Multi-Adaptive Galerkin Methods for ODEs

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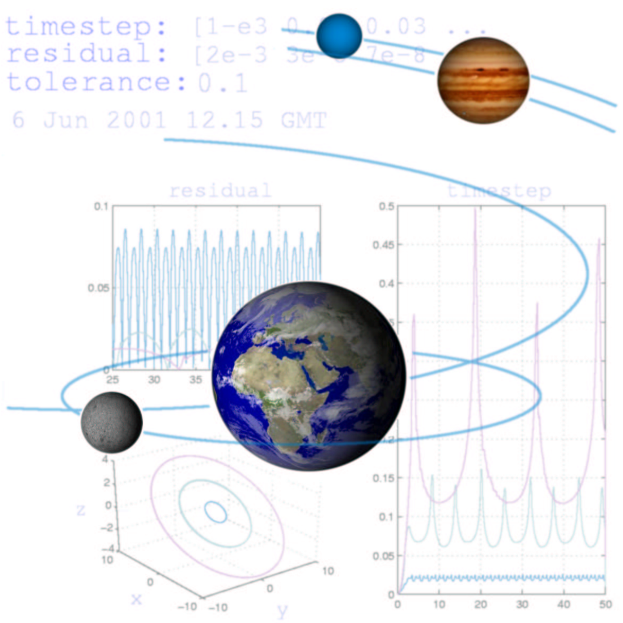
Chalmers Finite Element Center

An Introductory Example

The Solar System

- Moon: $T = 1/12$
- Pluto: $T = 250$
- Multiple time-scales
- Individual time-steps
- Multi-Adaptivity!



Objective

Solve the ODE initial value problem

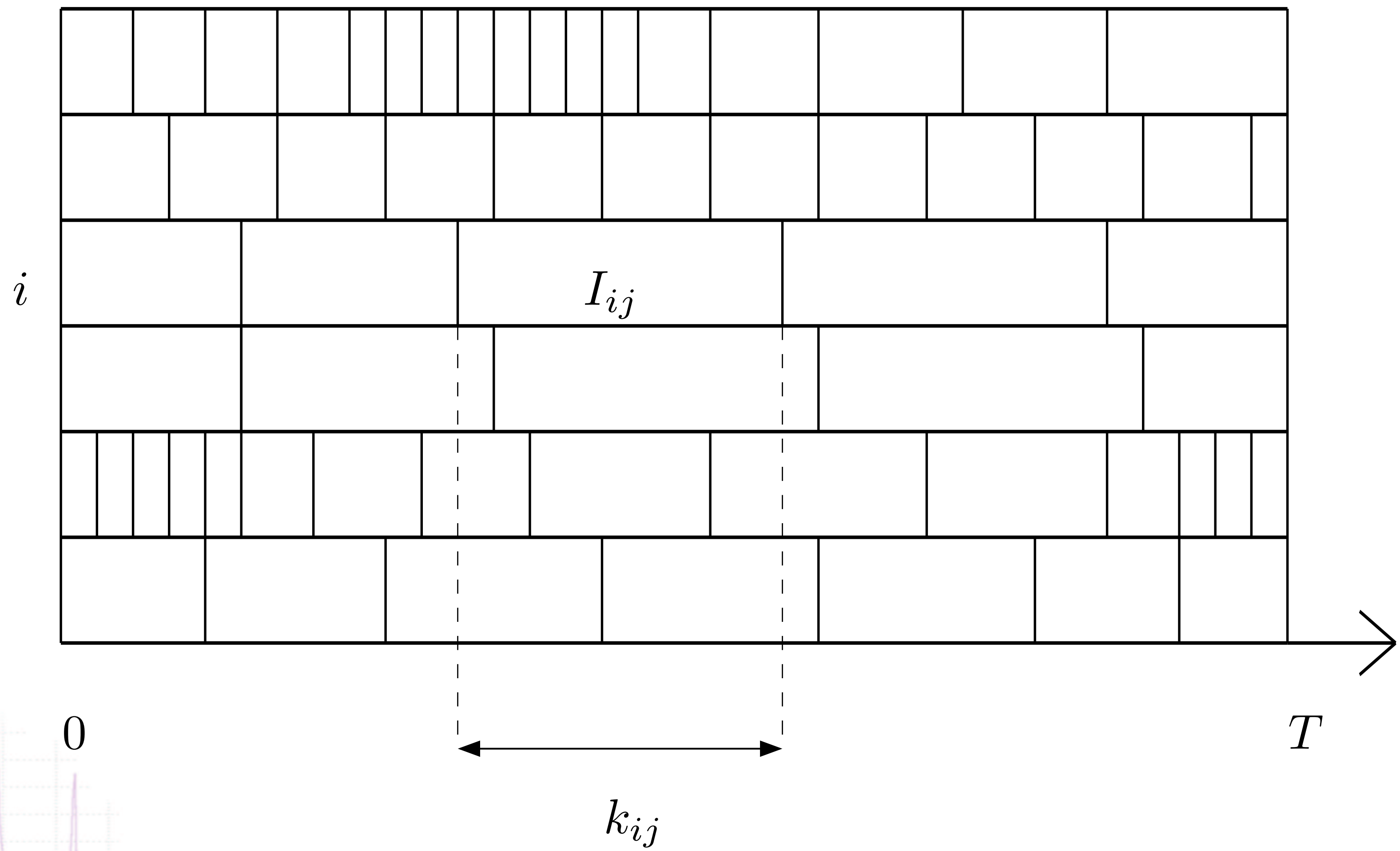
$$\begin{cases} \dot{u}(t) &= f(u(t),t),\ t \in (0,T], \\ u(0) &= u_0, \end{cases}$$

for $u : [0,T] \to \mathbb{R}^N$ with adaptive and individual time-steps for the different components $u_i(t)$ to achieve *efficient* and *reliable* control of the global error at time $t = T$.

