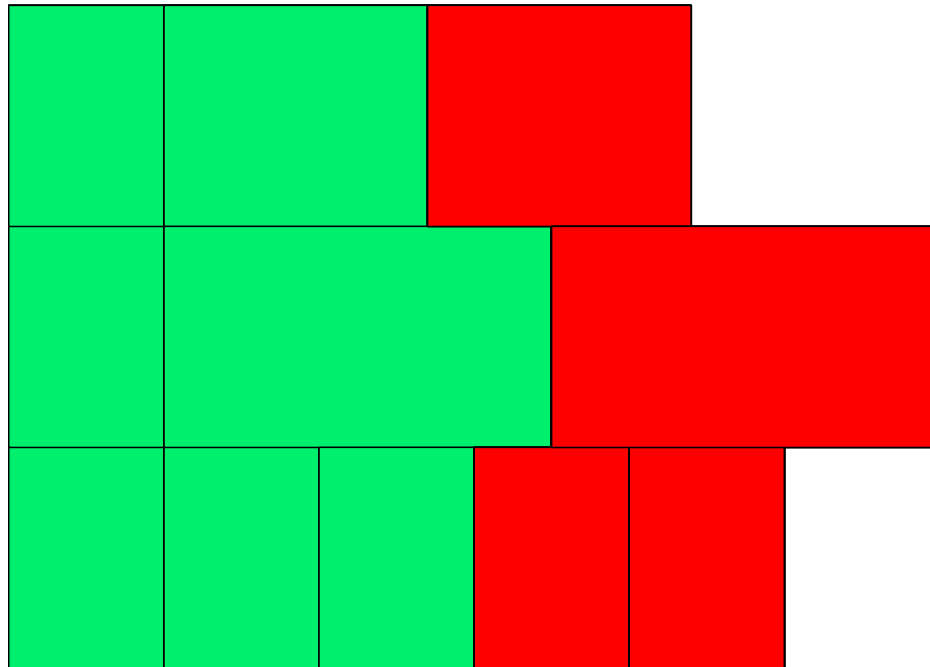
Action: Redoing steps



$t = 0.05$
 $n = 3$

Multi-adaptive Time-Stepping

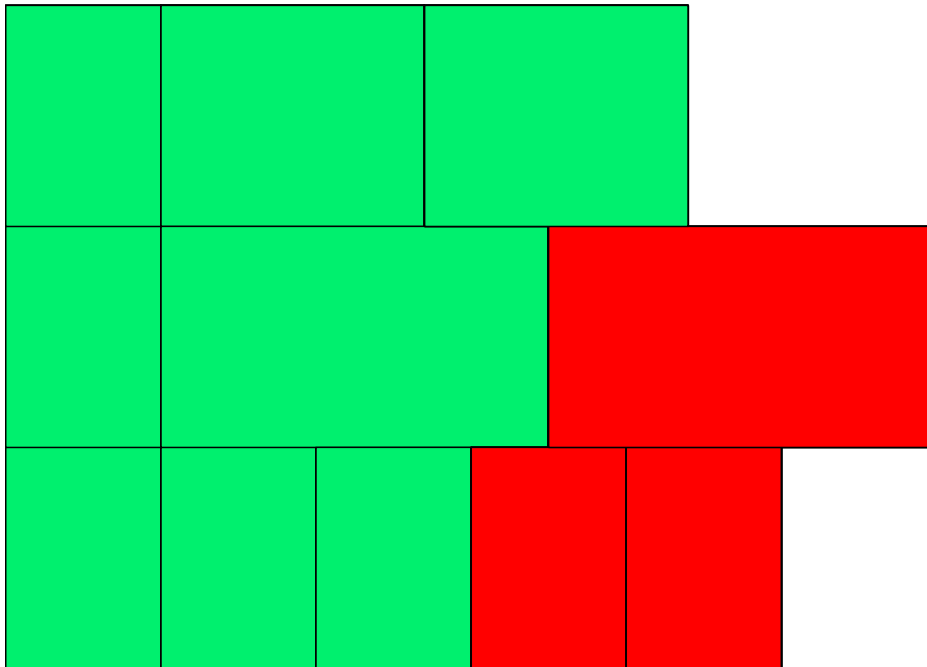
Action: Redoing steps



$t = 0.06$
 $n = 3.25$

Multi-adaptive Time-Stepping

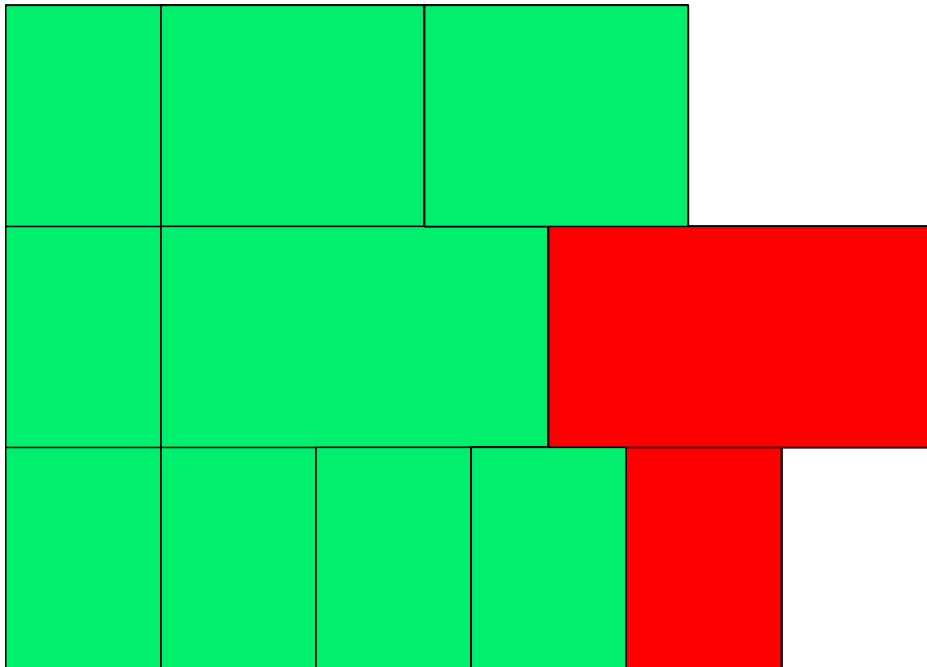
Action: Clearing steps



$t = 0.044$
 $n = 3.5$

Multi-adaptive Time-Stepping

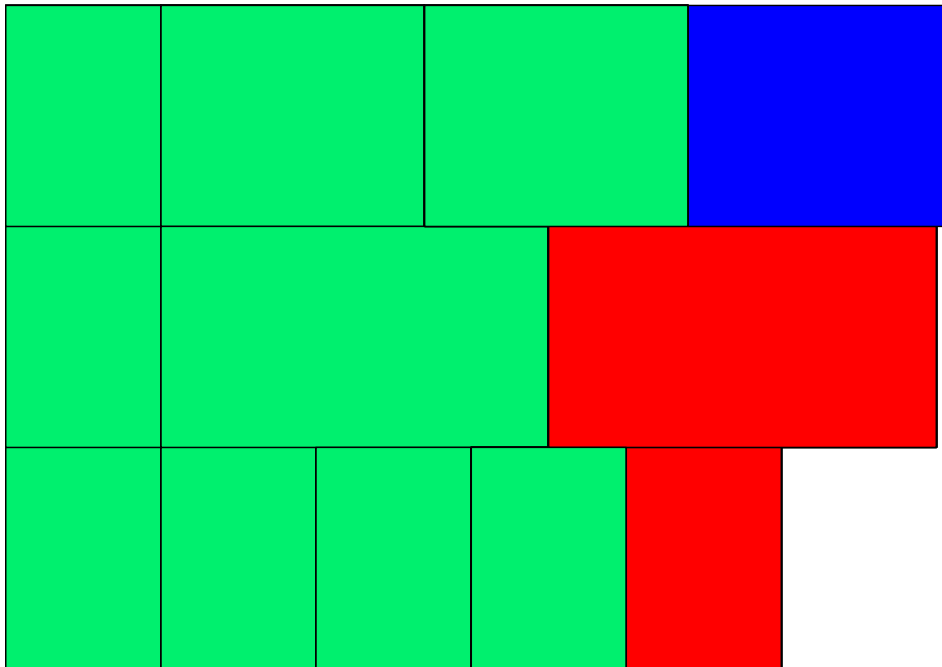
Action: Clearing steps



$t = 0.04$
 $n = 3.5$

Multi-adaptive Time-Stepping

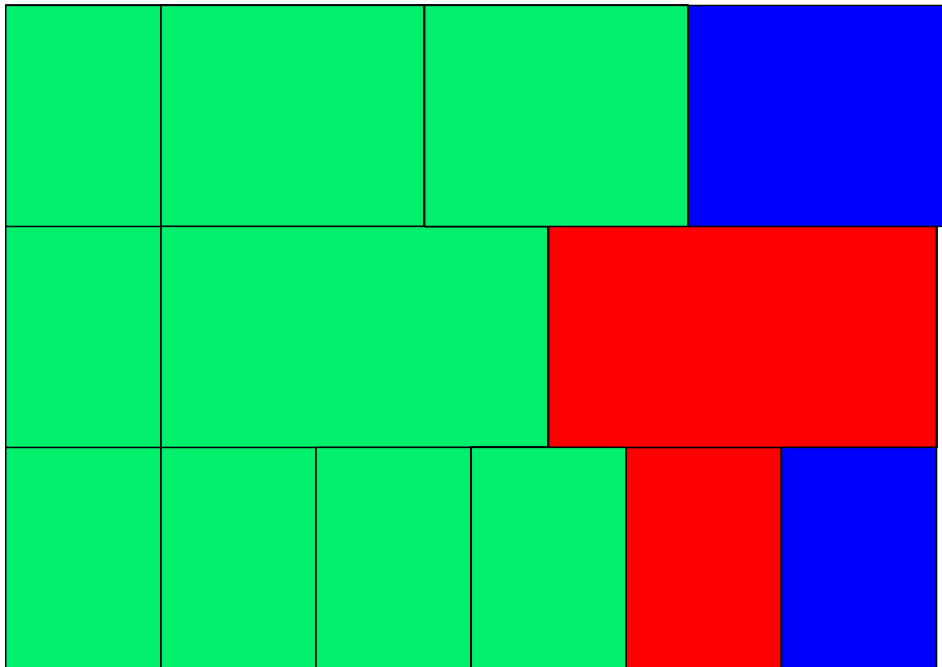
Action: Covering



$t = 0.061$
 $n = 3.5$

Multi-adaptive Time-Stepping

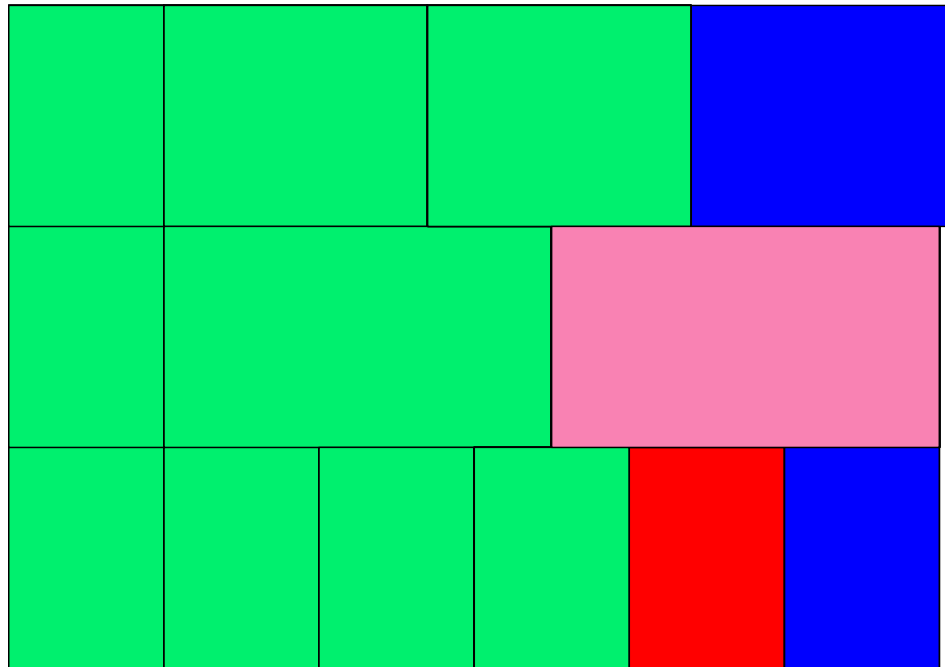
Action: Covering



$t = 0.06$
 $n = 3.5$

Multi-adaptive Time-Stepping

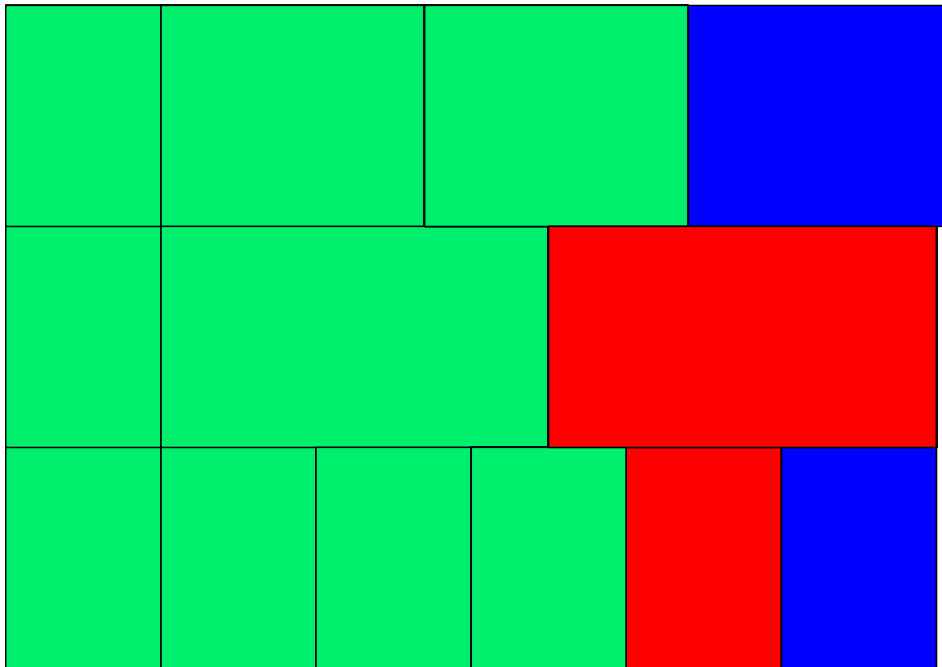
Action: Redoing steps



$t = 0.06$
 $n = 3.5$

Multi-adaptive Time-Stepping

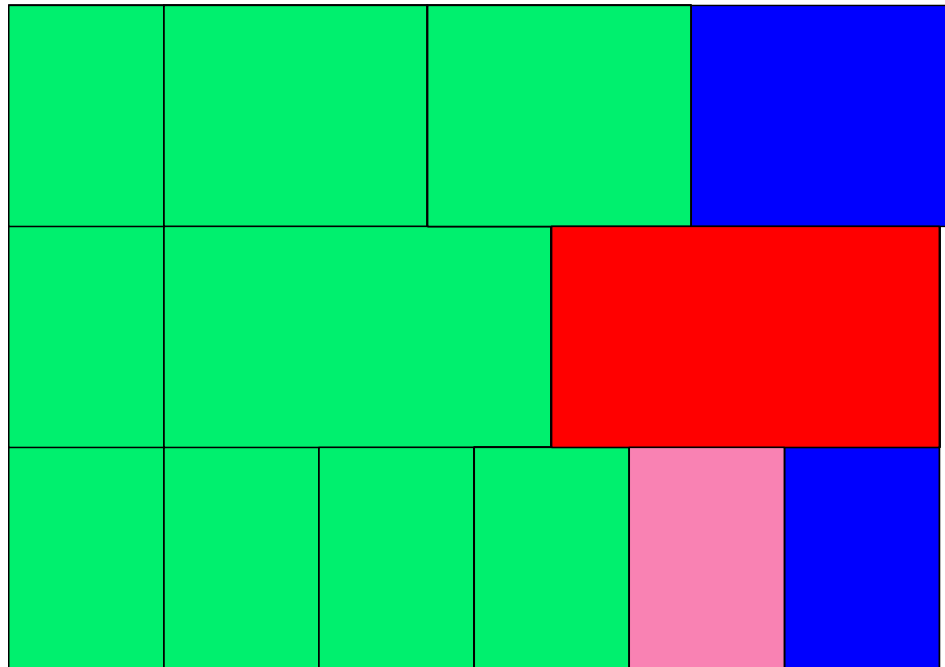
Action: Redoing steps



$t = 0.06$
 $n = 1$

Multi-adaptive Time-Stepping

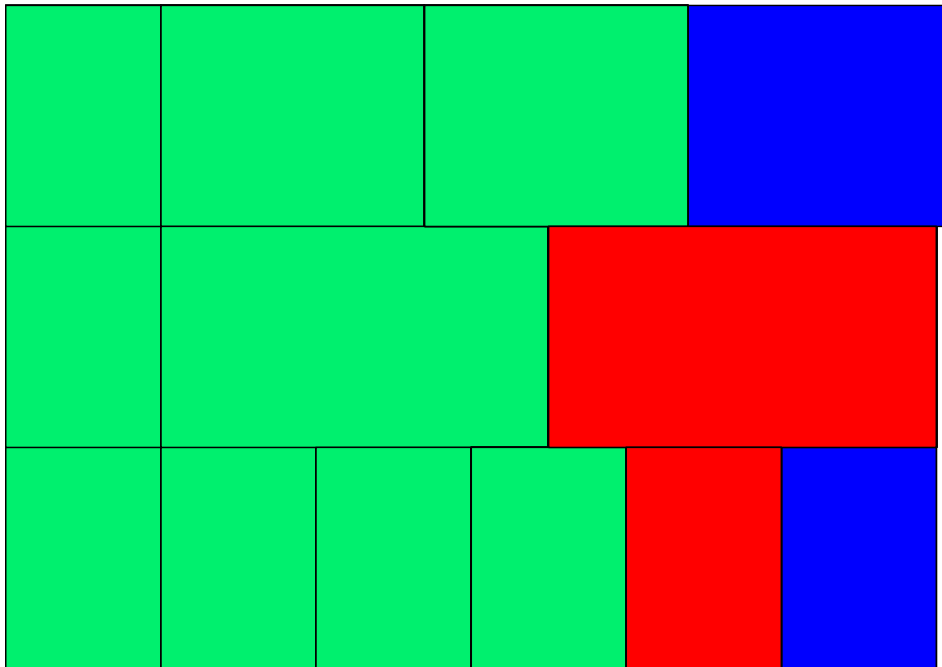
Action: Redoing steps



$t = 0.06$
 $n = 1$

Multi-adaptive Time-Stepping

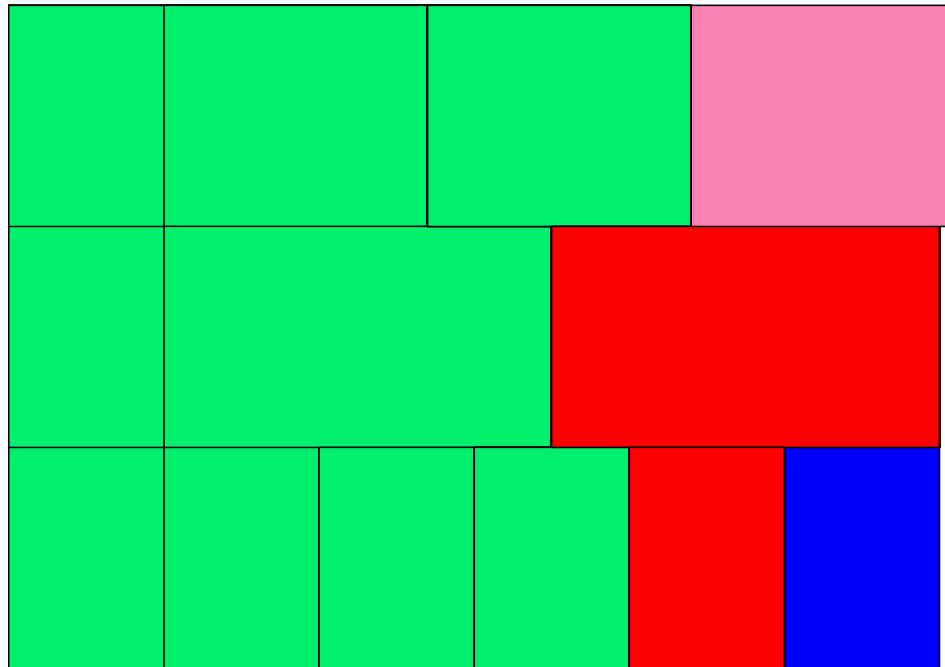
Action: Redoing steps



$t = 0.05$
 $n = 1$

Multi-adaptive Time-Stepping

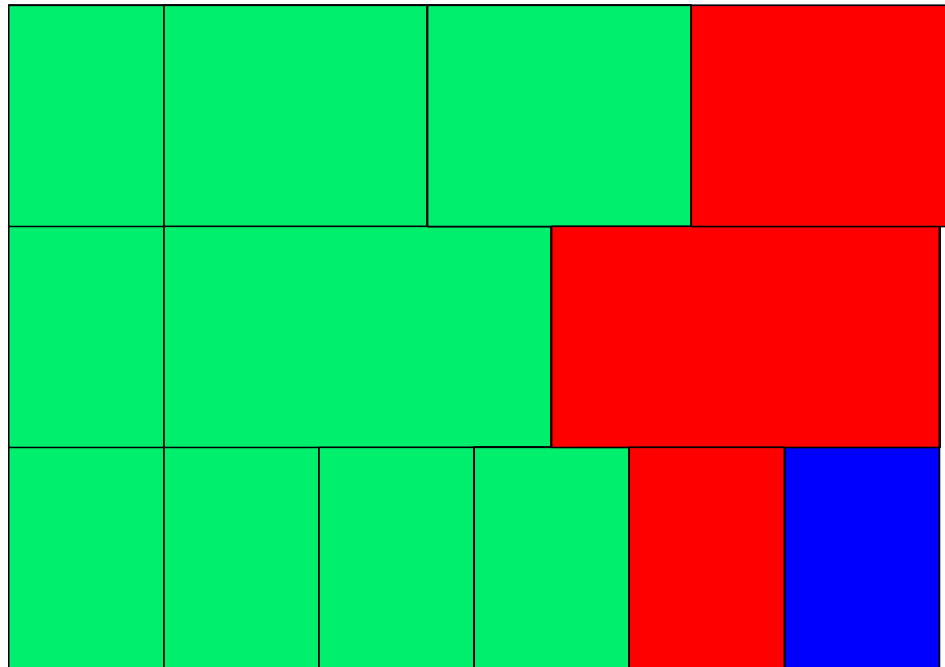
Action: Redoing steps



$t = 0.05$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.061$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.061$
 $n = 1$

Multi-adaptive Time-Stepping

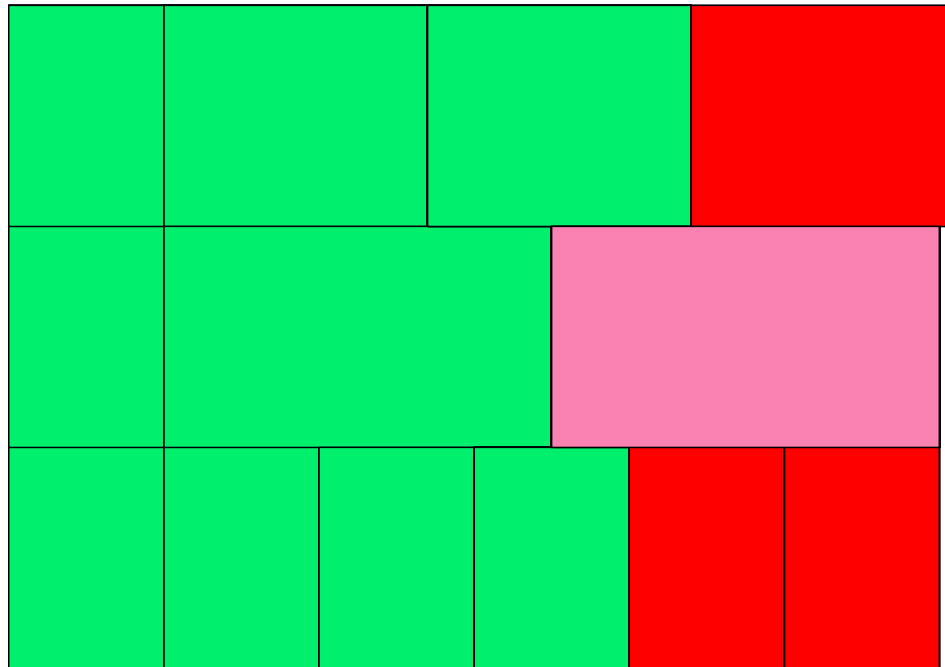
Action: Redoing steps



$t = 0.06$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.06$
 $n = 1$

Multi-adaptive Time-Stepping

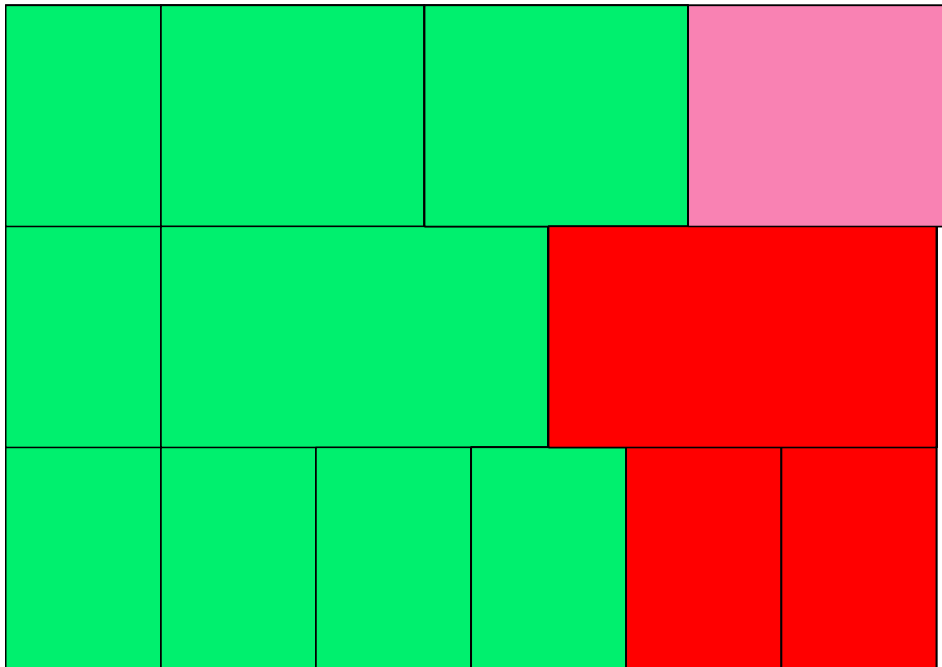
Action: Redoing steps



$t = 0.06$
 $n = 2$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.06$
 $n = 2$

Multi-adaptive Time-Stepping

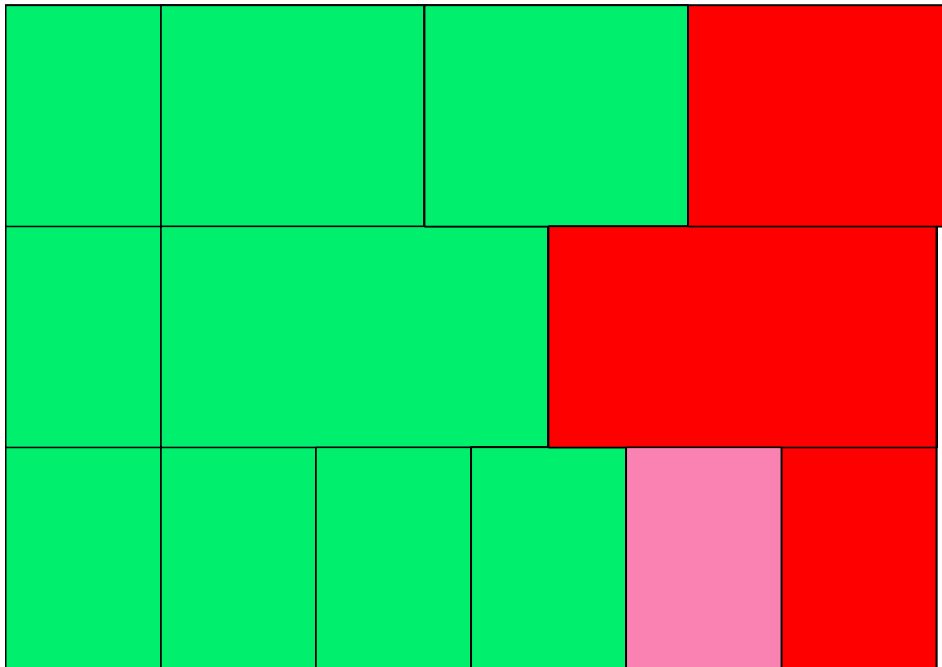
Action: Redoing steps



$t = 0.061$
 $n = 2$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.061$
 $n = 2$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.05$
 $n = 2.5$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.05$
 $n = 2.5$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.06$
 $n = 2.5$