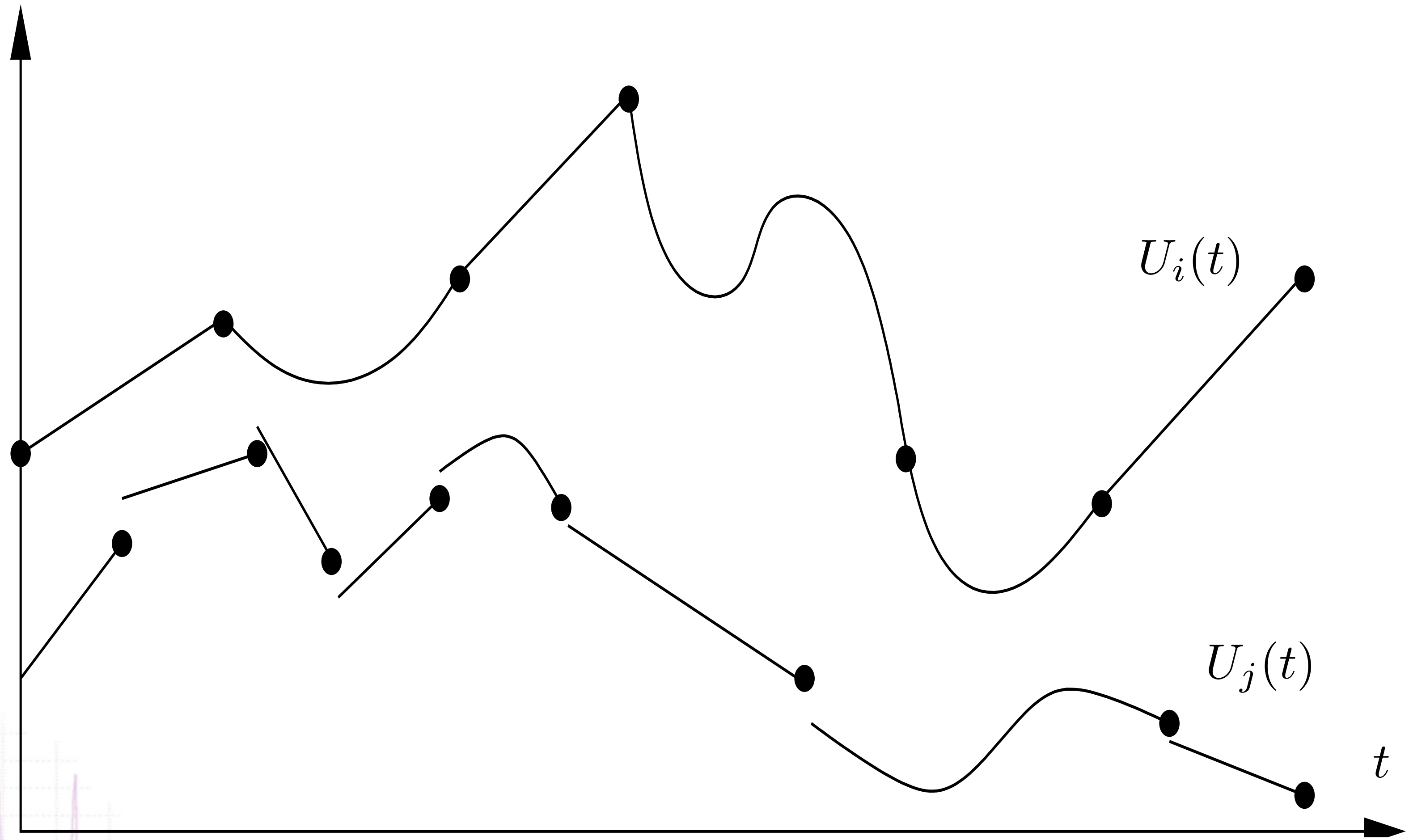
Key Features

- Individual time-steps, quadrature and order
- Global error control
- Iterative methods
- Implementation of high order (≤ 50)
- Applications

Individual Time-Steps



Individual Piecewise Polynomials



Ordinary Galerkin

Ordinary Galerkin $\mathrm{cG}(q)$ for $\dot{u} = f$:

$$\int_0^T (\dot{U}, v)\, dt = \int_0^T (f(U, \cdot), v)\, dt \;\forall v \in W,$$

with $U \in V$, $U(0) = u_0$ and the trial and test spaces defined as

$$\begin{aligned} V &= \{v \in C^N([0,T]) : v_i|_{I_j} \in \mathcal{P}^q(I_j)\}, \\ W &= \{v : v_i|_{I_j} \in \mathcal{P}^{q-1}(I_j)\}. \end{aligned}$$

Multi-Adaptive Galerkin

Multi-adaptive Galerkin $\mathrm{mcG}(q)$:

$$\int_0^T (\dot{U}, v)\, dt = \int_0^T (f(U,\cdot), v)\, dt \ \forall v \in W,$$

with $U \in V$, $U(0) = u_0$ and the trial and test spaces now defined as

$$\begin{aligned} V &= \{v \in C^N([0,T]) : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})\}, \\ W &= \{v : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}-1}(I_{ij})\}. \end{aligned}$$

The Discrete Equations

Making an Ansatz for U_i on I_{ij},

$$U_i(t) = \sum_{n=0}^{q_{ij}} \xi_{ijn} \lambda_n^{[q_{ij}]}(\tau_{ij}(t)),$$

we end up with

$$\xi_{ijn} = \xi_0 + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t))\, f_i(U,t)\, dt,$$

for certain weight functions $\{w_n^{[q]}\} \subset \mathcal{P}^{q-1}(0,1)$.

The Discontinuous Method

Similarly for the multi-adaptive version of the discontinuous method, $\mathrm{mdG}(q)$, we obtain

$$\xi_{ijm} = \xi_{ij0}^{-} + \int_{I_{ij}} w_m^{[q_{ij}]}(\tau_{ij}(t))\; f_i(U,t)\; dt,$$

for certain weight functions $\{w_n^{[q]}\} \subset \mathcal{P}^q(0,1)$.

Properties

- $\mathrm{mcG}(q)$ conserves energy if $k_{U_i} = k_{V_i}$.
- $\mathrm{mdG}(q)$ is B-stable:

$$\|U(\bar{t}^-) - V(\bar{t}^-)\| \leq \|U(0^-) - V(0^-)\|,$$

if f is monotone, i.e.

$$(f(u,\cdot) - f(v,\cdot), u - v) \leq 0 \; \forall u, v \in \mathbb{R}^N.$$

A Priori Error Estimates

- $\mathrm{mcG}(q)$ is of order $2q$ (locally $2q_{ij}$):

$$\|e\| \leq \int_0^T (k^{2q}, w)\; dt,$$

- $\mathrm{mdG}(q)$ is of order $2q+1$ (locally $2q_{ij}+1$):

$$\|e\| \leq \int_0^T (k^{2q+1}, w)\; dt.$$

A Posteriori Error Estimates

$$\|e\| \leq E_G + E_C + E_Q$$

- E_G: Galerkin error
- E_C: Computational error
- E_Q: Quadrature error

The Dual Problem

$$\begin{cases} -\dot{\varphi}(t) & = & J^*(u(t), U(t), t)\varphi(t) + g(t), \;\; t \in [0, T), \\ \varphi(T) & = & \varphi_T, \end{cases}$$

where

$$J^*(v_1, v_2, \cdot) = \left(\int_0^1 \frac{\partial f}{\partial u}(sv_1 + (1-s)v_2, \cdot)\; ds \right)^* .$$

The Galerkin Error: E_G

- Many different variants
- More or less local/global:

$$
\begin{aligned}
E_G &= \left| \int_0^T (R, \varphi - \pi_k \varphi)\, dt \right| \\
&= \left| \sum_{i=1}^N \sum_{j=1}^{M_i} \int_{I_{ij}} R_i(U)(\varphi_i - \pi_k \varphi_i)\, dt \right| \\
&\leq \sum_{i=1}^N \sum_{j=1}^{M_i} C_{q_{ij}-1} k_{ij}^{q_{ij}+1}\; r_{ij} s_{ij}^{[q_{ij}]} \\
&\leq \sum_{i=1}^N S_i^{[q]} \max_{[0,T]} \{ C_{q_i-1} k_i^{q_i} r_i \} \\
&\leq S^{[q],1} \sqrt{N} \max_{i,[0,T]} \{ C_{q_i-1} k_i^{q_i} r_i \}
\end{aligned}
$$

The Computational Error: E_C

$$E_C \approx \sum_{i=1}^{N} \bar{S}_i^{[0]} \max_{[0,T]} |R_i^{\mathcal{C}}|,$$

where

$$R_i^{\mathcal{C}} = \frac{1}{k_{ij}} \left[U(t_{ij}) - U(t_{i,j-1}) - \int_{I_{ij}} f_i(U, \cdot)\, dt \right]$$

is the *discrete residual*.

The Quadrature Error: E_Q

$$E_Q \approx \sum_{i=1}^{N} \bar{S}_i^{[0]} \max_{[0,T]} |R_i^Q|,$$

where

$$R_i^Q = \frac{1}{k_{ij}} \left[\tilde{\int}_{I_{ij}} f_i(U,\cdot)\; dt - \int_{I_{ij}} f_i(U,\cdot)\; dt \right]$$

is the *quadrature residual*.