Multi-adaptive Time-Stepping

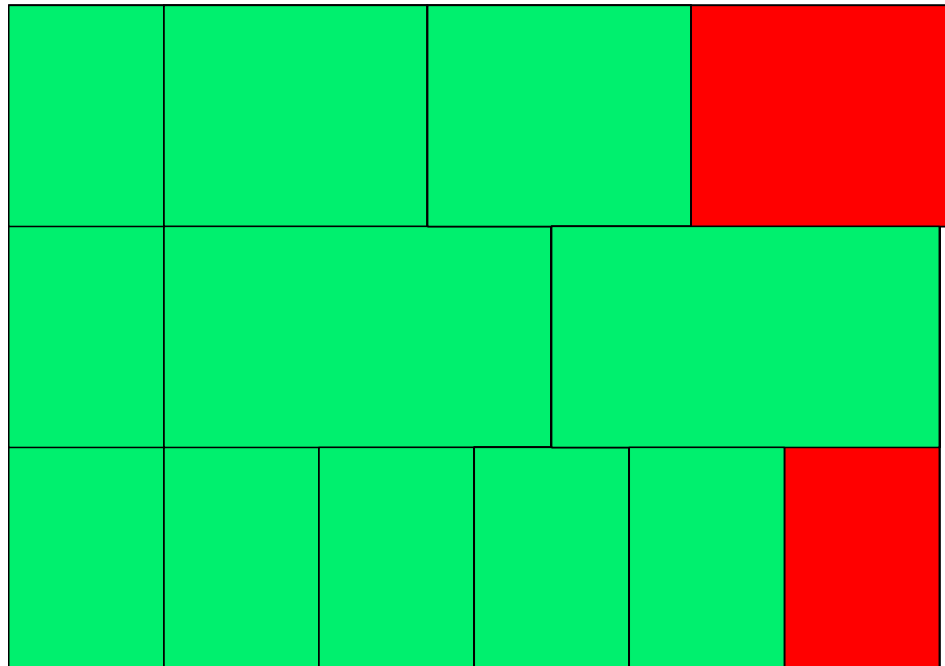
Action: Clearing steps



$t = 0.06$
 $n = 3$

Multi-adaptive Time-Stepping

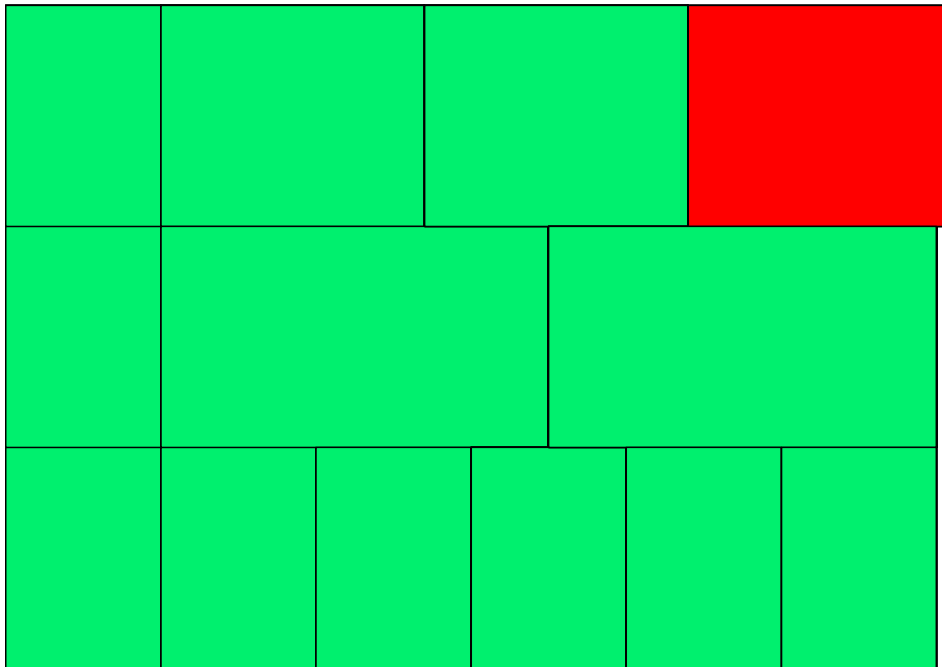
Action: Clearing steps



$t = 0.05$
 $n = 3$

Multi-adaptive Time-Stepping

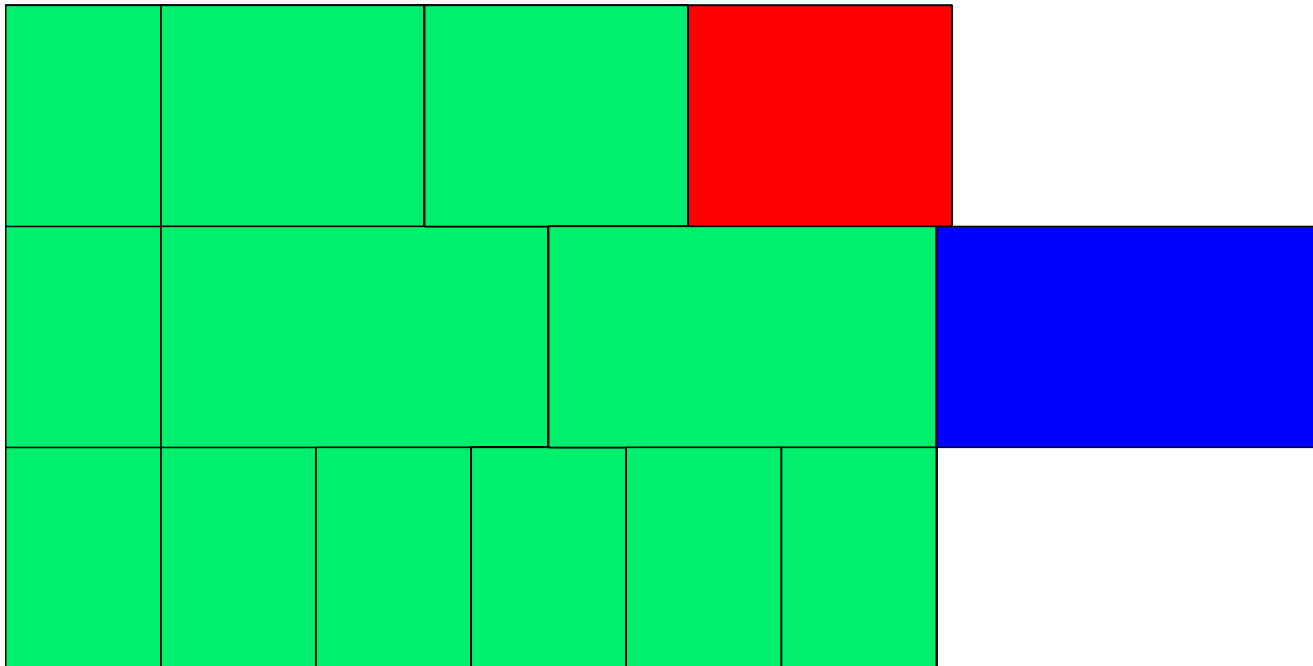
Action: Clearing steps



$t = 0.06$
 $n = 3$

Multi-adaptive Time-Stepping

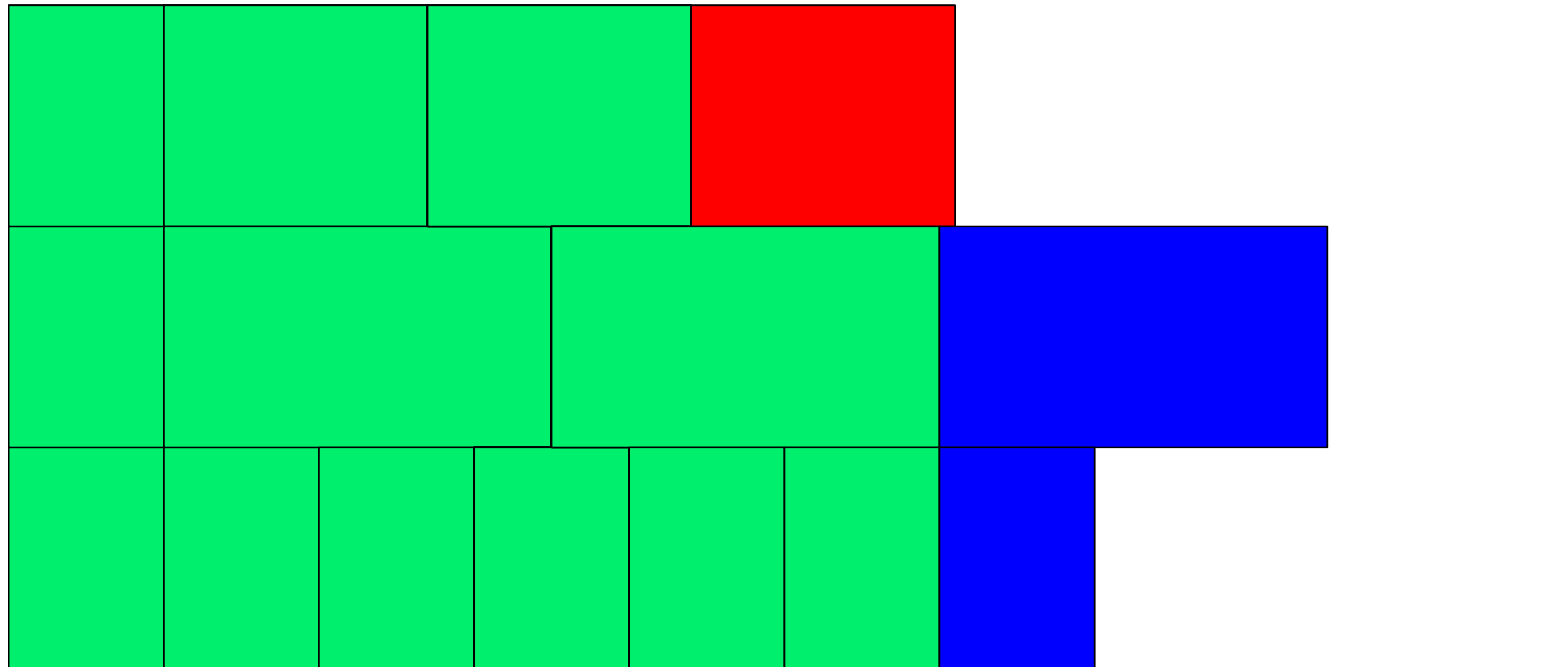
Action: Covering



$t = 0.085$
 $n = 3$

Multi-adaptive Time-Stepping

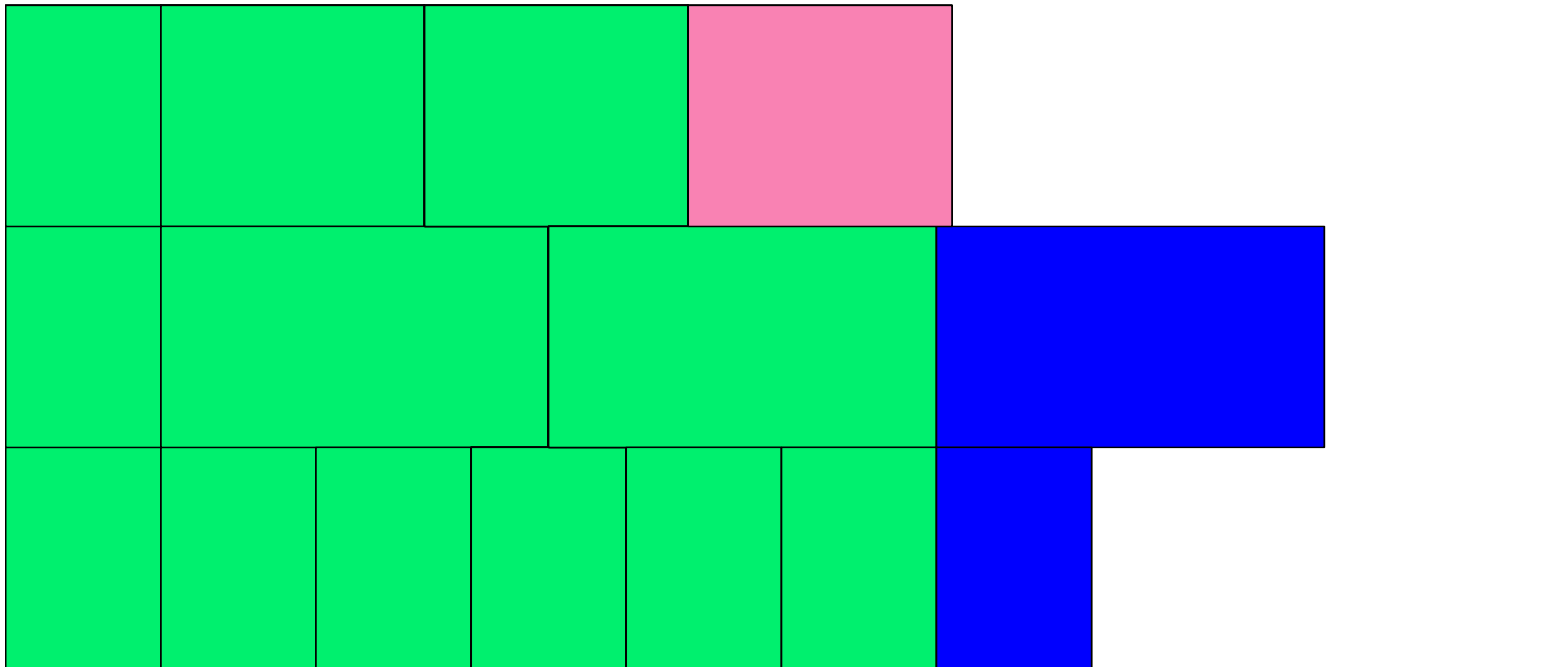
Action: Covering



$t = 0.07$
 $n = 3$

Multi-adaptive Time-Stepping

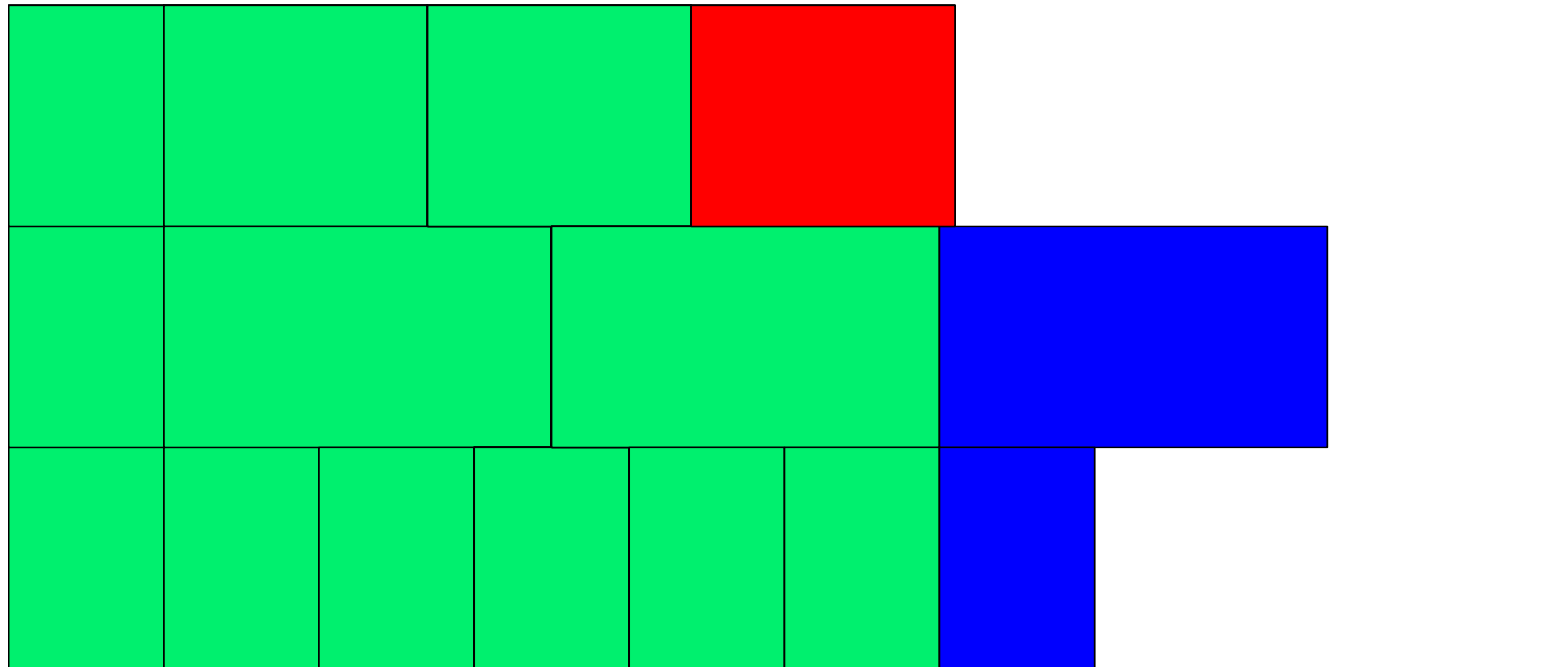
Action: Redoing steps



$t = 0.07$
 $n = 3$

Multi-adaptive Time-Stepping

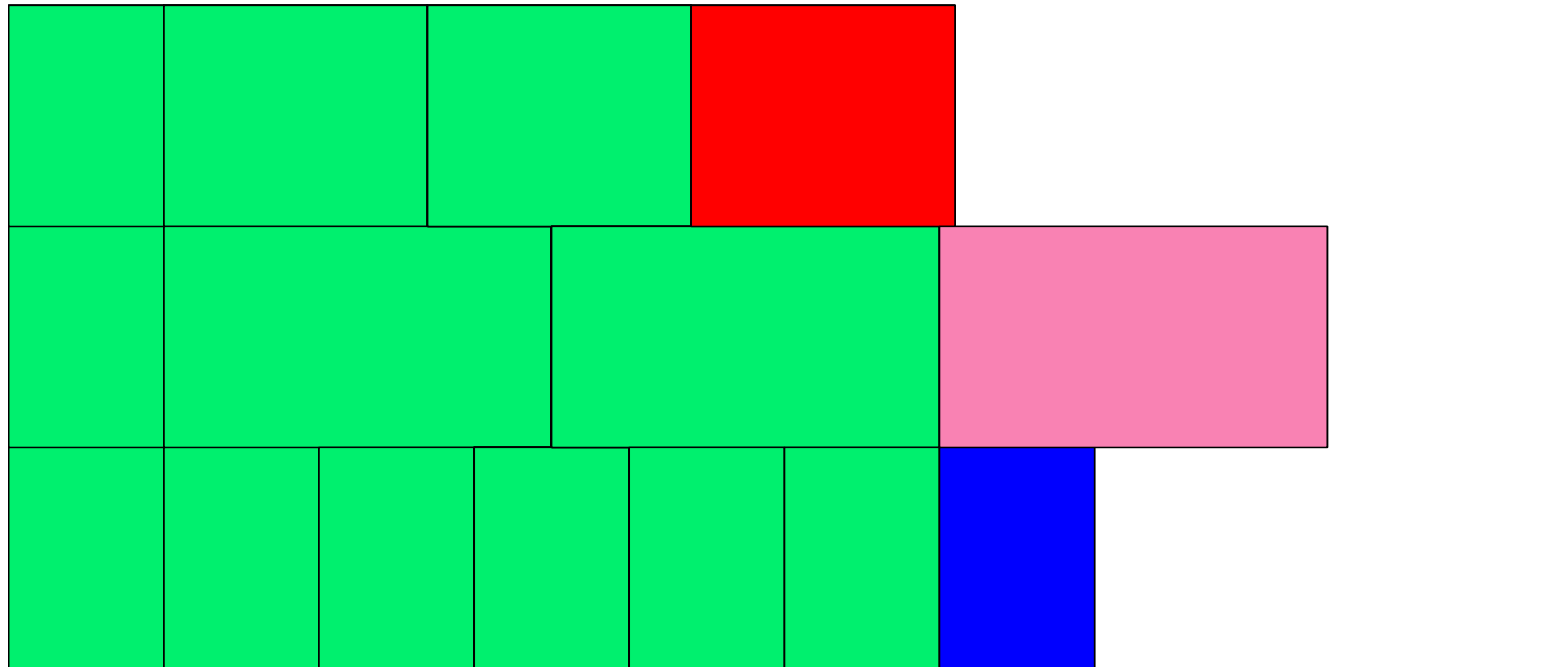
Action: Redoing steps



$t = 0.061$
 $n = 1$

Multi-adaptive Time-Stepping

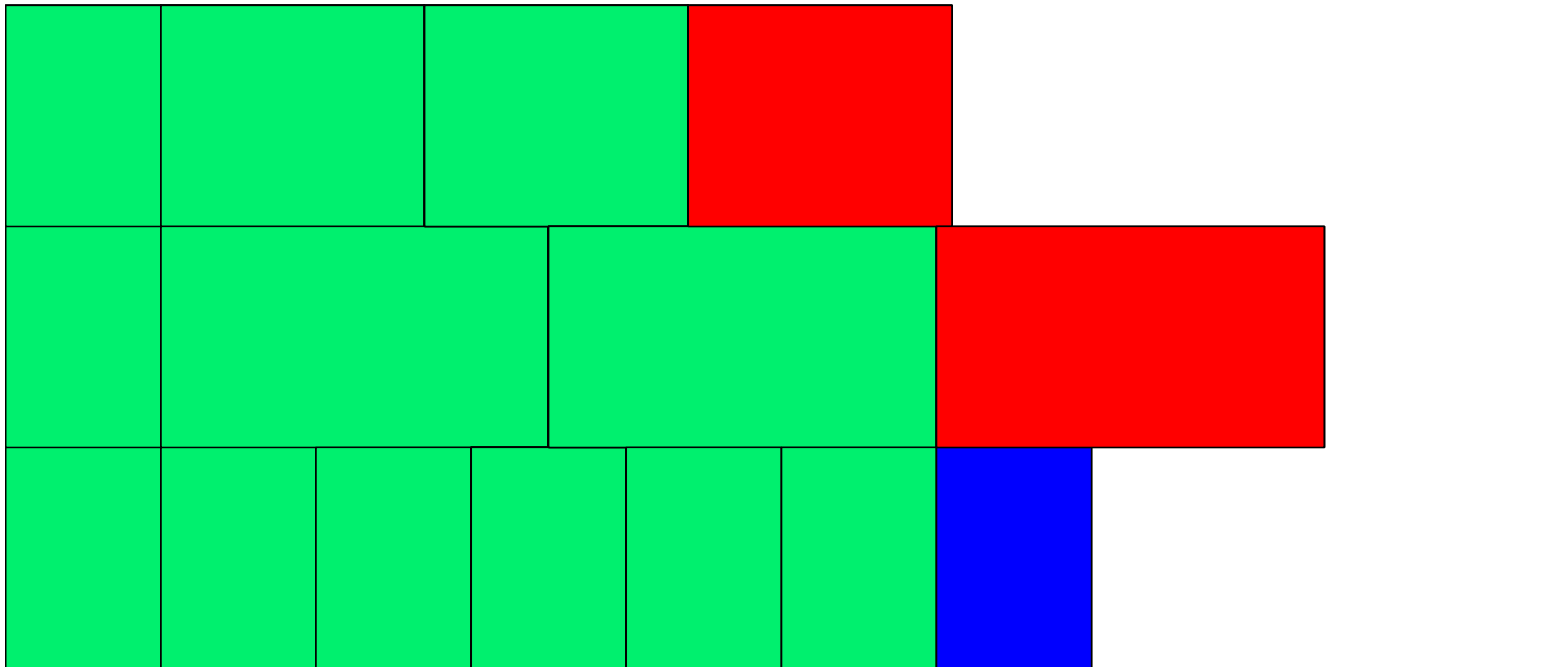
Action: Redoing steps



$t = 0.061$
 $n = 1$

Multi-adaptive Time-Stepping

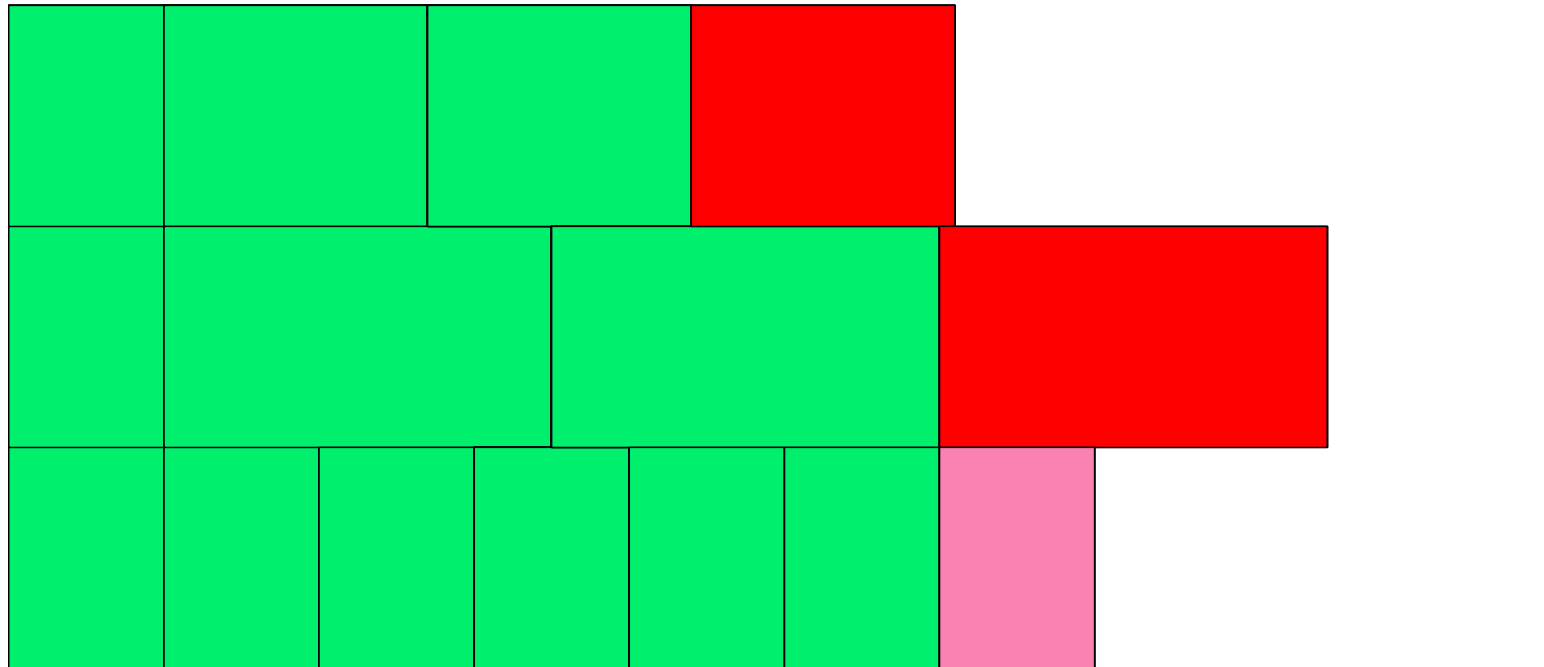
Action: Redoing steps



$t = 0.085$
 $n = 1$

Multi-adaptive Time-Stepping

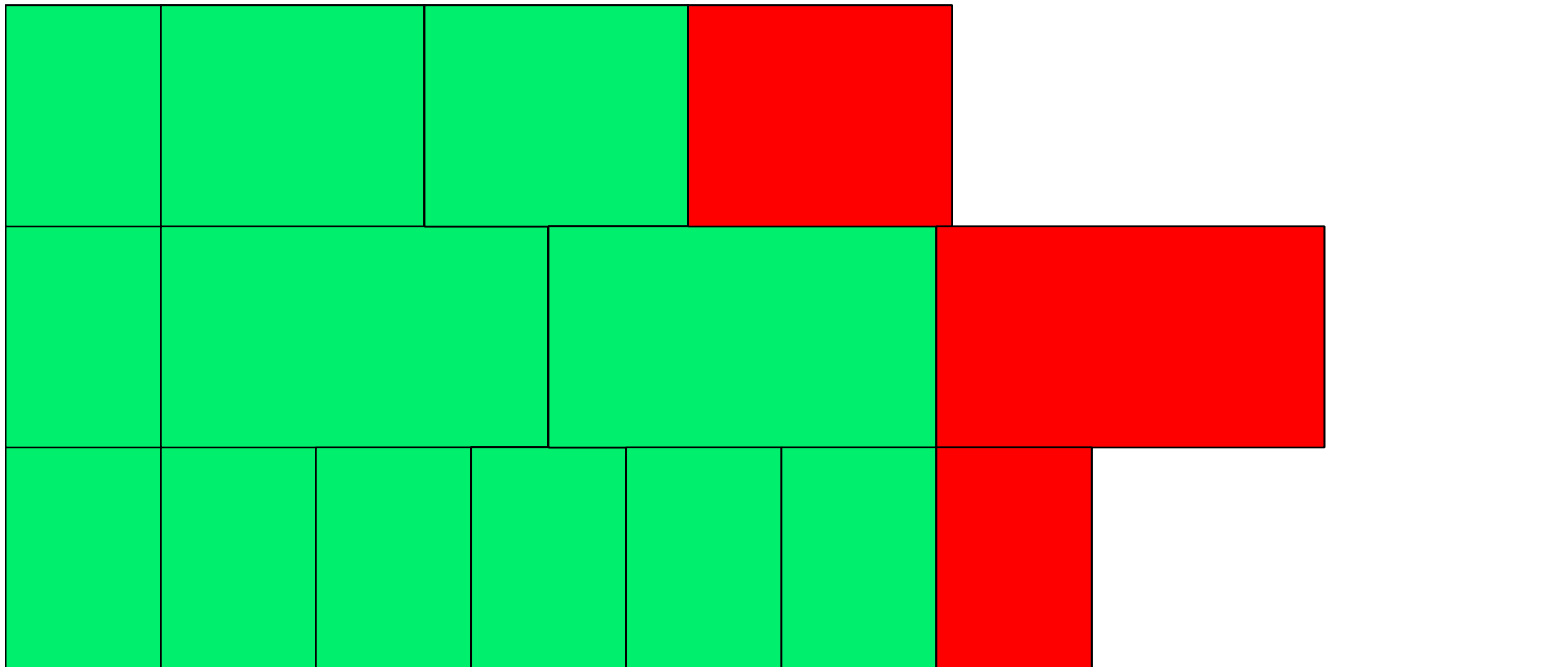
Action: Redoing steps



$t = 0.085$
 $n = 1$

Multi-adaptive Time-Stepping

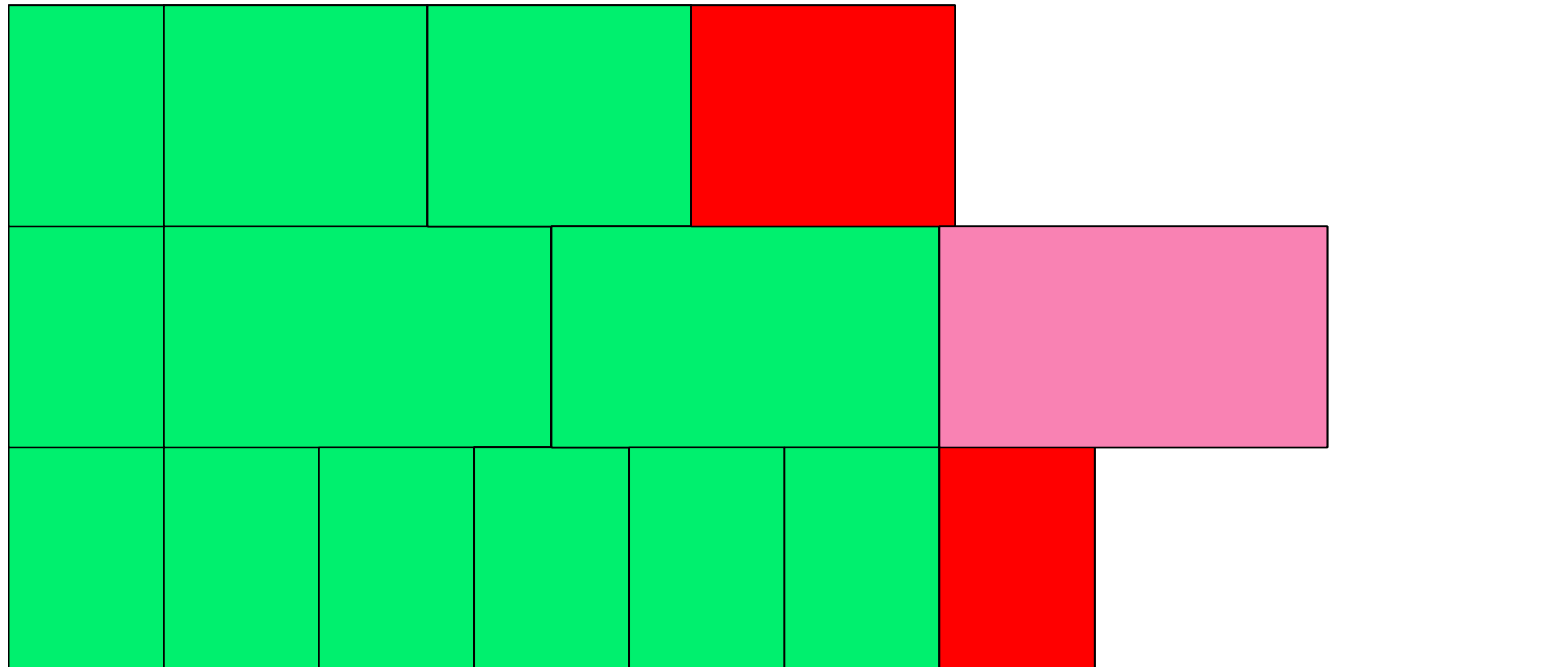
Action: Redoing steps



$t = 0.07$
 $n = 1$

Multi-adaptive Time-Stepping

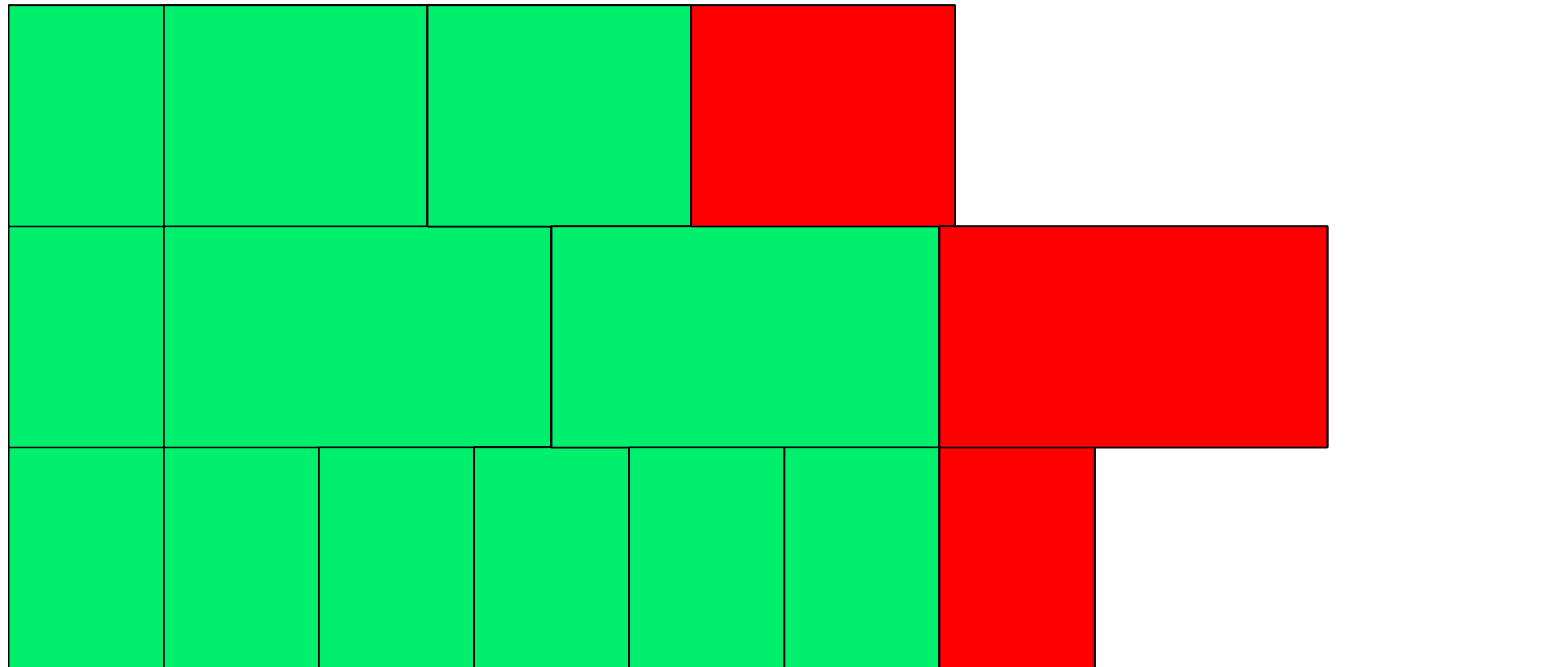
Action: Redoing steps



$t = 0.07$
 $n = 1$

Multi-adaptive Time-Stepping

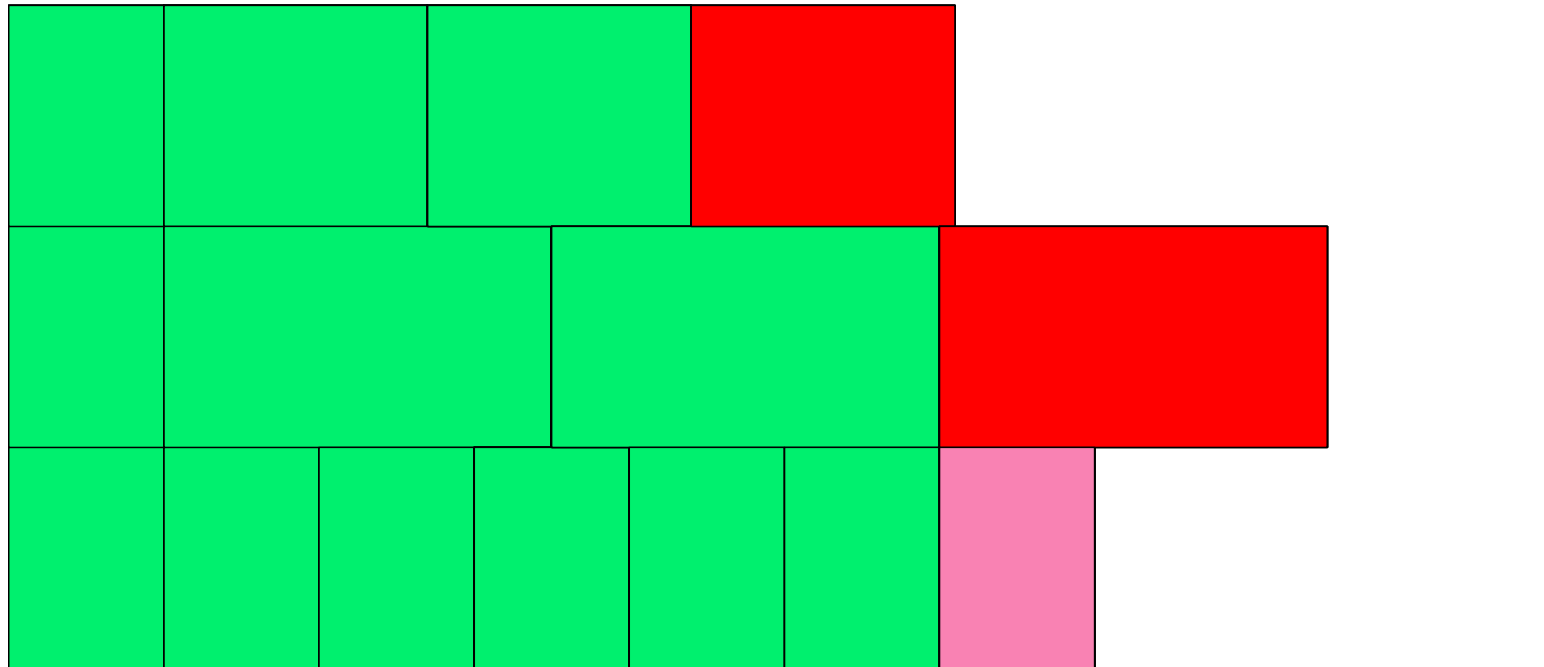
Action: Redoing steps



$t = 0.085$
 $n = 2$

Multi-adaptive Time-Stepping

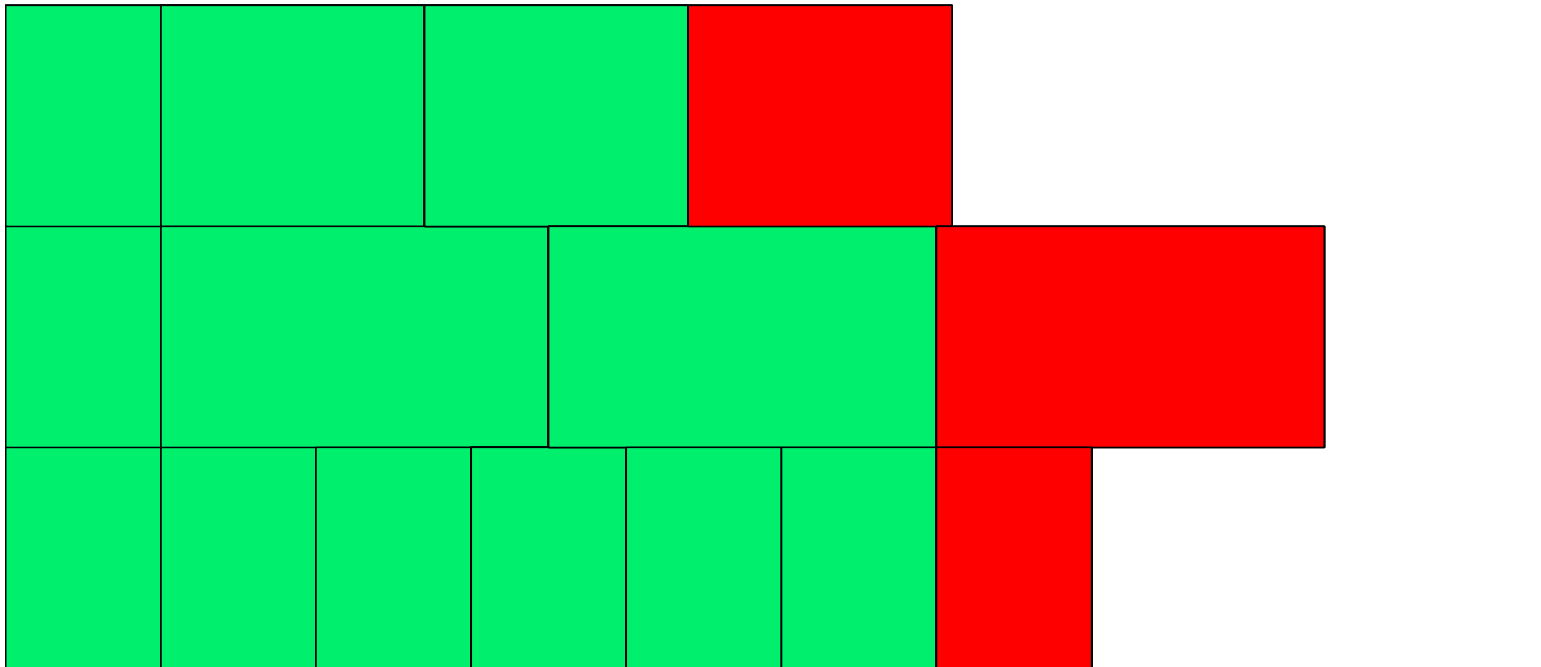
Action: Redoing steps



$t = 0.085$
 $n = 2$

Multi-adaptive Time-Stepping

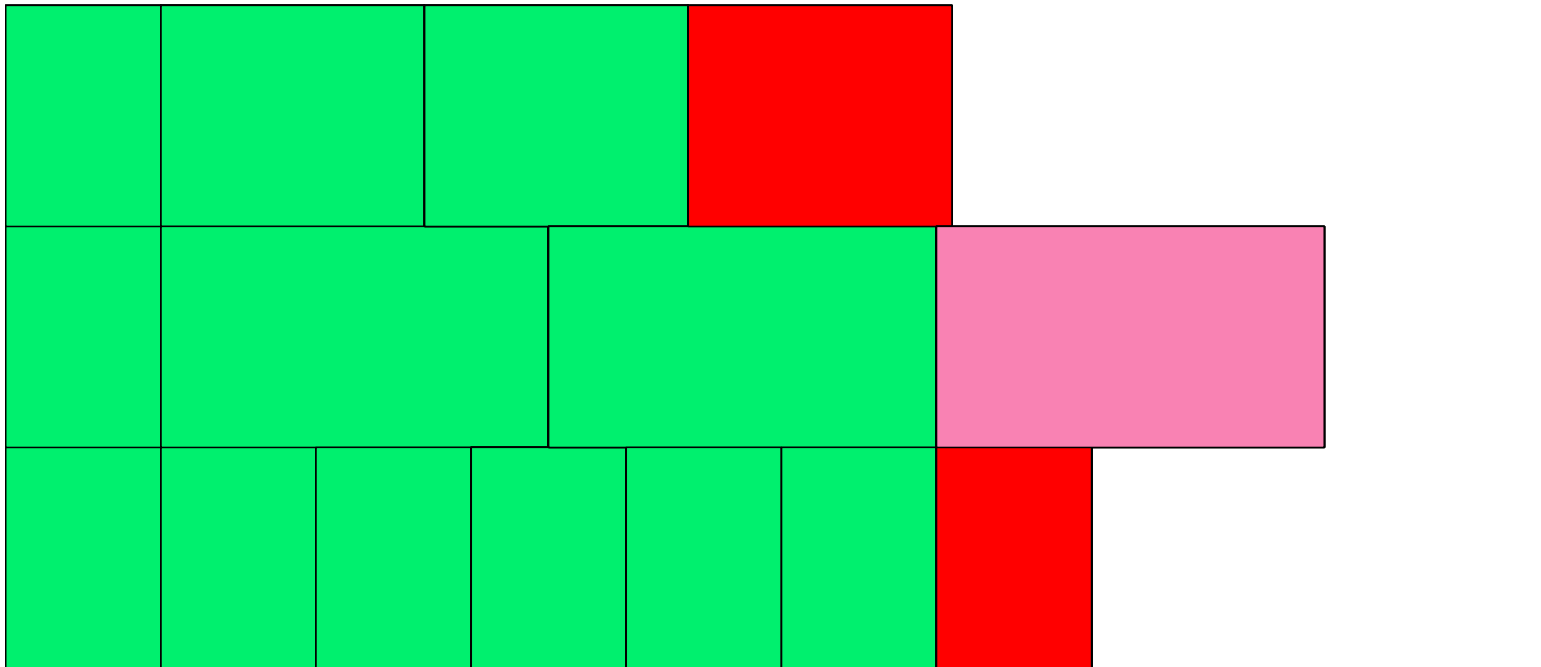
Action: Redoing steps



$t = 0.07$
 $n = 2.3333$

Multi-adaptive Time-Stepping

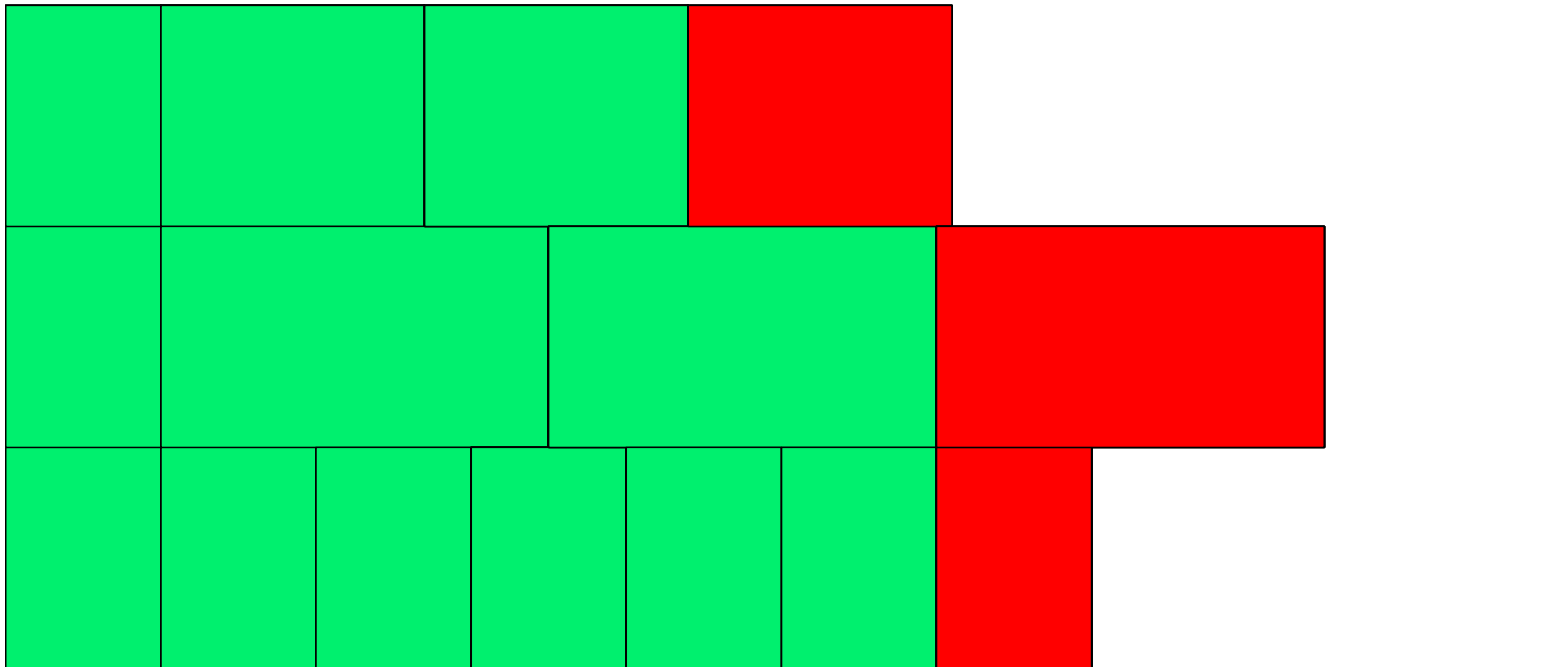
Action: Redoing steps



$t = 0.07$
 $n = 2.3333$

Multi-adaptive Time-Stepping

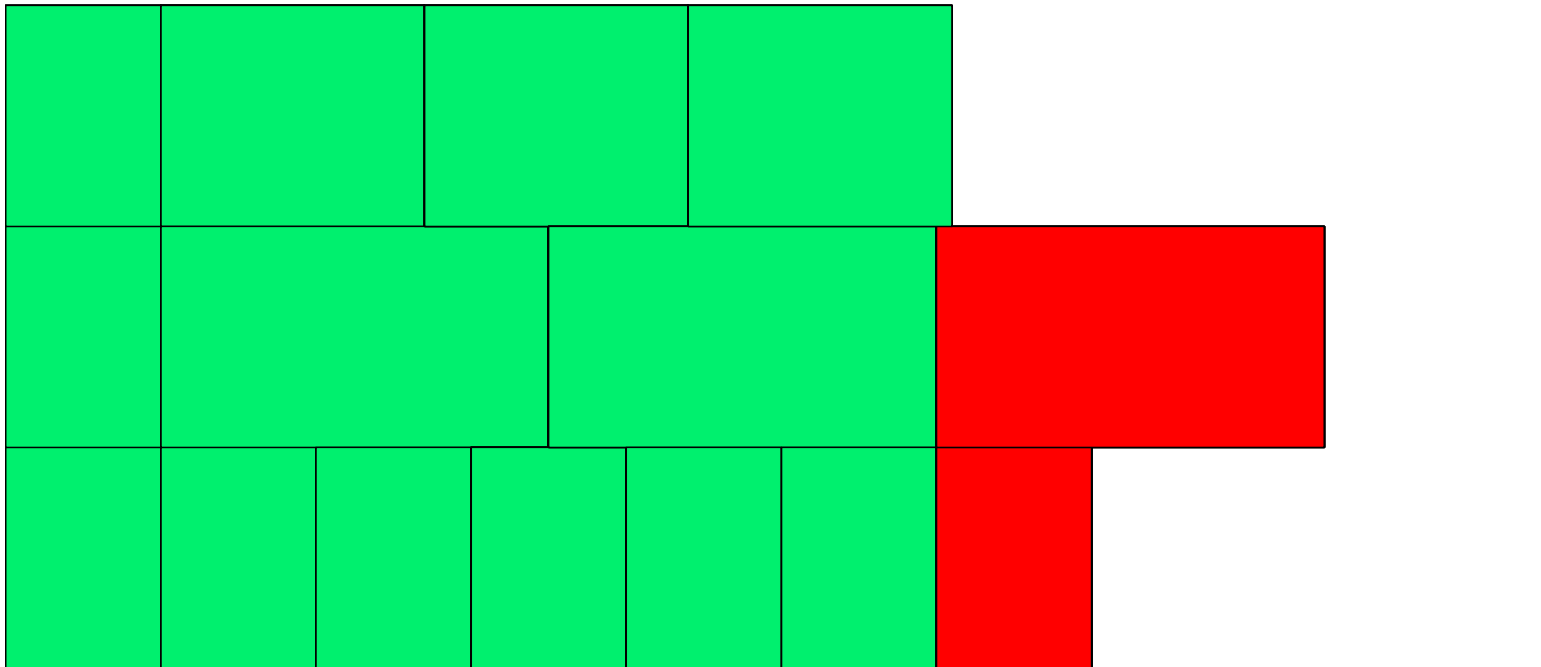
Action: Redoing steps



$t = 0.085$
 $n = 2.6667$

Multi-adaptive Time-Stepping

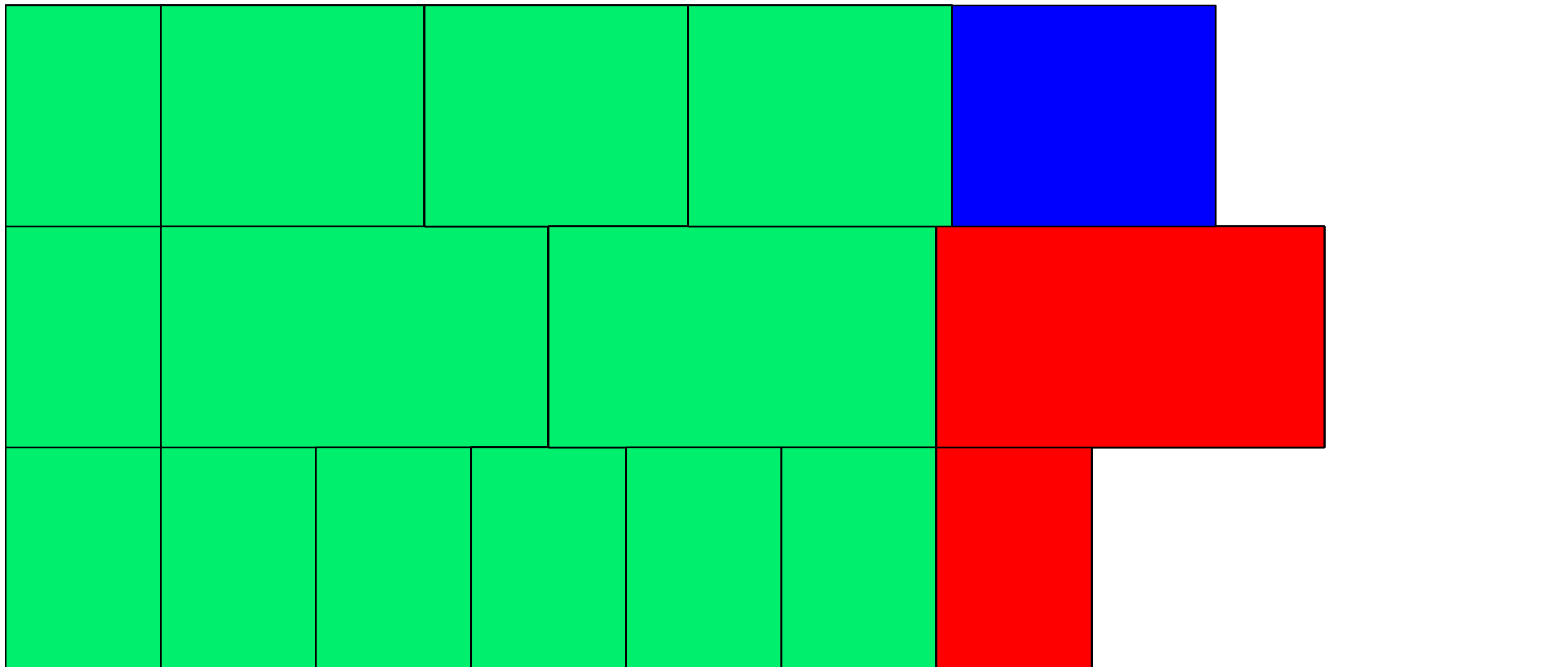
Action: Clearing steps



$t = 0.061$
 $n = 3$

Multi-adaptive Time-Stepping

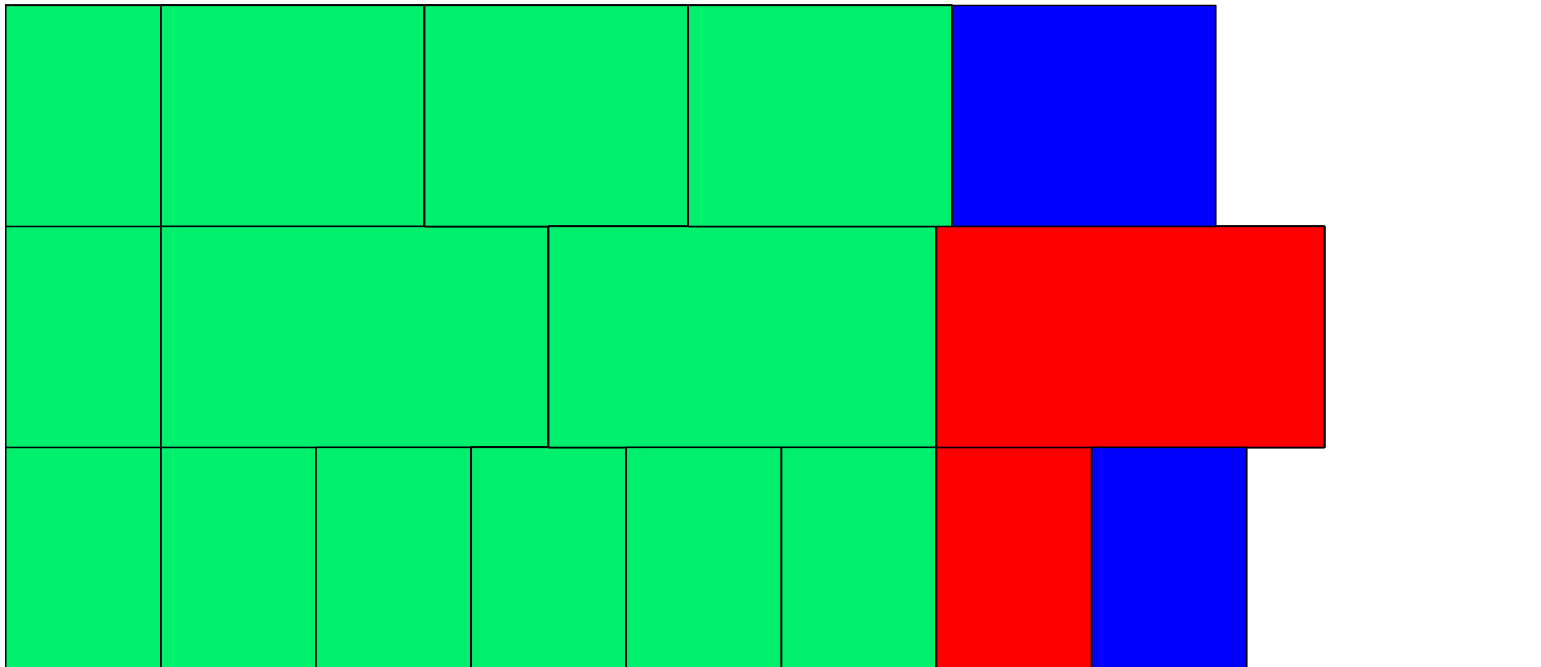
Action: Covering



$t = 0.078$
 $n = 3$

Multi-adaptive Time-Stepping

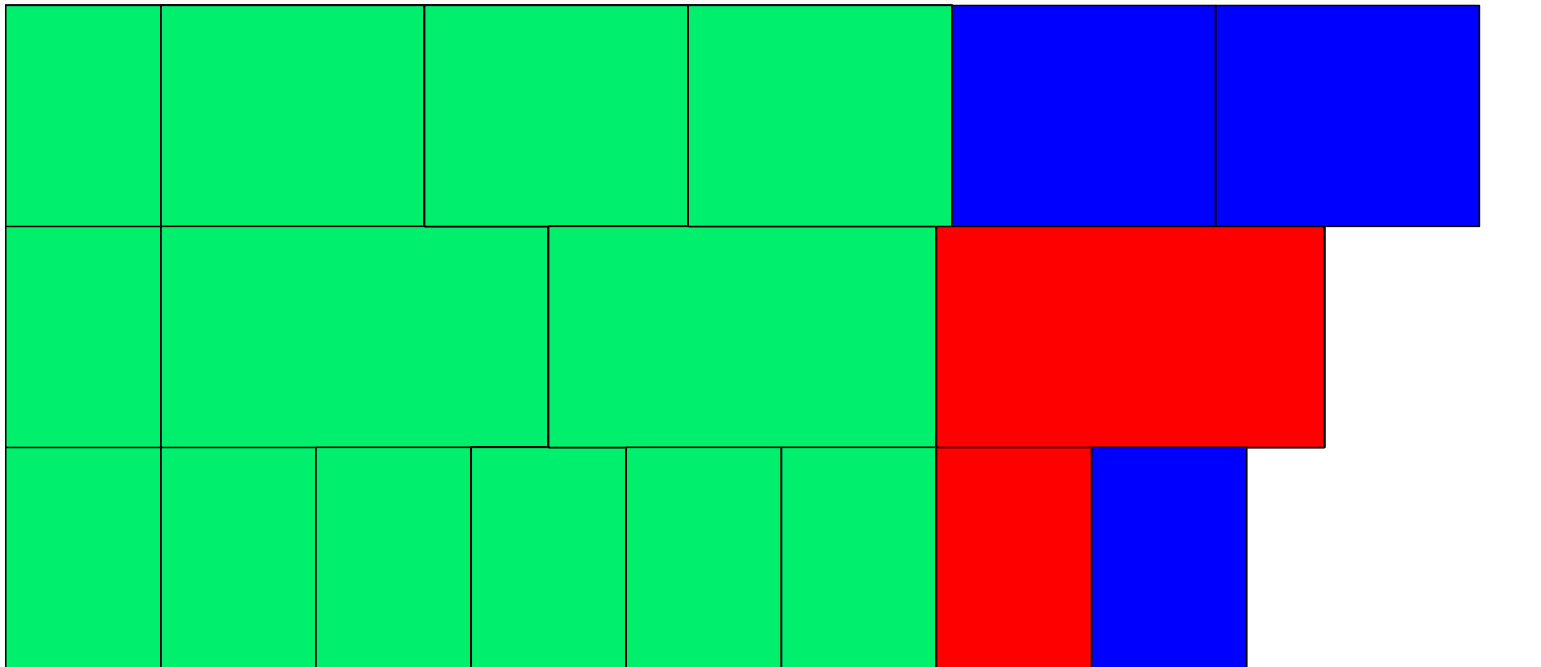
Action: Covering



$t = 0.08$
 $n = 3$

Multi-adaptive Time-Stepping

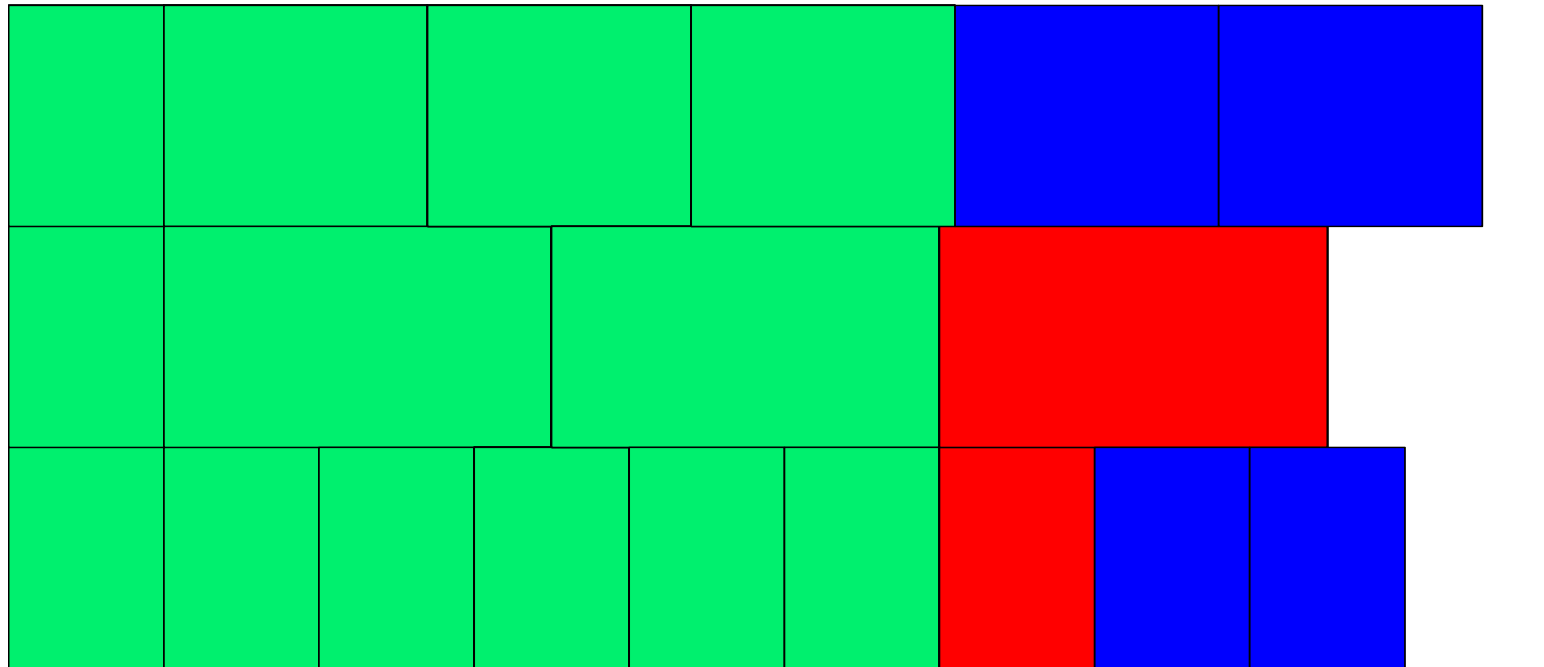
Action: Covering



$t = 0.095$
 $n = 3$