Solving the Discrete Equations

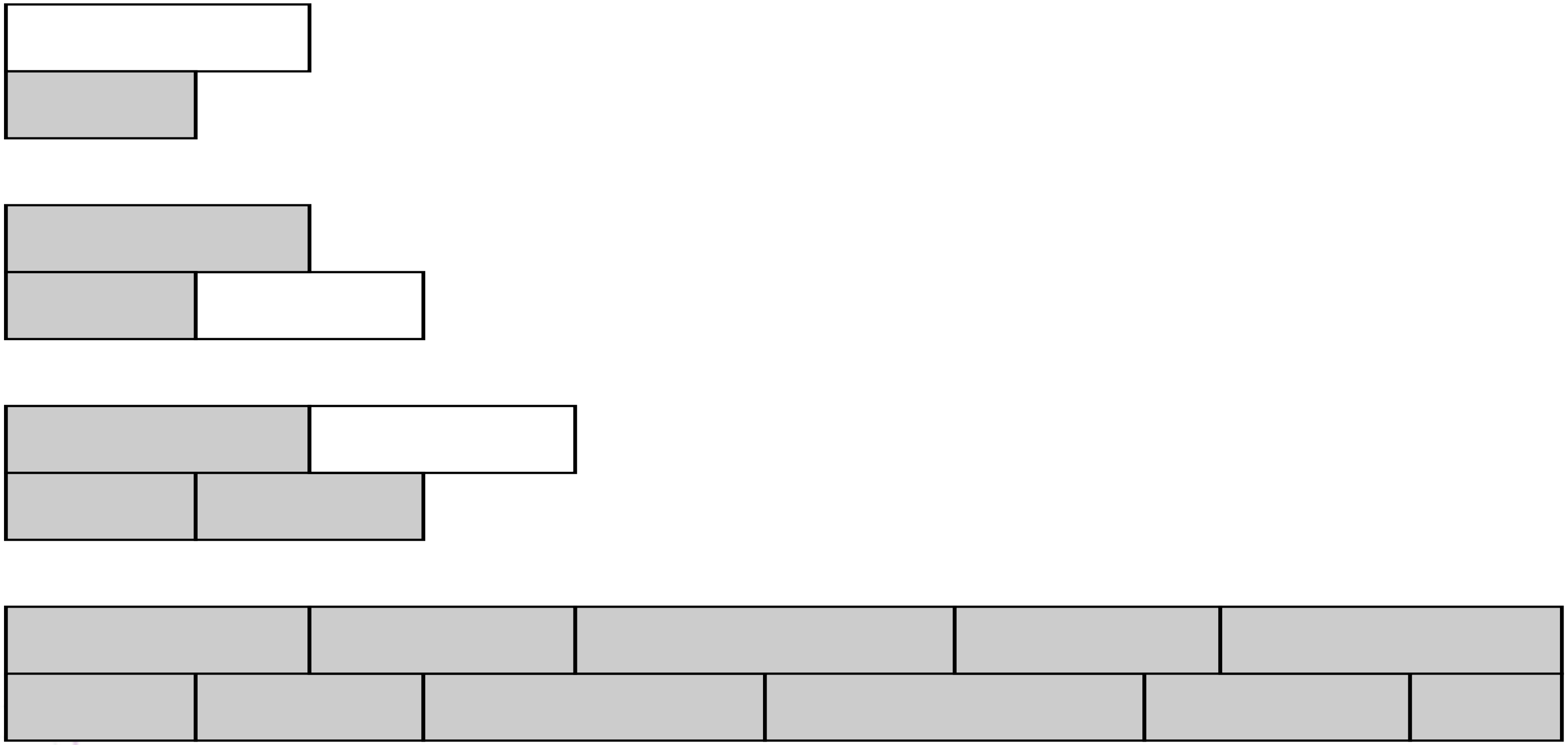
$$\xi_{ijm} = \xi_{ij0} + k_{ij} \sum_{n=0}^{q_{ij}} w_{mn}^{[q_{ij}]} \, f_i(U(\tau_{ij}^{-1}(s_n^{[q_{ij}]})), \tau_{ij}^{-1}(s_n^{[q_{ij}]}))$$

An Iterative Method

- Arrange elements in *time-slabs*
- Gauss-Seidel type iterations
- Diagonal Newton
- Control discrete residual and E_C
- Adaptive quadrature

A Simple Strategy

The last component steps first.



Multi-Adaptive Time-Stepping

Action: Redoing steps

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t = 0.01
n = 1

Multi-Adaptive Time-Stepping

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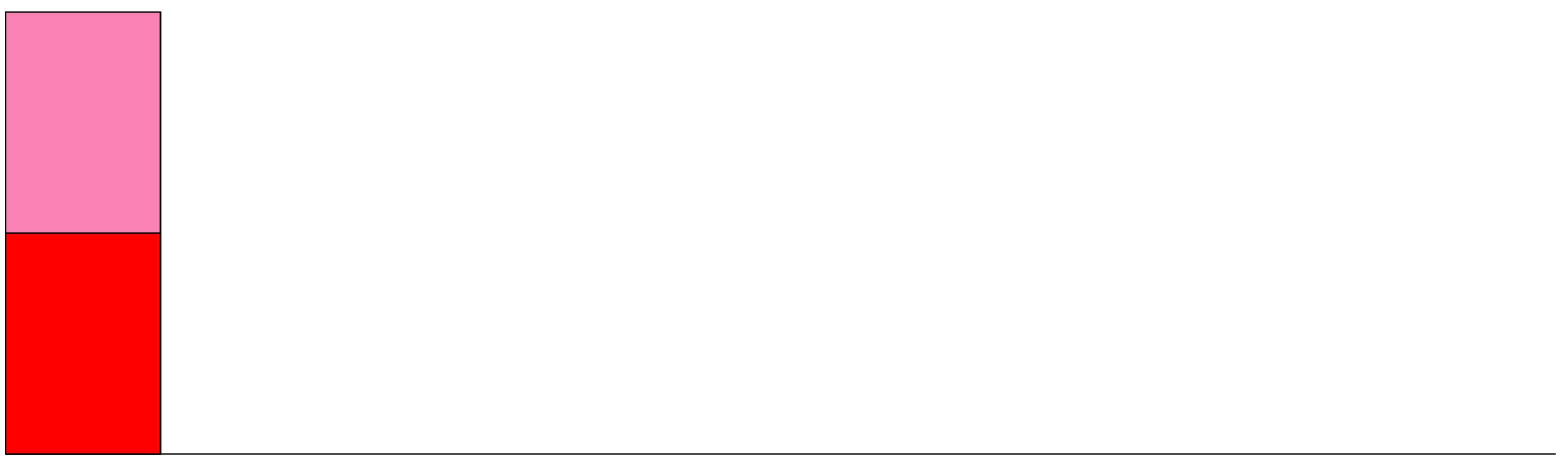

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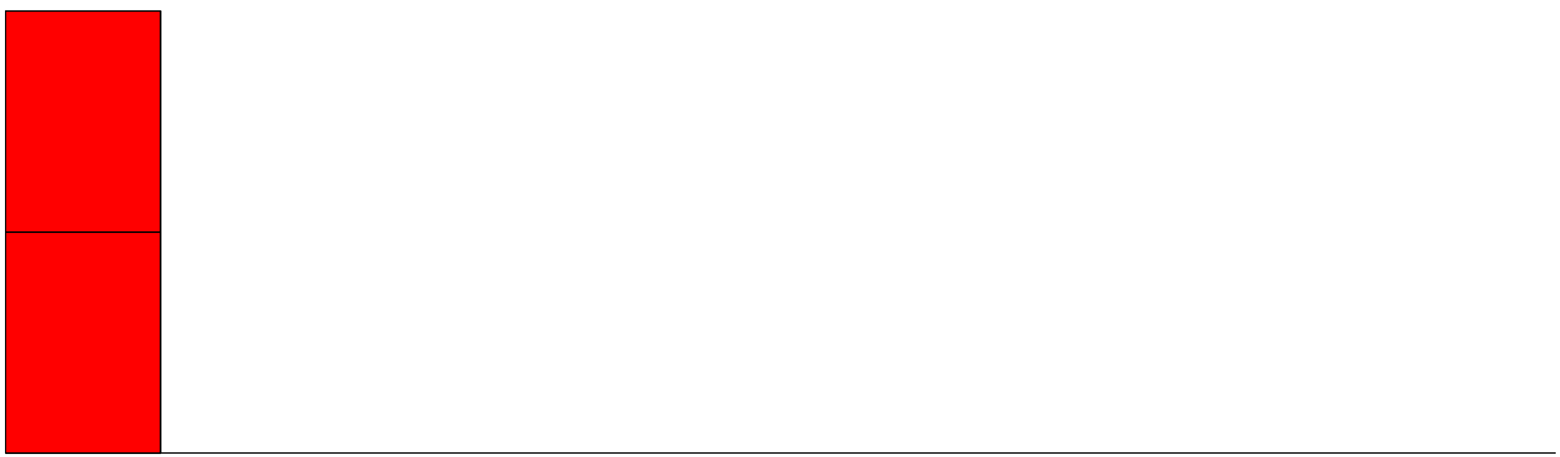