Multi-adaptive Time-Stepping

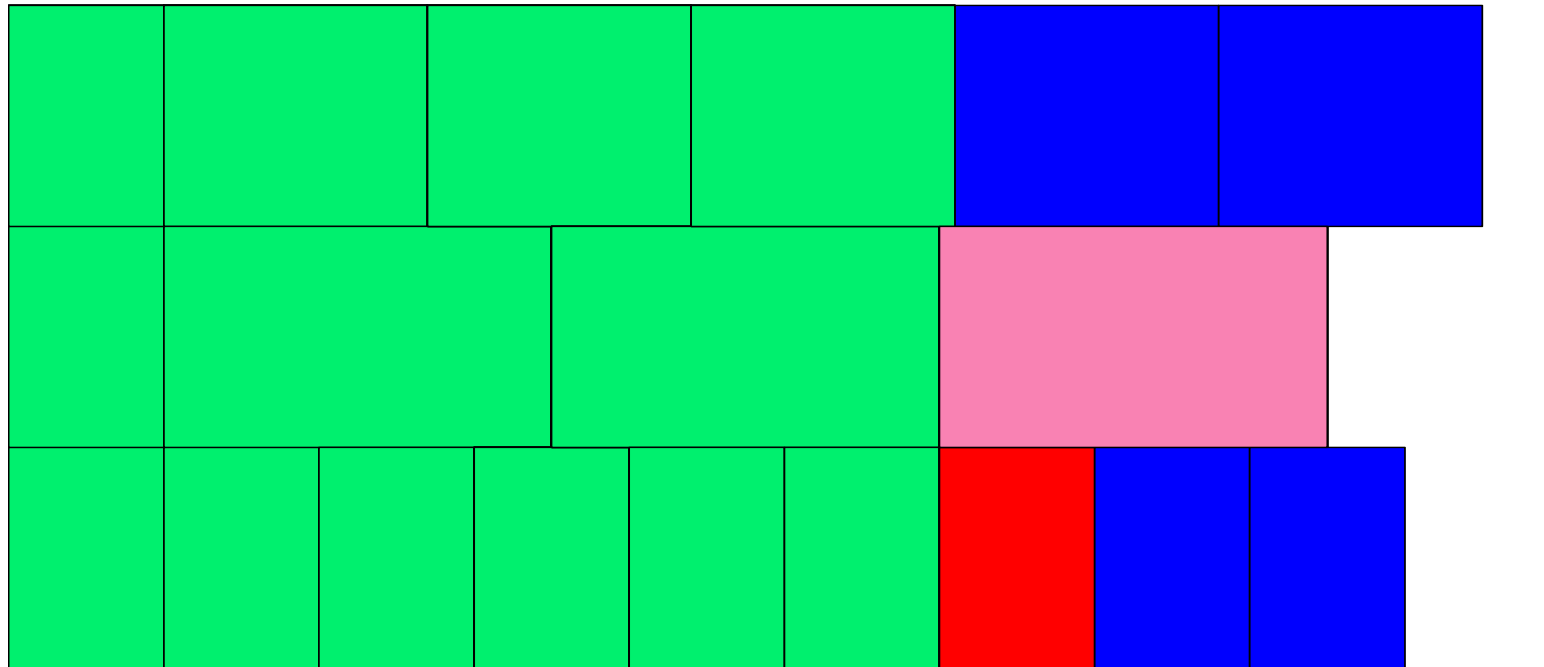
Action: Covering



$t = 0.09$
 $n = 3$

Multi-adaptive Time-Stepping

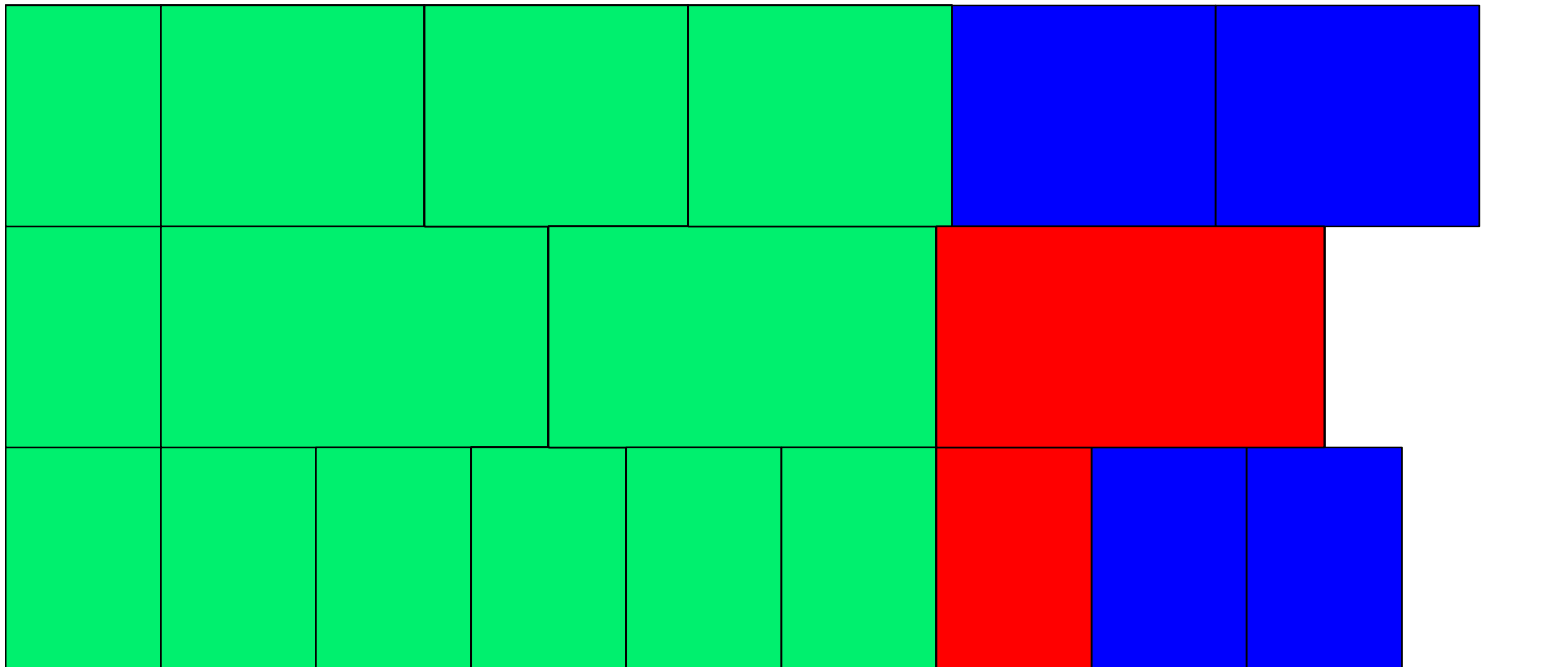
Action: Redoing steps



$t = 0.09$
 $n = 3$

Multi-adaptive Time-Stepping

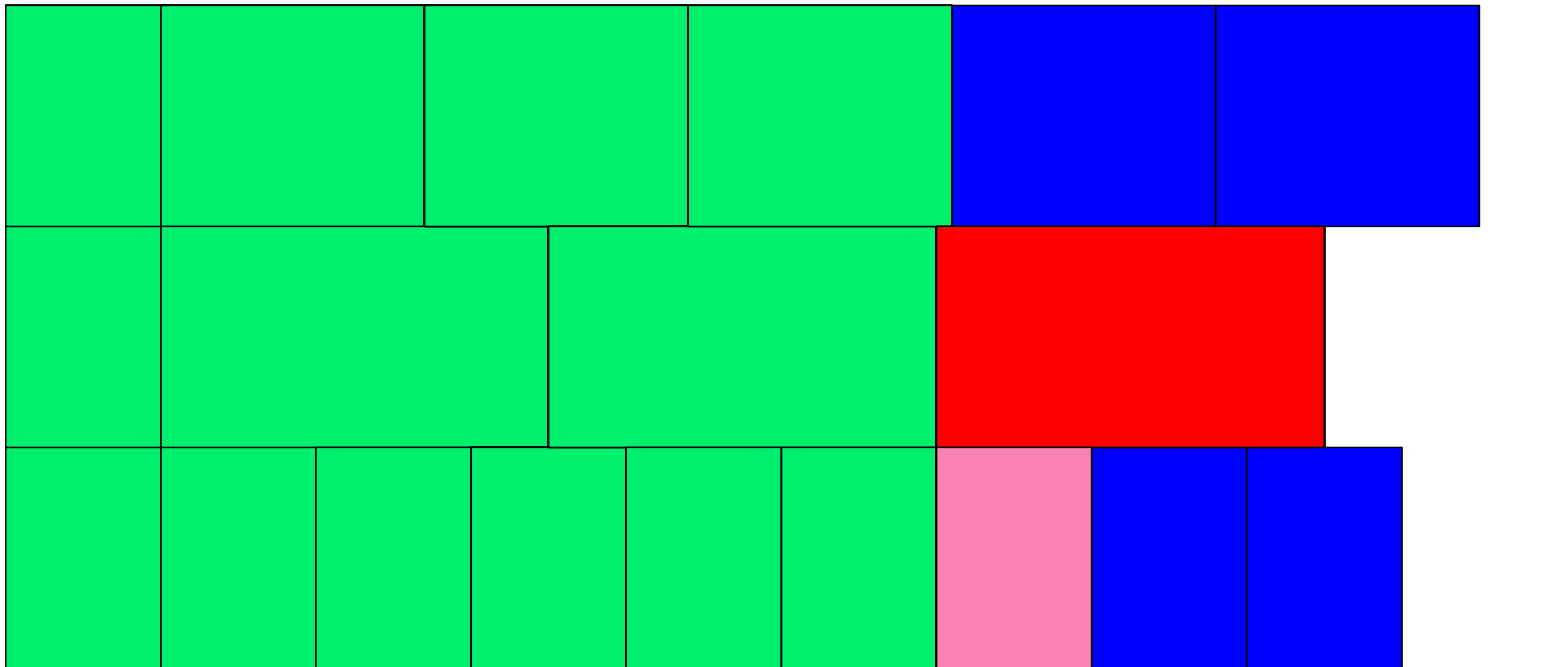
Action: Redoing steps



$t = 0.085$
 $n = 1$

Multi-adaptive Time-Stepping

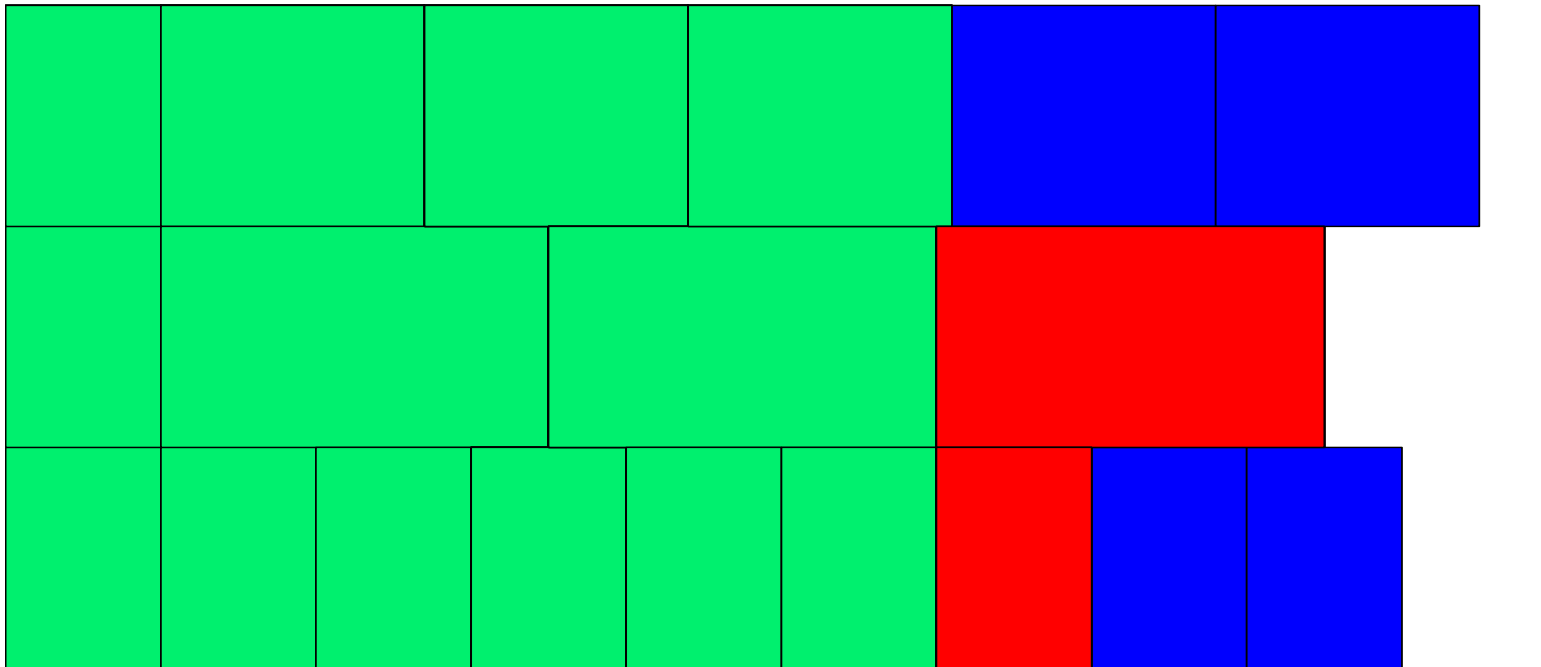
Action: Redoing steps



$t = 0.085$
 $n = 1$

Multi-adaptive Time-Stepping

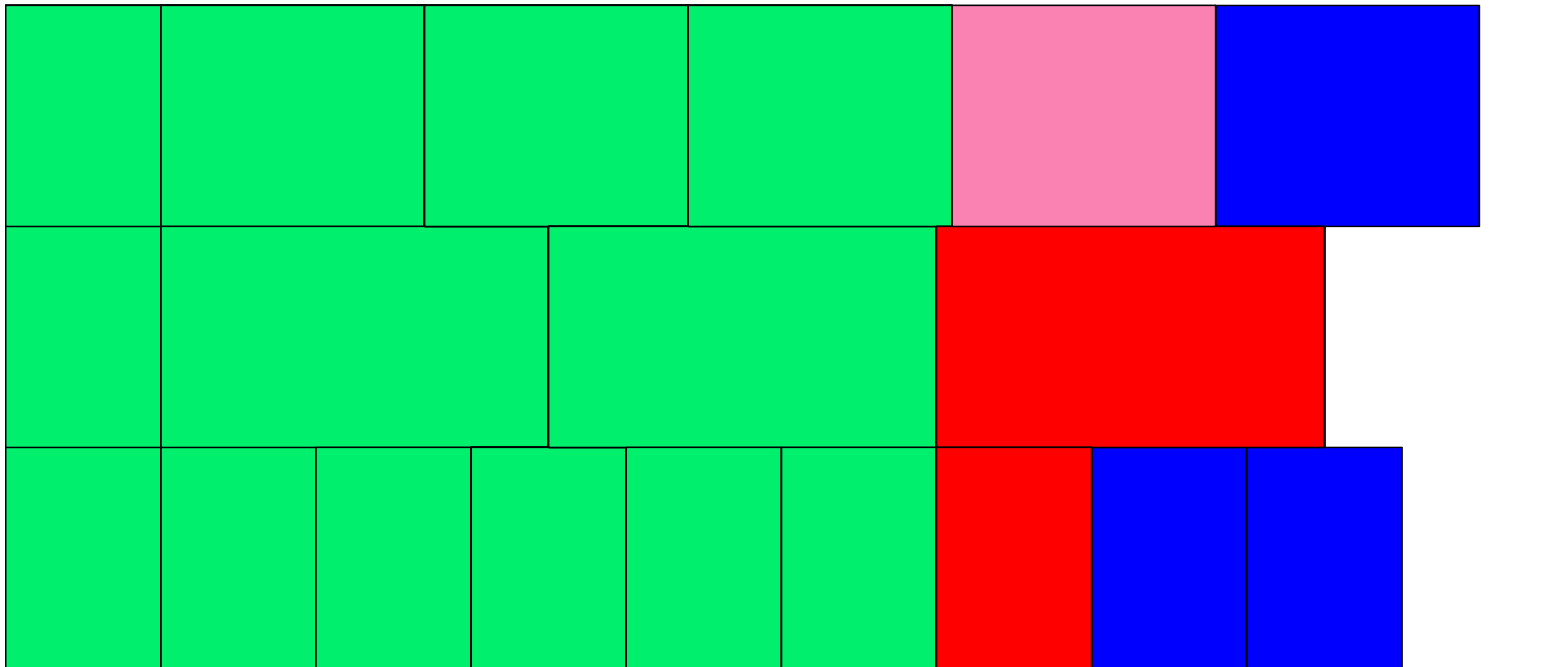
Action: Redoing steps



$t = 0.07$
 $n = 1$

Multi-adaptive Time-Stepping

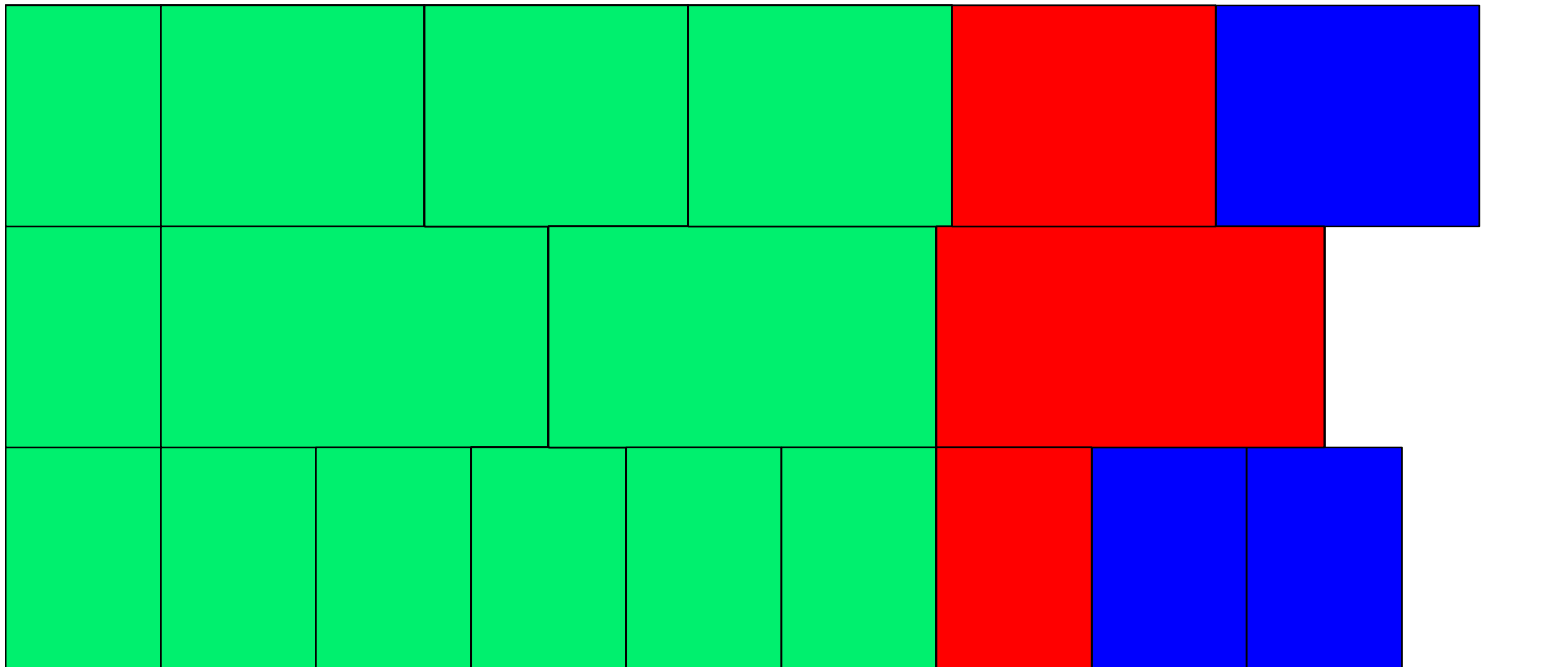
Action: Redoing steps



$t = 0.07$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.078$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.078$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.08$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.08$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.095$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.095$
 $n = 1$

Multi-adaptive Time-Stepping

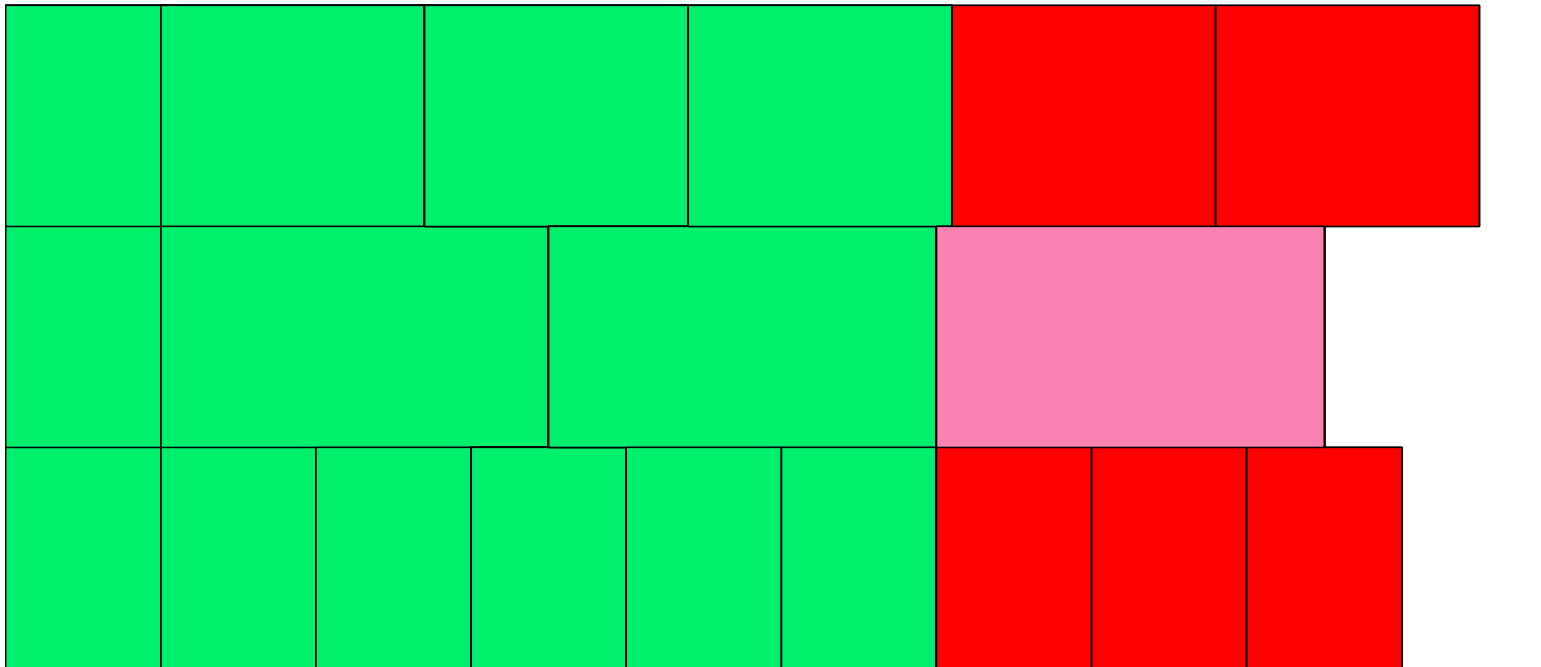
Action: Redoing steps



$t = 0.09$
 $n = 1$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.09$
 $n = 1$

Multi-adaptive Time-Stepping

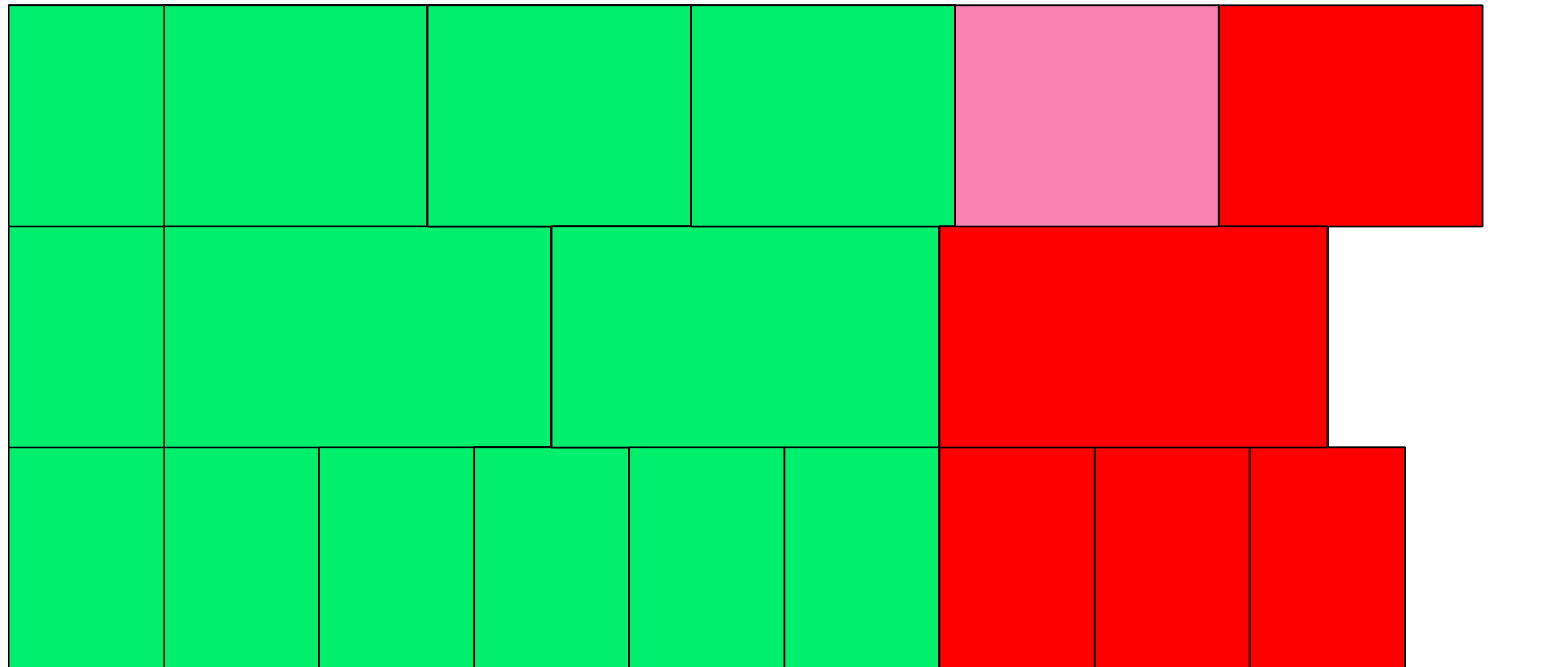
Action: Redoing steps



$t = 0.085$
 $n = 2$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.085$
 $n = 2$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.078$
 $n = 2$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.078$
 $n = 2$

Multi-adaptive Time-Stepping

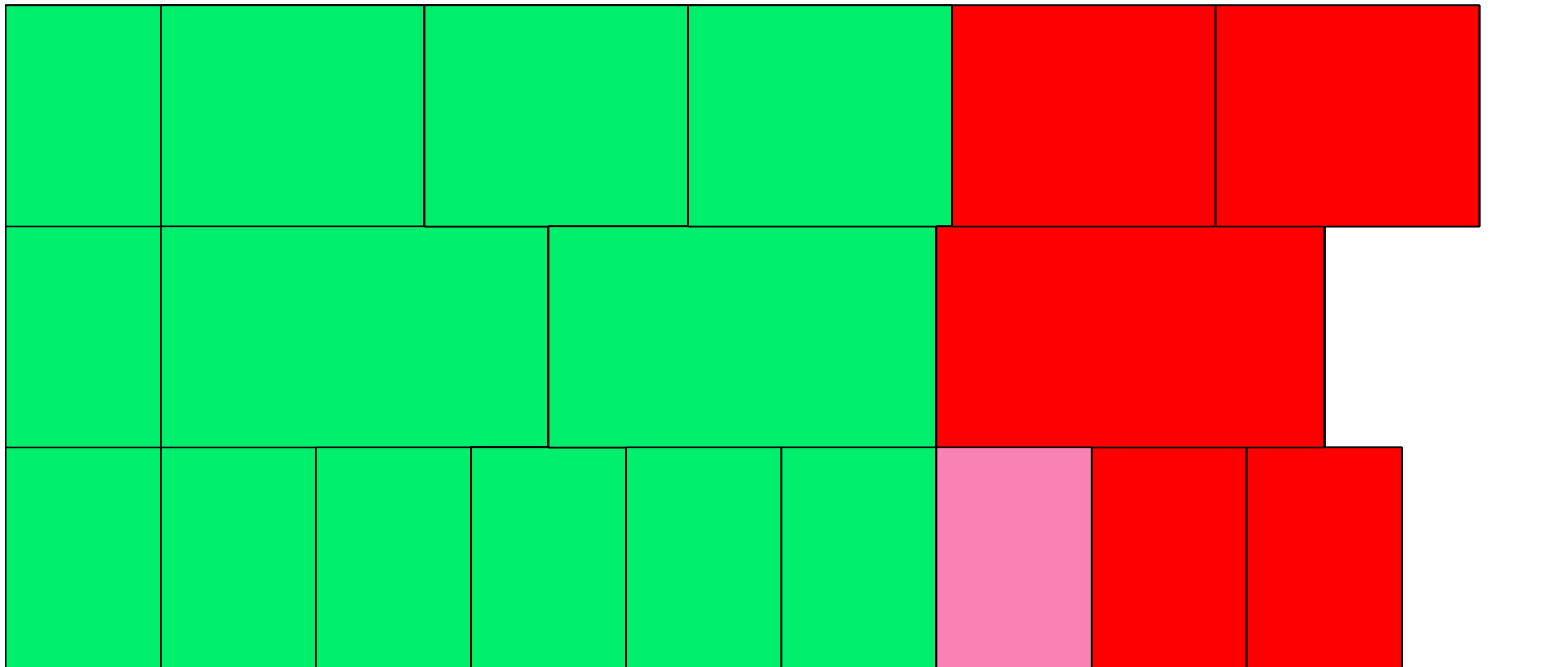
Action: Redoing steps



$t = 0.095$
 $n = 2$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.095$
 $n = 2$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.07$
 $n = 2.5$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.07$
 $n = 2.5$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.08$
 $n = 2.5$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.08$
 $n = 2.5$

Multi-adaptive Time-Stepping

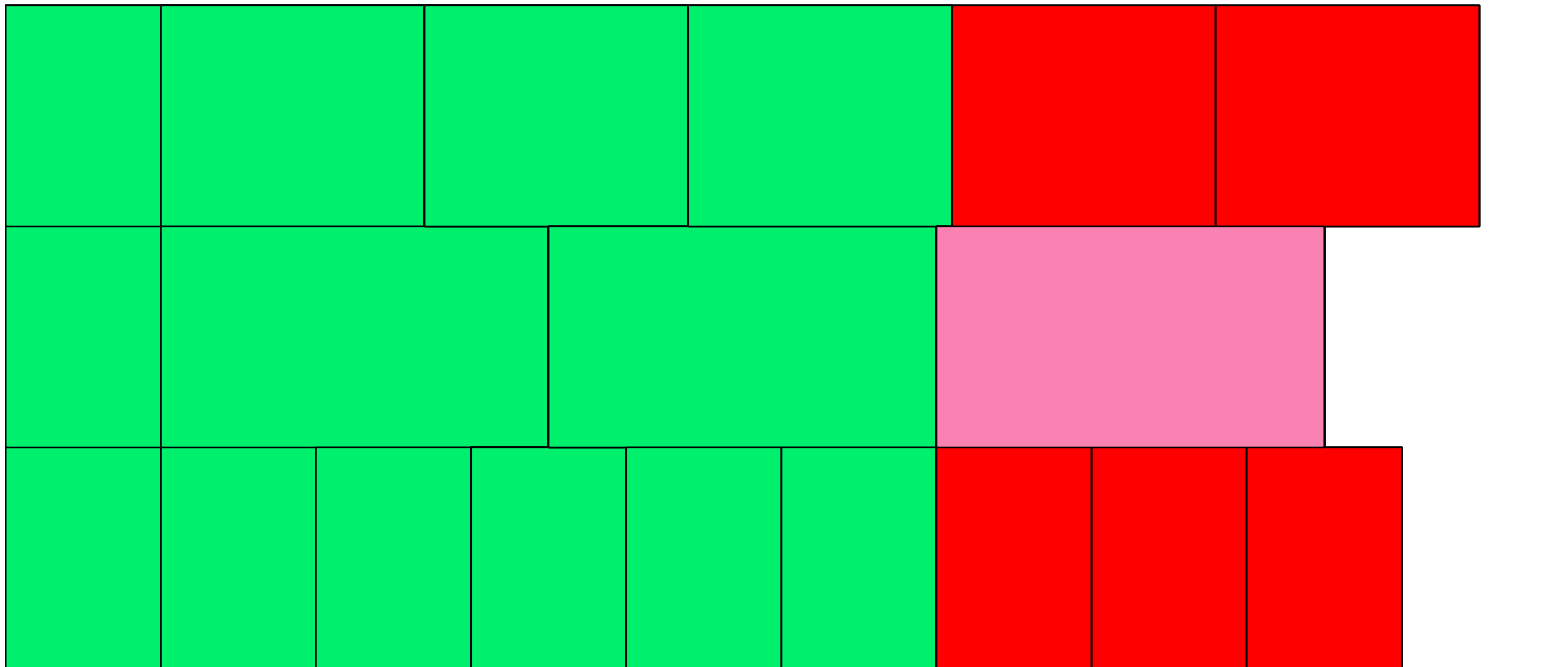
Action: Redoing steps



$t = 0.09$
 $n = 2.5$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.09$
 $n = 2.5$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.085$
 $n = 3$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.085$
 $n = 3$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.095$
 $n = 3$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.095$
 $n = 3$