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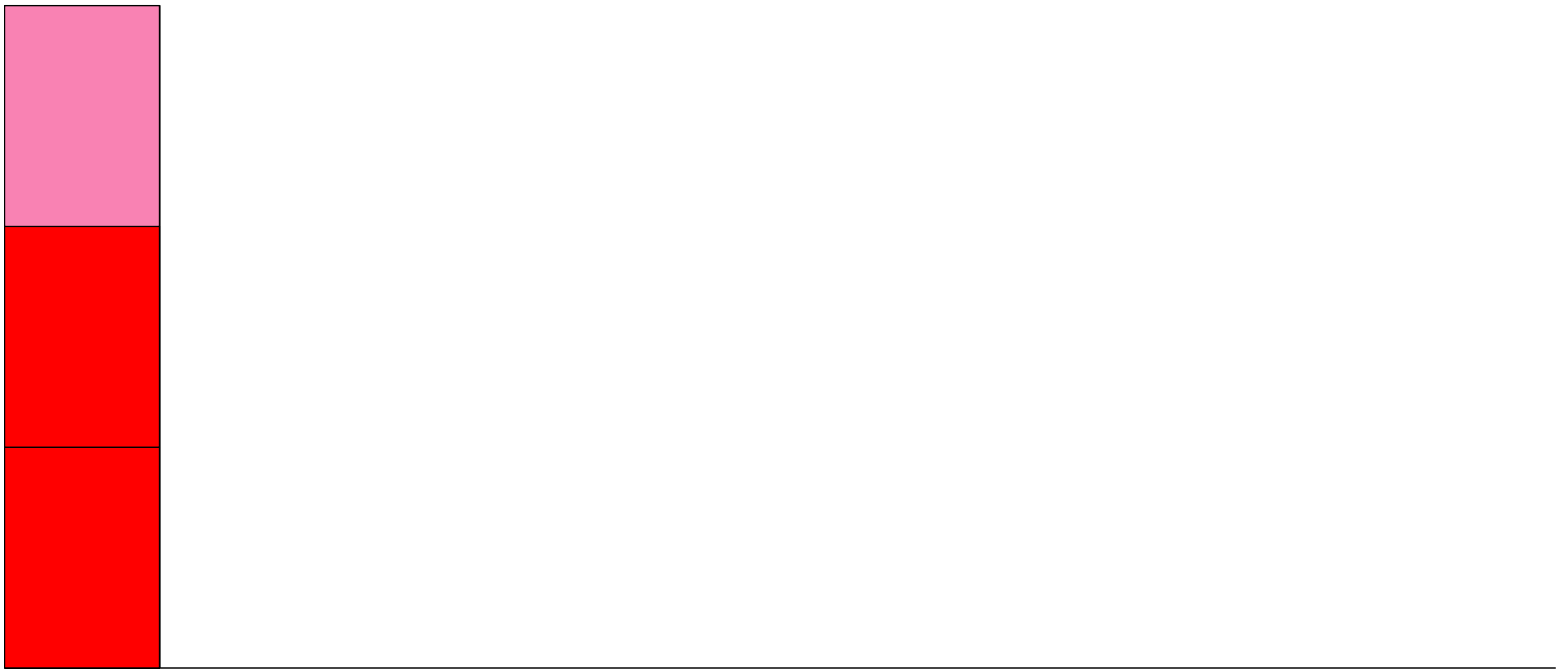

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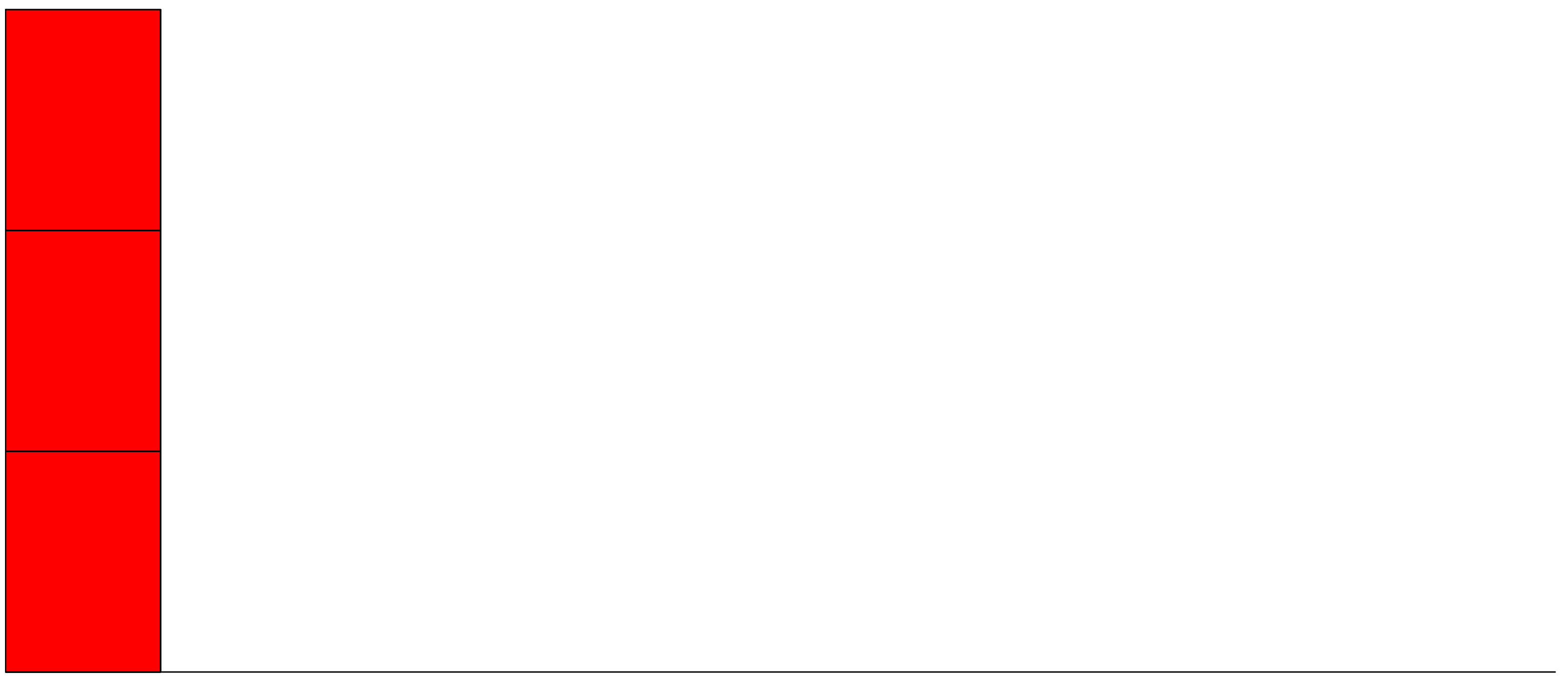

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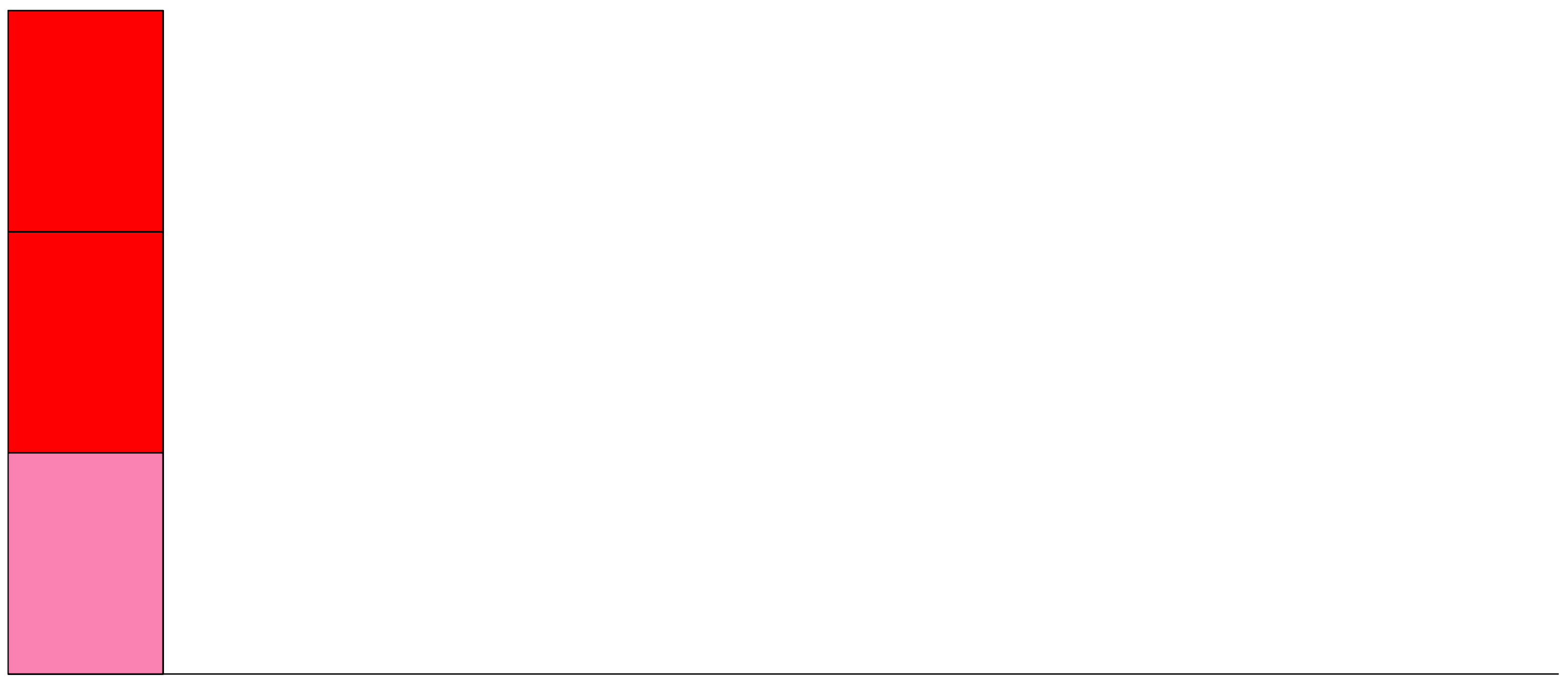