Multi-adaptive Time-Stepping

Action: Redoing steps



$t = 0.09$
 $n = 3.3333$

Multi-adaptive Time-Stepping

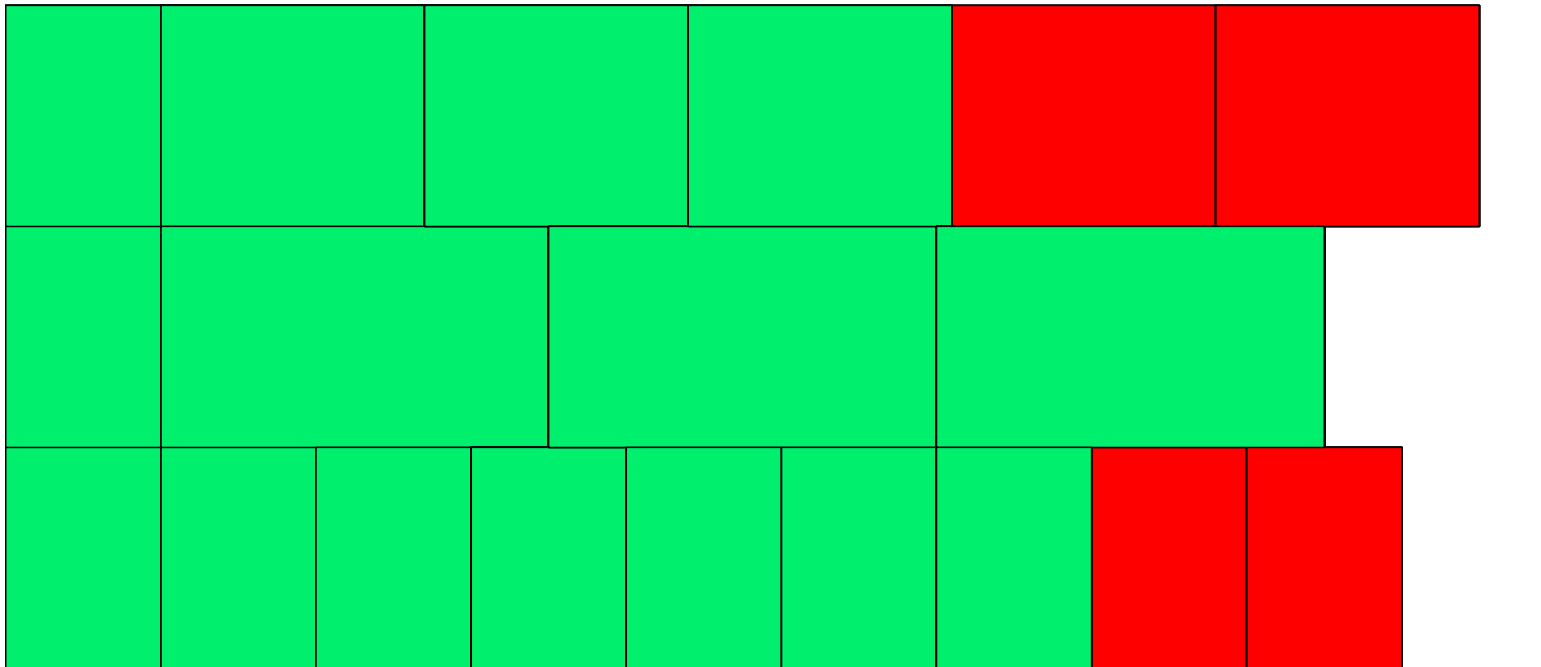
Action: Clearing steps



$t = 0.085$
 $n = 3.5$

Multi-adaptive Time-Stepping

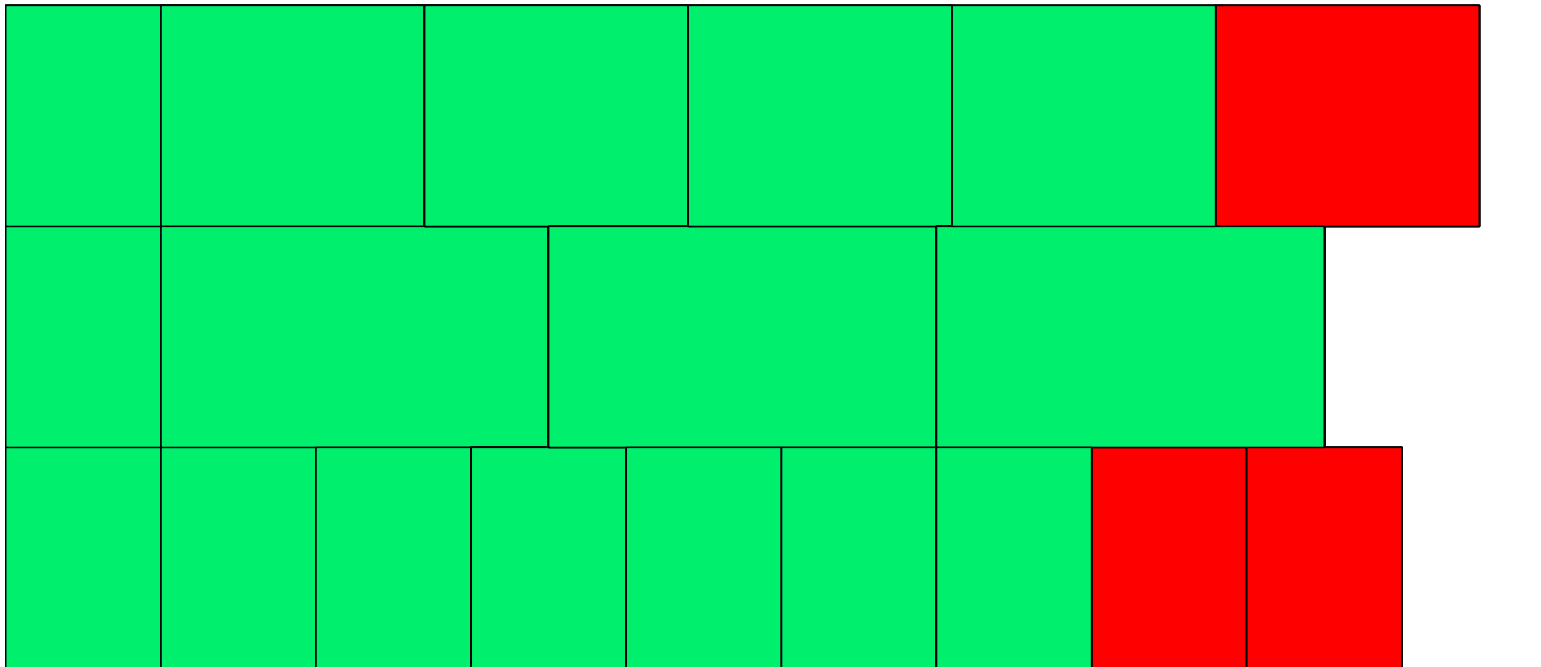
Action: Clearing steps



$t = 0.07$
 $n = 3.5$

Multi-adaptive Time-Stepping

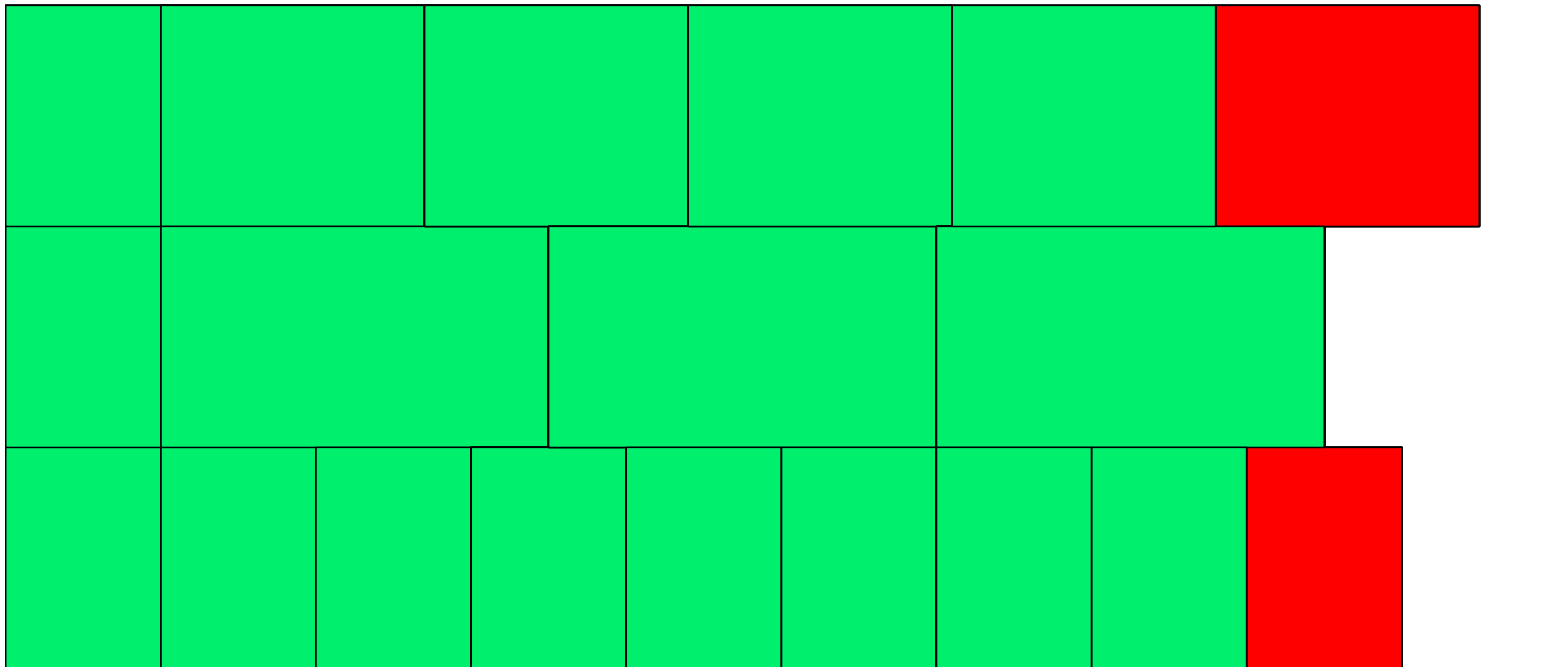
Action: Clearing steps



$t = 0.078$
 $n = 3.5$

Multi-adaptive Time-Stepping

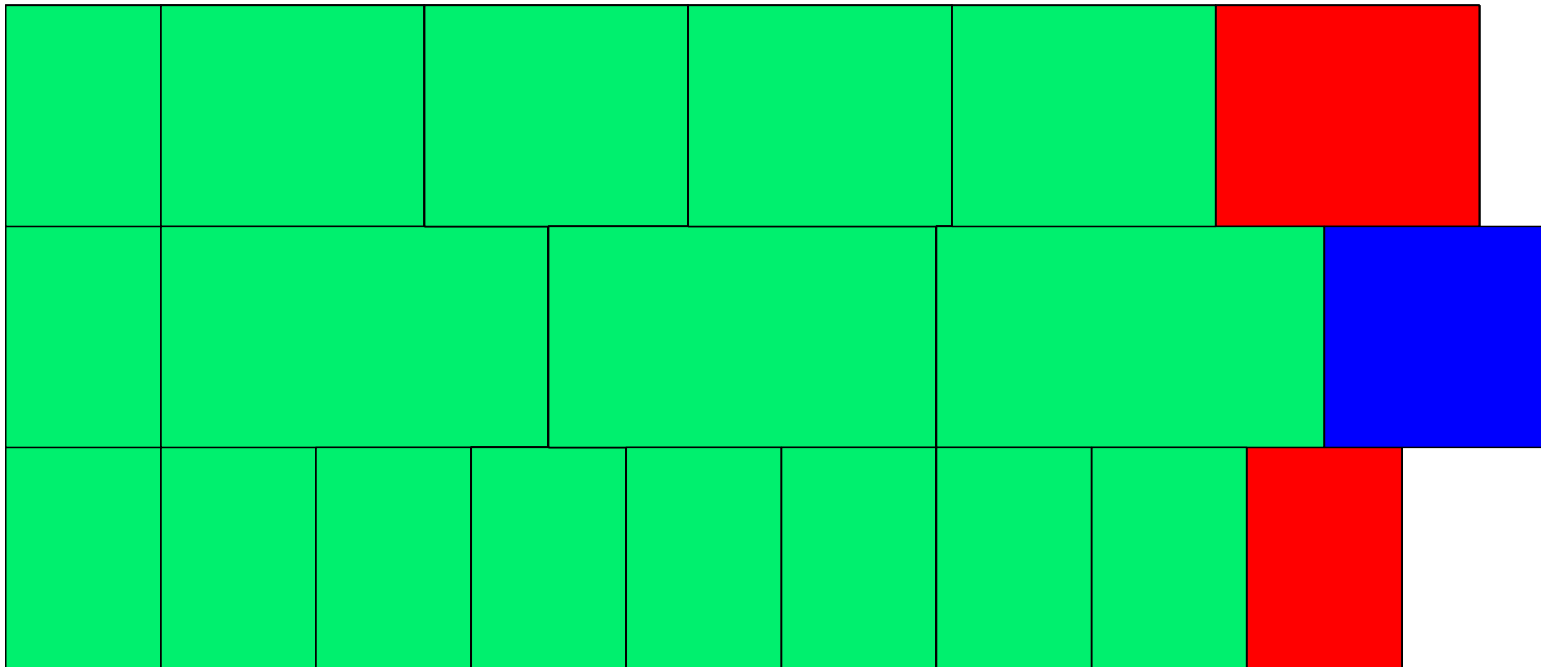
Action: Clearing steps



$t = 0.08$
 $n = 3.5$

Multi-adaptive Time-Stepping

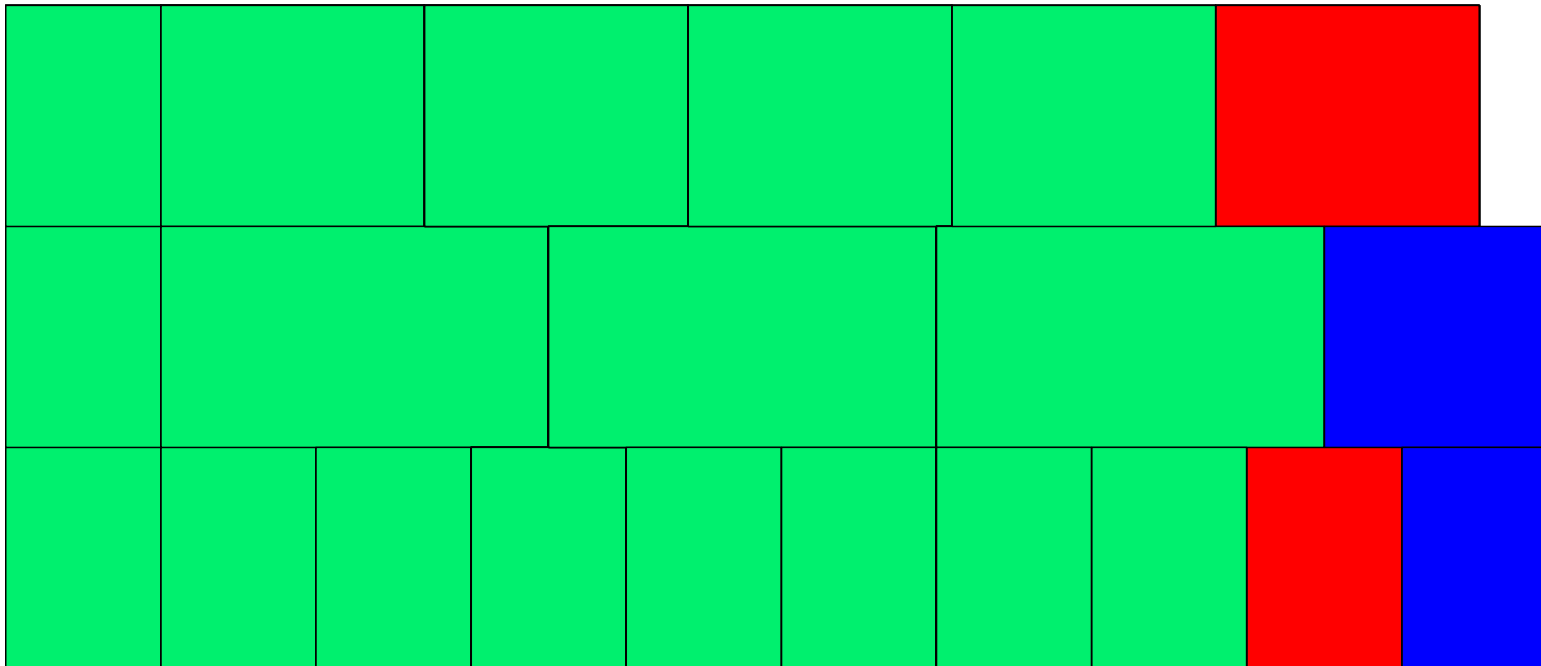
Action: Covering



$t = 0.1$
 $n = 3.5$

Multi-adaptive Time-Stepping

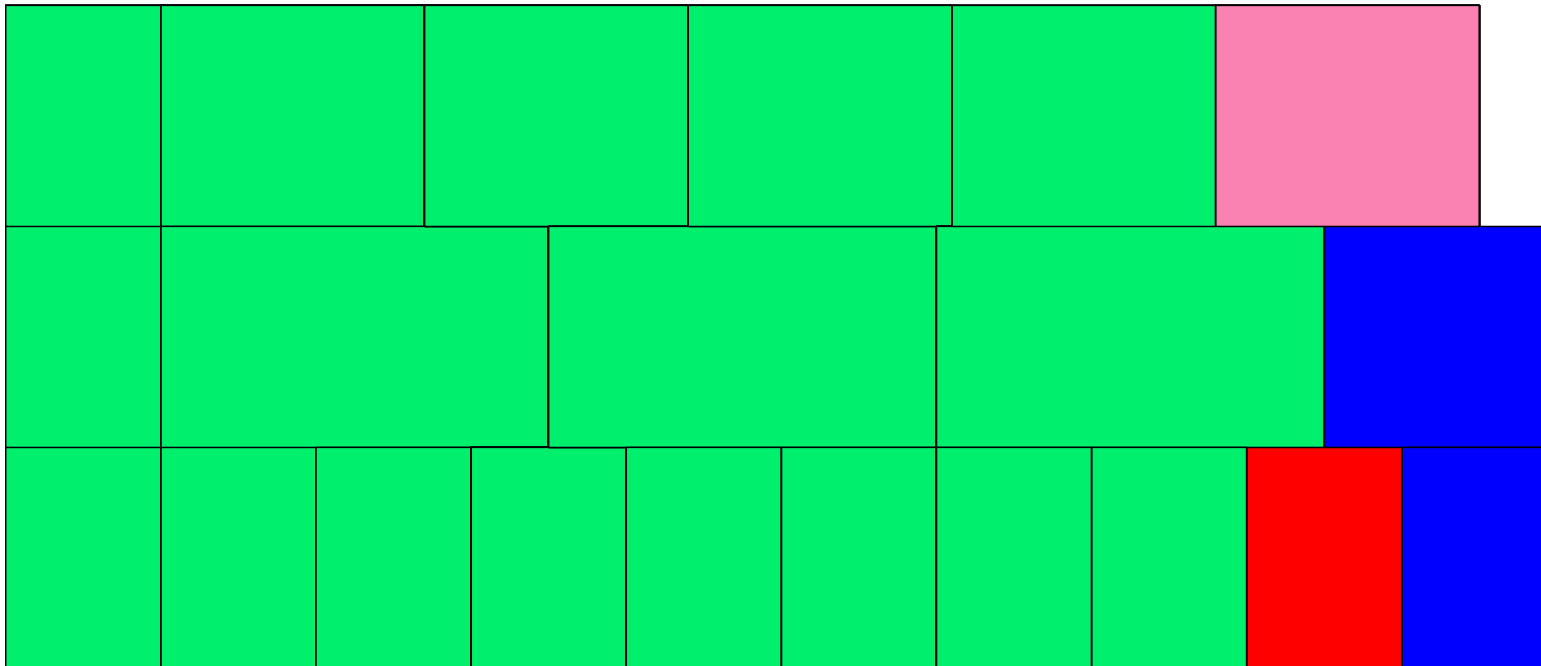
Action: Covering



$t = 0.1$
 $n = 3.5$

Multi-adaptive Time-Stepping

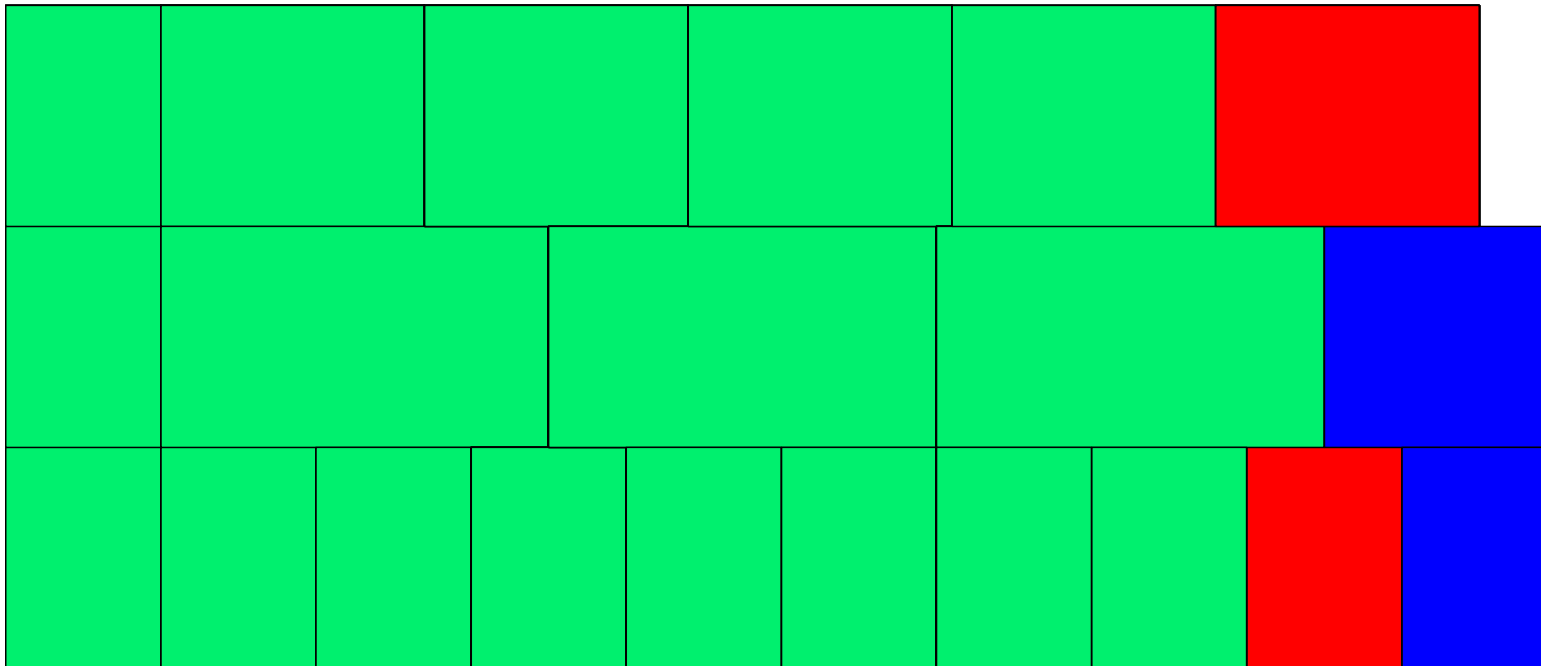
Action: Redoing steps



$t = 0.1$
 $n = 3.5$

Multi-adaptive Time-Stepping

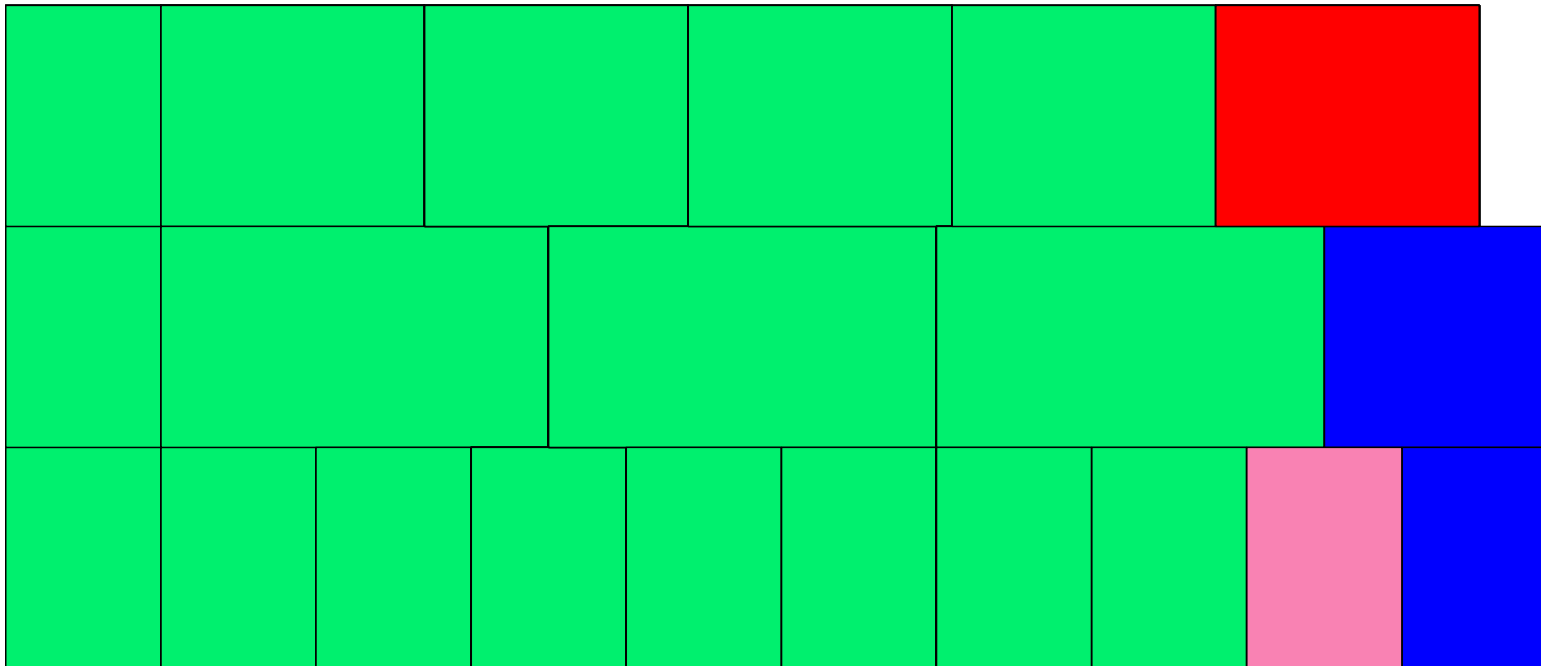
Action: Redoing steps



$t = 0.095$
 $n = 1$

Multi-adaptive Time-Stepping

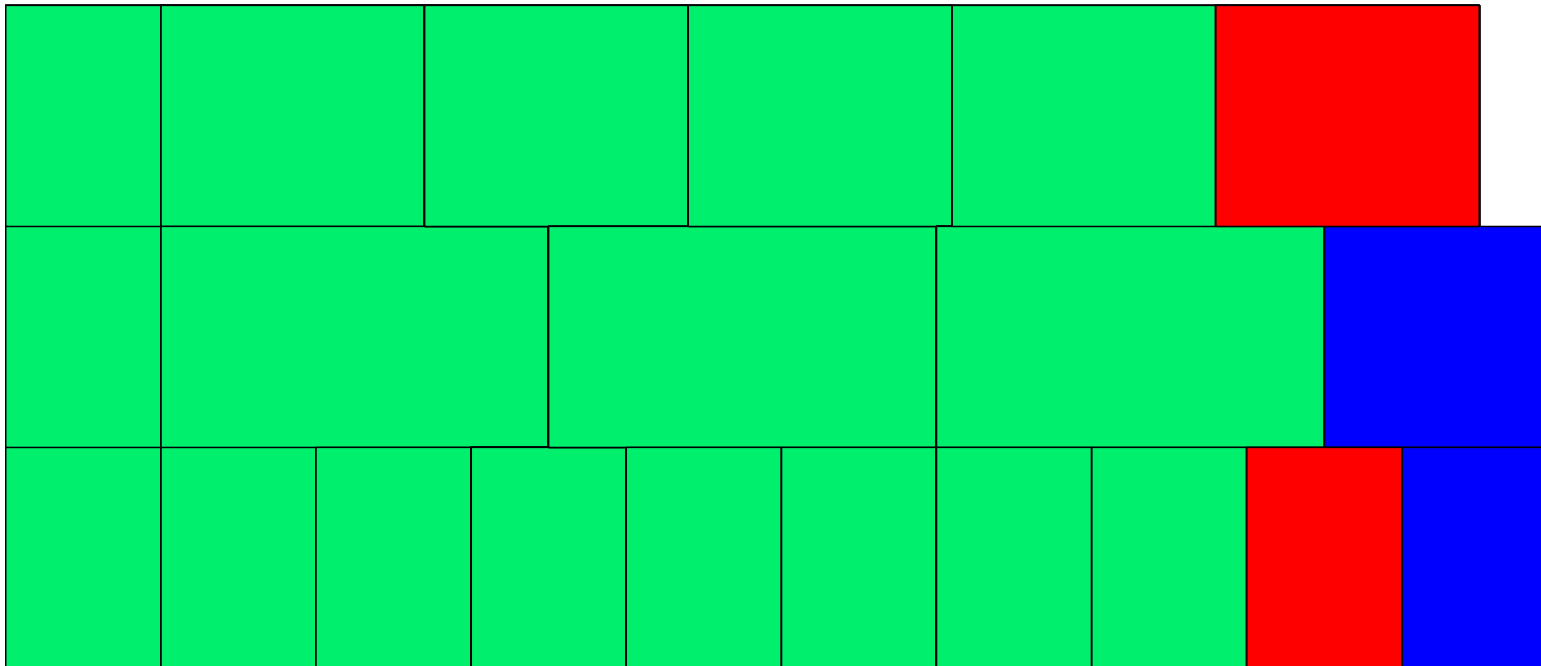
Action: Redoing steps



$t = 0.095$
 $n = 1$

Multi-adaptive Time-Stepping

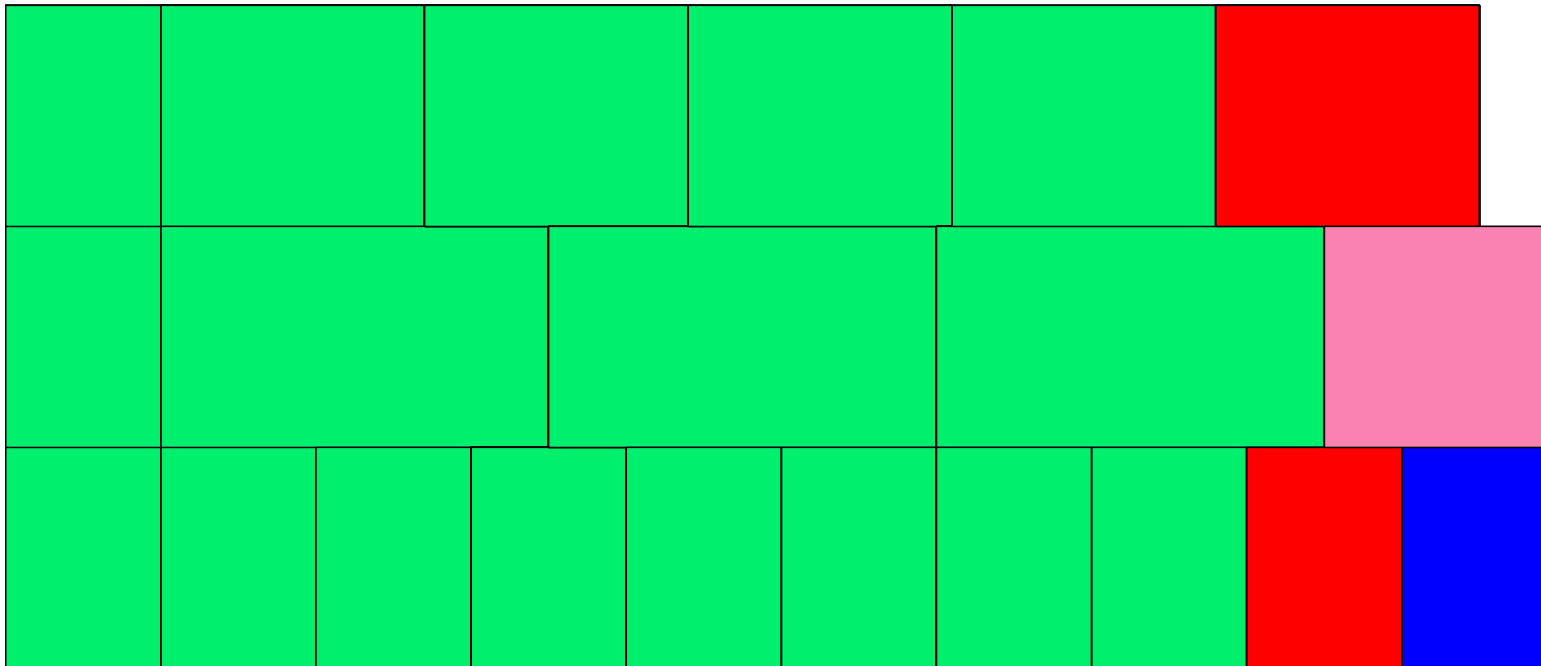
Action: Redoing steps



$t = 0.09$
 $n = 1$

Multi-adaptive Time-Stepping

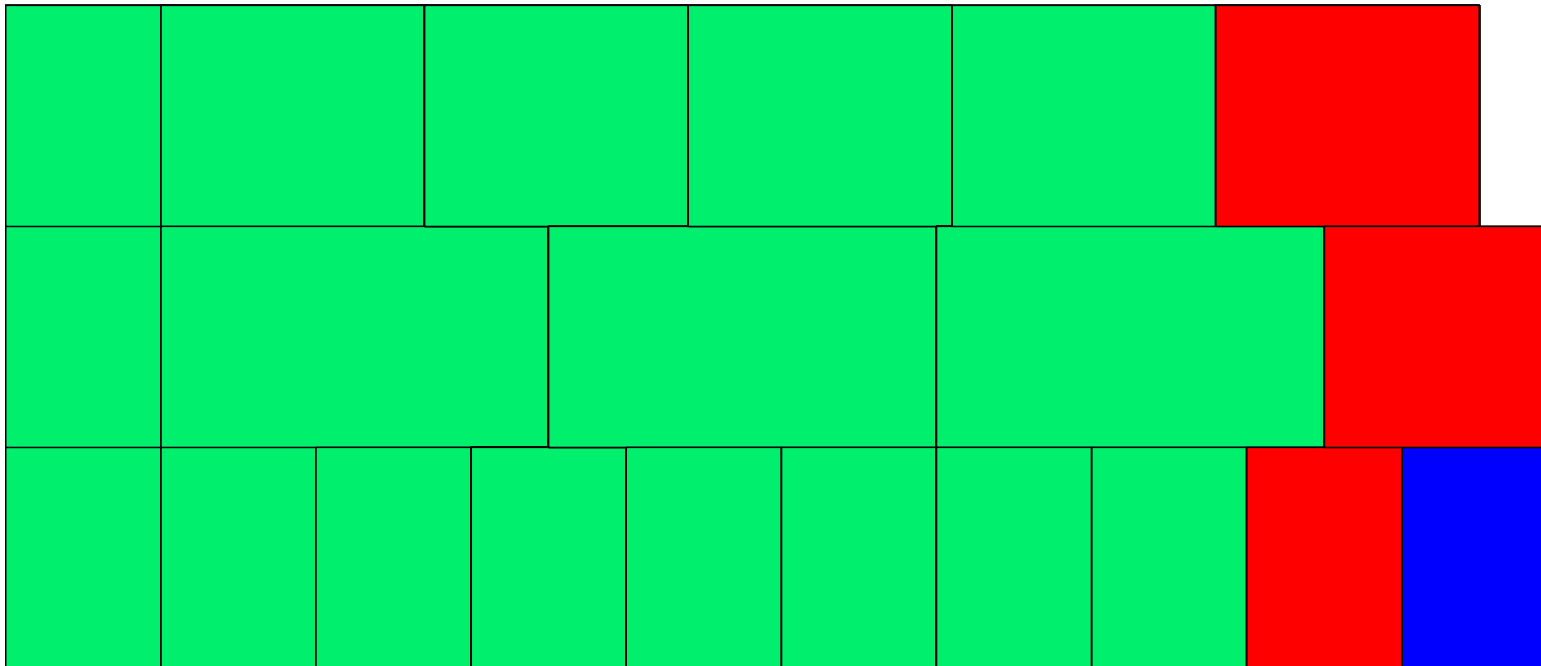
Action: Redoing steps



$t = 0.09$
 $n = 1$

Multi-adaptive Time-Stepping

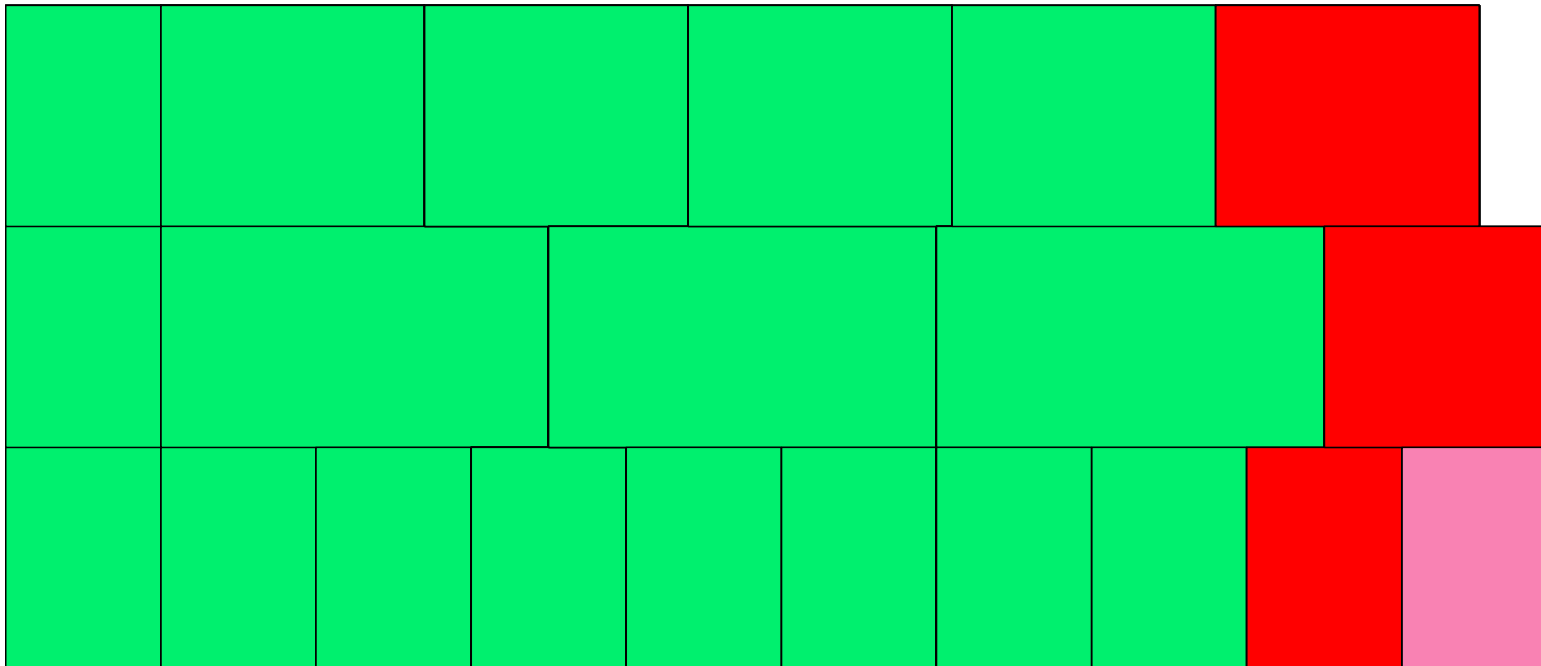
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

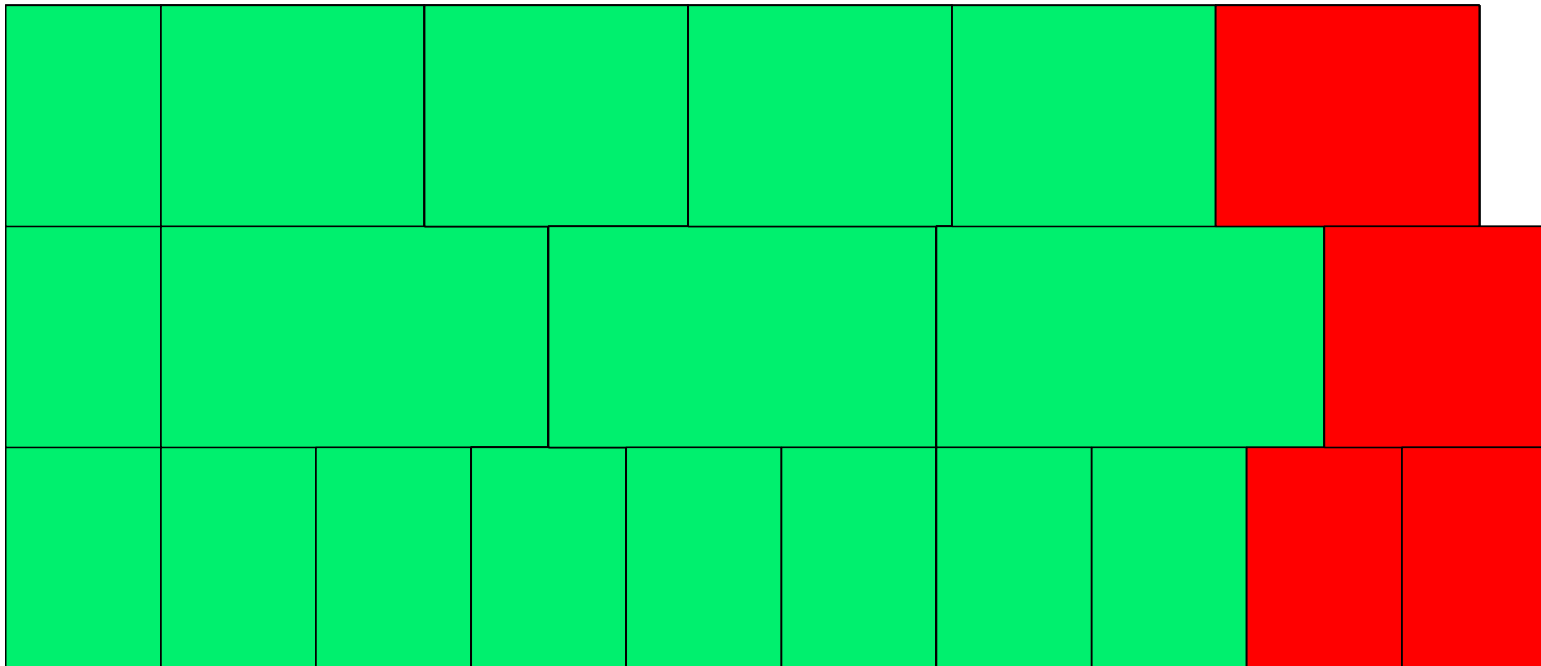
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

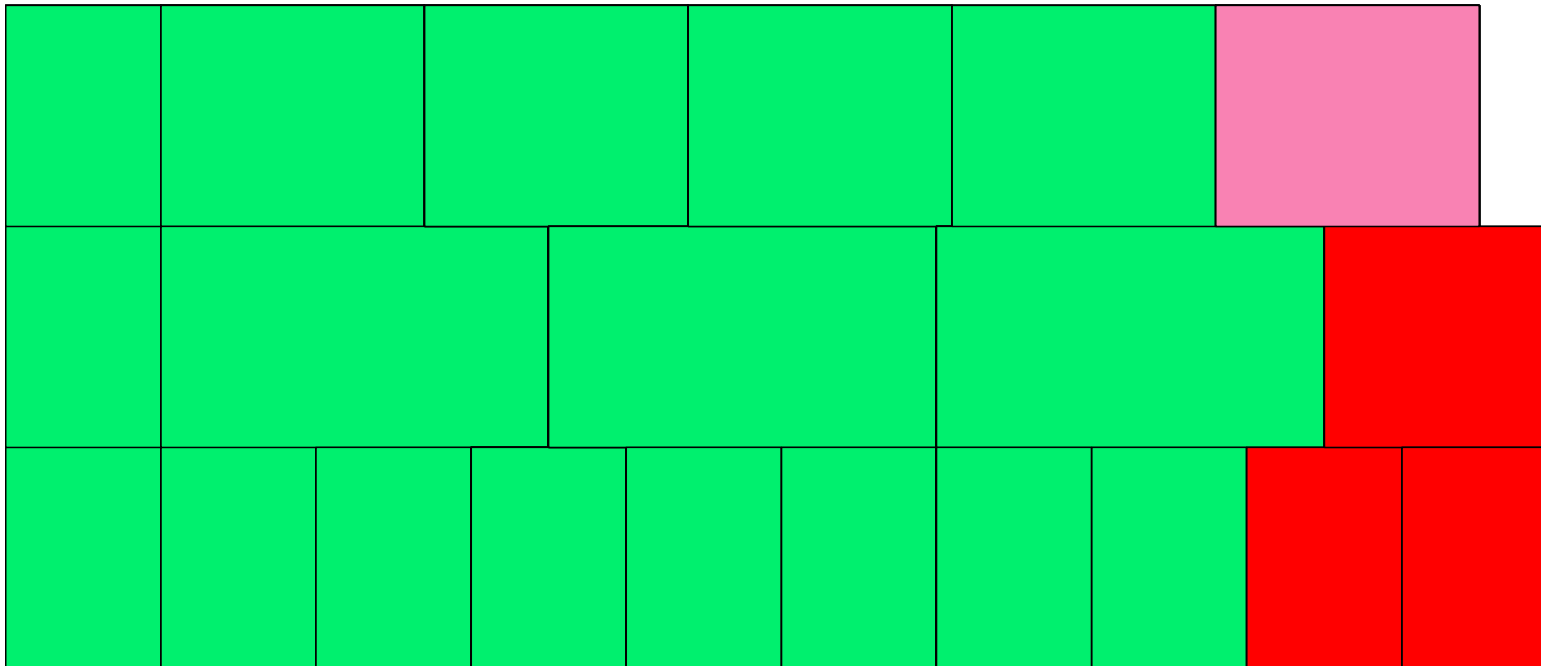
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

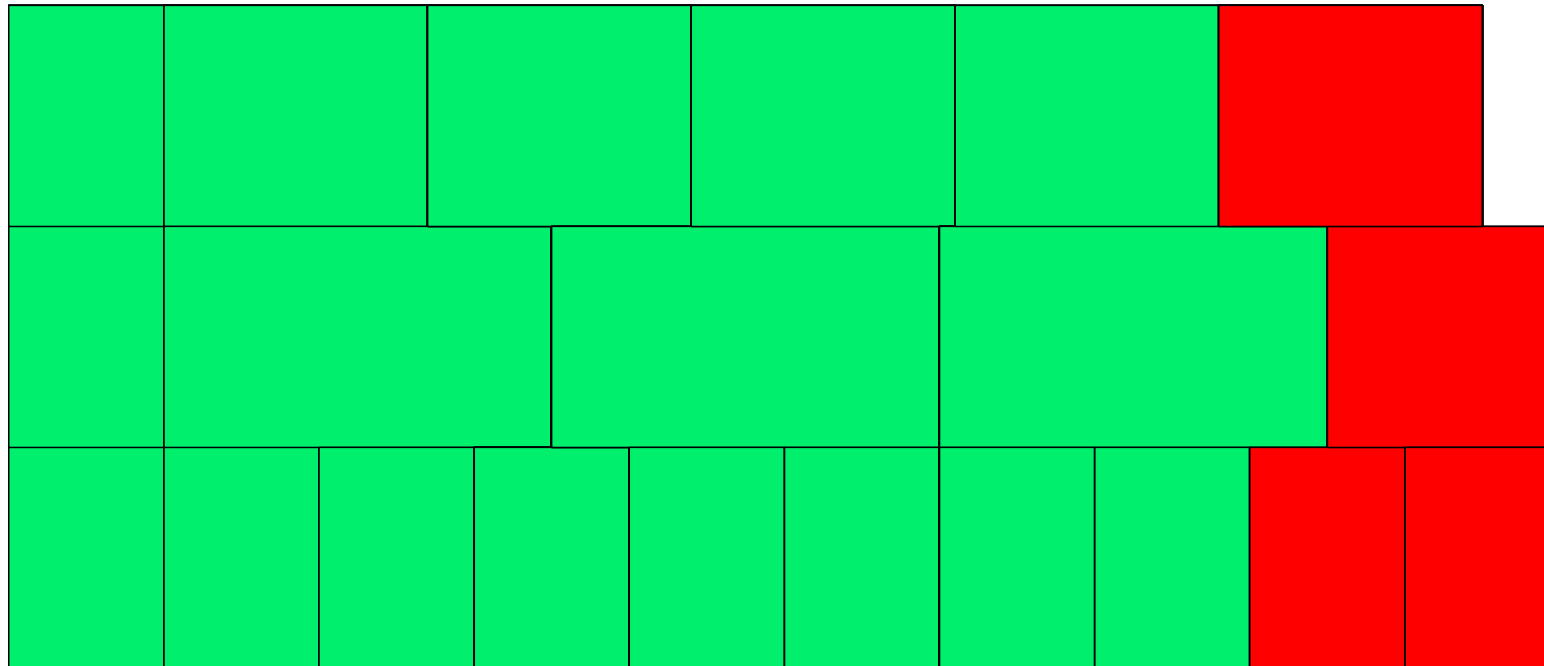
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

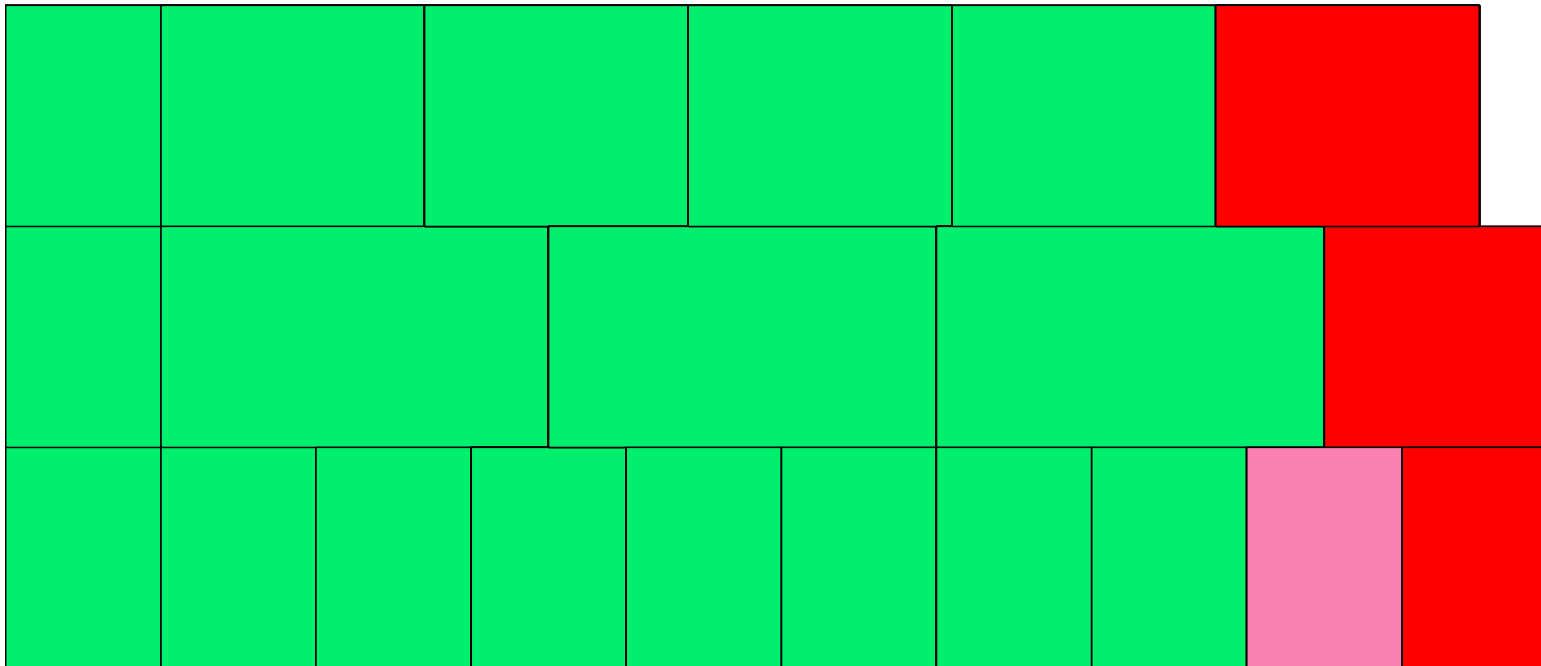
Action: Redoing steps



$t = 0.095$
 $n = 2$

Multi-adaptive Time-Stepping

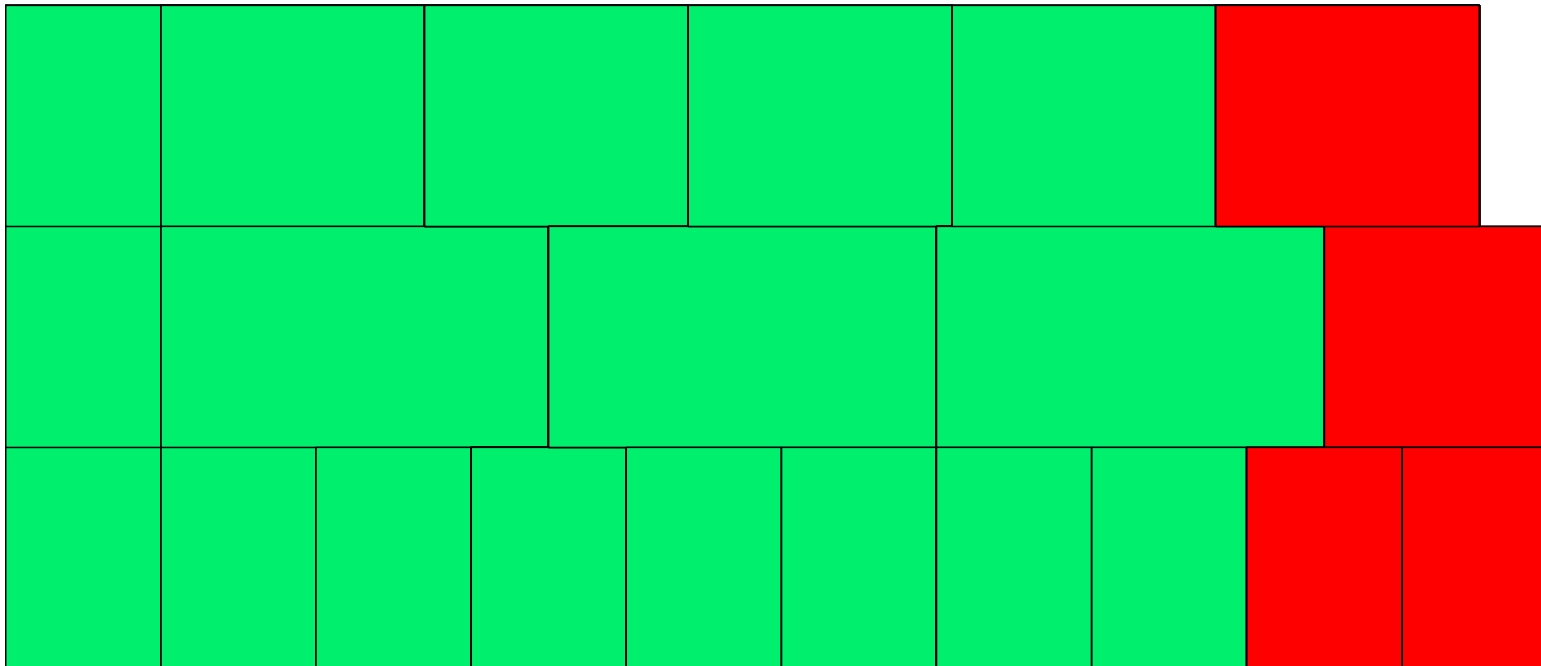
Action: Redoing steps



$t = 0.095$
 $n = 2$

Multi-adaptive Time-Stepping

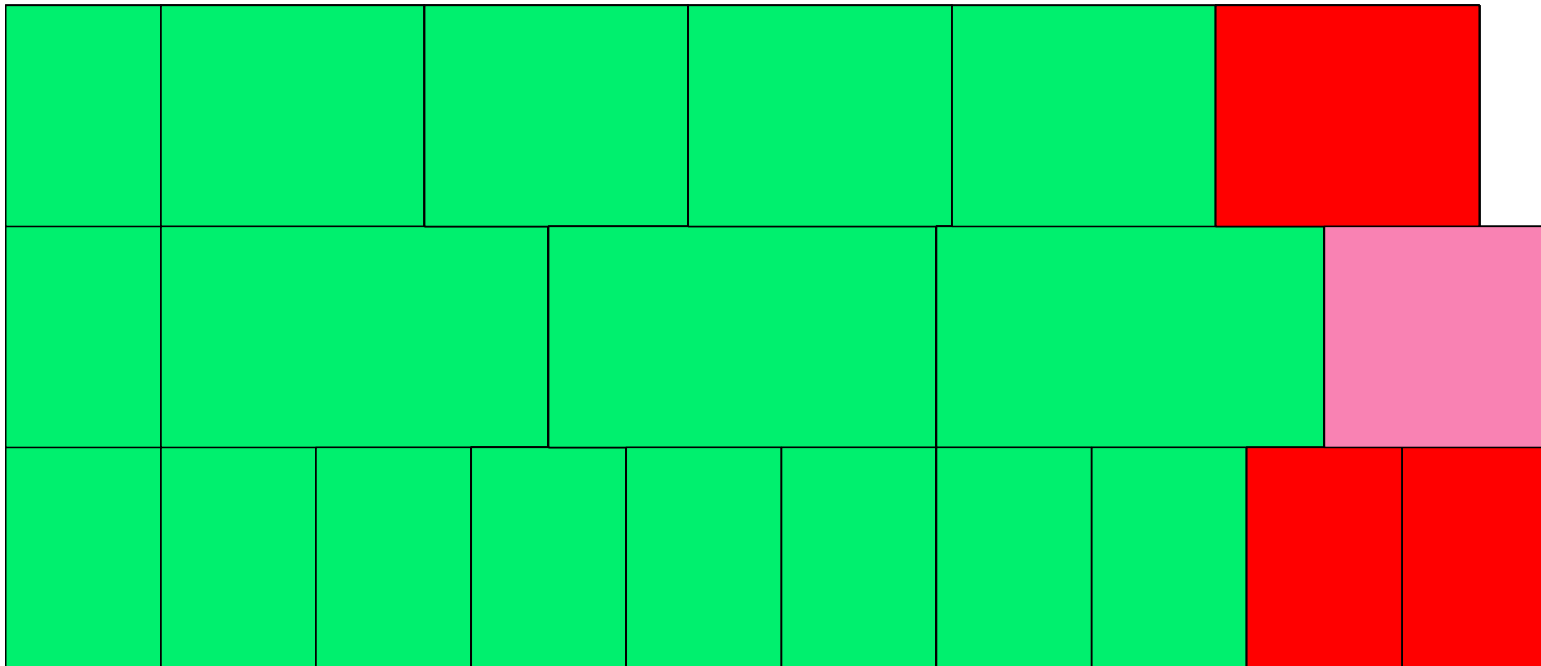
Action: Redoing steps



$t = 0.09$
 $n = 2$

Multi-adaptive Time-Stepping

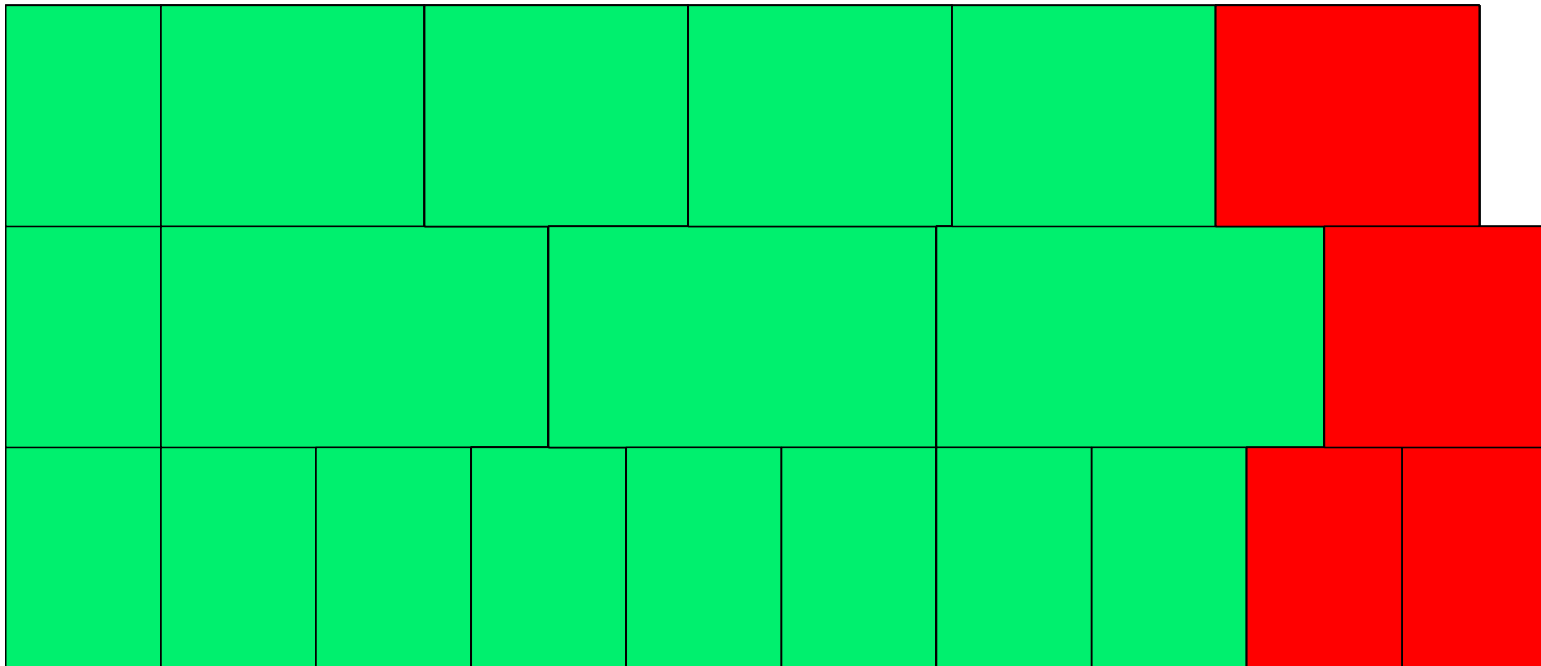
Action: Redoing steps



$t = 0.09$
 $n = 2$

Multi-adaptive Time-Stepping

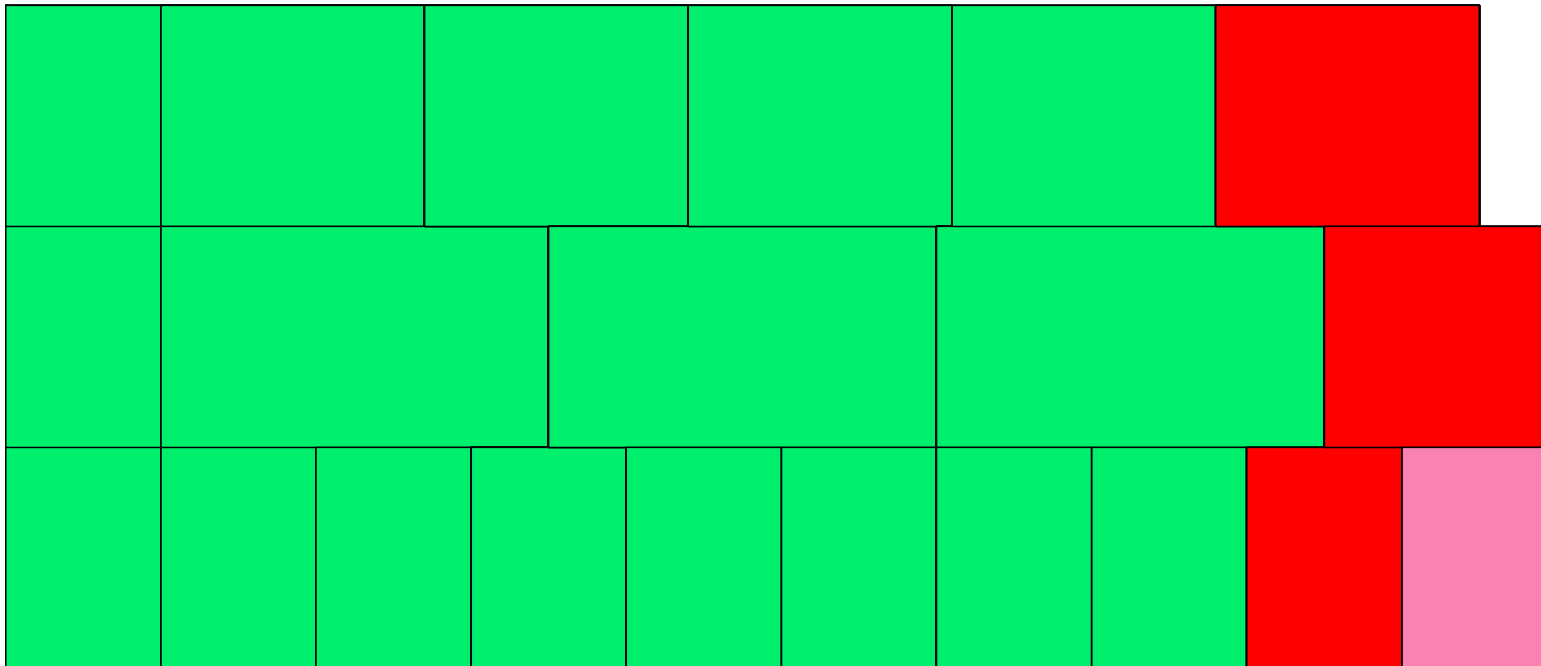
Action: Redoing steps



$t = 0.1$
 $n = 2$

Multi-adaptive Time-Stepping

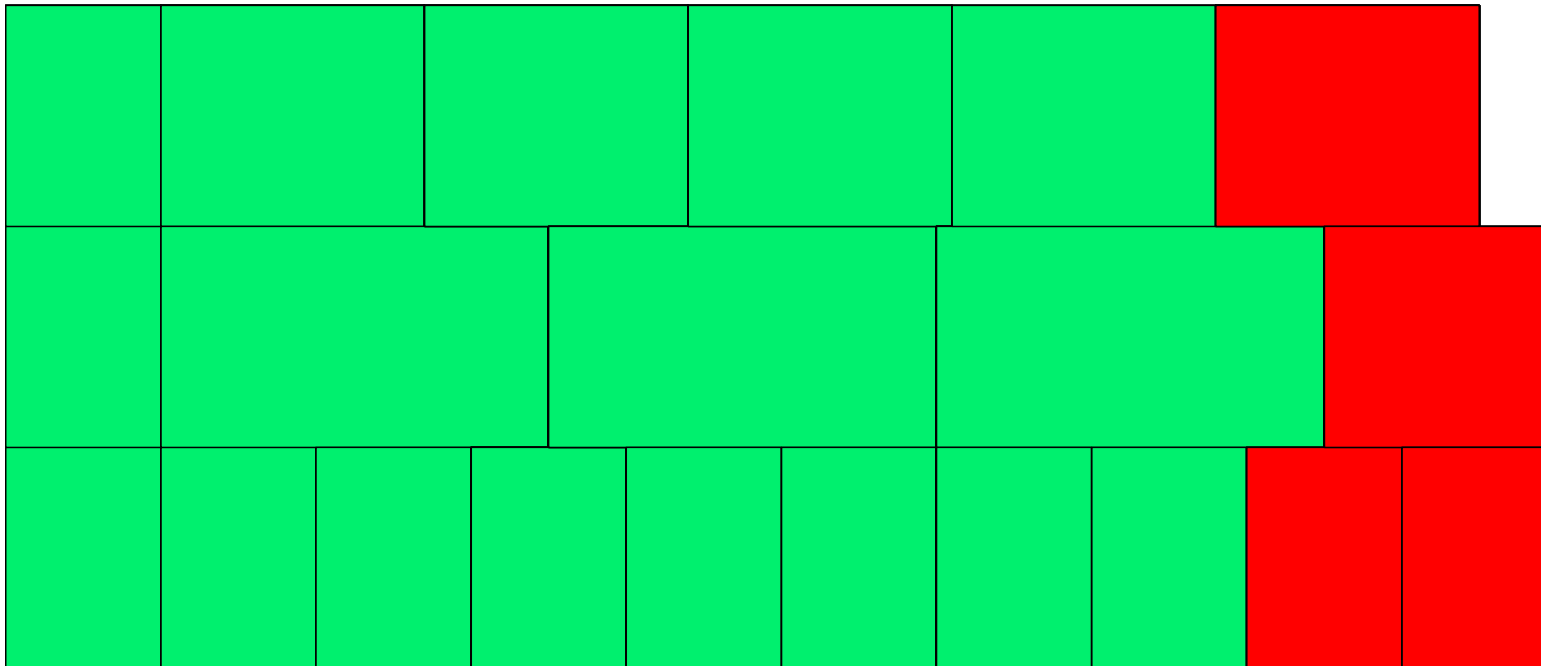
Action: Redoing steps



$t = 0.1$
 $n = 2$

Multi-adaptive Time-Stepping

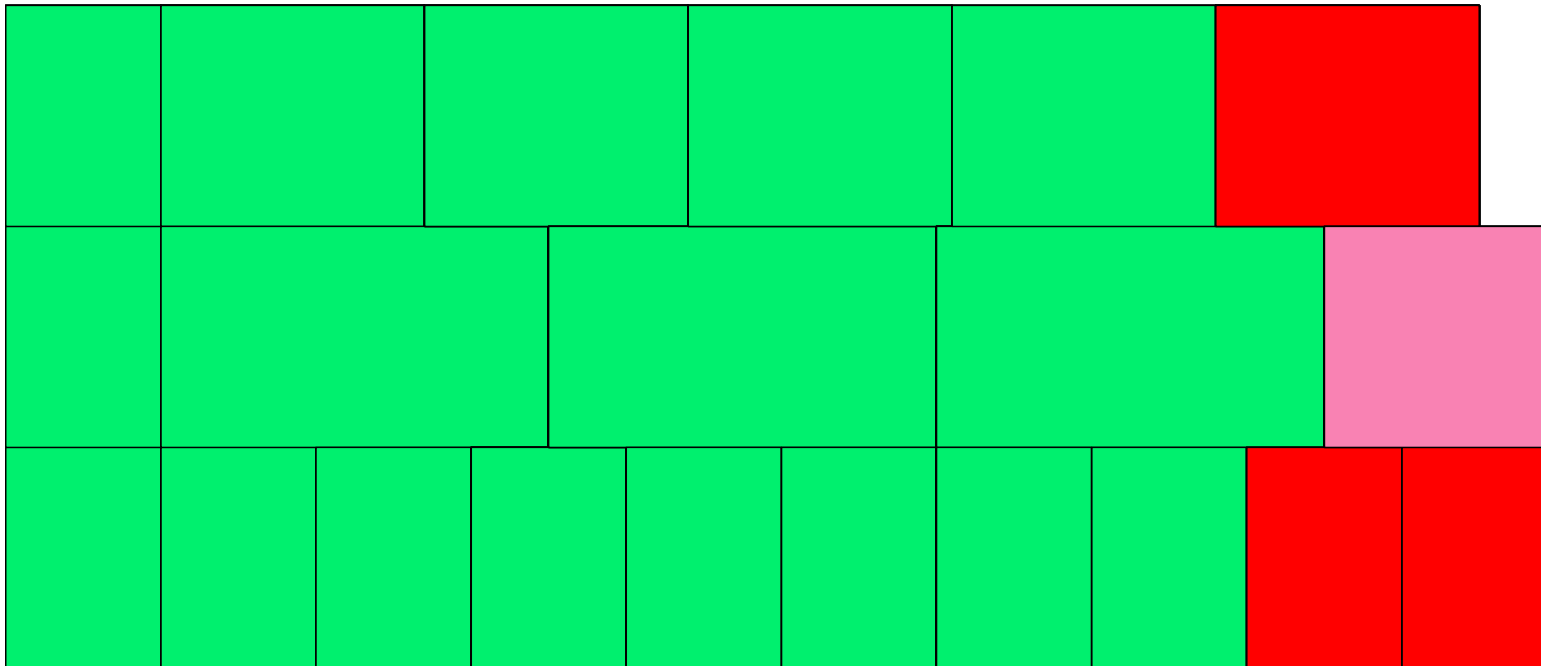
Action: Redoing steps



$t = 0.1$
 $n = 2.75$

Multi-adaptive Time-Stepping

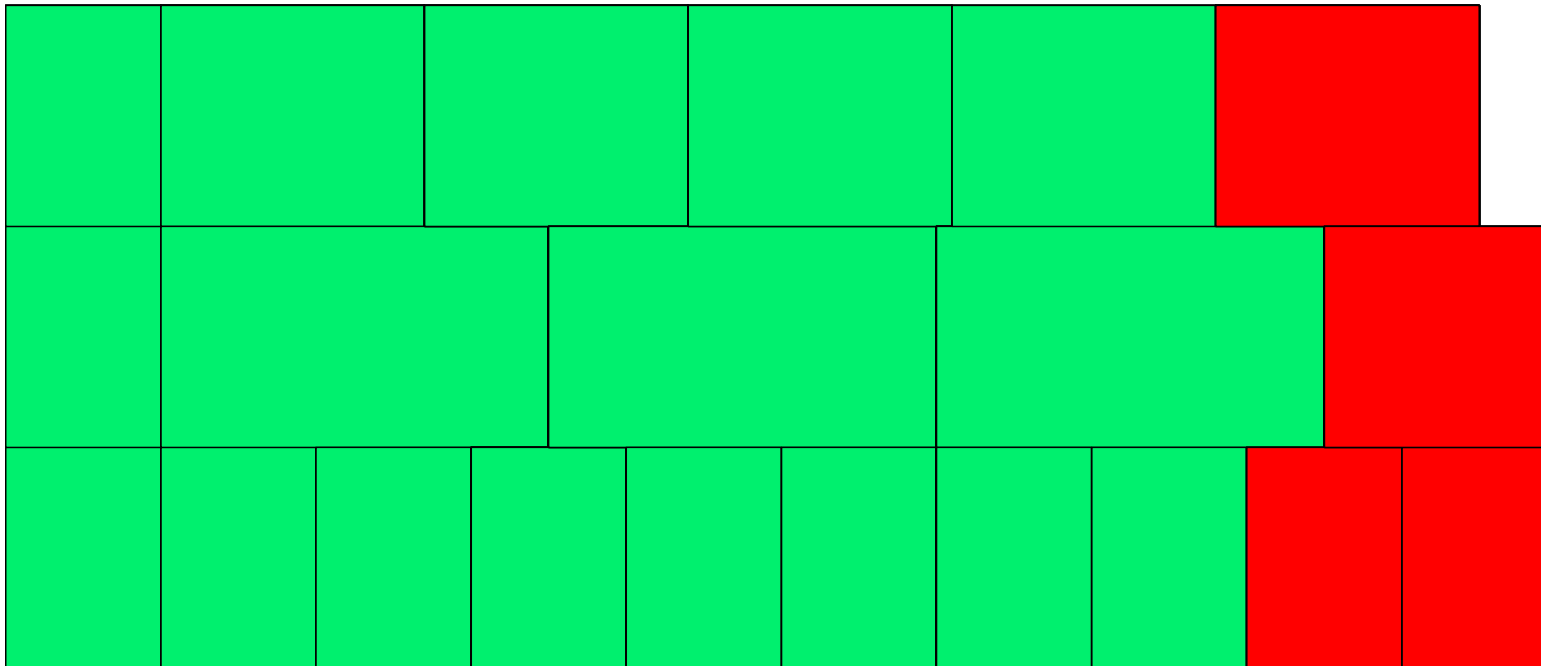
Action: Redoing steps



$t = 0.1$
 $n = 2.75$

Multi-adaptive Time-Stepping

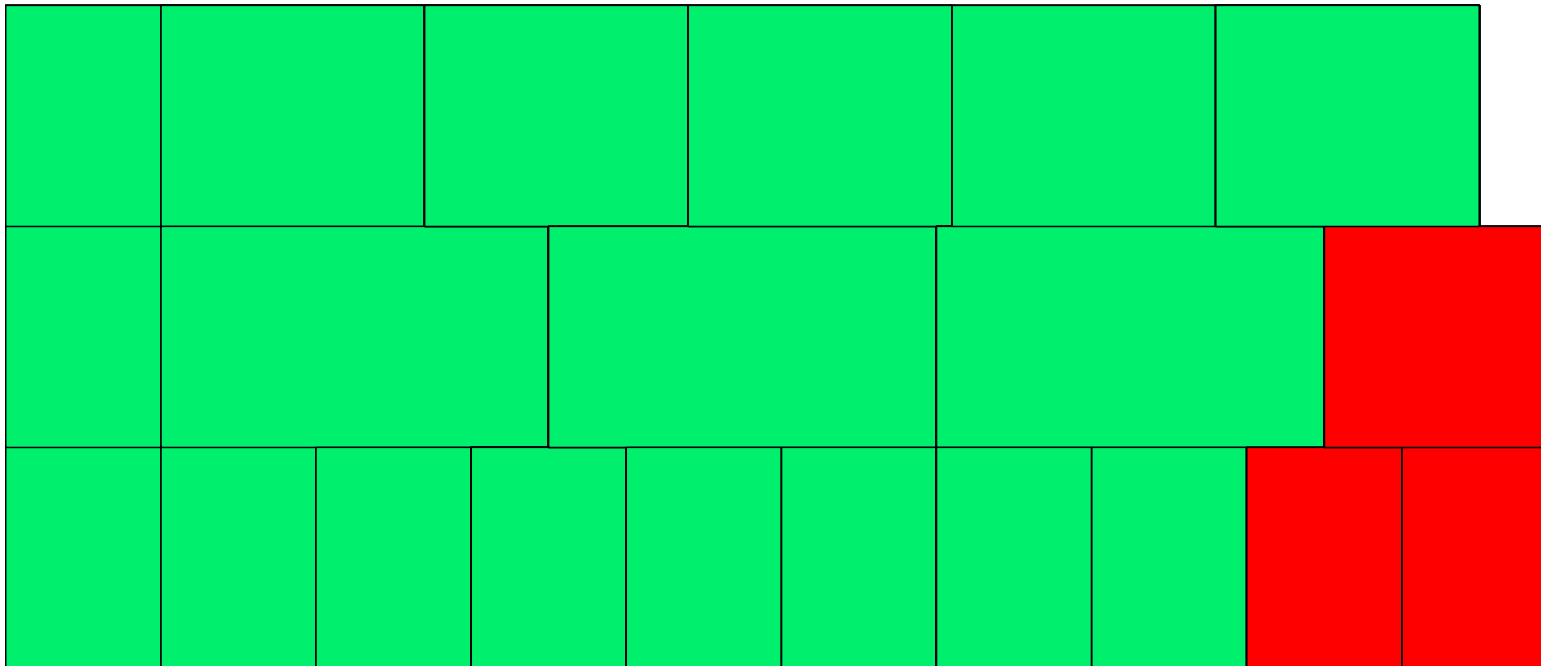
Action: Redoing steps



$t = 0.1$
 $n = 3$

Multi-adaptive Time-Stepping

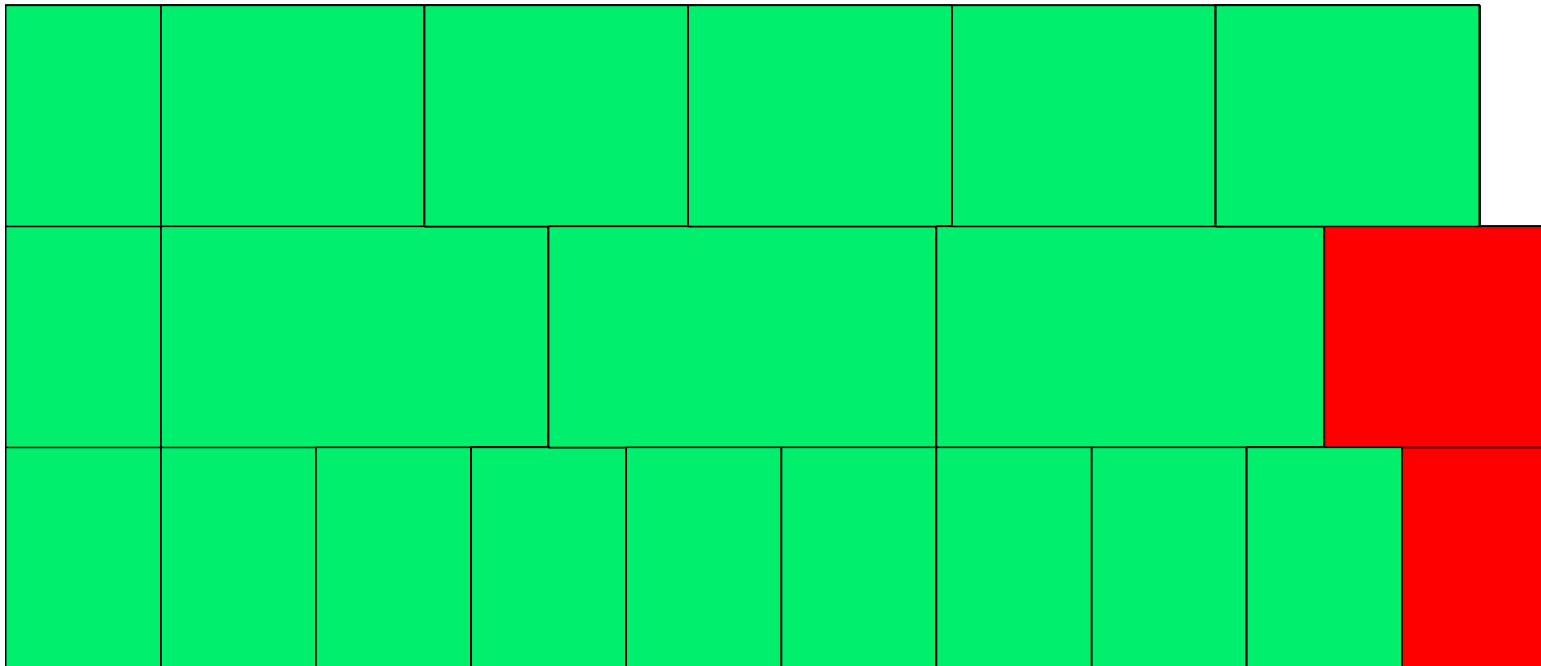
Action: Clearing steps



$t = 0.095$
 $n = 3.25$

Multi-adaptive Time-Stepping

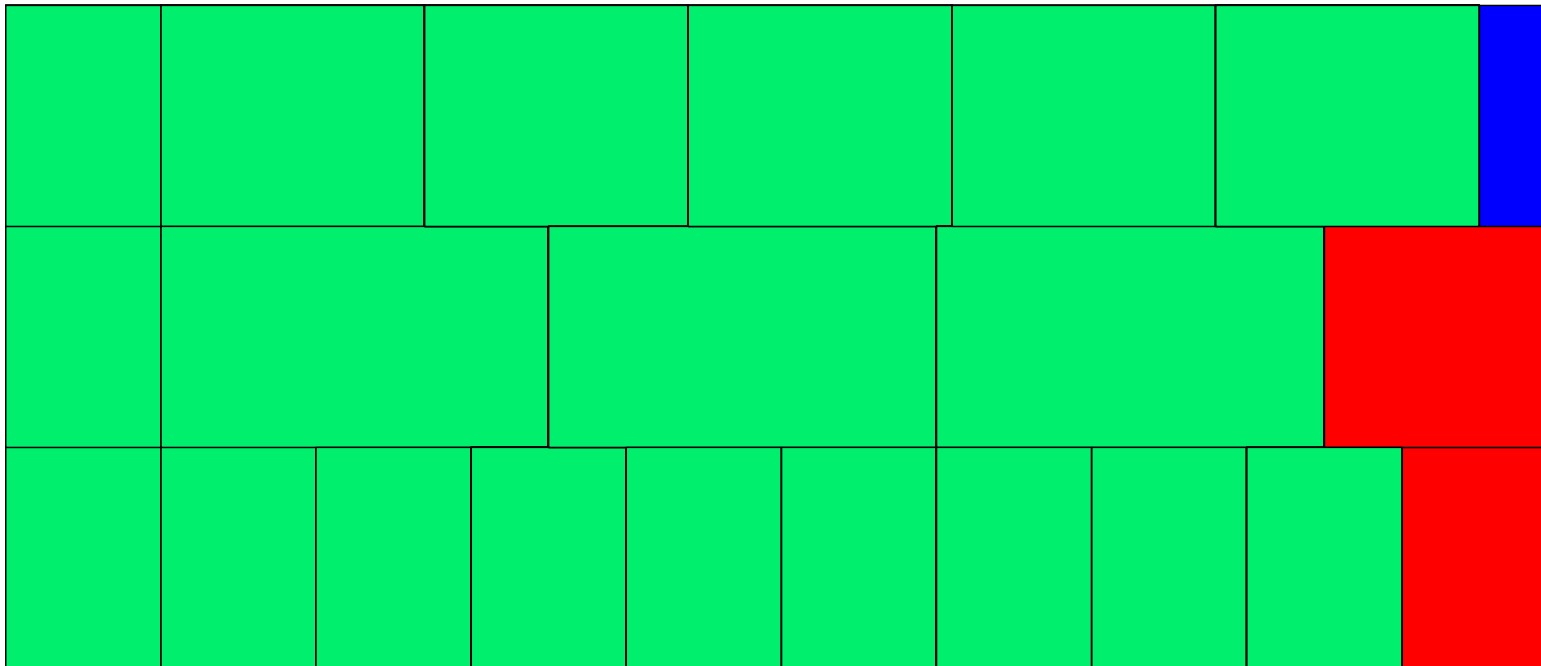
Action: Clearing steps



$t = 0.09$
 $n = 3.25$

Multi-adaptive Time-Stepping

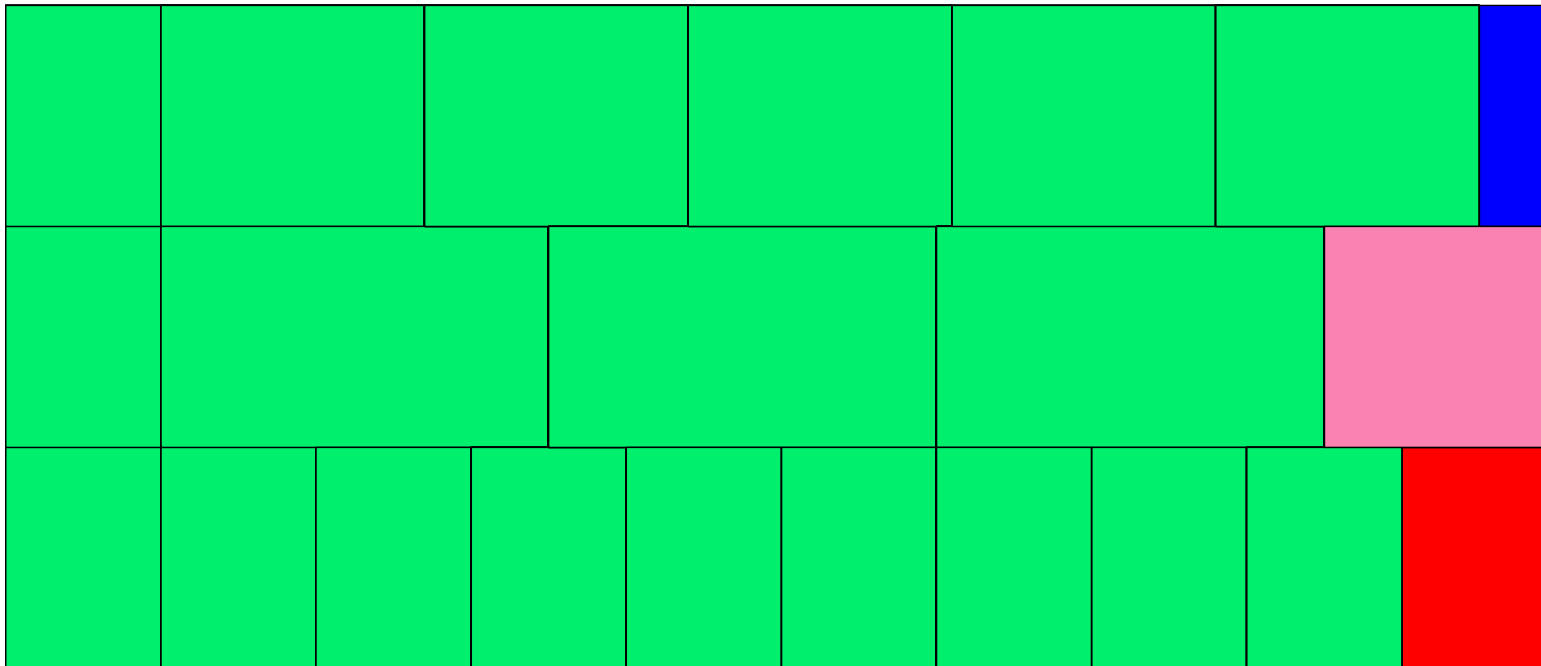
Action: Covering



$t = 0.1$
 $n = 3.25$

Multi-adaptive Time-Stepping

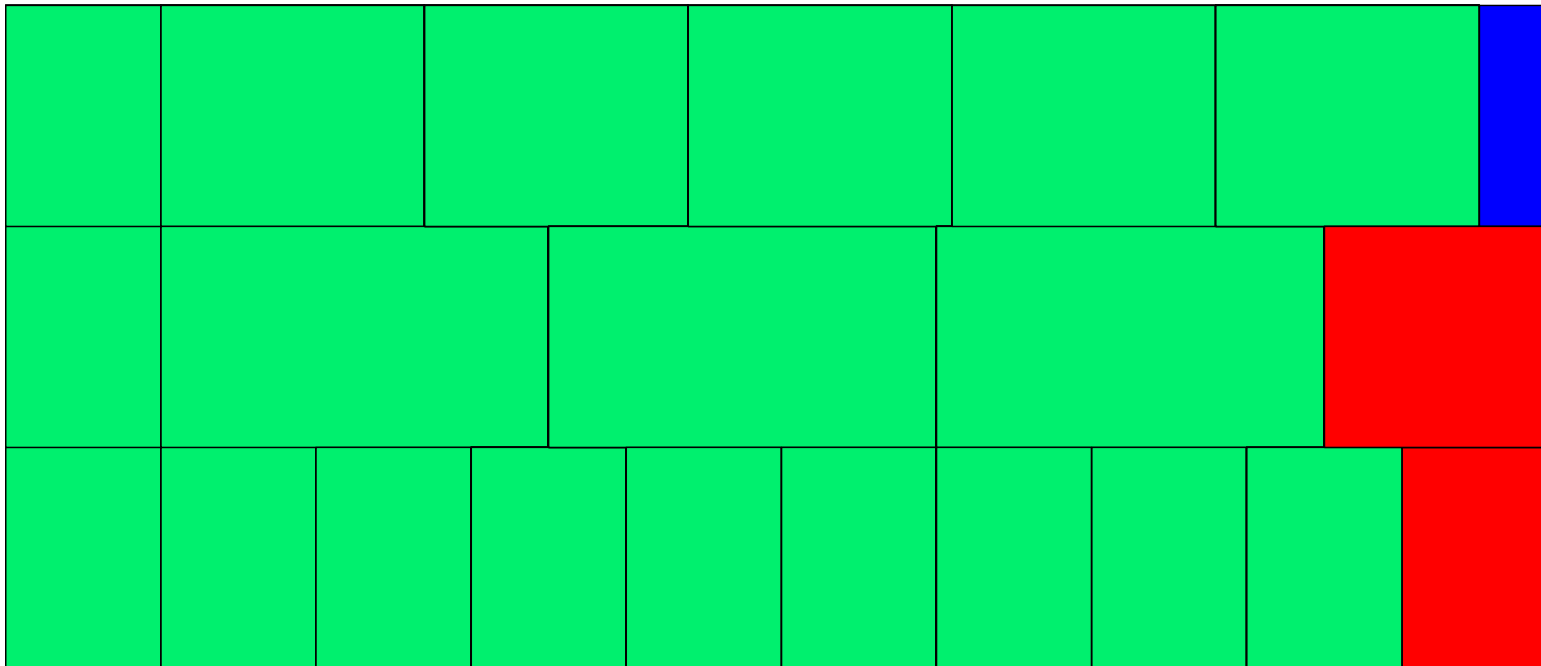
Action: Redoing steps



$t = 0.1$
 $n = 3.25$

Multi-adaptive Time-Stepping

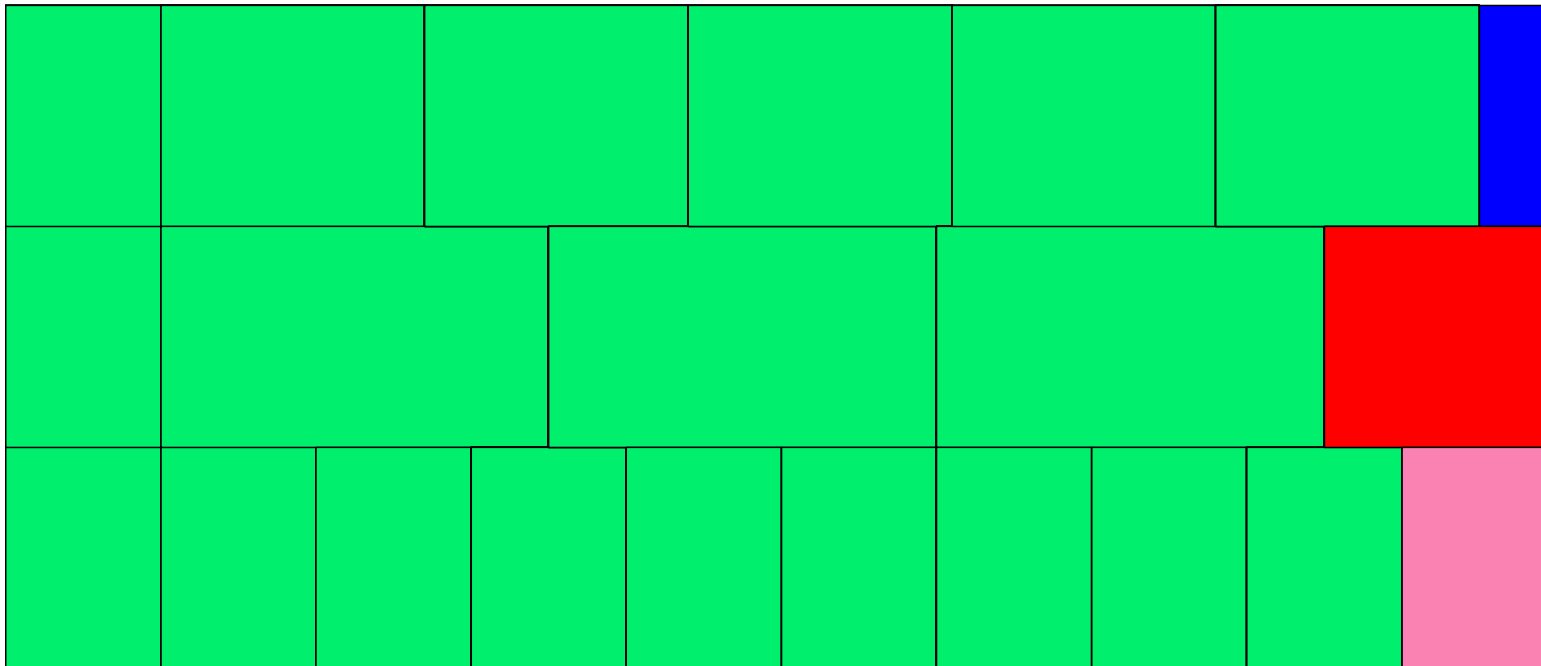
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

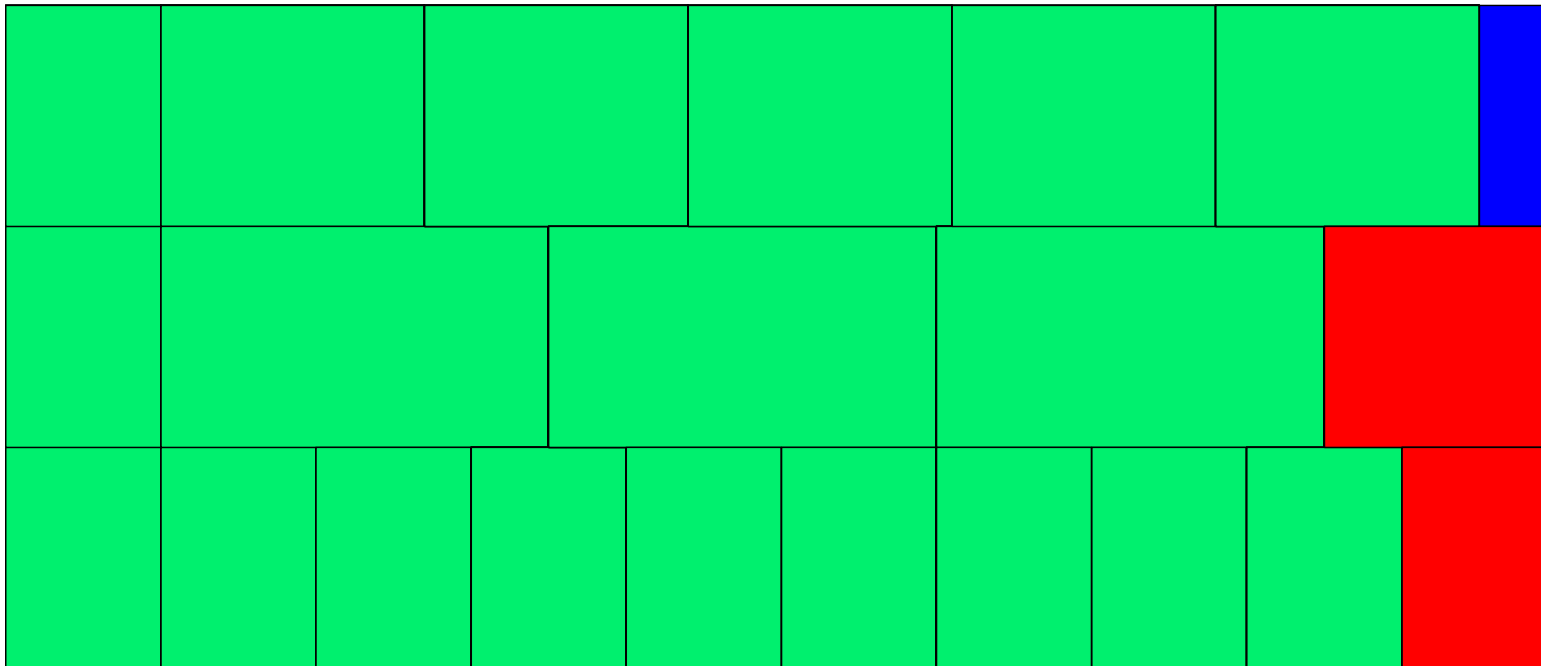
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

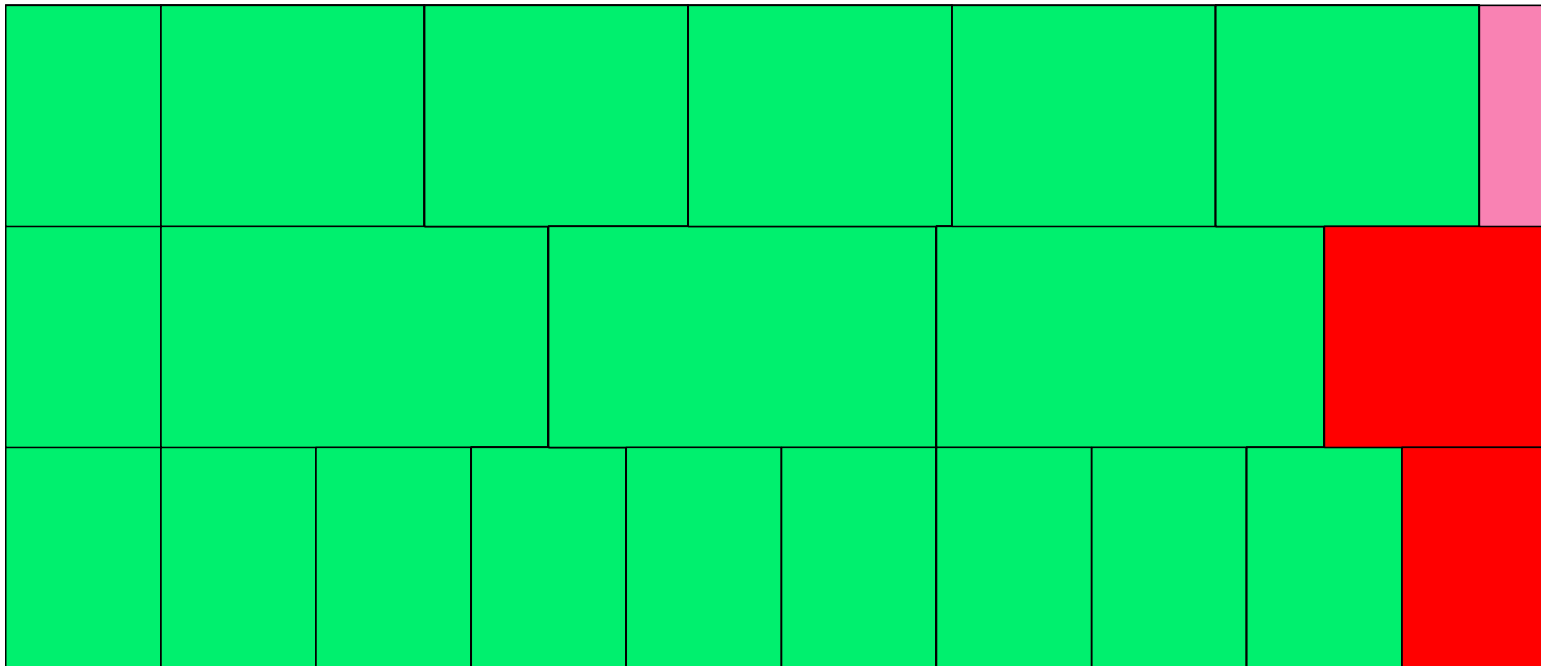
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

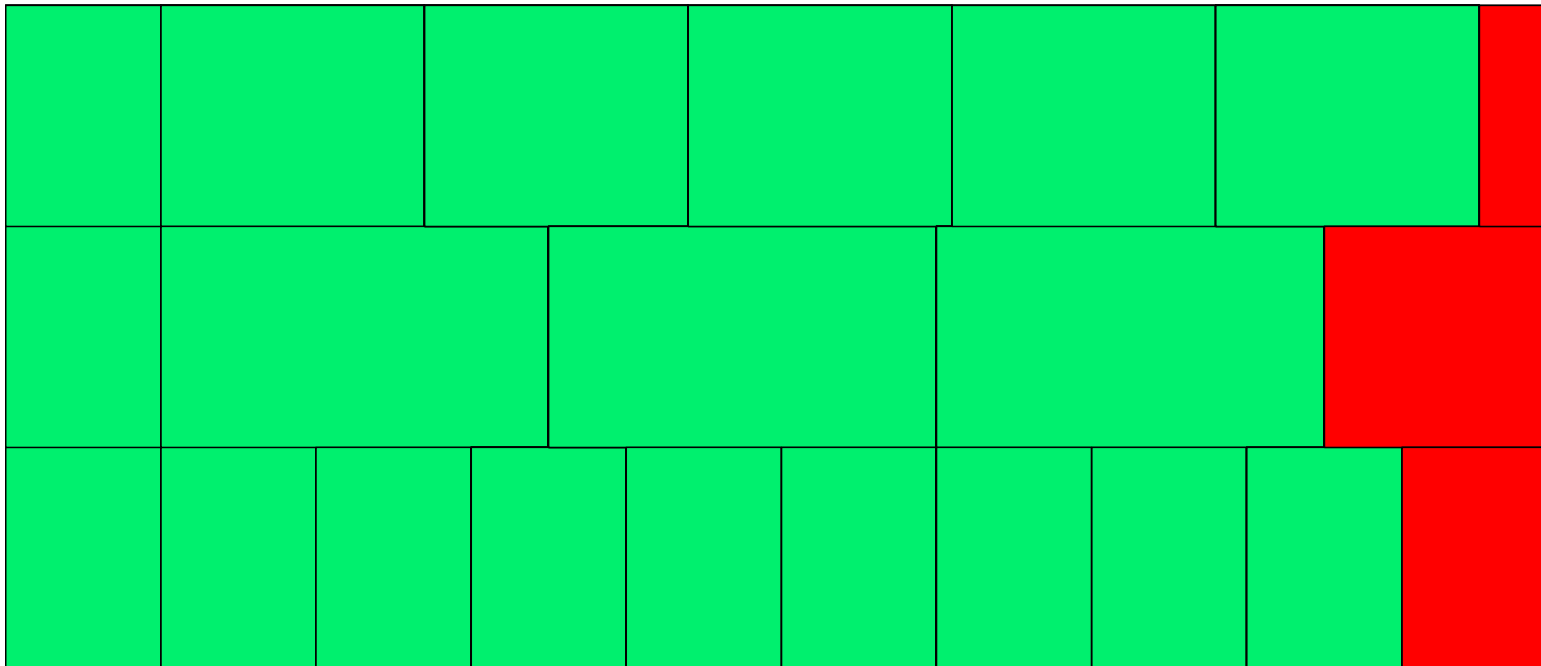
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

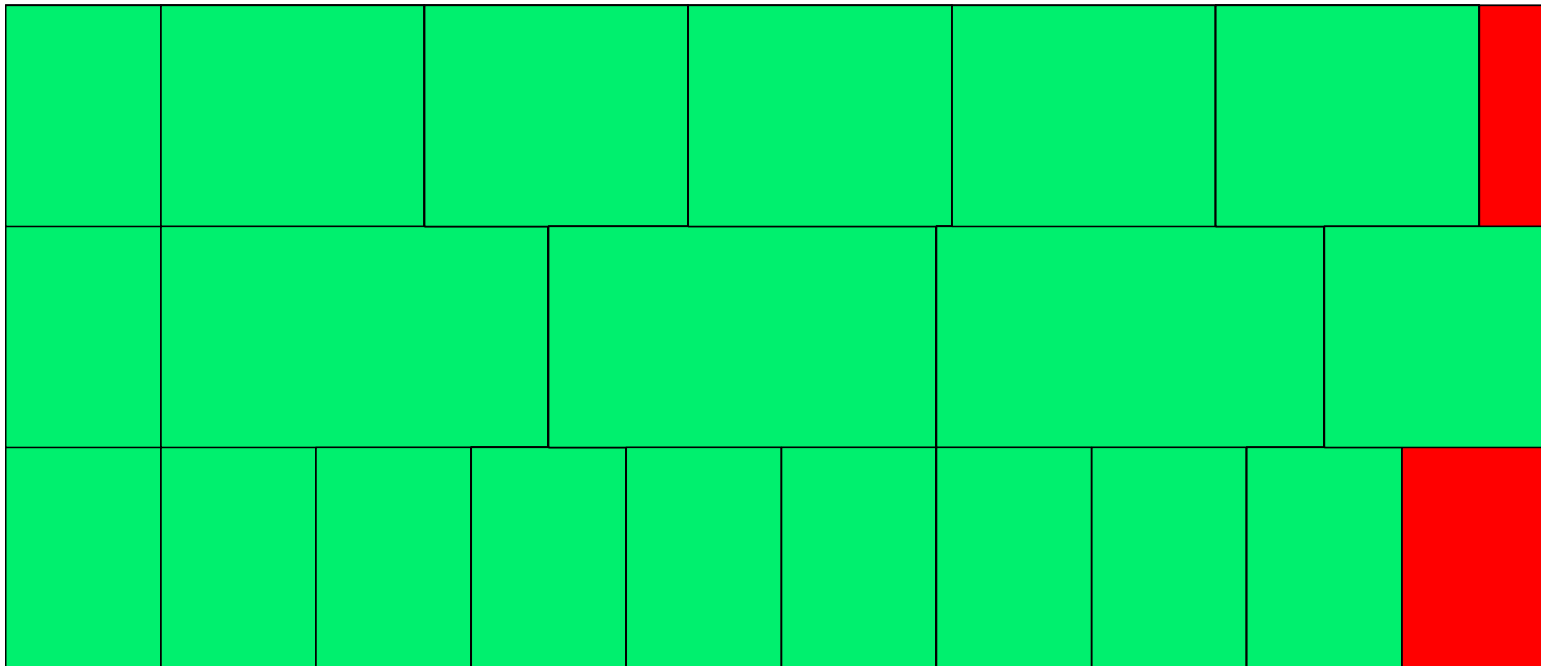
Action: Redoing steps



$t = 0.1$
 $n = 1$

Multi-adaptive Time-Stepping

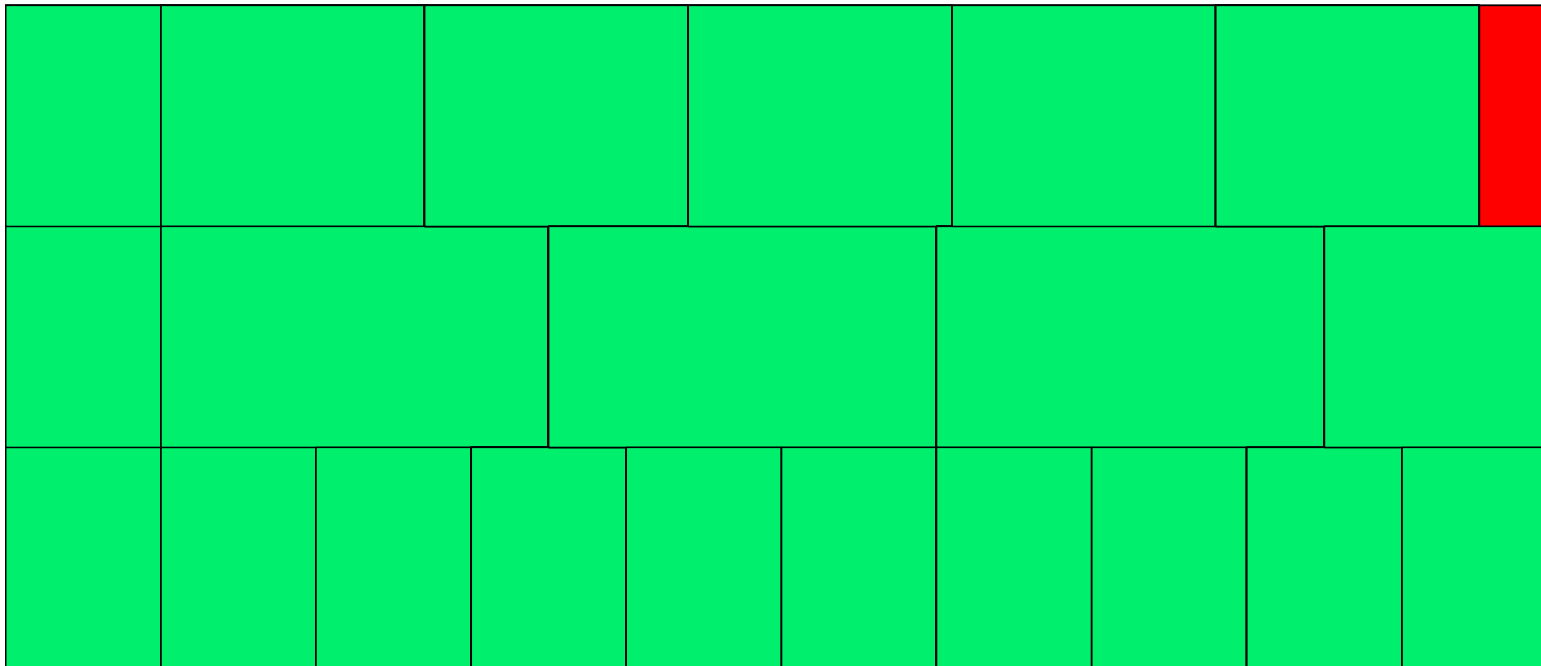
Action: Clearing steps



$t = 0.1$
 $n = 2$

Multi-adaptive Time-Stepping

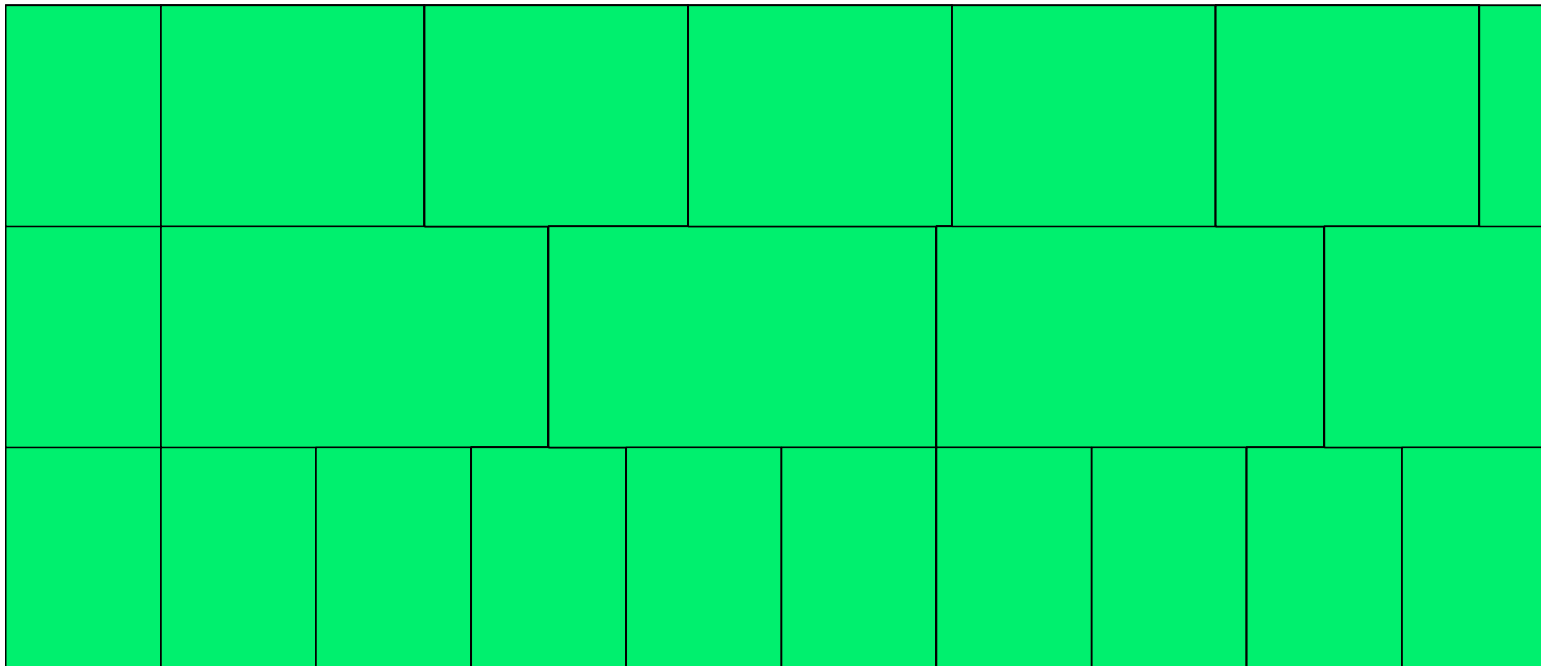
Action: Clearing steps



$t = 0.1$
 $n = 2$

Multi-adaptive Time-Stepping

Done



$t = 0.1$
 $n = 2$

first

General Order

- The $\text{mcG}(q)$ method is of order $2q$, i.e. locally of order $2q_{ij}$.
- The $\text{mdG}(q)$ method is of order $2q + 1$, i.e. locally of order $2q_{ij} + 1$.