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n = 1

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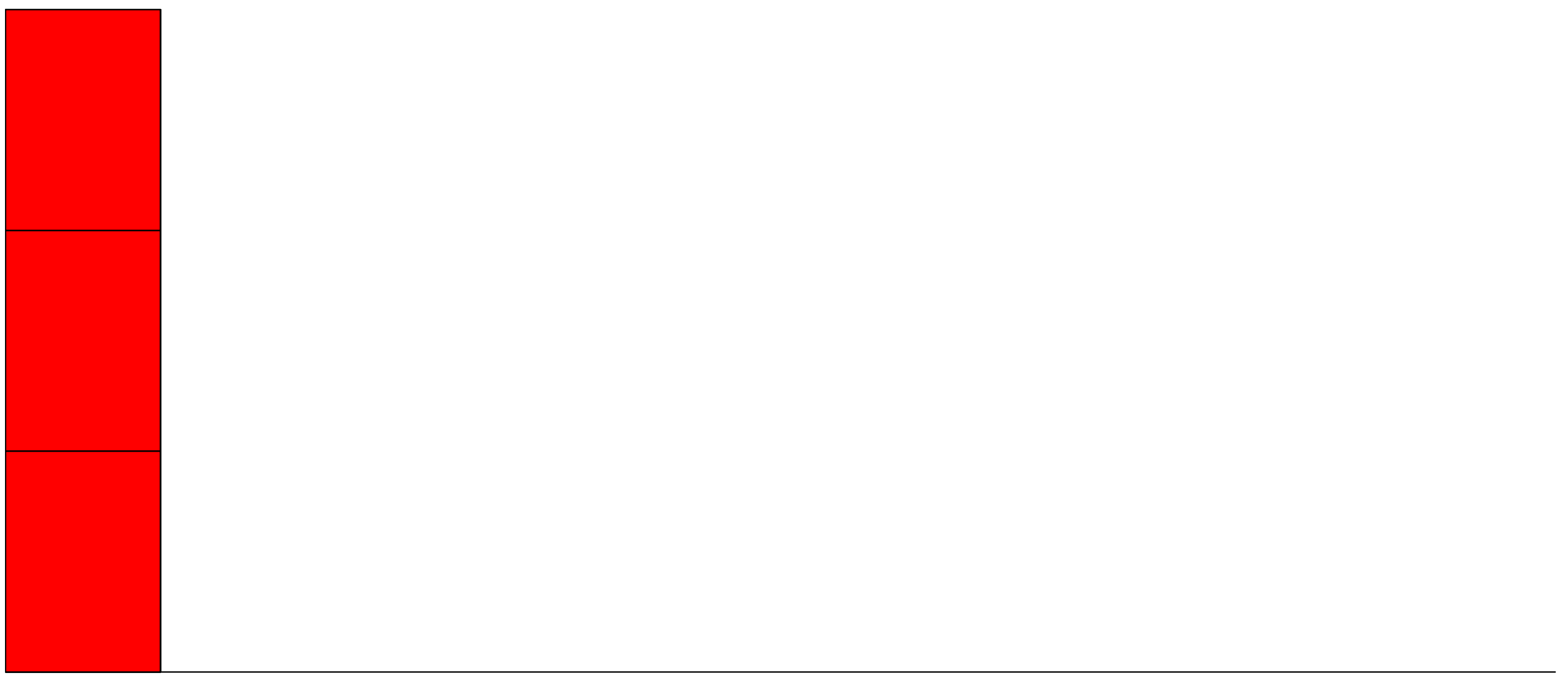
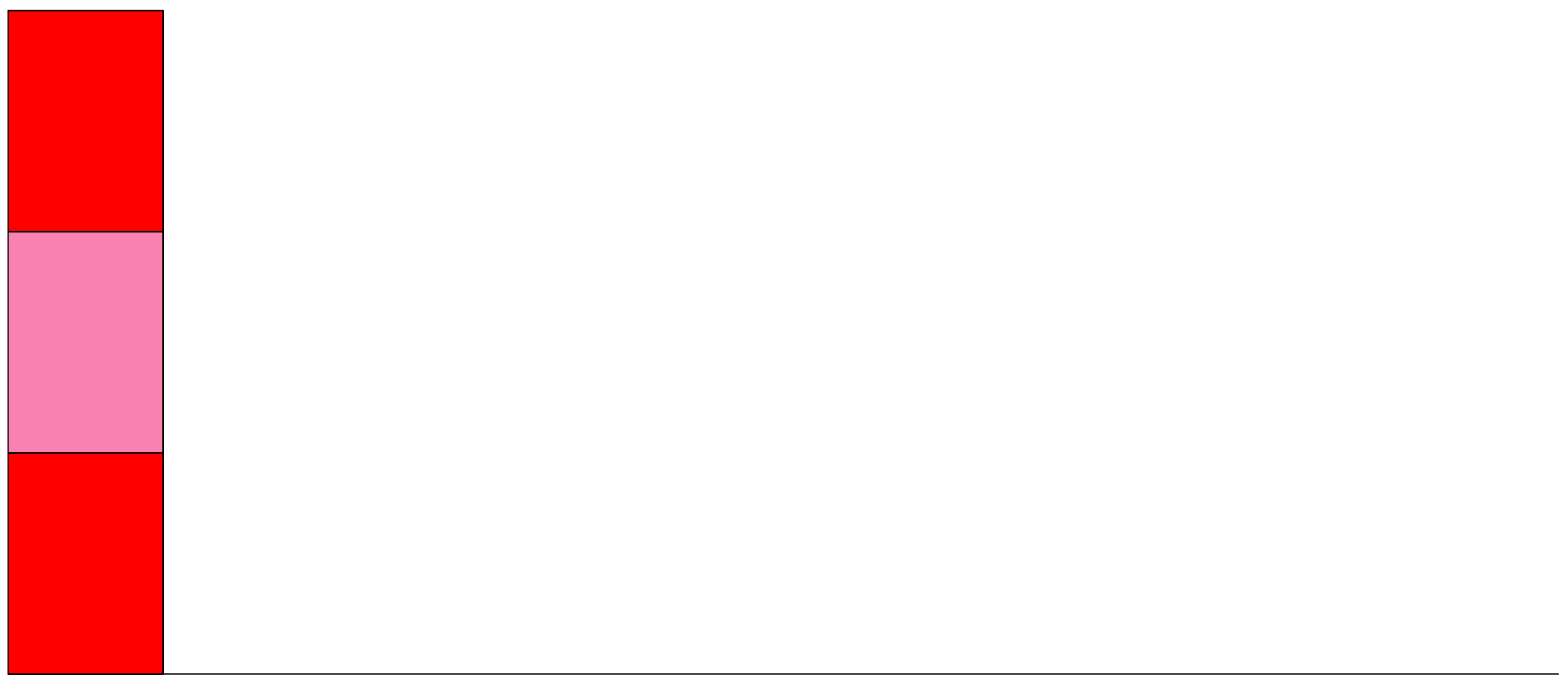


t = 0.01
n = 2

Multi-Adaptive Time-Stepping

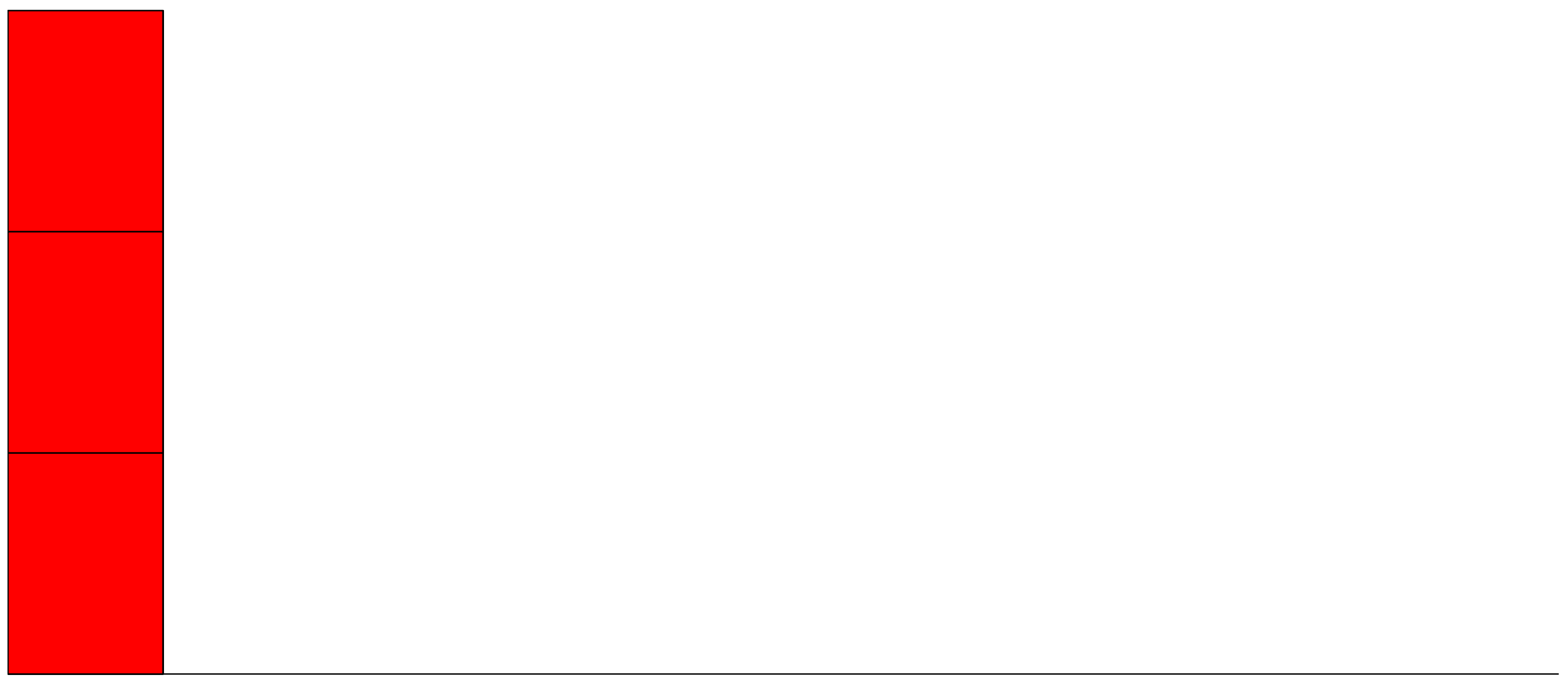
Action: Redoing steps



t = 0.01
n = 2

Multi-Adaptive Time-Stepping

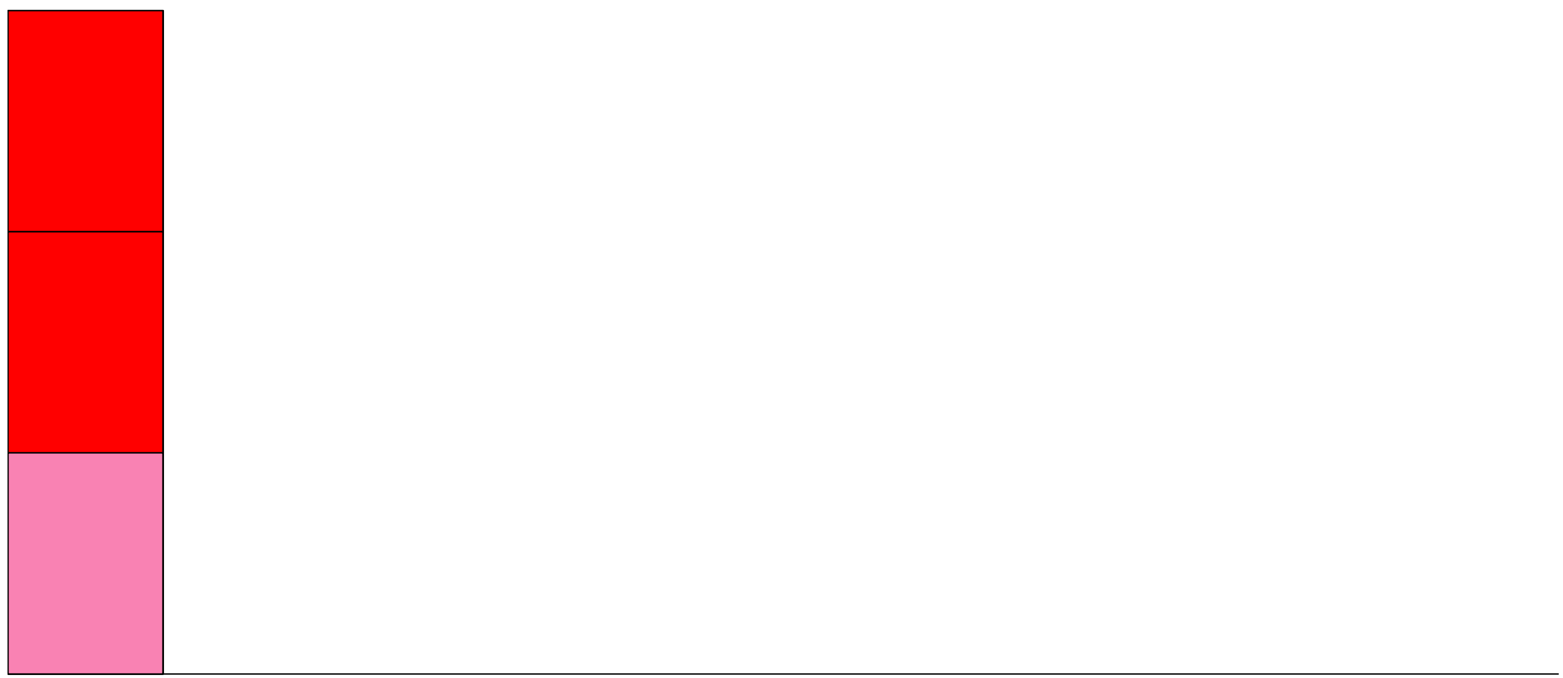
Action: Redoing steps



t = 0.01
n = 2

Multi-Adaptive Time-Stepping