Simple Galerkin thus gives methods of arbitrary order with individual timesteps, such as e.g. the 40:th order method $\text{mcG}(20)$.

Error Estimates

- A priori error estimates,

$$\|e\| \leq \int_0^T (k^{2q}, w) dt,$$

$$\|e\| \leq \int_0^T (k^{2q+1}, w) dt.$$

- A posteriori error estimates,

$$\|e\| \leq E_G + E_C + E_Q.$$

Galerkin Errors

- Many different variants
- More or less local/global
- Two extremes (for the continuous method):

$$E_G = \left| \int_0^T (R, \varphi - \pi_k \varphi) dt \right| \leq \sum_{i=1}^N S_i \max\{C_{q_i} k_i^{q_i} |R_i|\}$$

Computational Errors

$$E_C \approx \sum_{i=1}^N \bar{S}_i^{[0]} \max_{[0,T]} |R_i^C|,$$

where

$$R_i^C = \frac{1}{k_{ij}} \left[U(t_{ij}) - U(t_{i,j-1}) - \int_{I_{ij}} f_i(U, \cdot) dt \right]$$

is the *discrete residual*. (For the discontinuous method, $t_{ij}, t_{i,j-1}$ are replaced by $t_{ij}^-, t_{i,j-1}^-$.)



Quadrature Errors

$$E_Q \approx \sum_{i=1}^N \bar{S}_i^{[0]} \max_{[0,T]} |R_i^Q|,$$

where

$$R_i^Q = \frac{1}{k_{ij}} \left[\int_{I_{ij}}^{\tilde{}} f_i(U, \cdot) dt - \int_{I_{ij}} f_i(U, \cdot) dt \right]$$

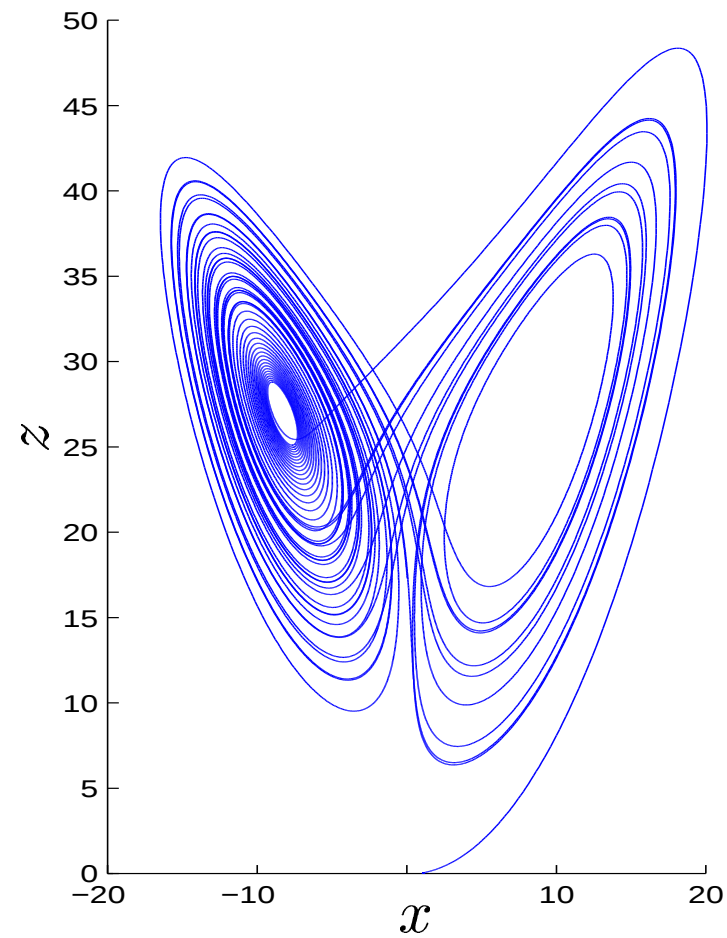
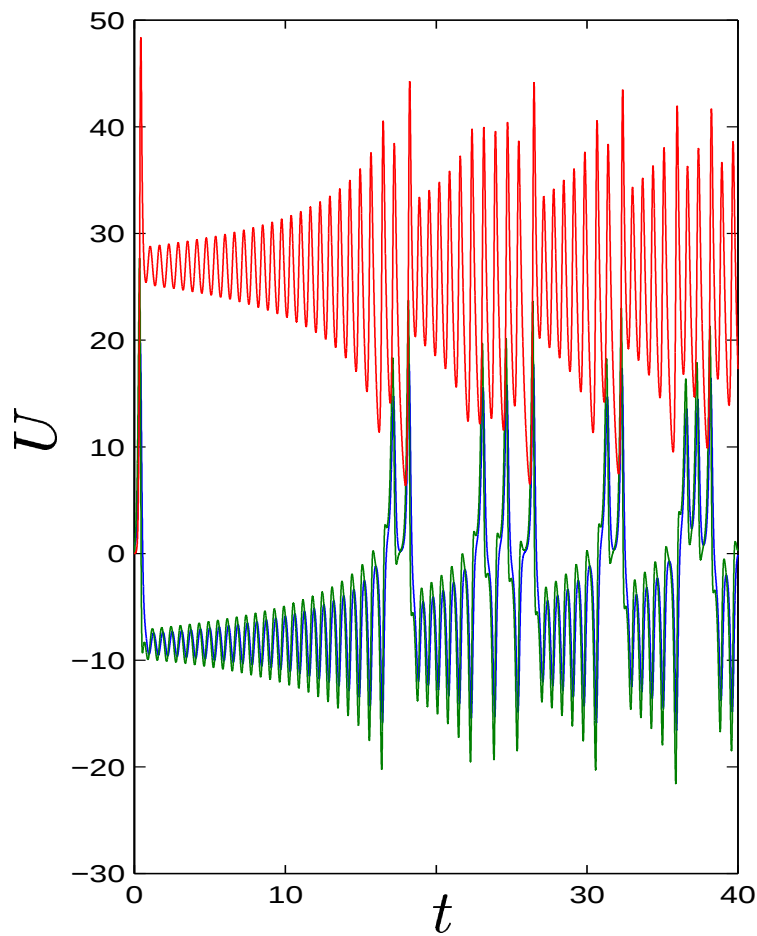
is the *quadrature residual*.

The Lorenz System

$$\begin{cases} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz, \end{cases}$$

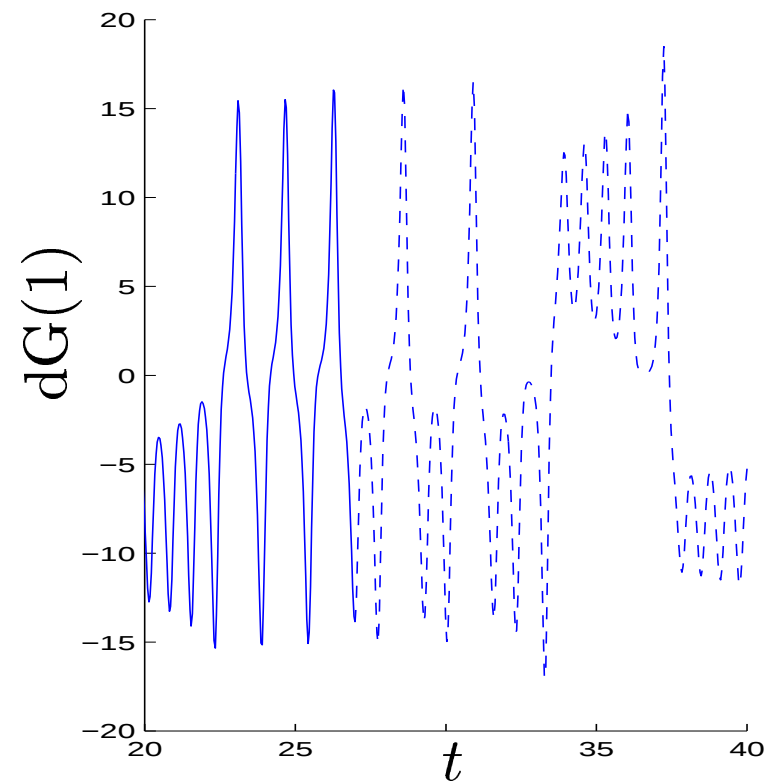
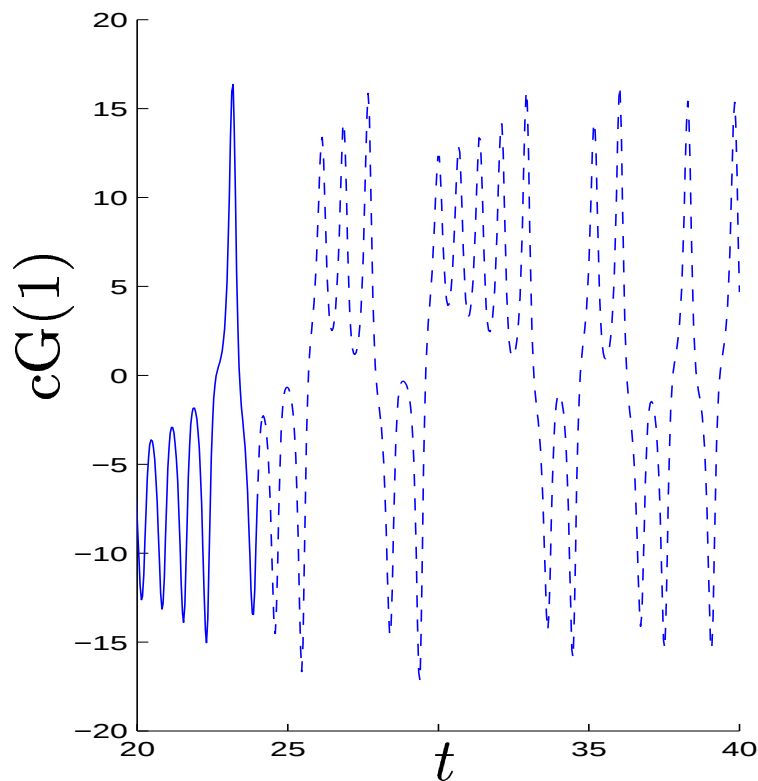
where we, as usual, take $(x_0, y_0, z_0) = (1, 0, 0)$,
 $\sigma = 10$, $b = 8/3$ and $r = 28$.

The Solution?



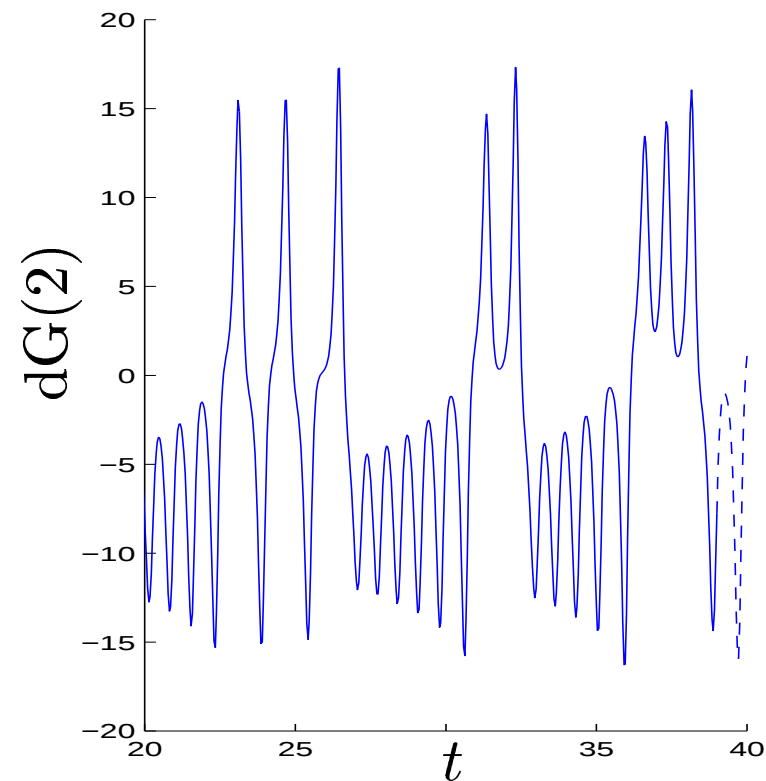
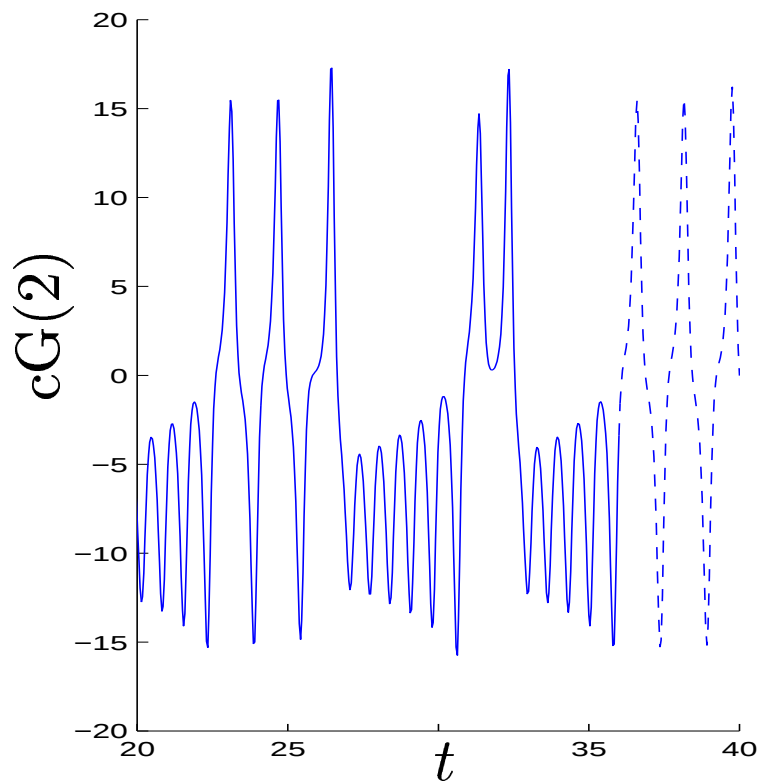
A Simple Experiment

- $T = 40$
- $k = 0.001$



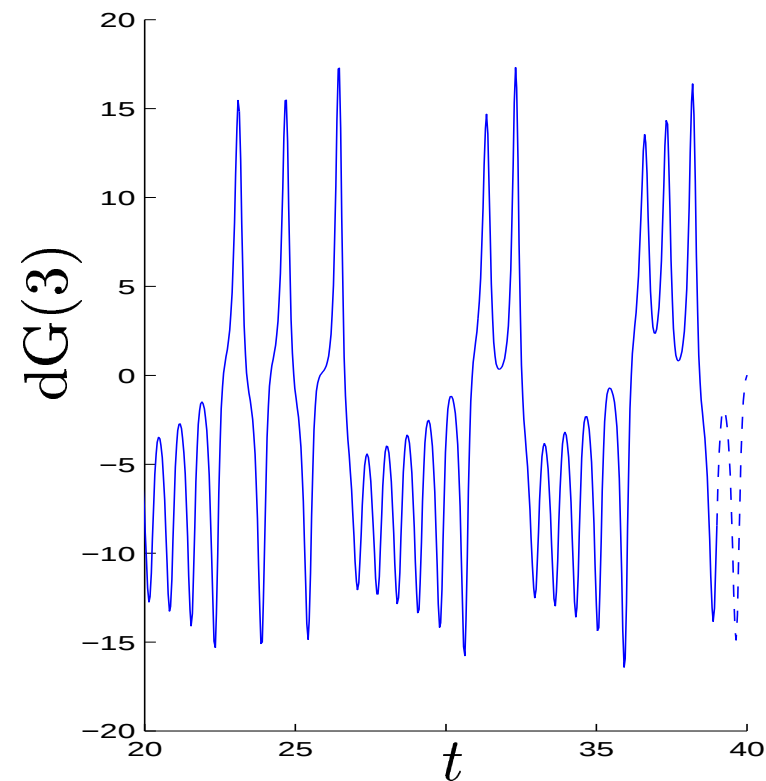
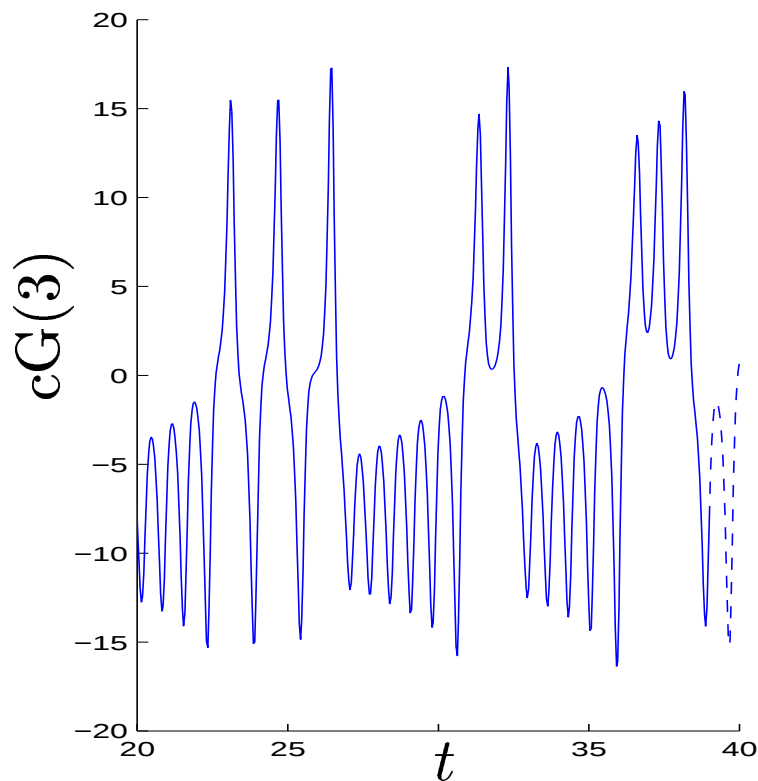
A Simple Experiment

- $T = 40$
- $k = 0.001$



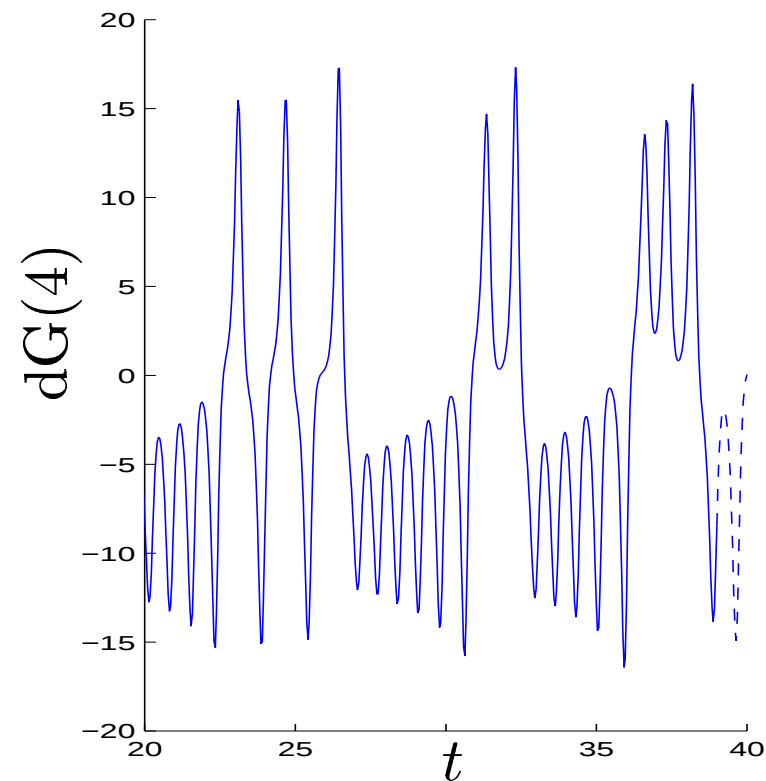
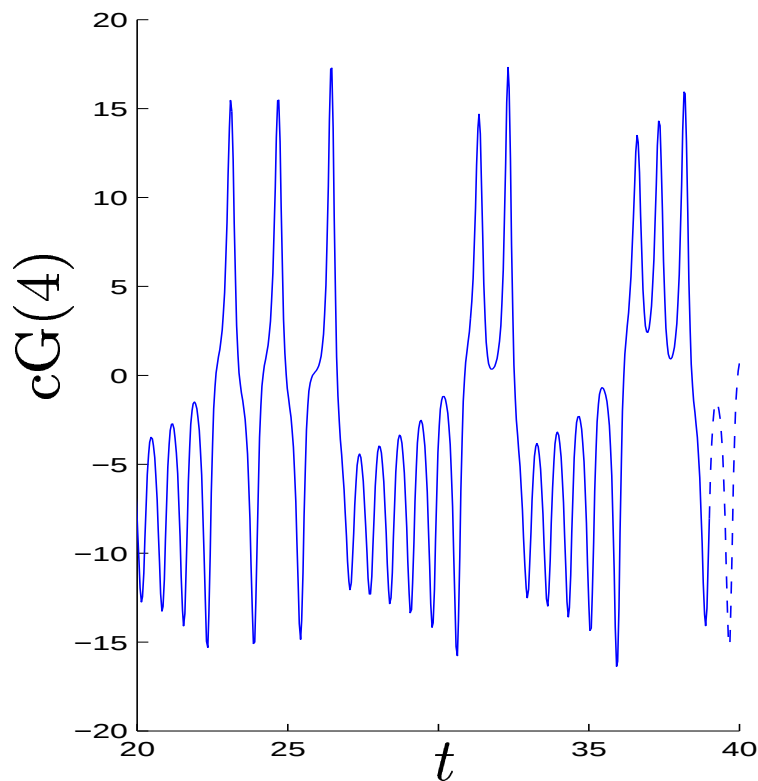
A Simple Experiment

- $T = 40$
- $k = 0.001$



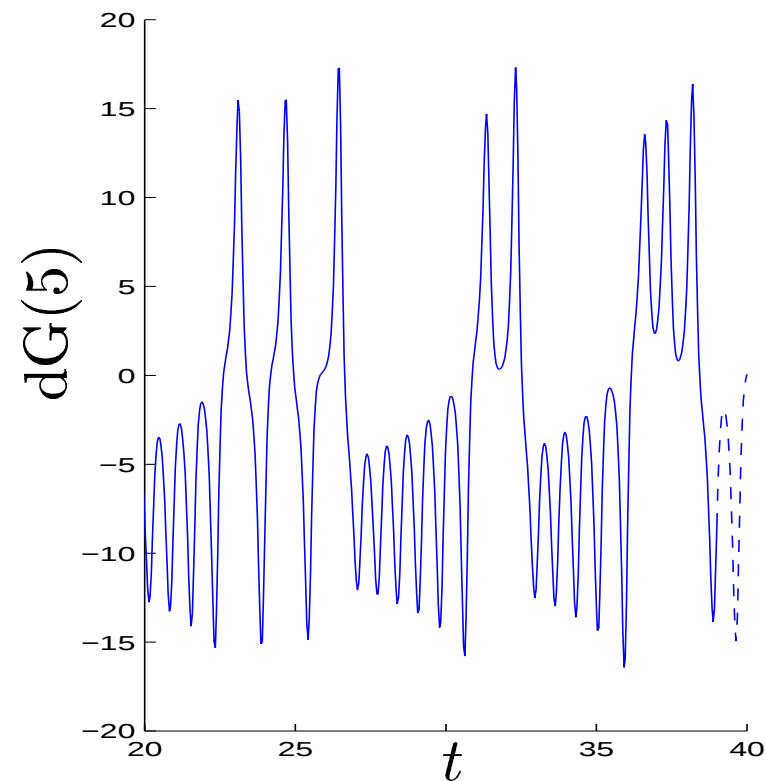
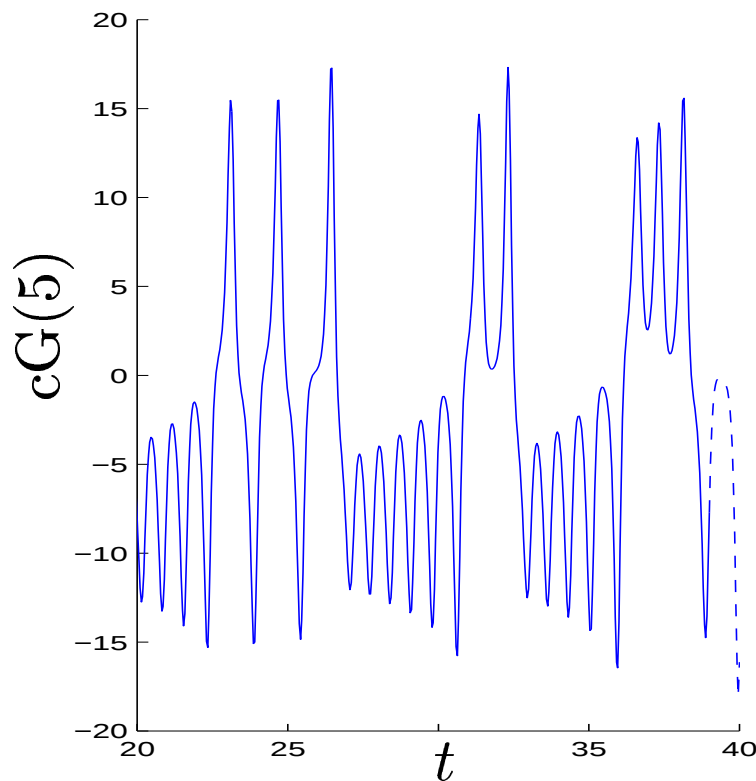
A Simple Experiment

- $T = 40$
- $k = 0.001$



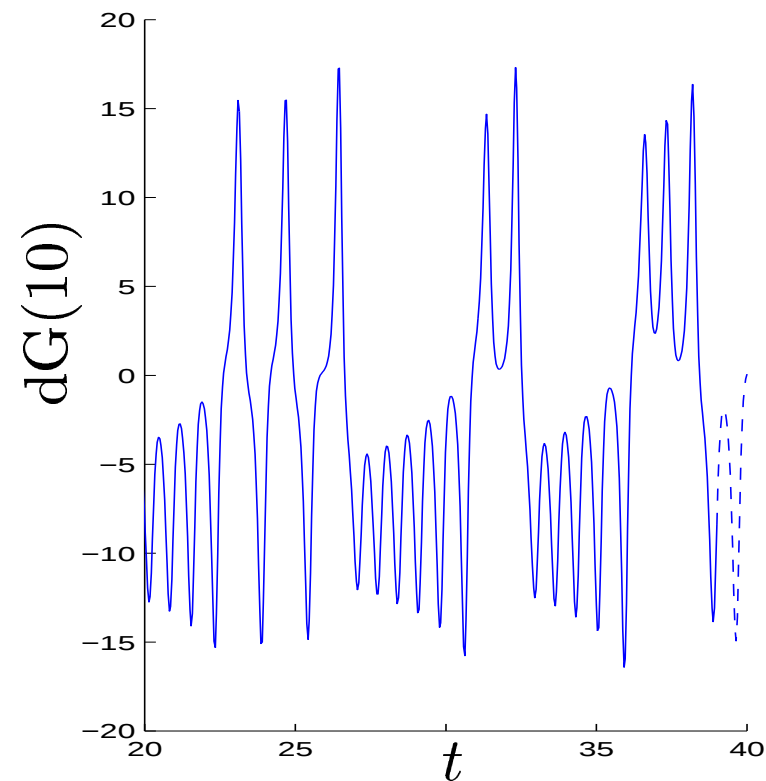
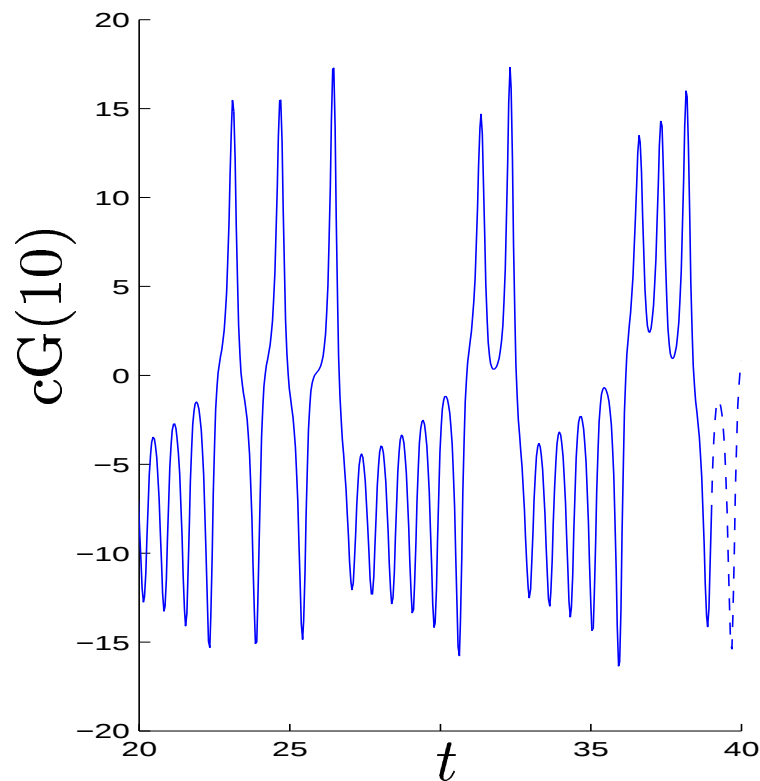
A Simple Experiment

- $T = 40$
- $k = 0.001$

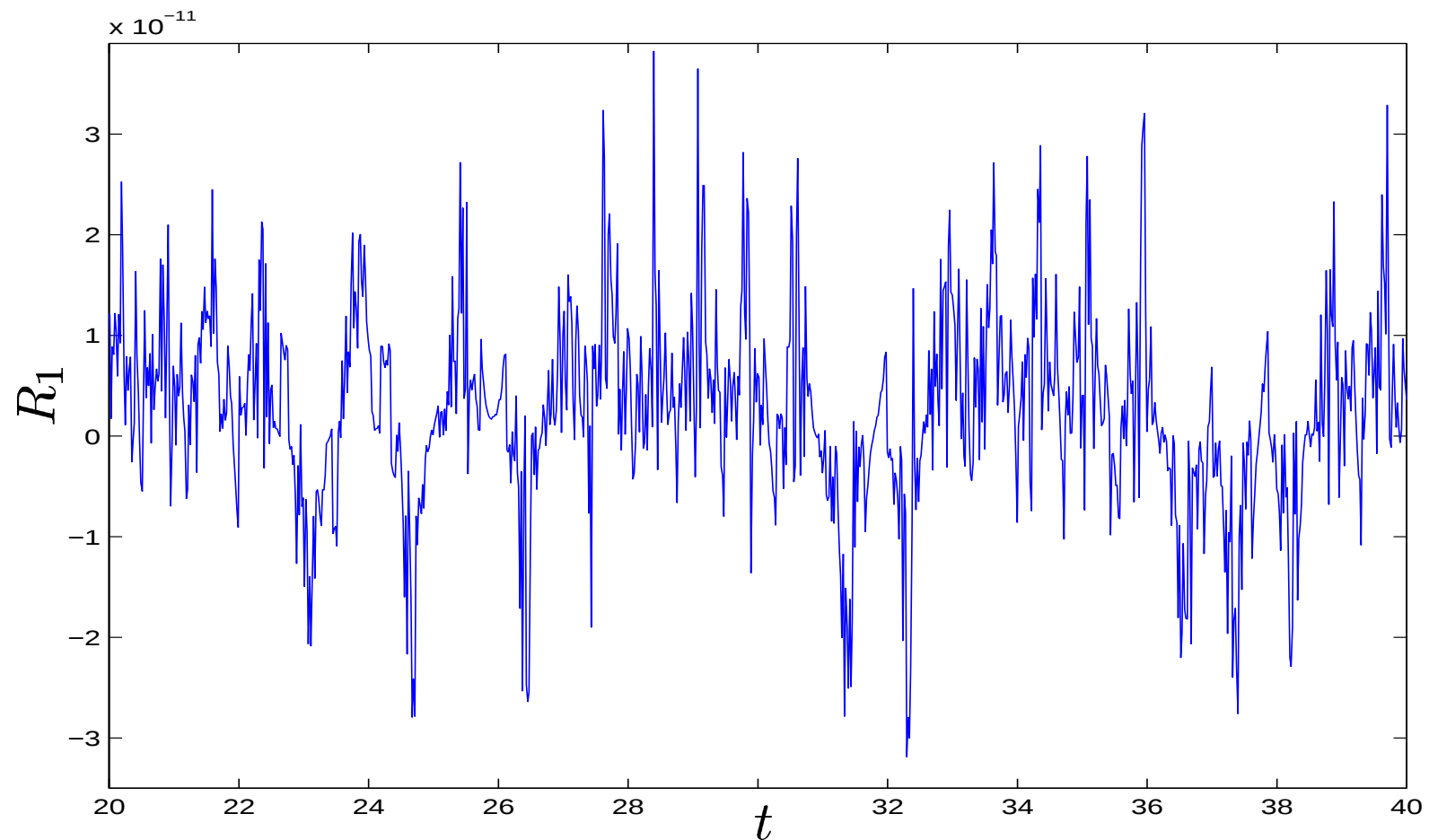


A Simple Experiment

- $T = 40$
- $k = 0.001$



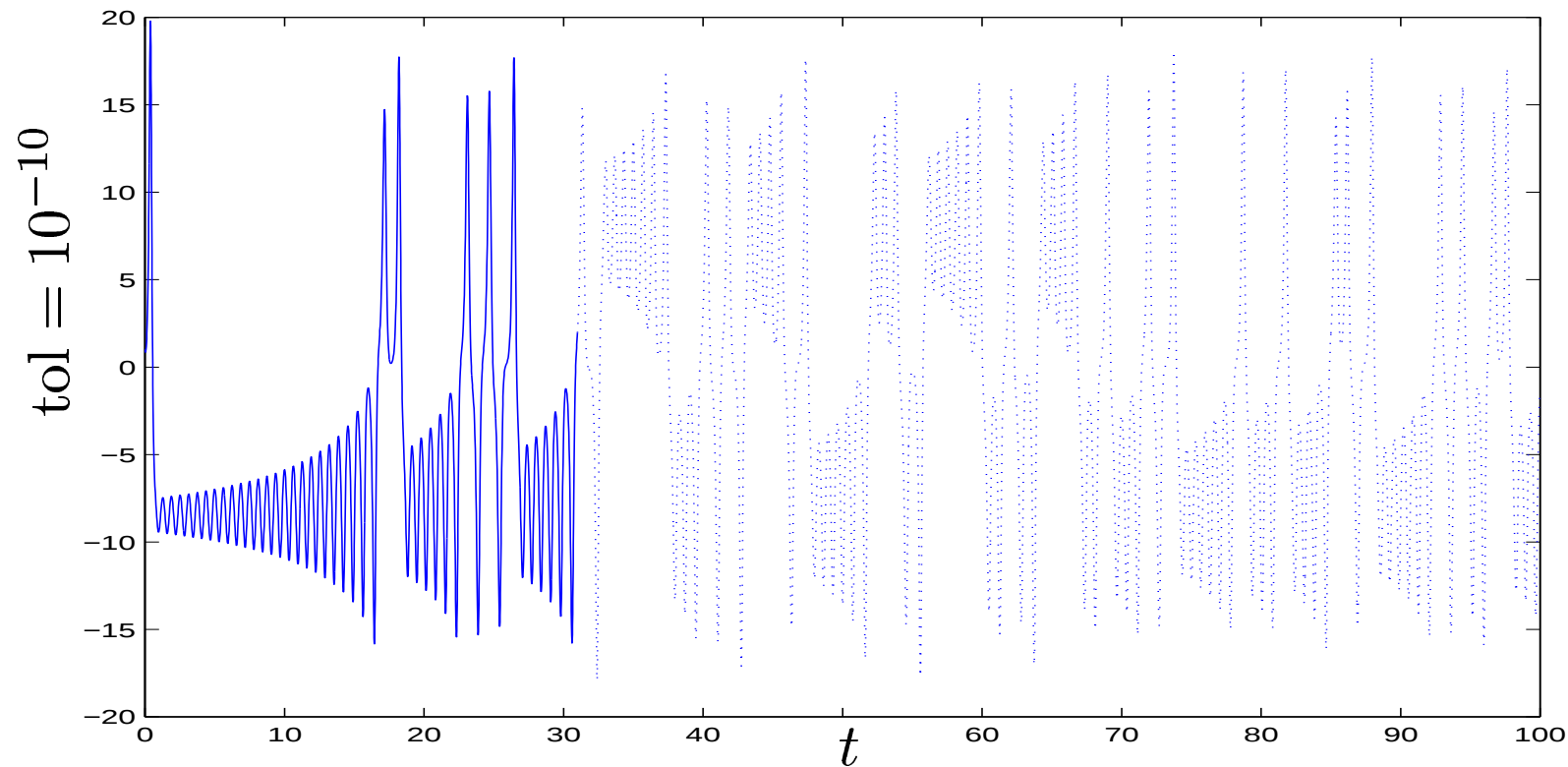
The Residual



[cG(5) residual with $k = 0.001$]

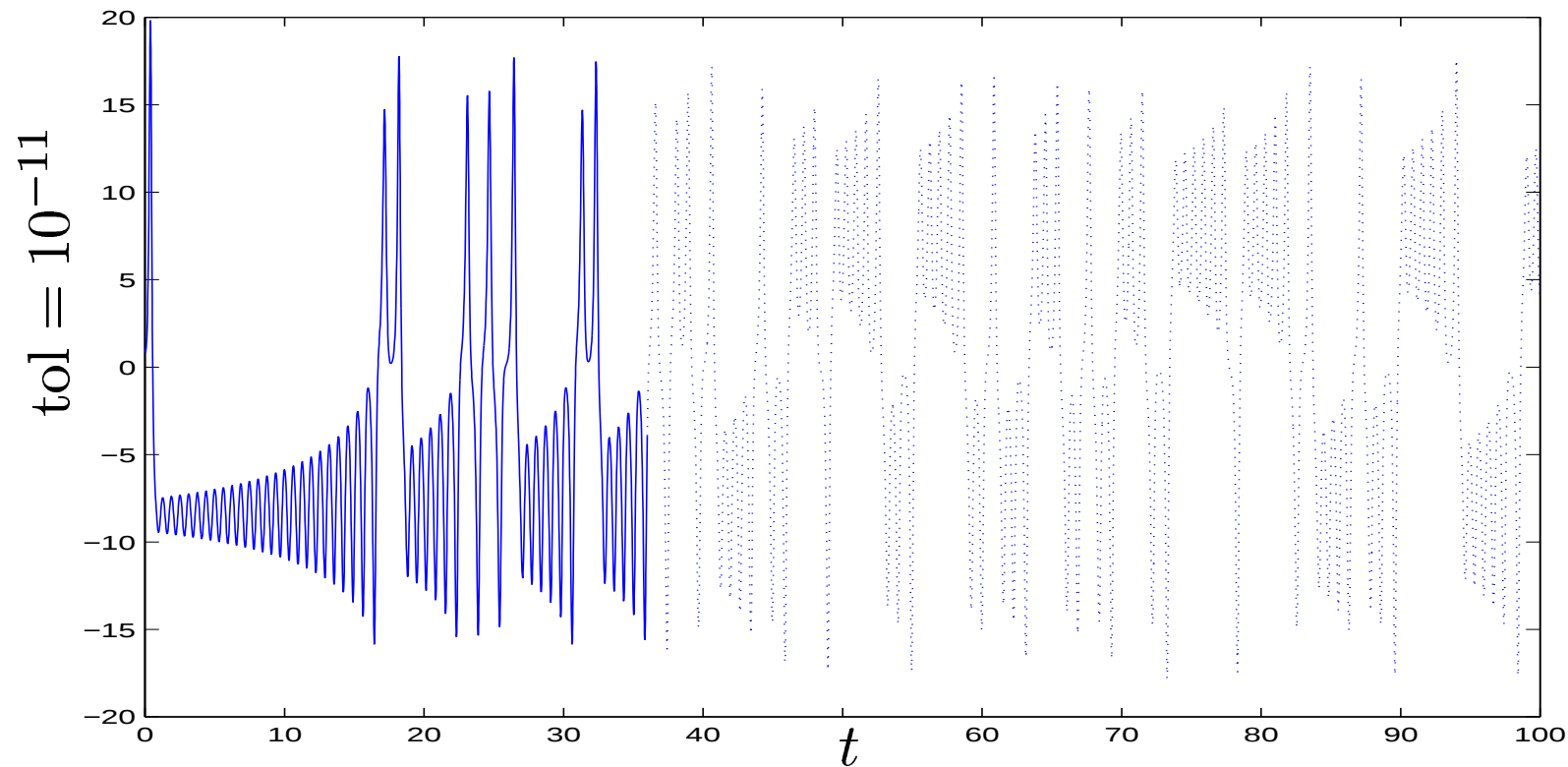
ode45

Trying the same thing with Matlabs ode45, we get the following results:



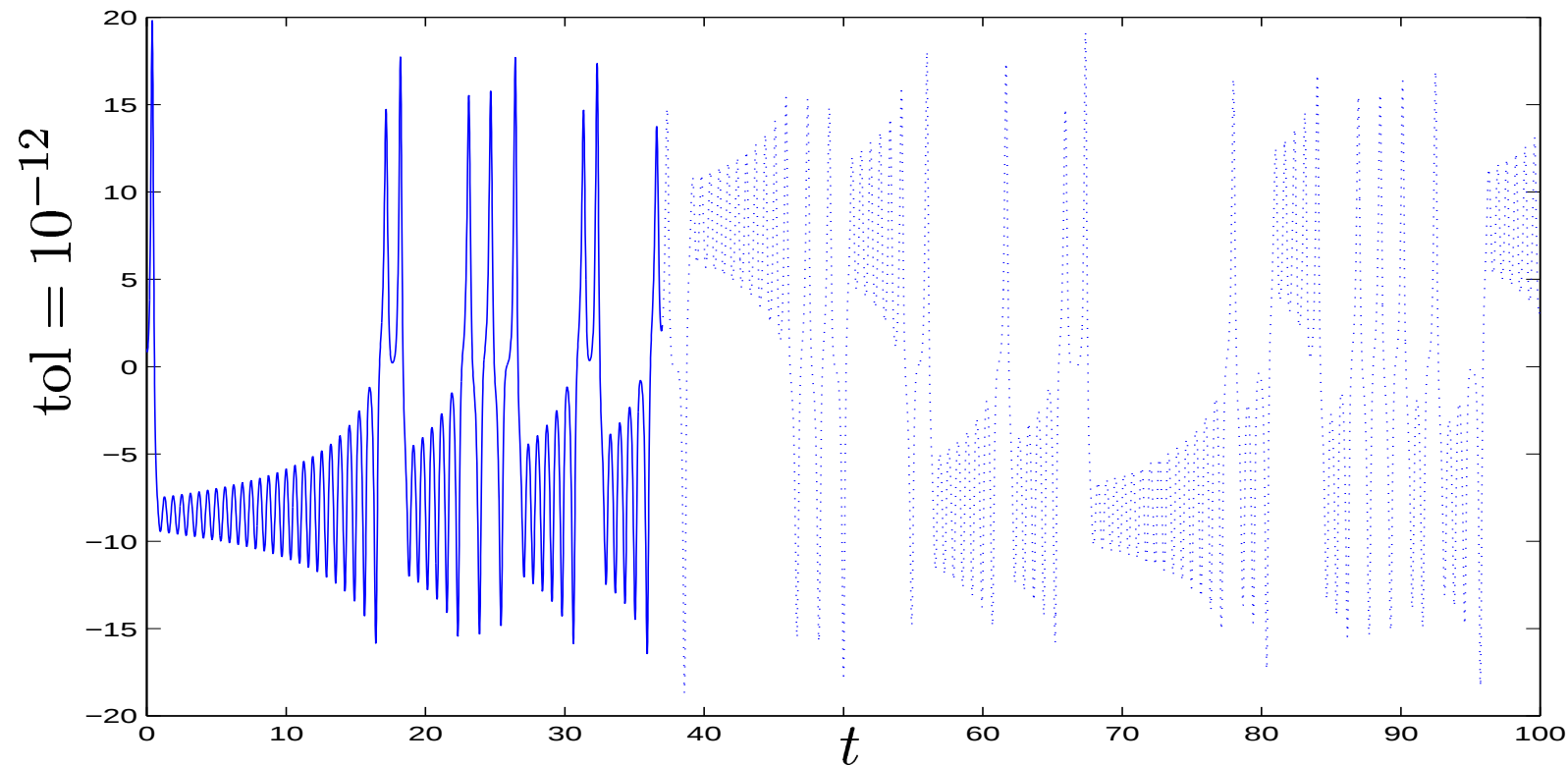
ode45

Trying the same thing with Matlabs ode45, we get the following results:



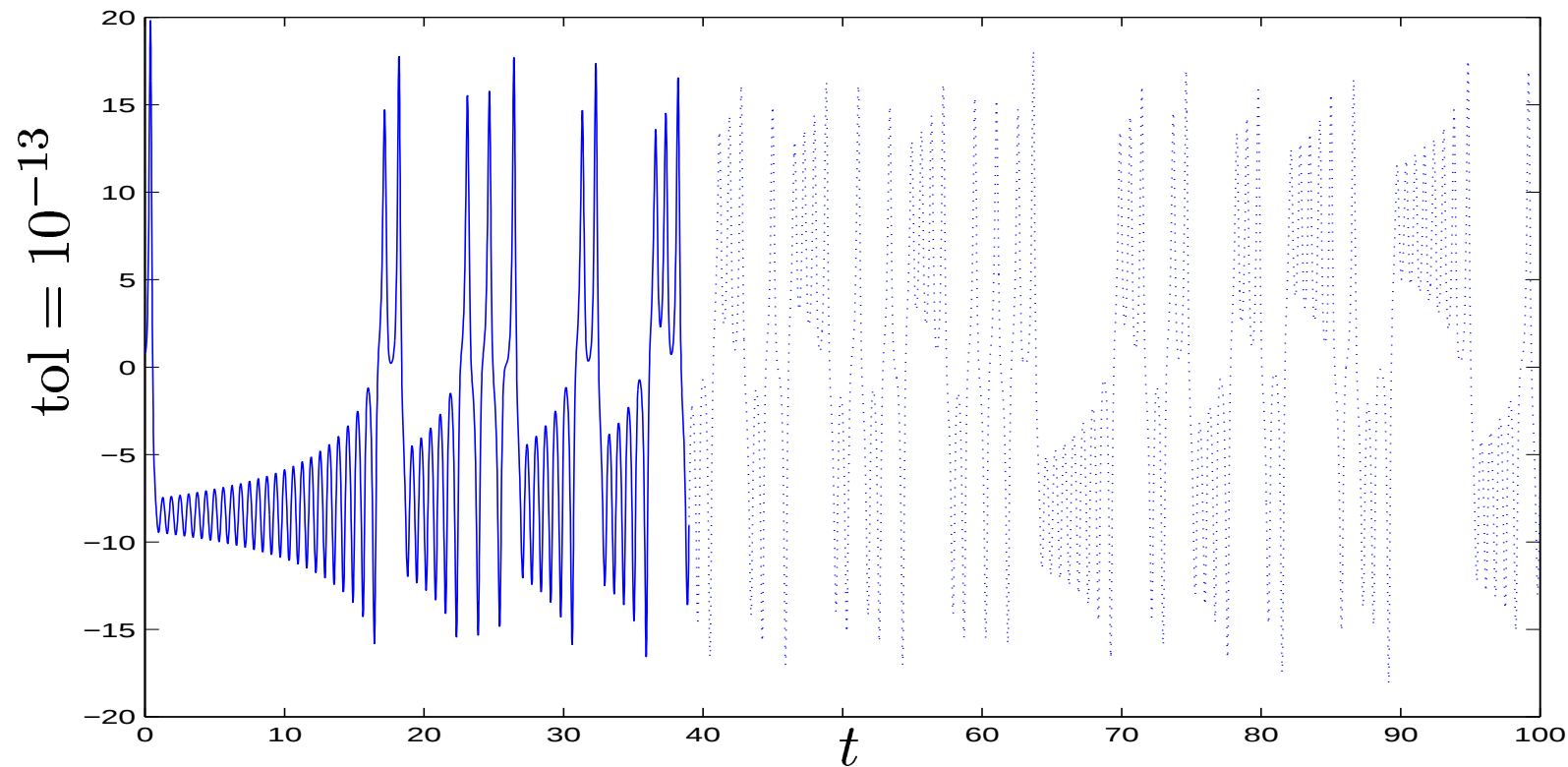
ode45

Trying the same thing with Matlabs ode45, we get the following results:



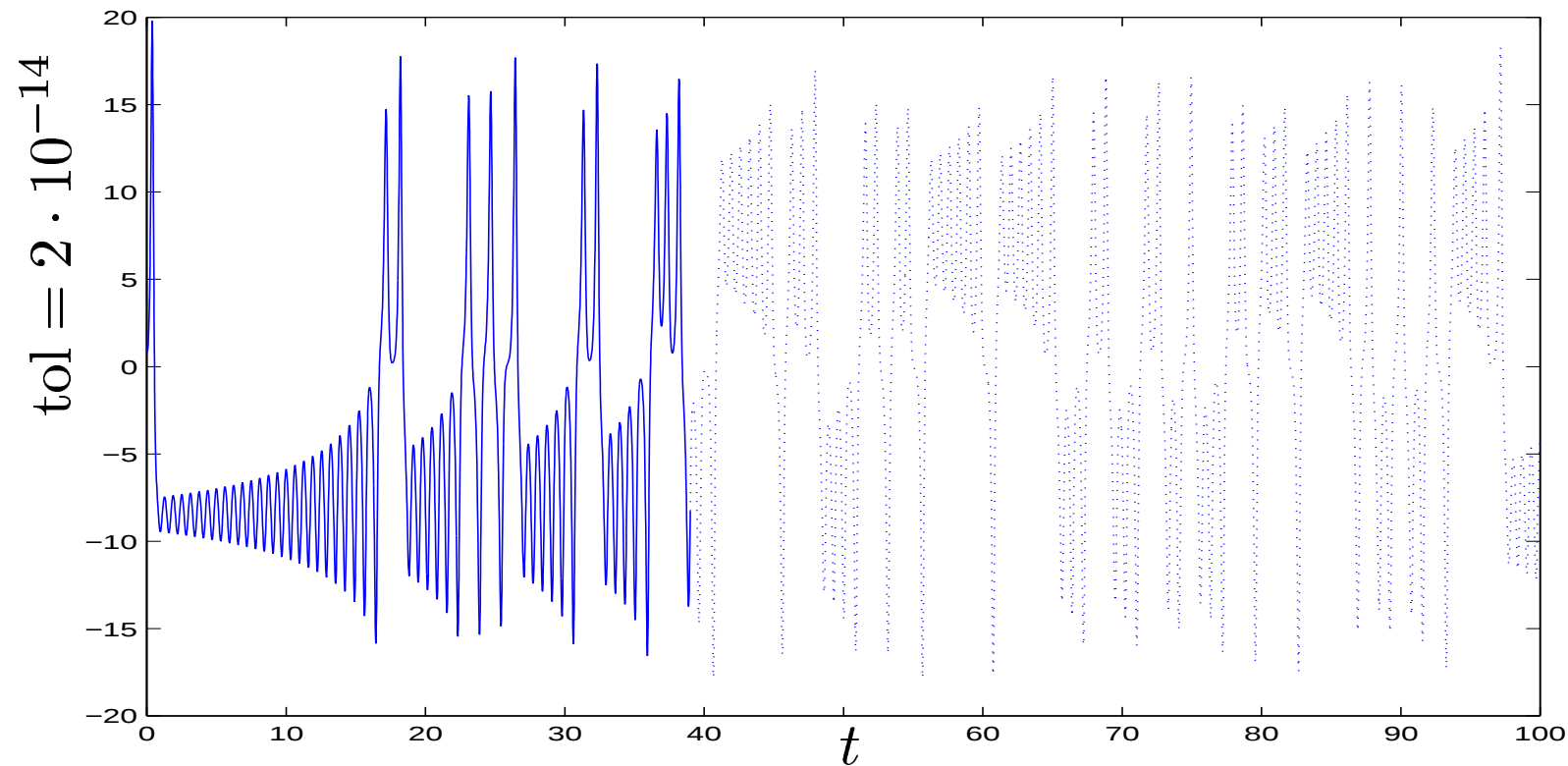
ode45

Trying the same thing with Matlabs ode45, we get the following results:



ode45

Trying the same thing with Matlabs ode45, we get the following results:



Getting Further

- Small Galerkin error
- Large computational error

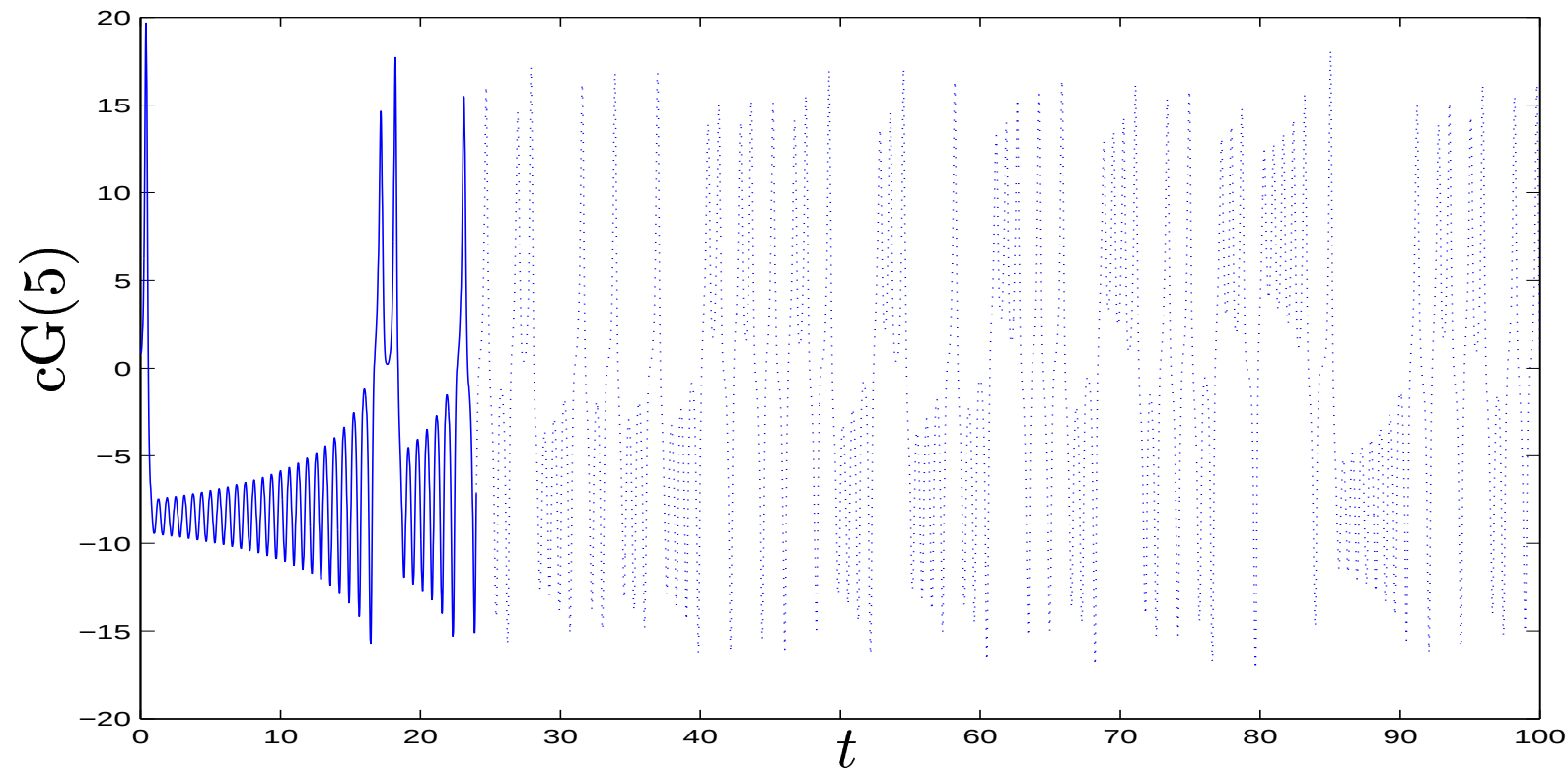
We must thus decrease the computational error, which means decreasing the discrete residual.

$$R^C \approx \frac{10^{-16}}{k}$$

This in turn means that we have to *increase* the timestep!

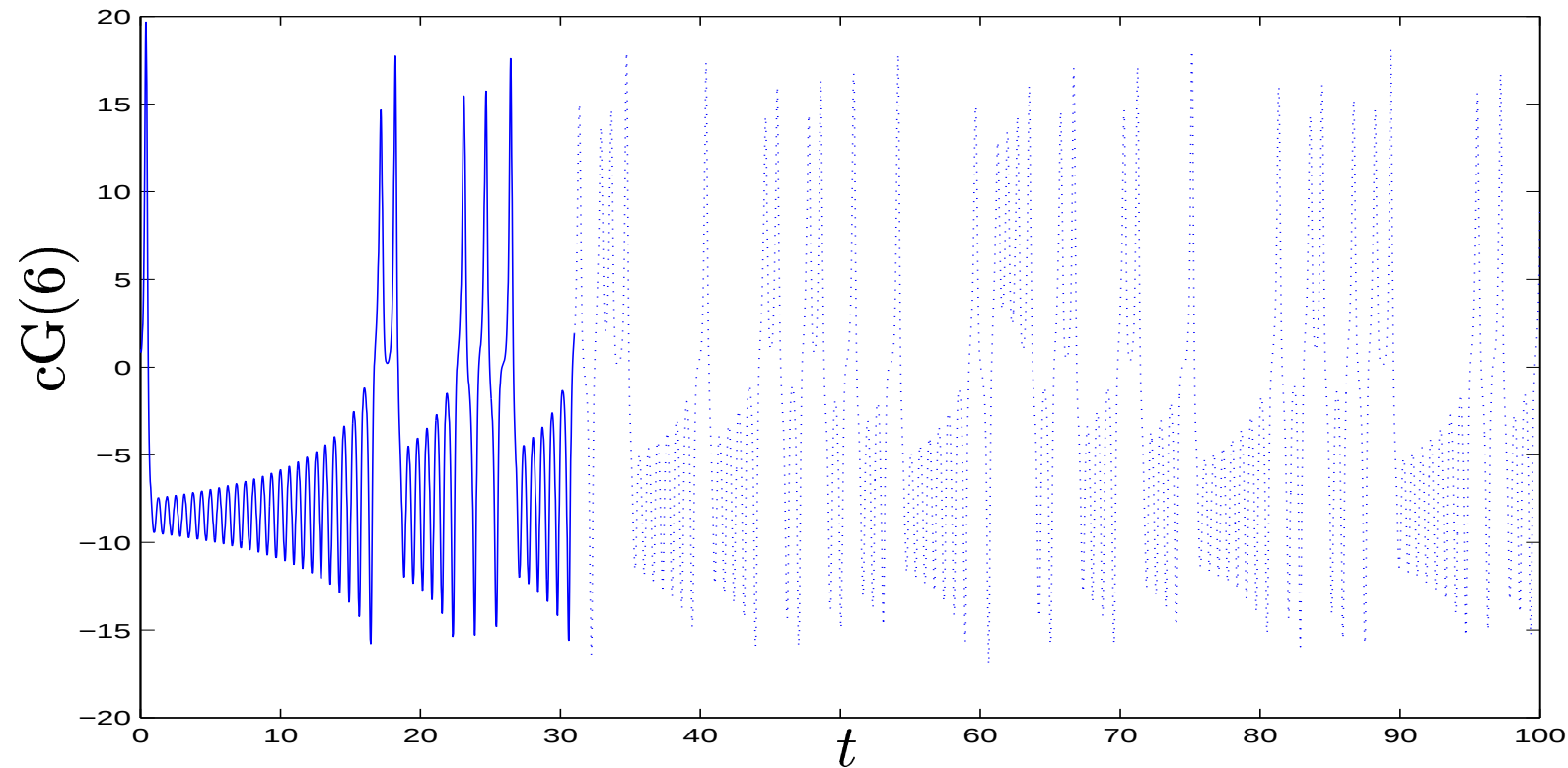
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



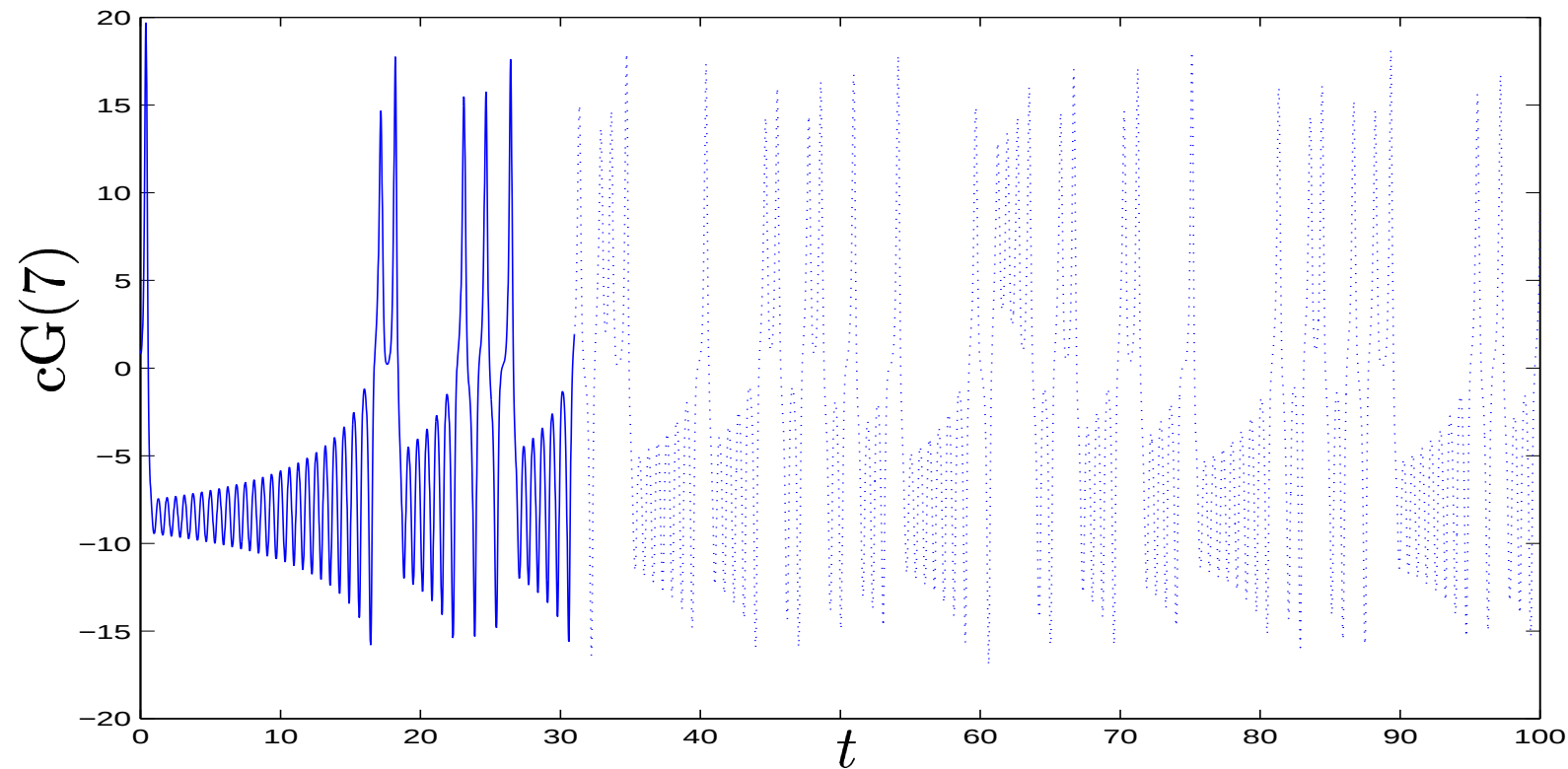
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



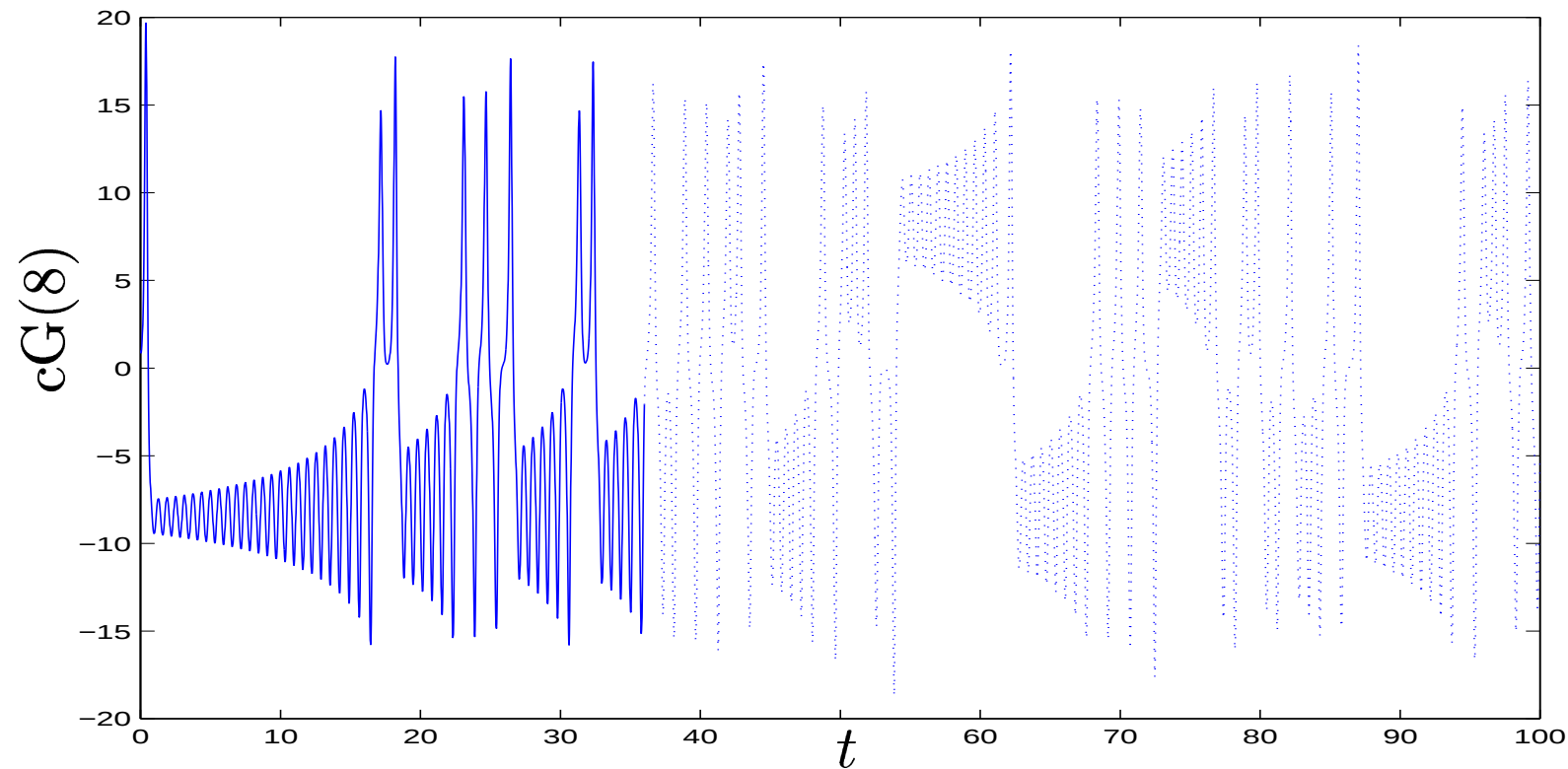
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



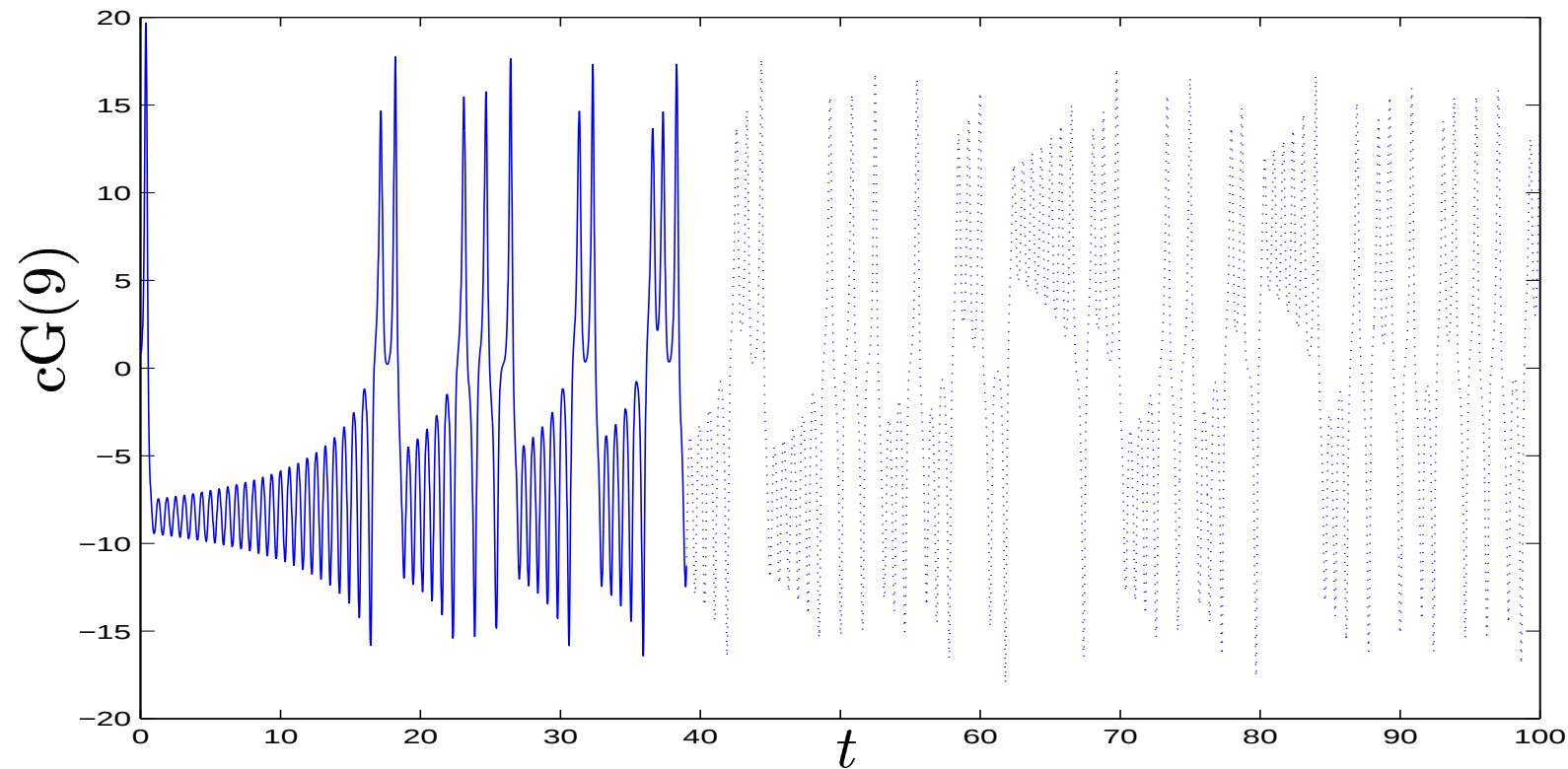
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



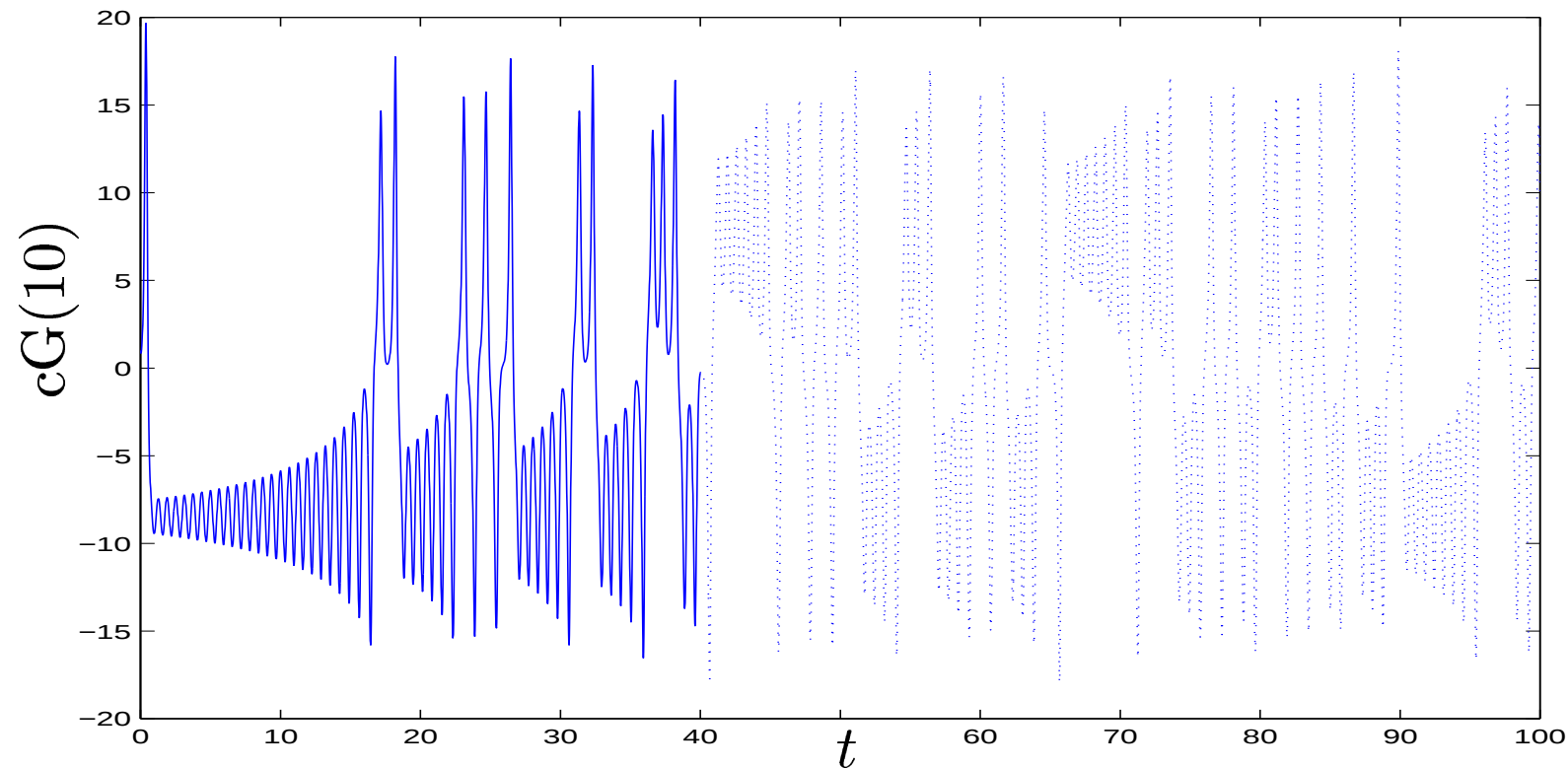
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



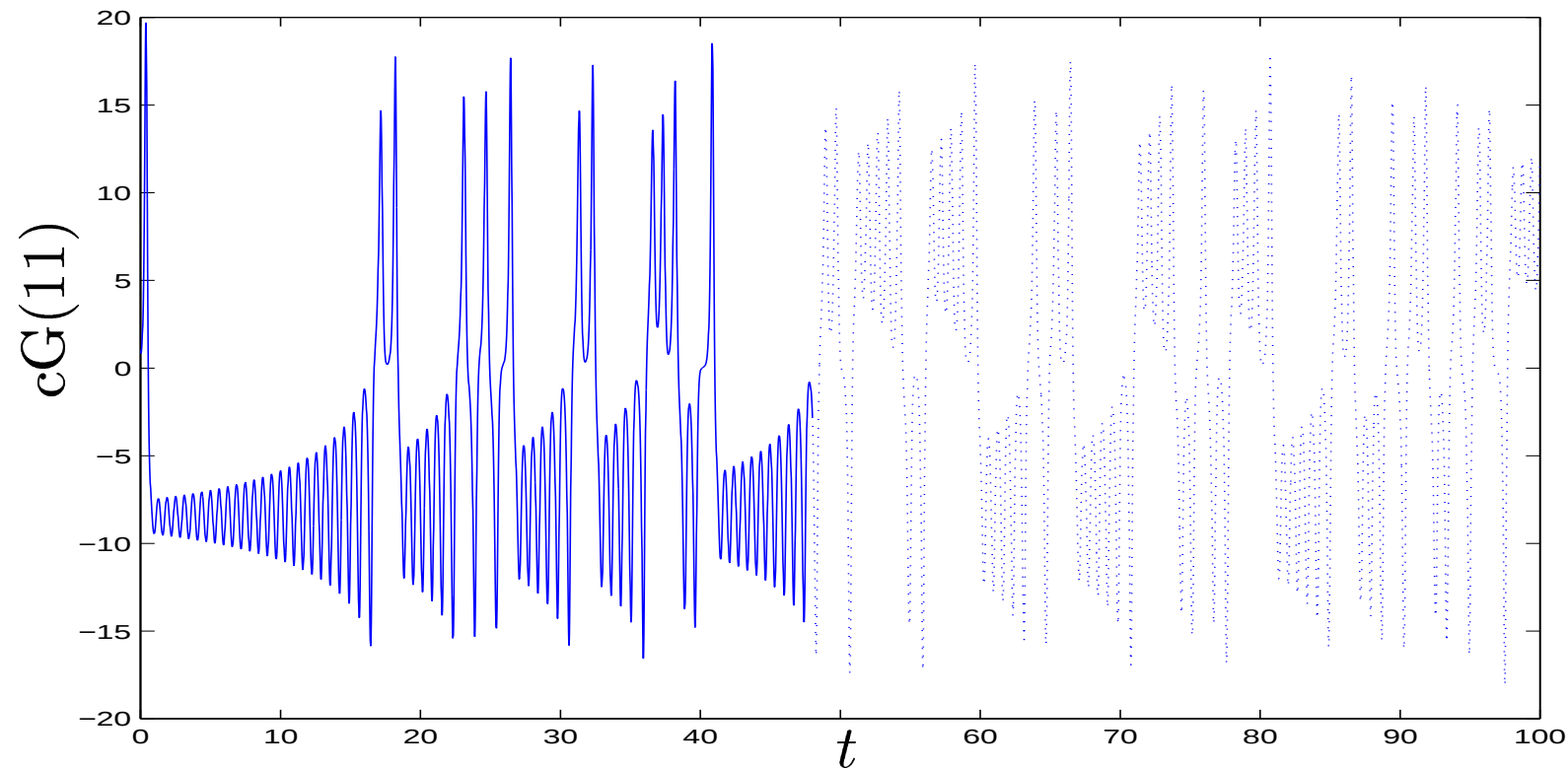
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



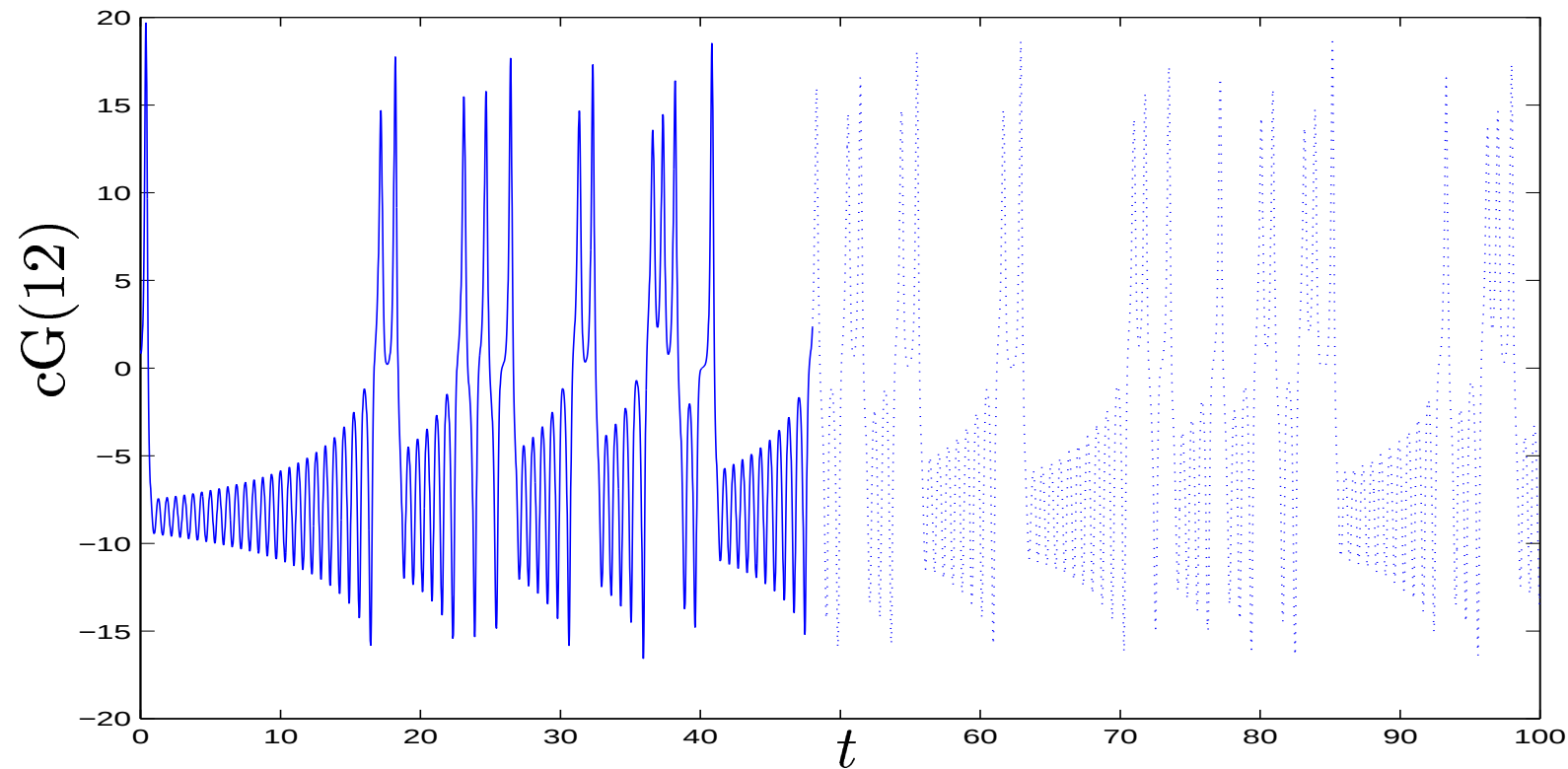
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



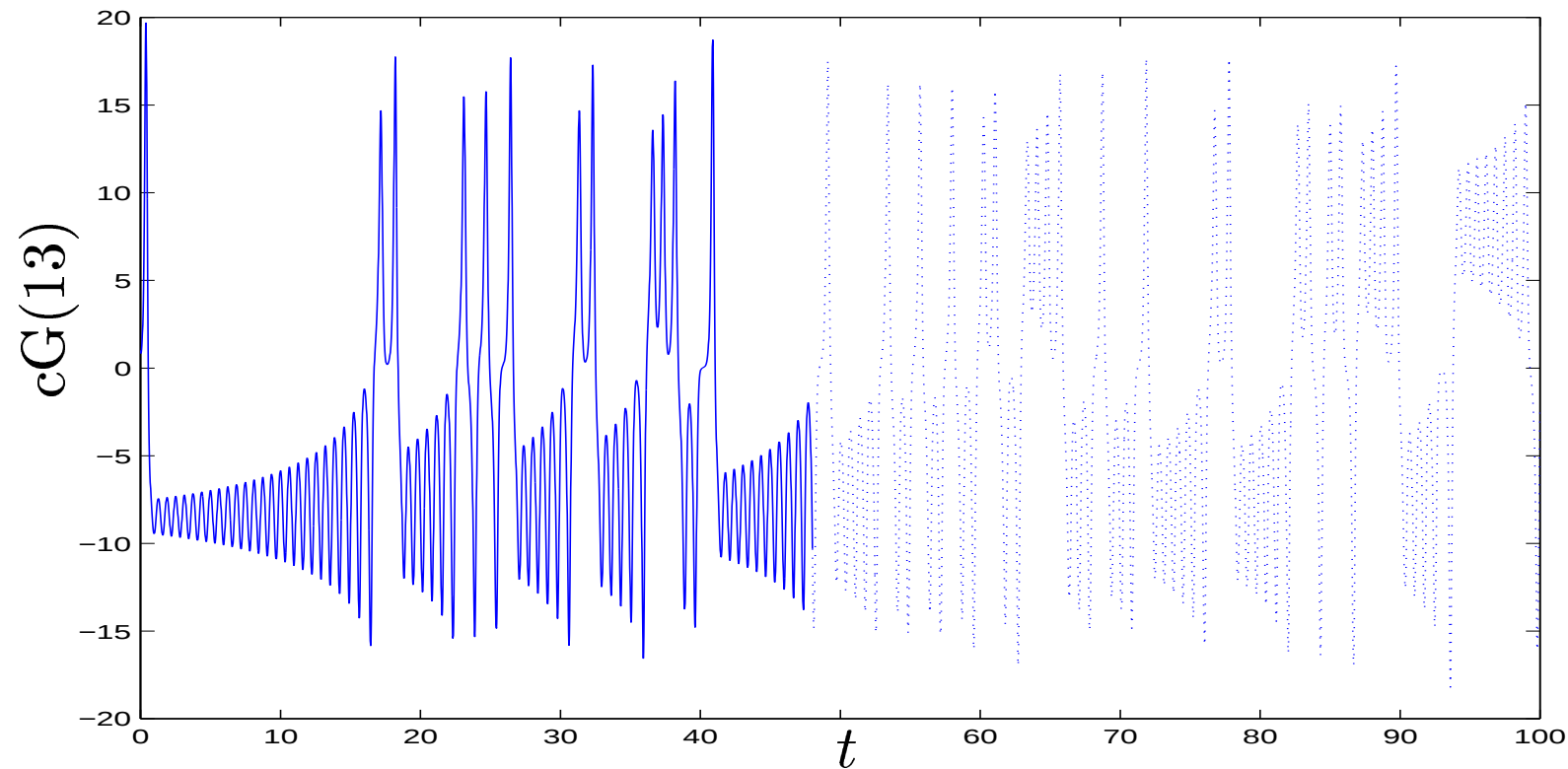
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



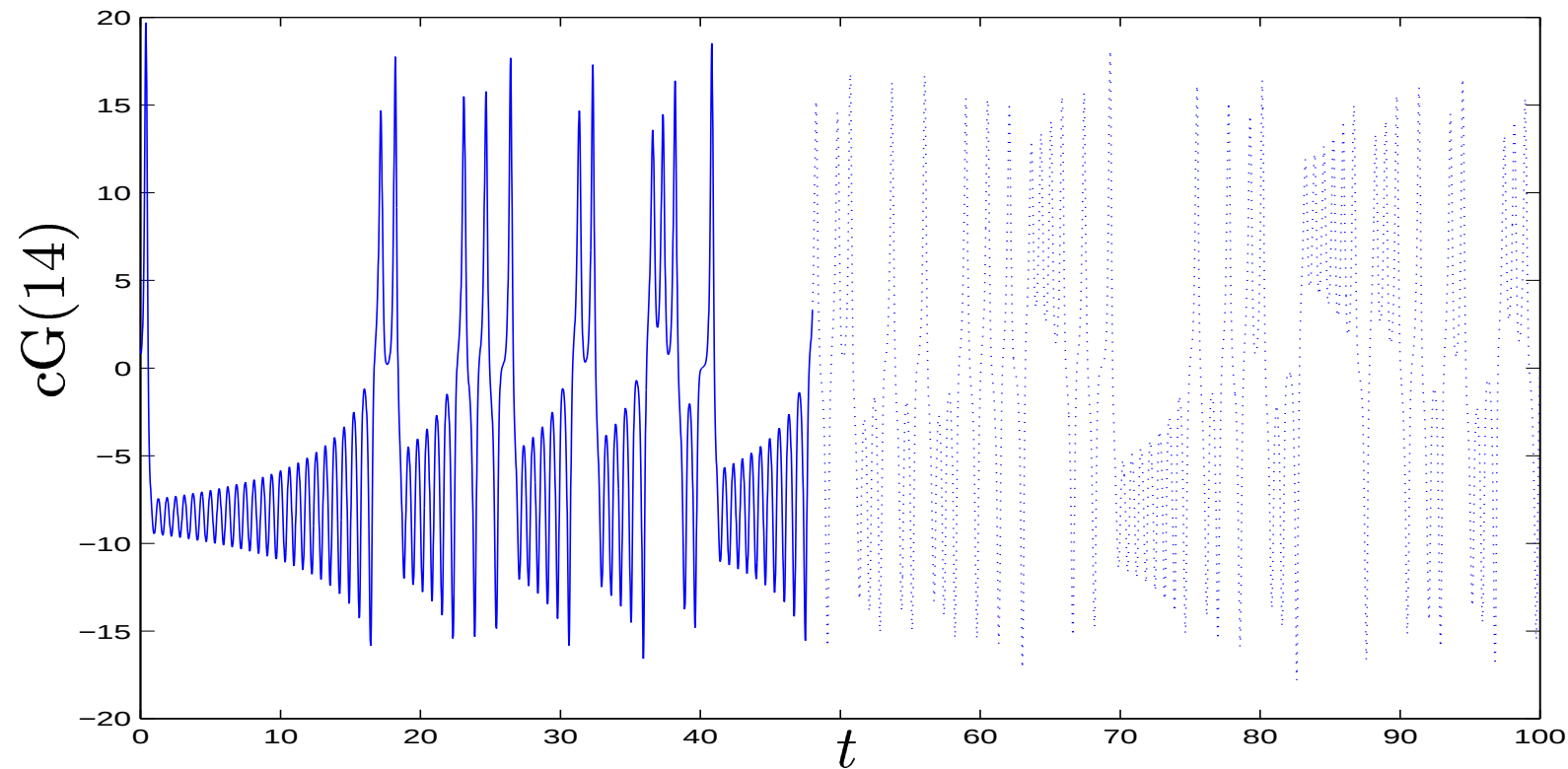
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



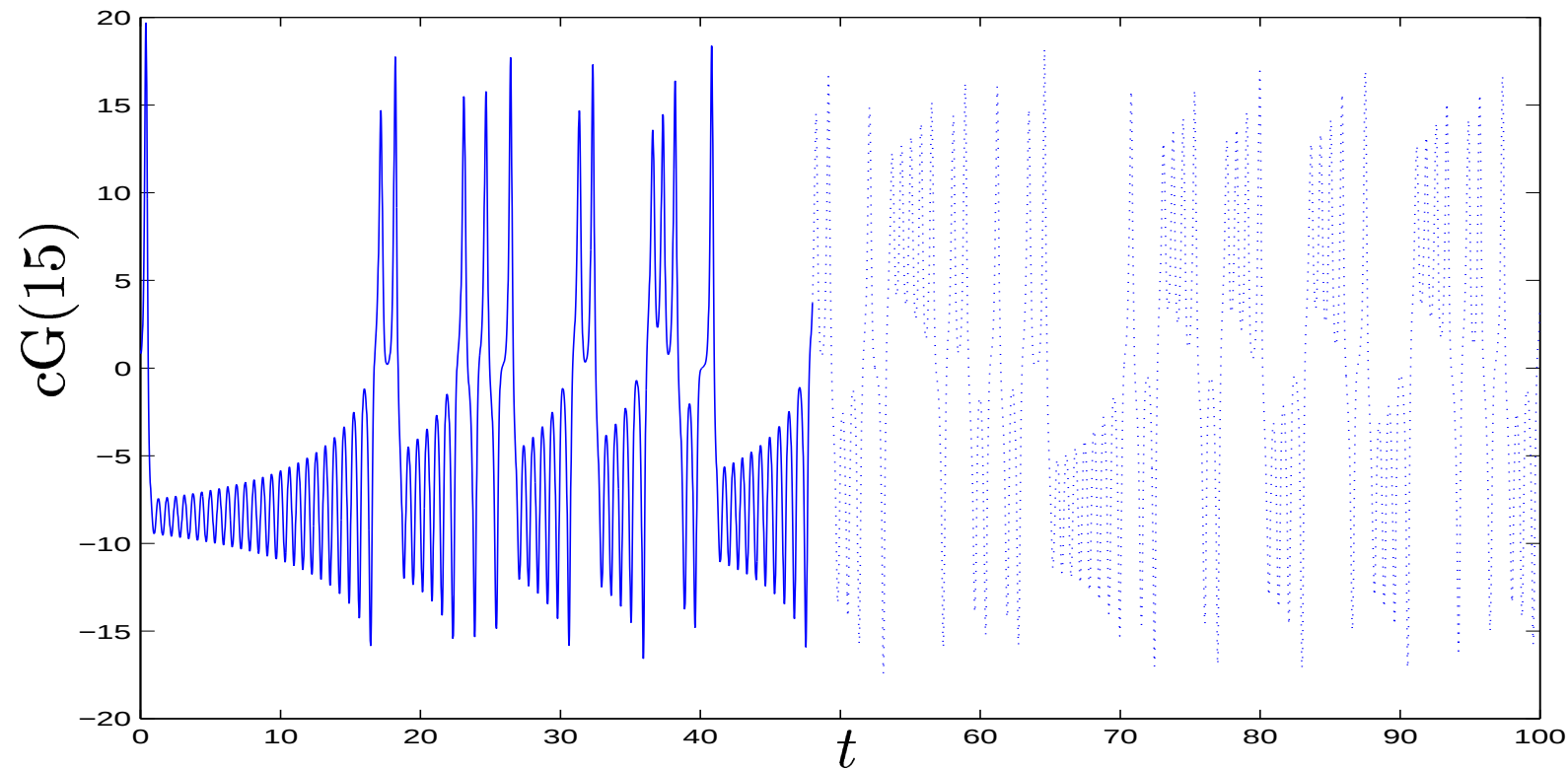
A Simple Experiment — *continued*

Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:

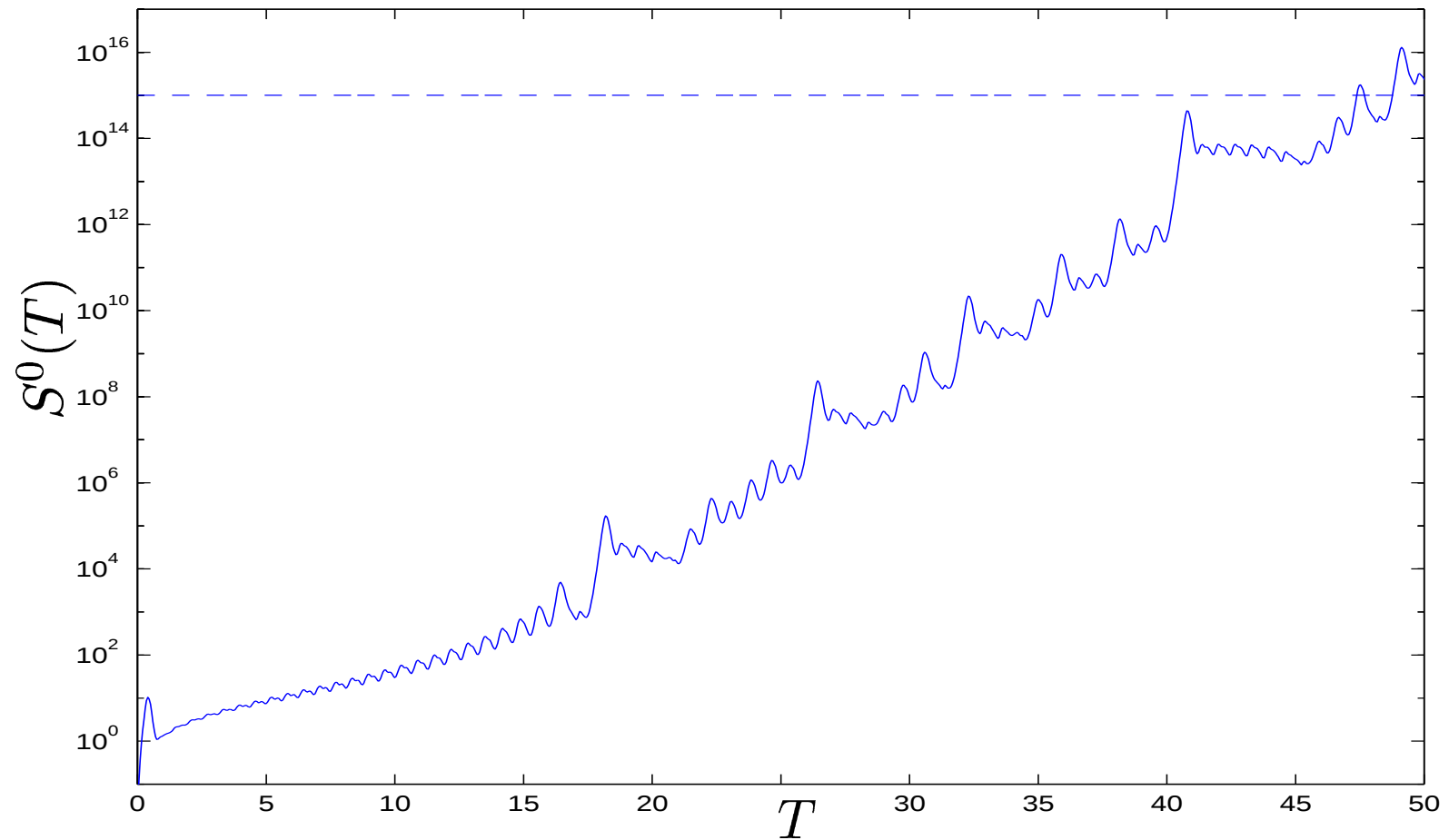


A Simple Experiment — *continued*

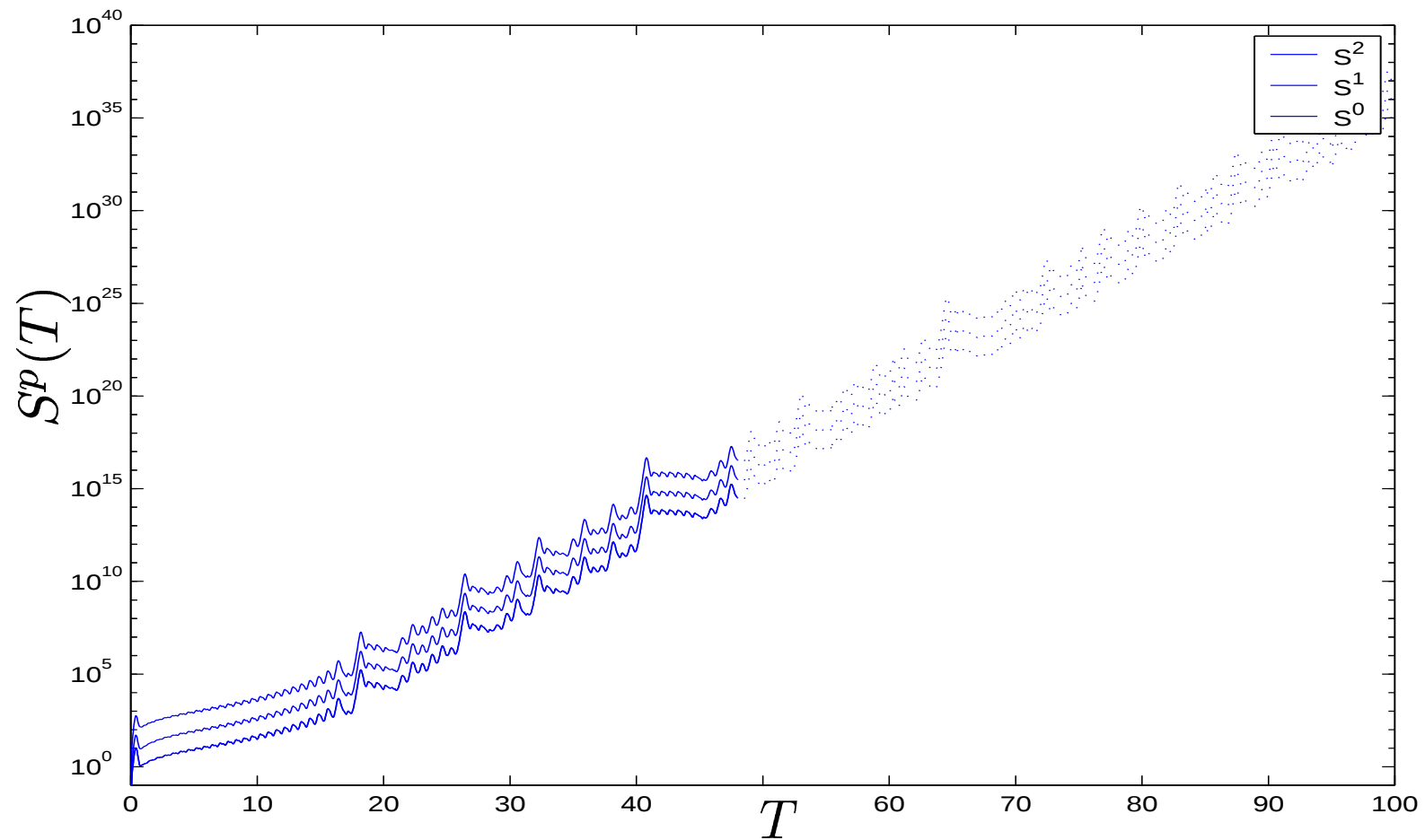
Increasing the timestep to $k = 0.1$, we get the following results when increasing the order:



The Stability Factor



Galerkin Stability Factors



A Simple Estimate

A simple estimate for the stability factors is

$$S^{[q]}(T) \approx 10^{q+T/3},$$

so that

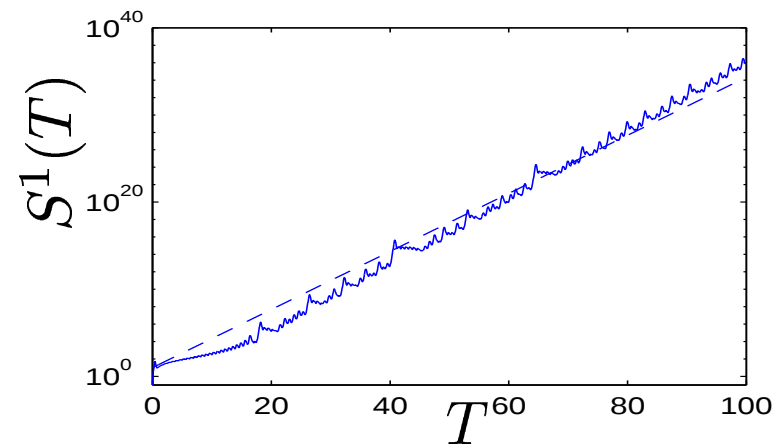
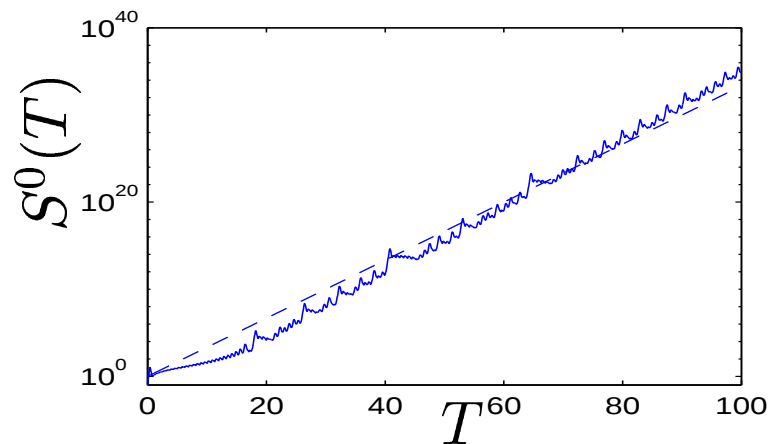
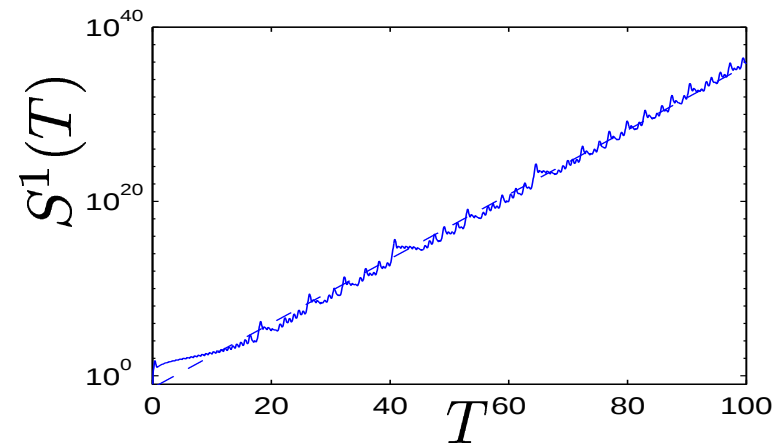
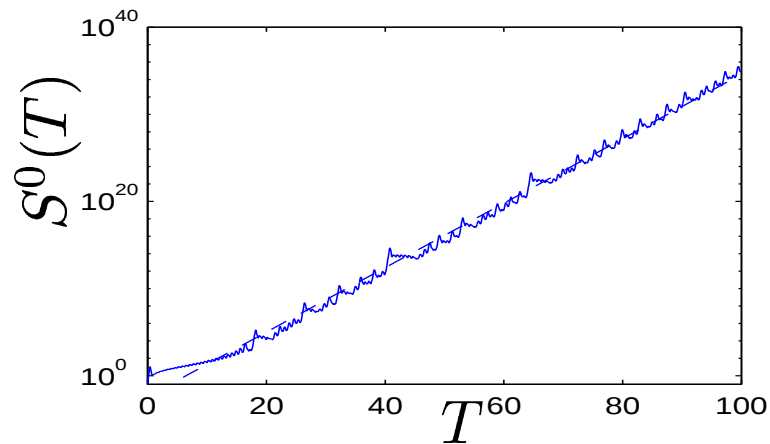
$$E_C(T) \approx \frac{10^{-16}}{k} 10^{T/3} = 10^{T/3-16}/k.$$

How far we get

Keeping $E_C < 1$ we find the maximum value of T :

- $T = 3 \cdot 13 = 39$ with $k = 0.001$,
- $T = 3 \cdot 15 = 45$ with $k = 0.1$,
- not much further with larger timestep, since soon we will have $k > T$.

The Approximation



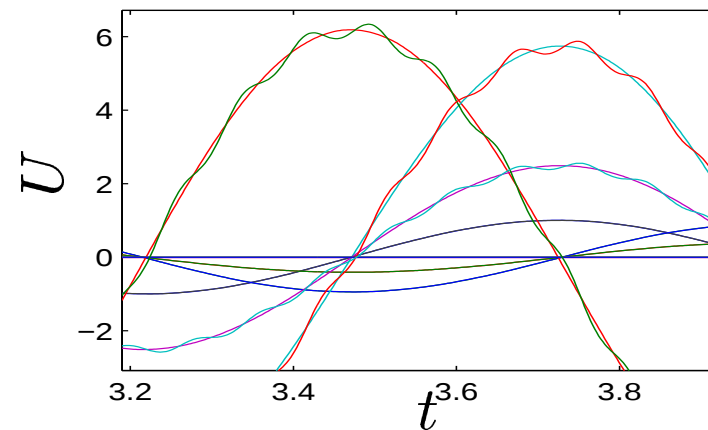
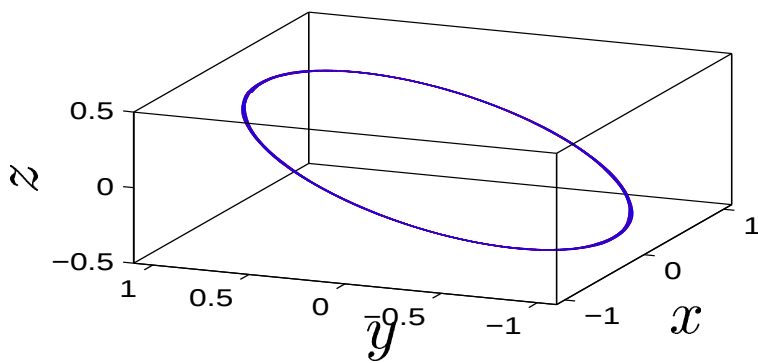
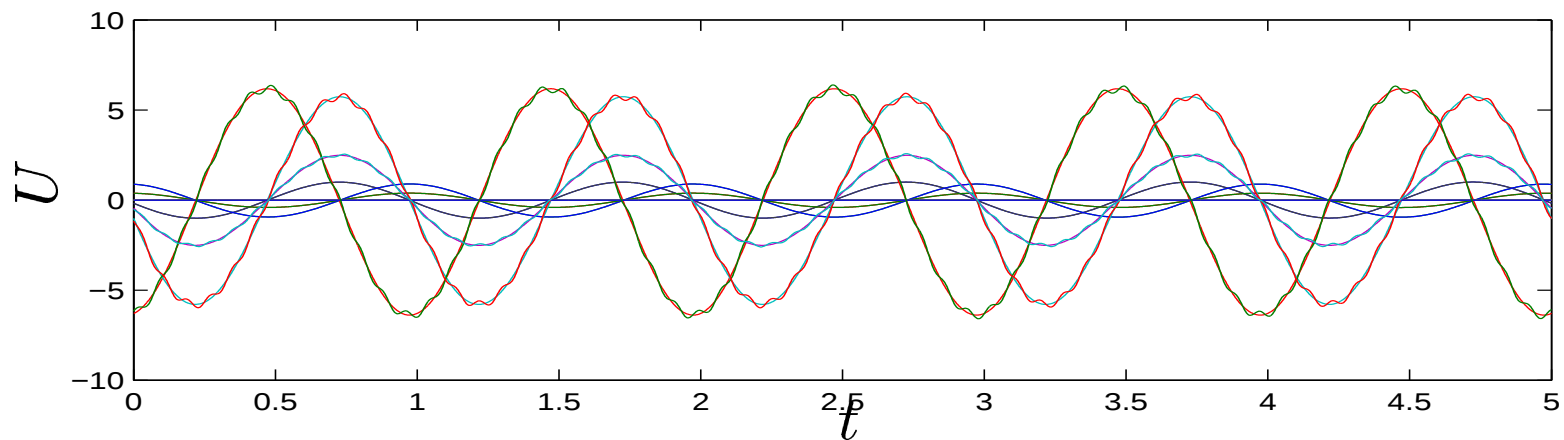
The Solar system

A problem of more practical relevance than the Lorenz system:

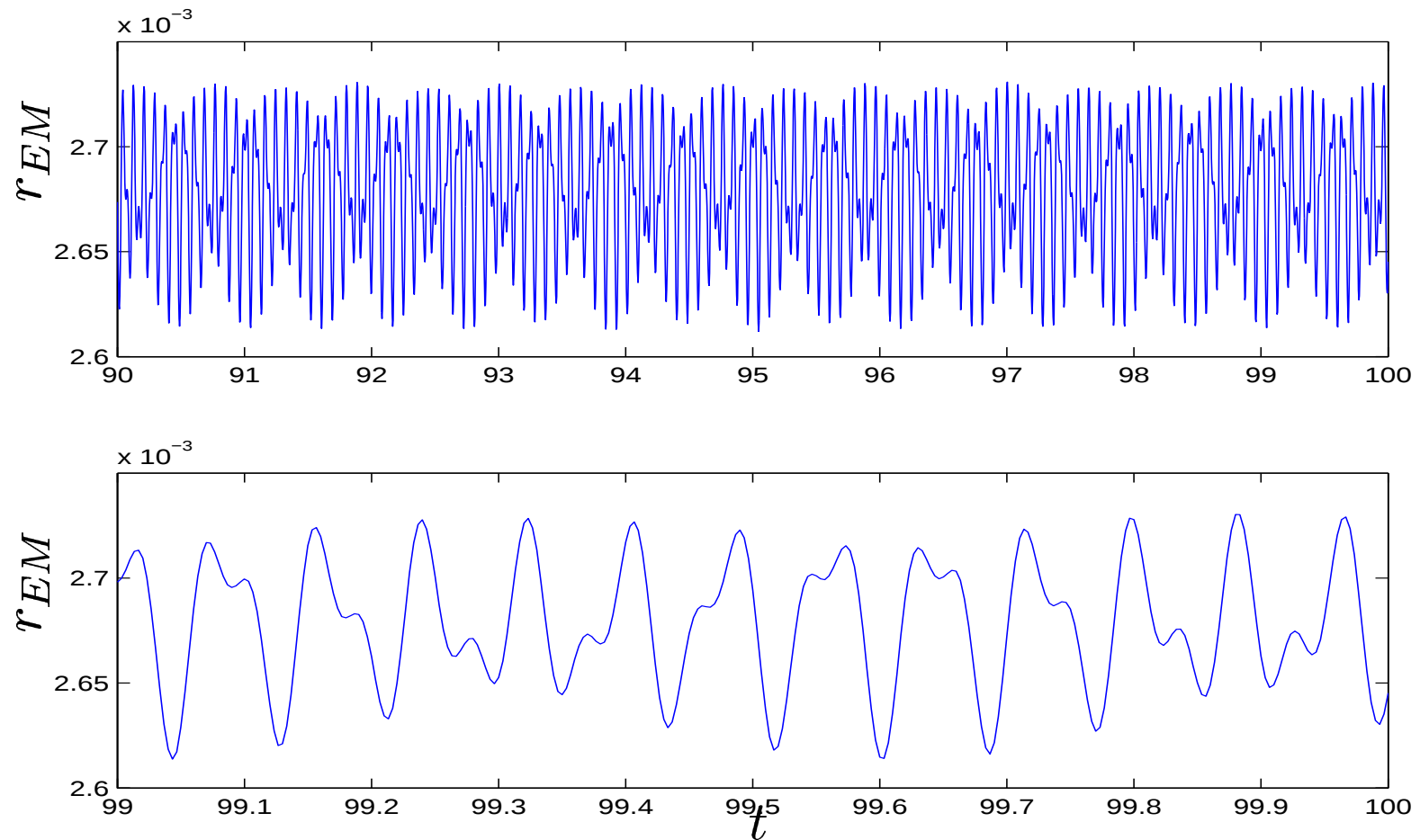
$$m_i \ddot{x}_i = \sum_{j \neq i} \frac{G m_i m_j}{|x_j - x_i|^3} (x_j - x_i);$$

force equals mass times acceleration.

The Earth and the Moon



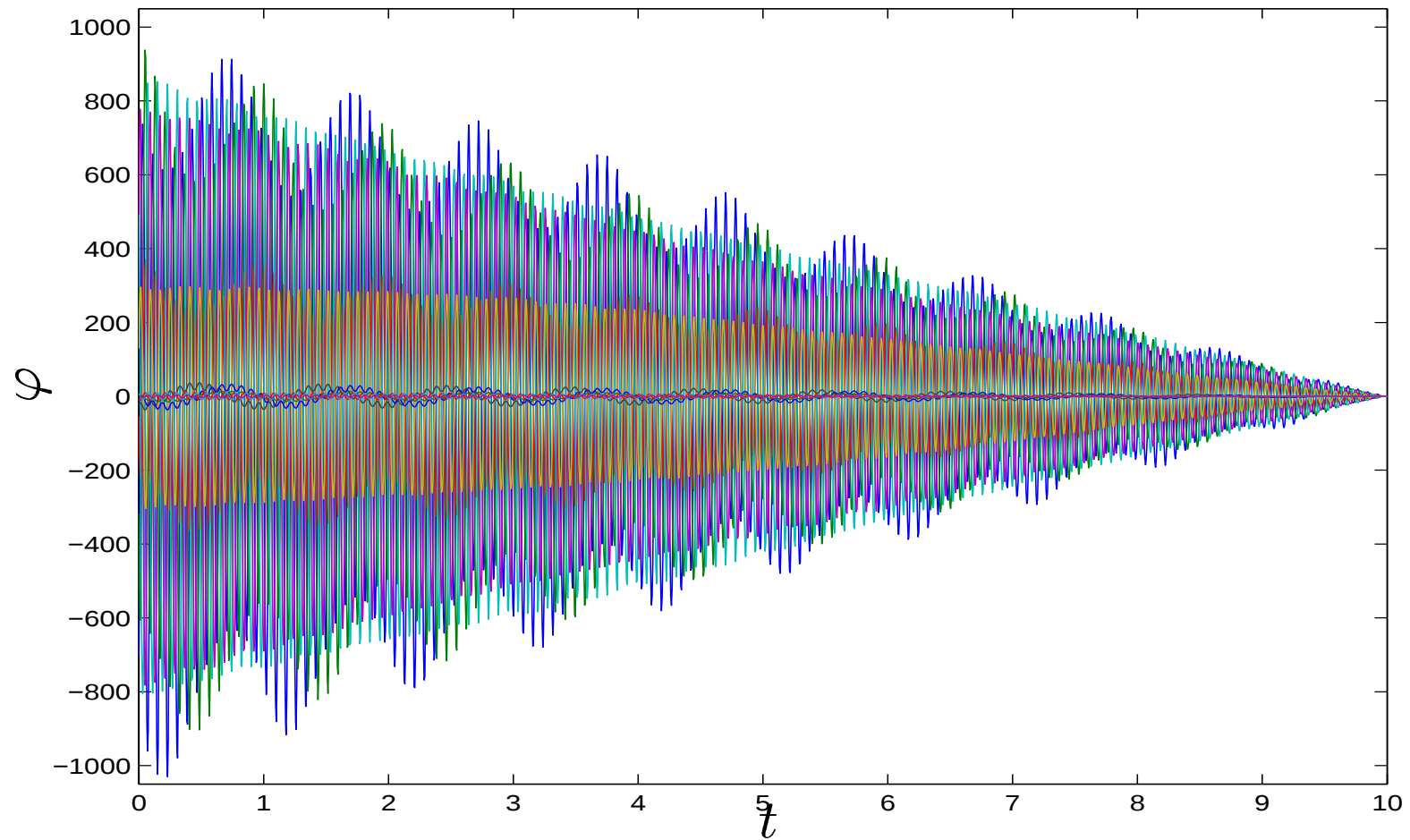
Distance Earth—Moon



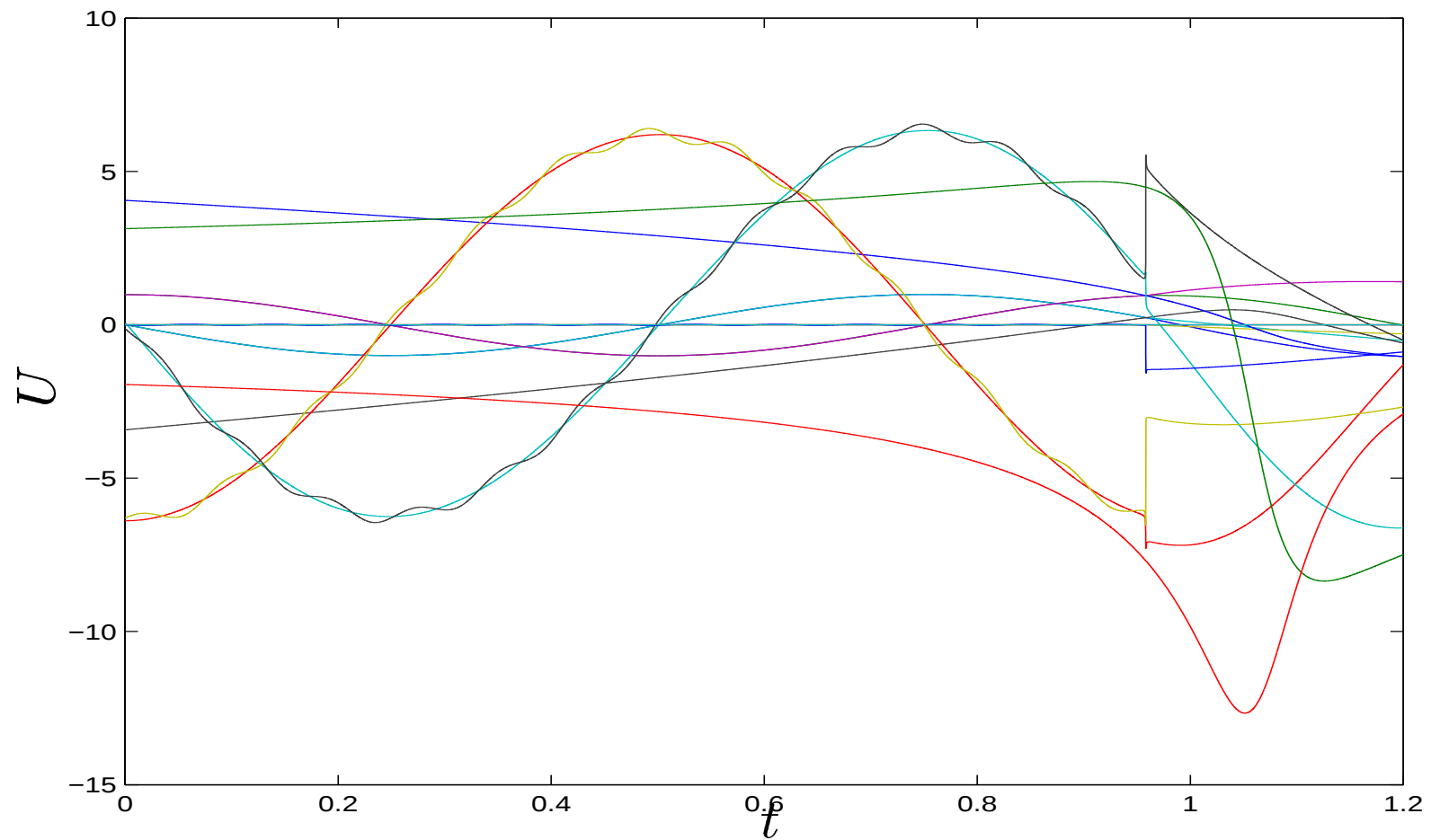
Comparable influences Sun/Earth

$$\frac{F_{EM}}{F_{SM}} = \frac{Gm_E m_M / R_{EM}^2}{Gm_S m_M / R_{SM}^2} = \frac{m_E}{m_S} \left(\frac{R_{SM}}{R_{EM}} \right)^2 \approx 0.5$$

The Dual

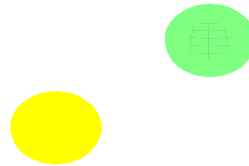


The Earth, the Moon and the Comet



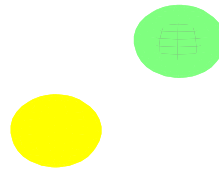
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Live from the set

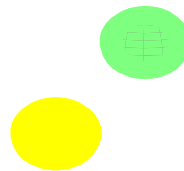


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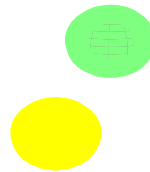


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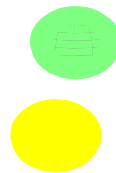


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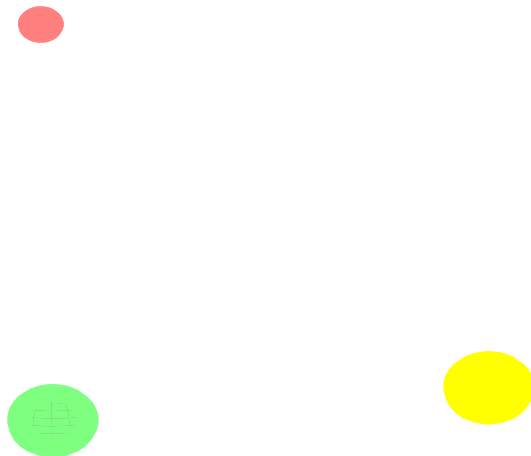


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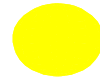
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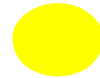
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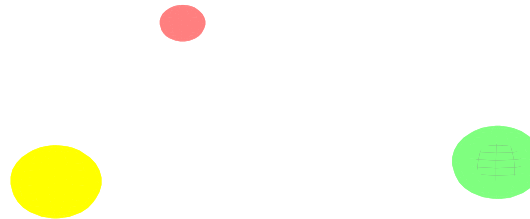
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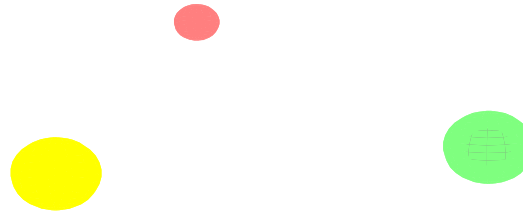


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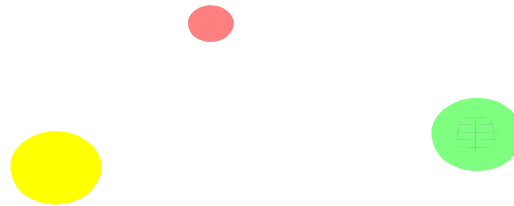


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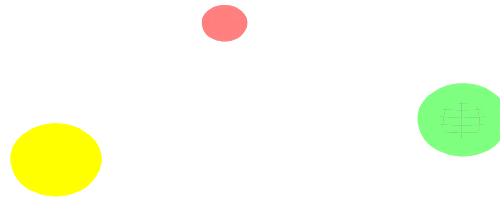
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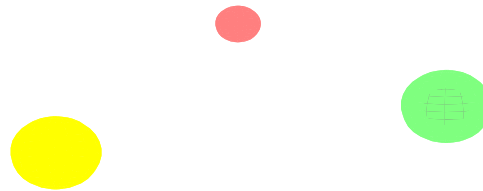
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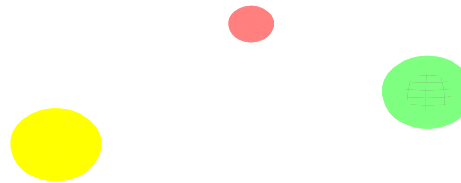
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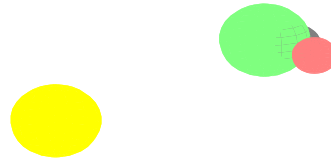
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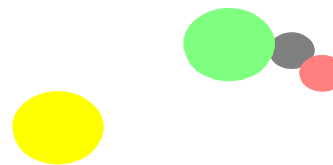
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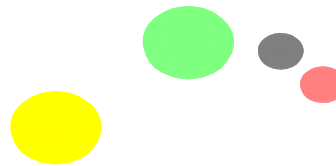
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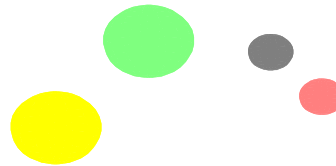
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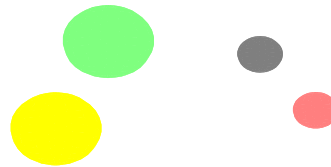
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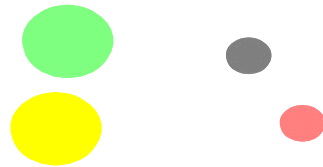
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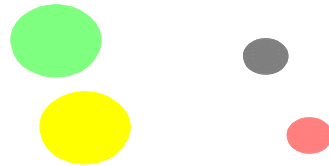
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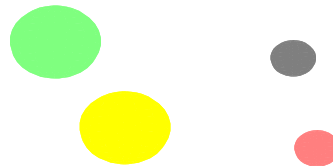
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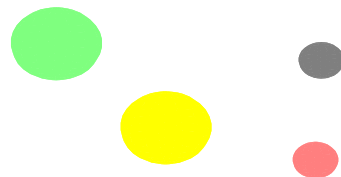
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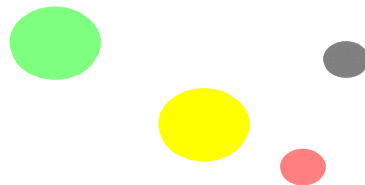
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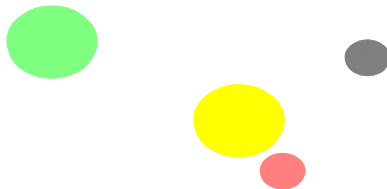
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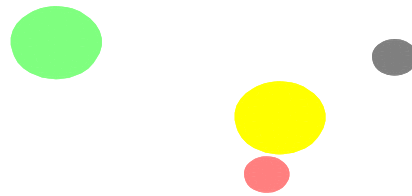


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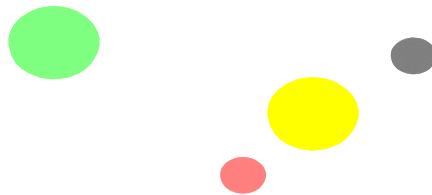
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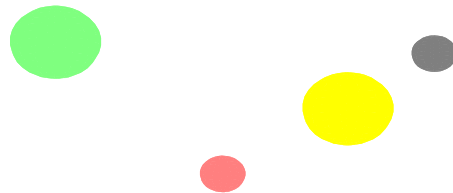


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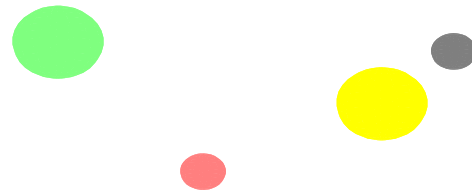


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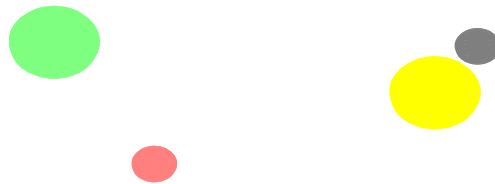


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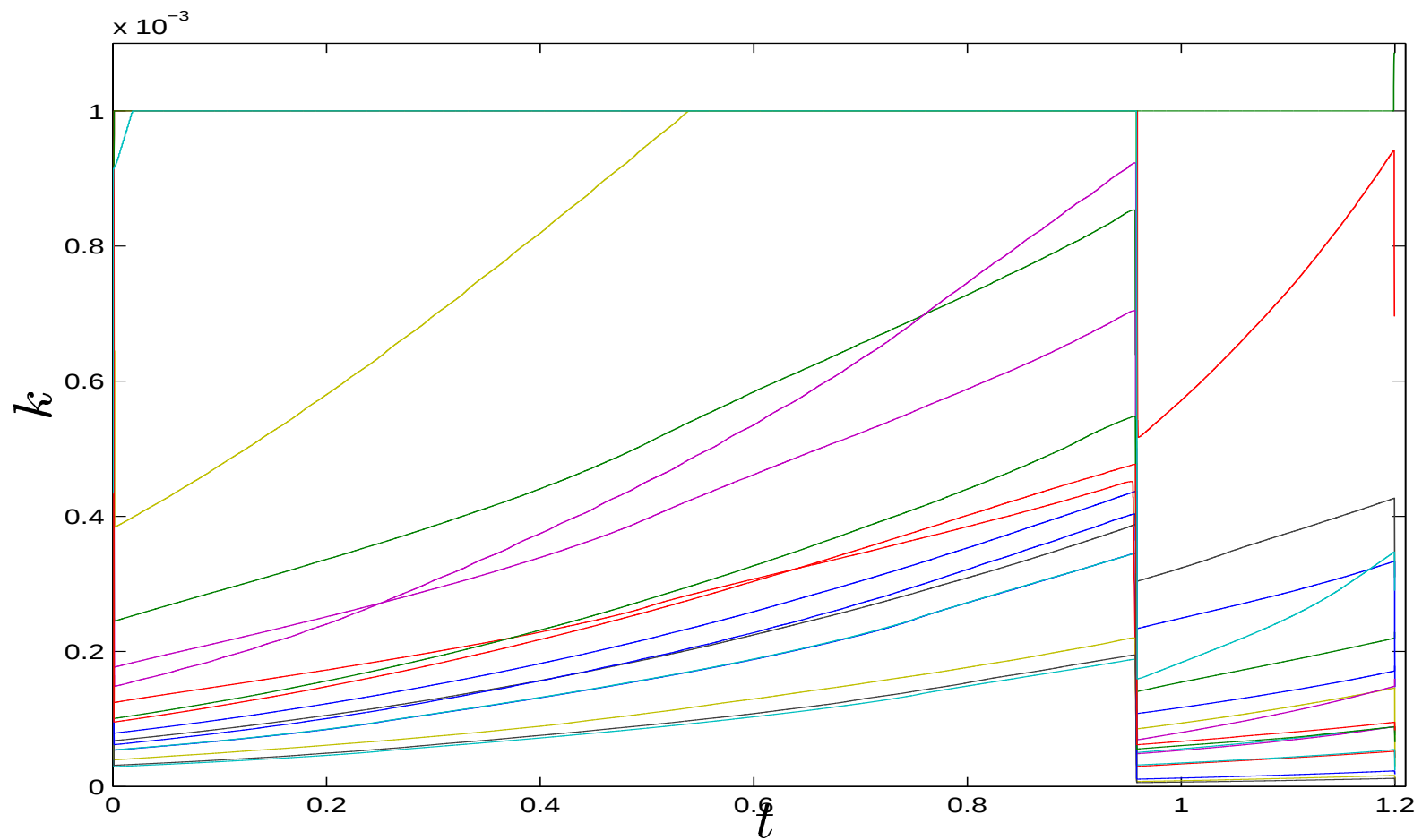
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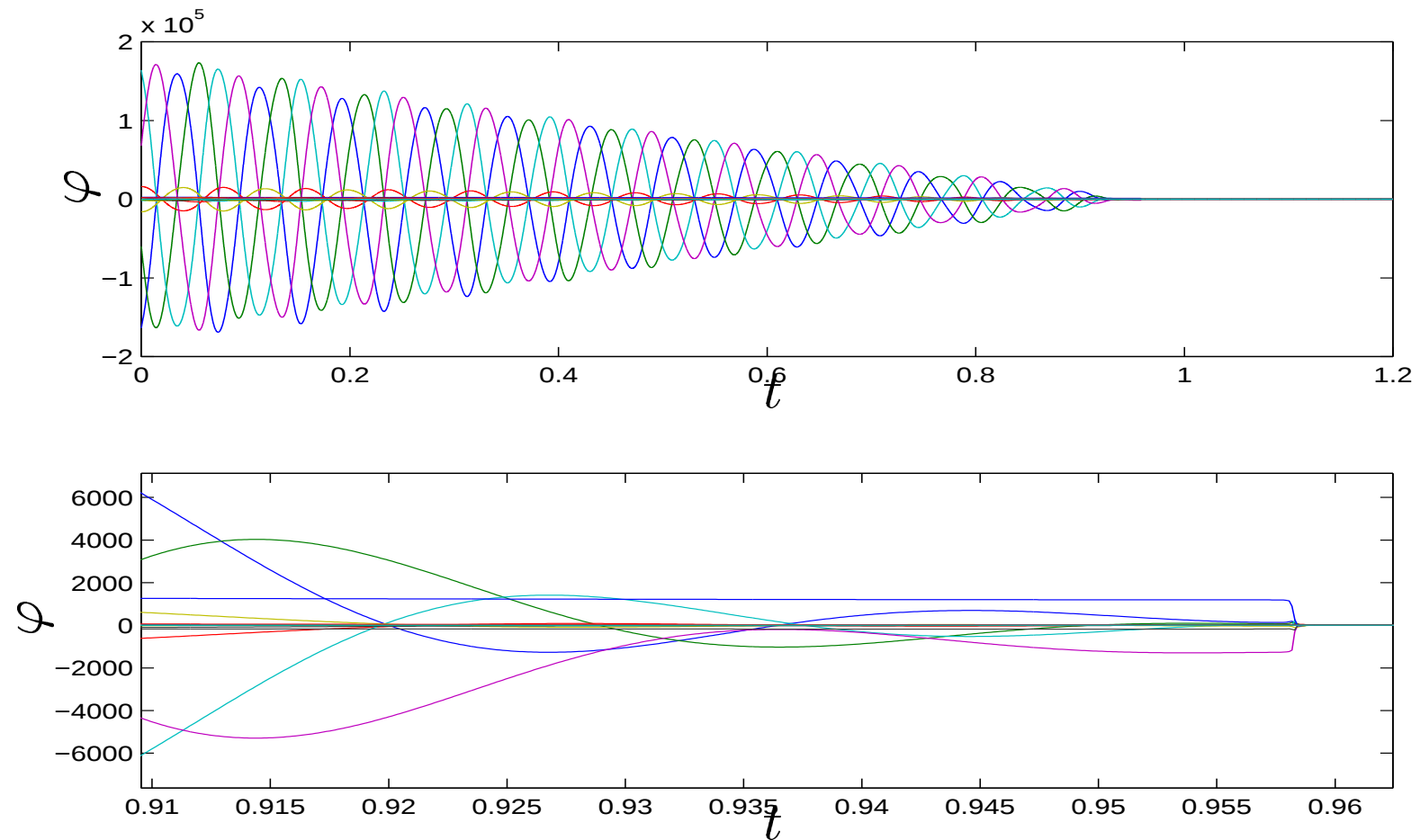


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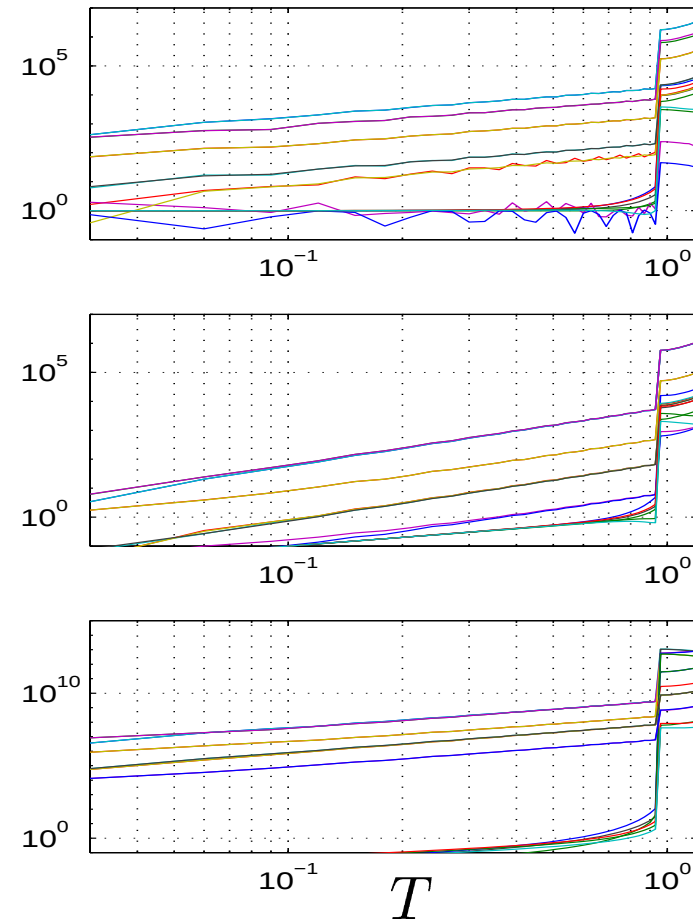
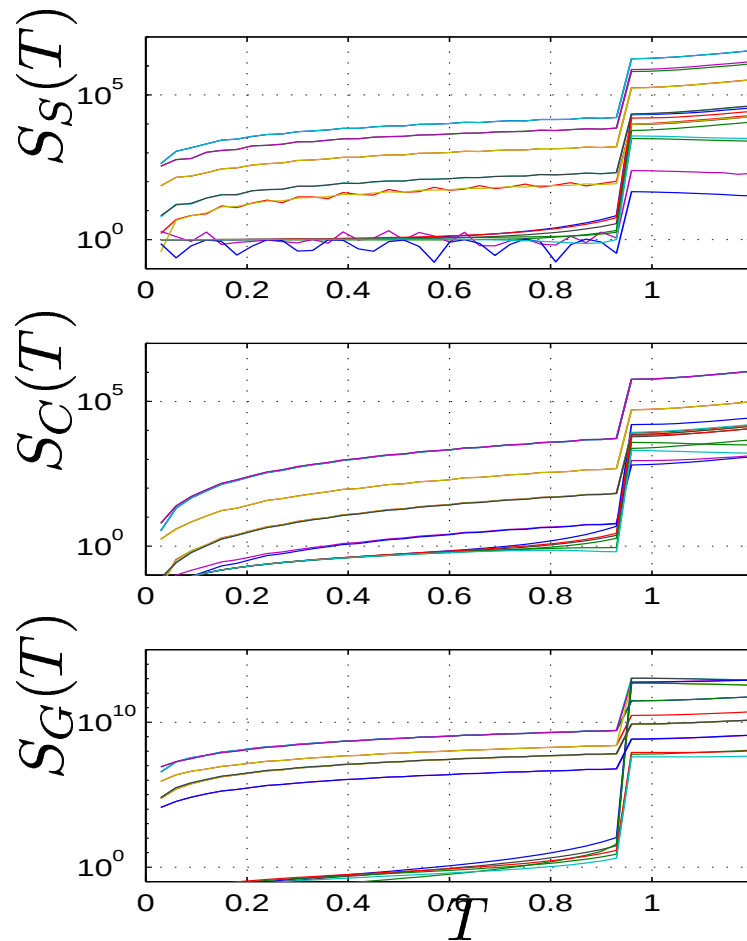
Timesteps



The Dual



Stability Factors



Conclusions

- There are several sources for the total error.
- Each of these are equally important, in that for a certain application any one of these may dominate.
- From computing the stability factors, we may see clearly which of the sources is dominant for a specific problem and tune the implementation accordingly.

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