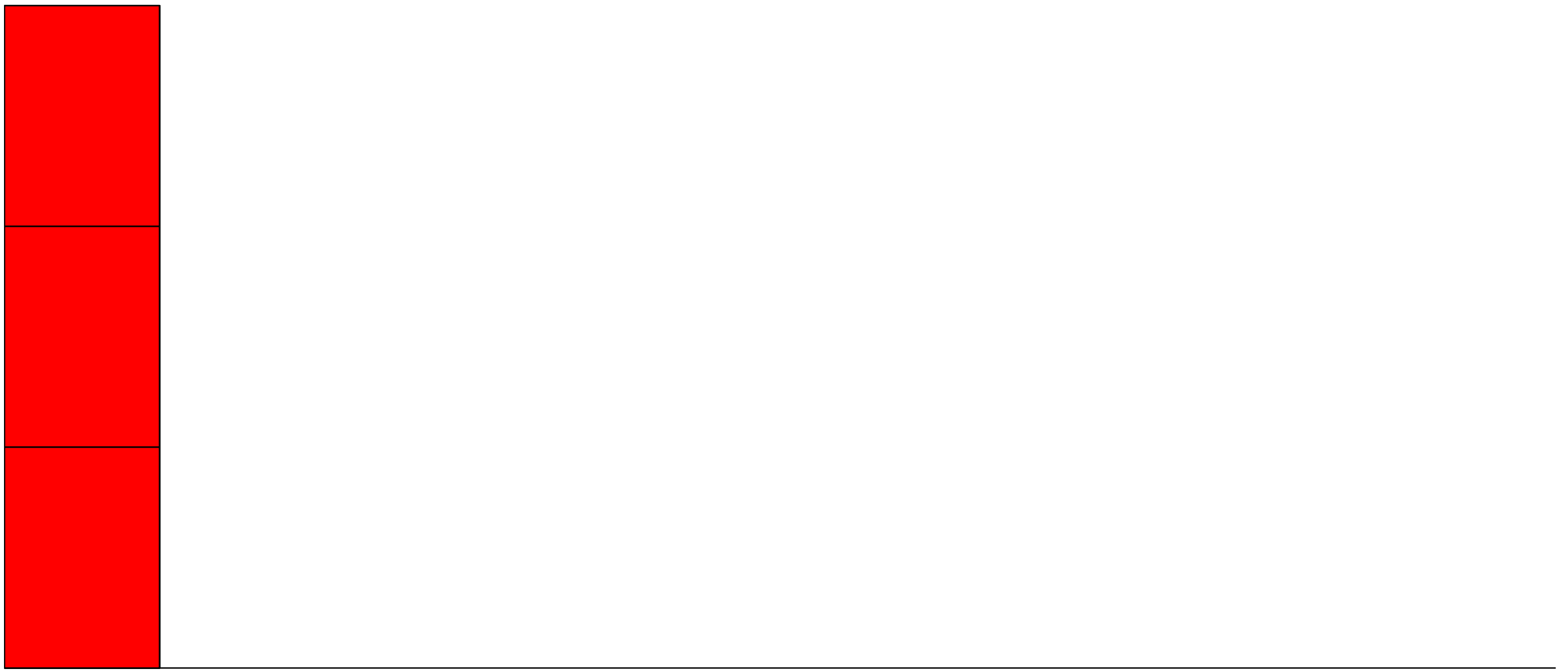
Action: Redoing steps



t = 0.01
n = 2

Multi-Adaptive Time-Stepping

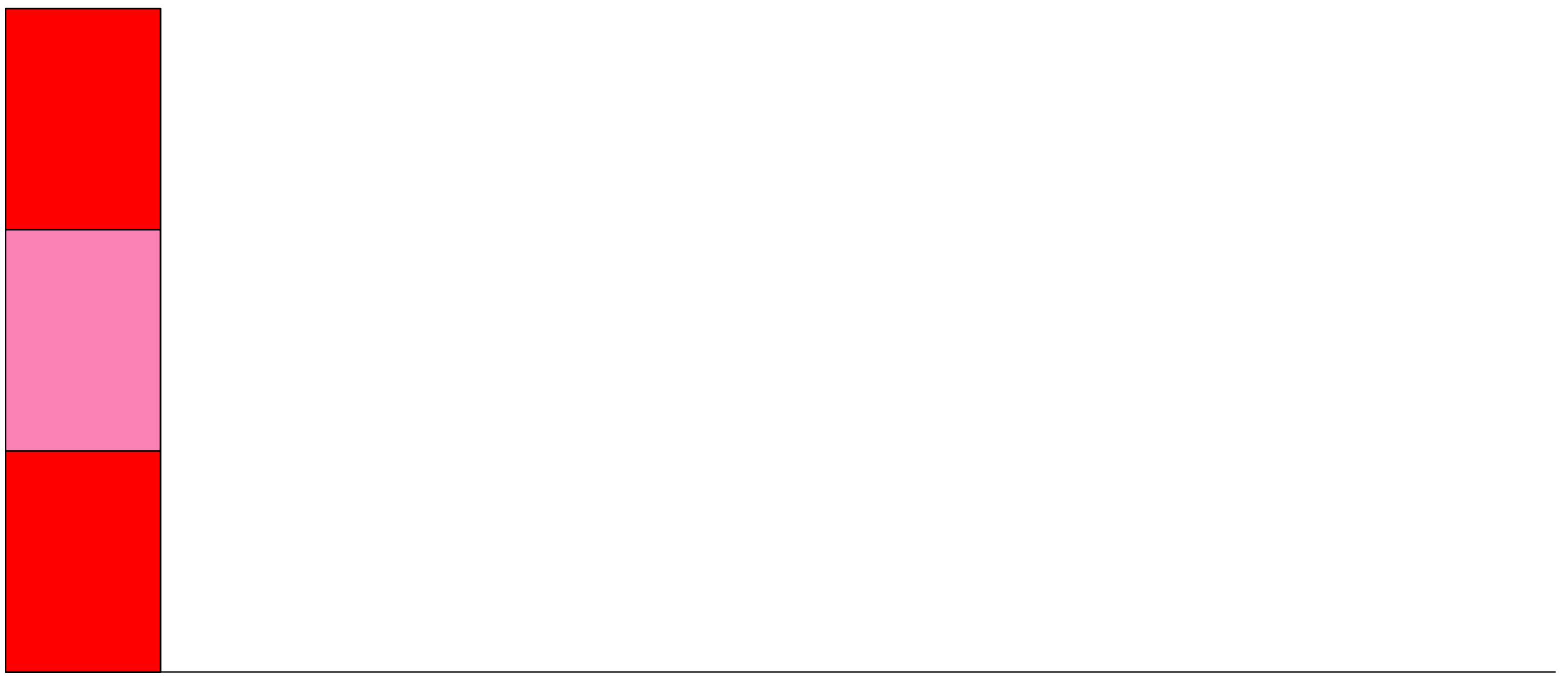
Action: Redoing steps



t = 0.01
n = 2.6667

Multi-Adaptive Time-Stepping

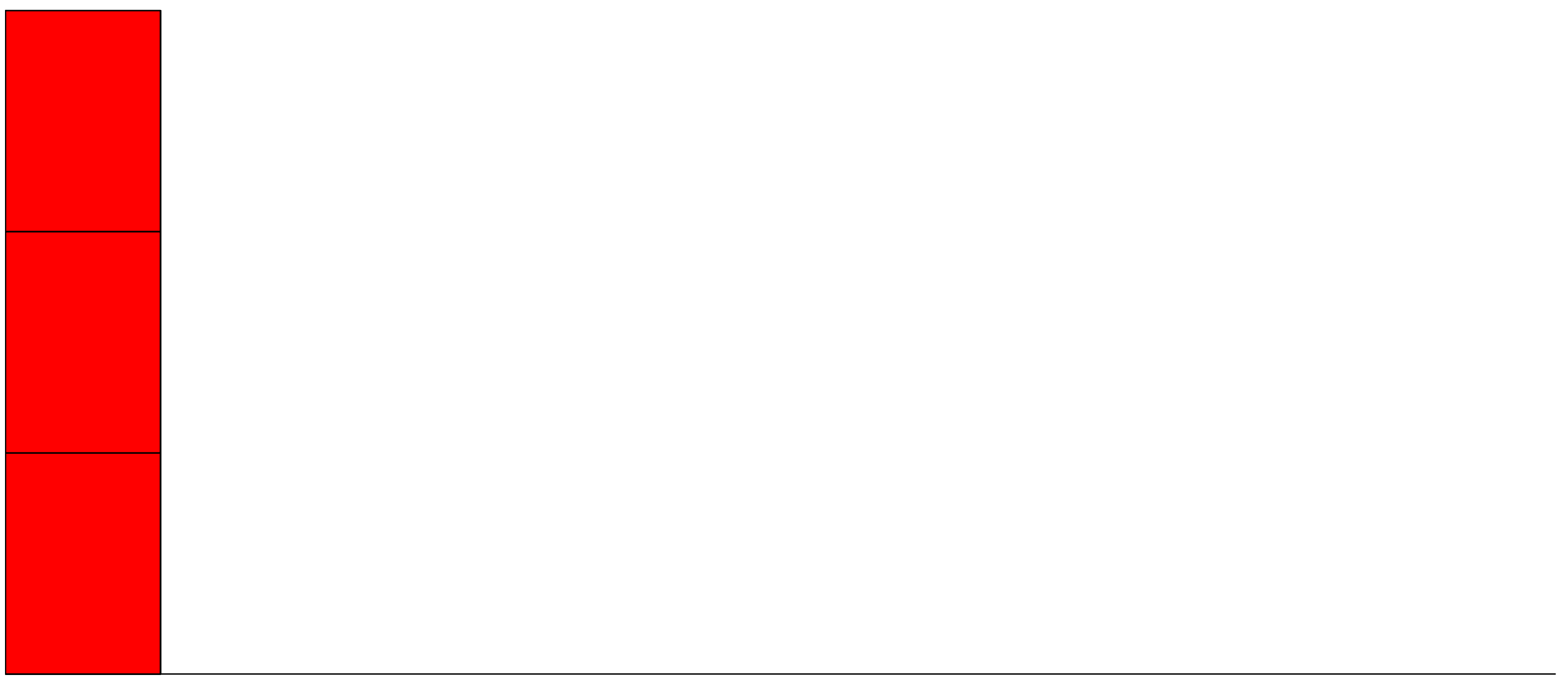
Action: Redoing steps



t = 0.01
n = 2.6667

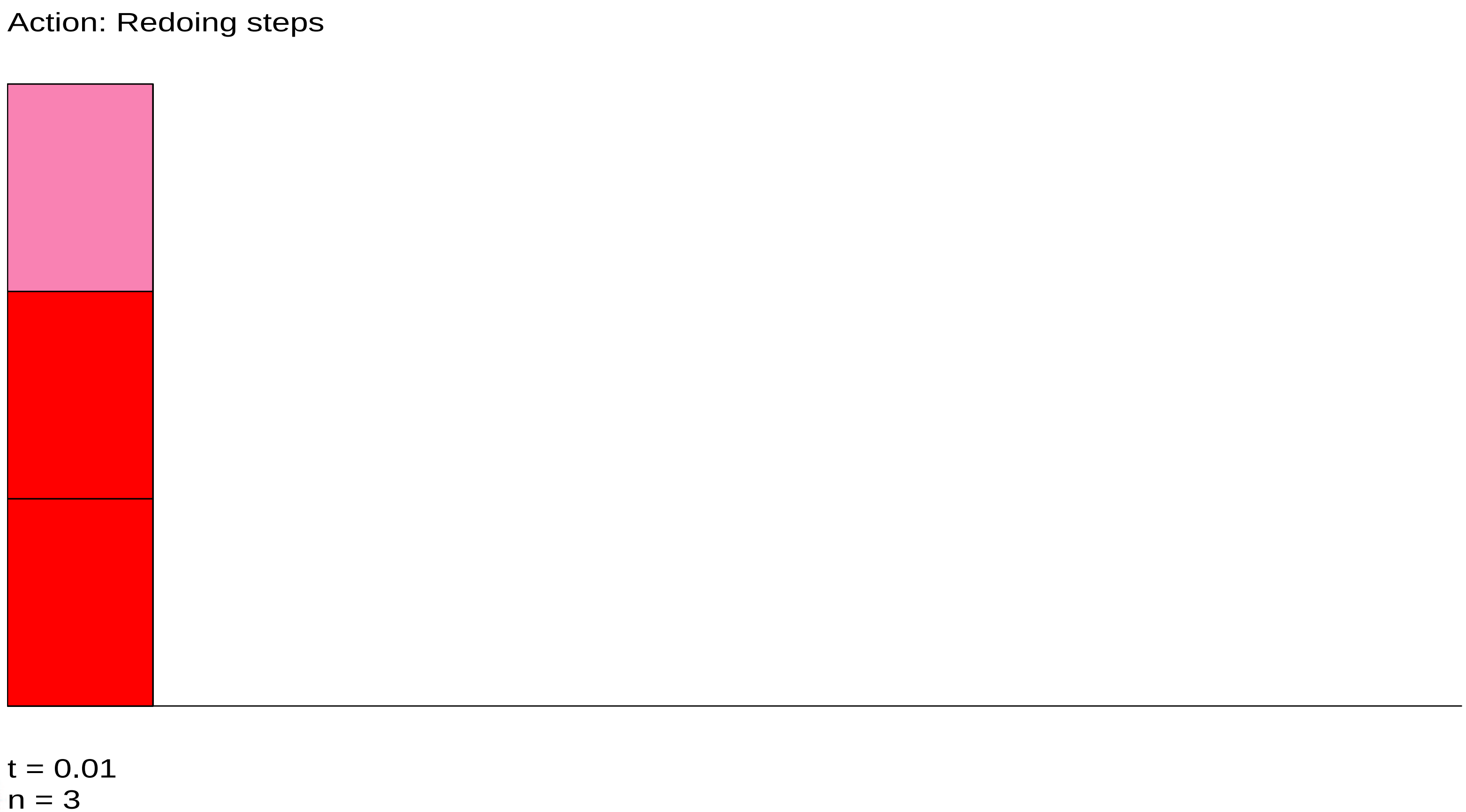
Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.01
n = 3

Multi-Adaptive Time-Stepping



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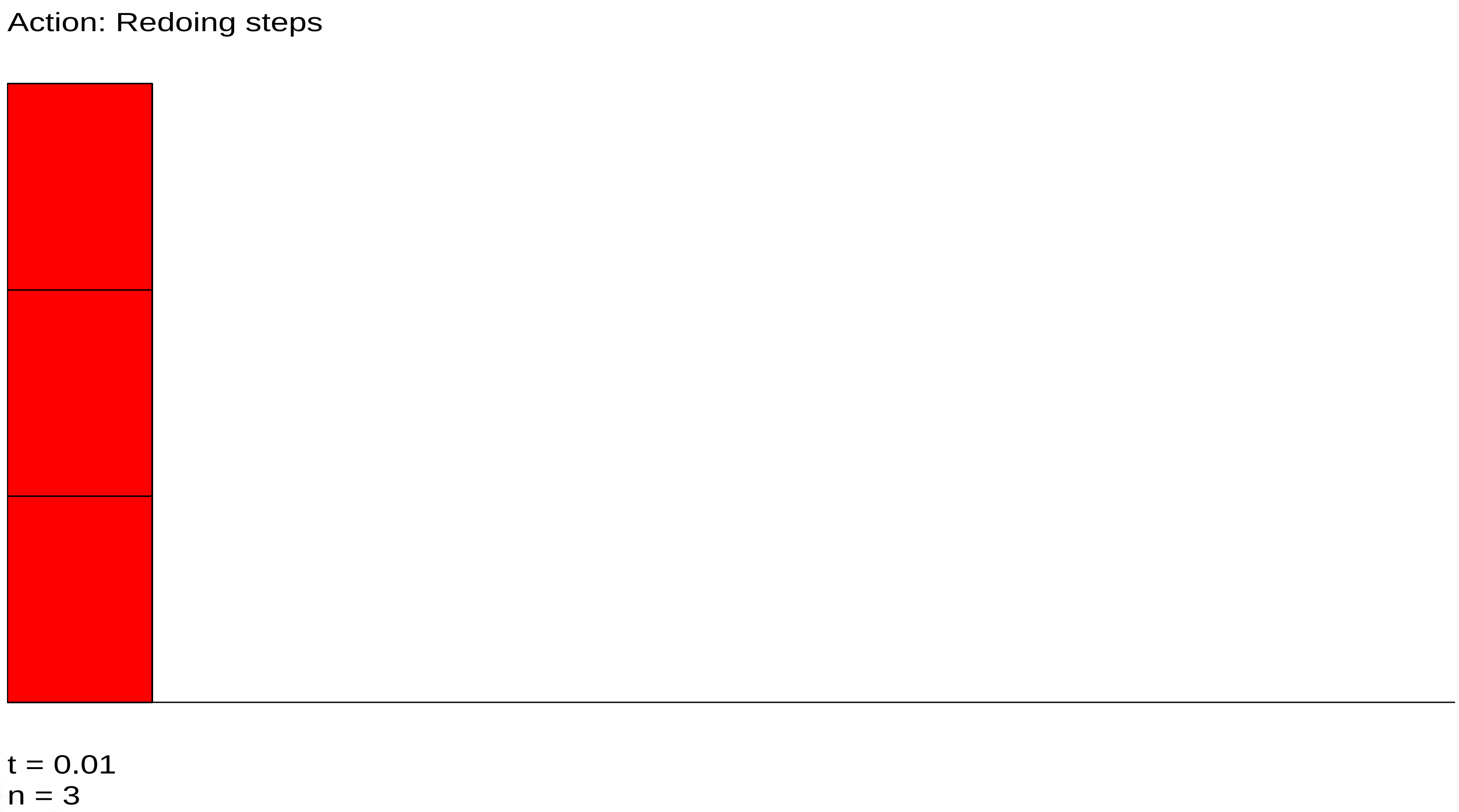

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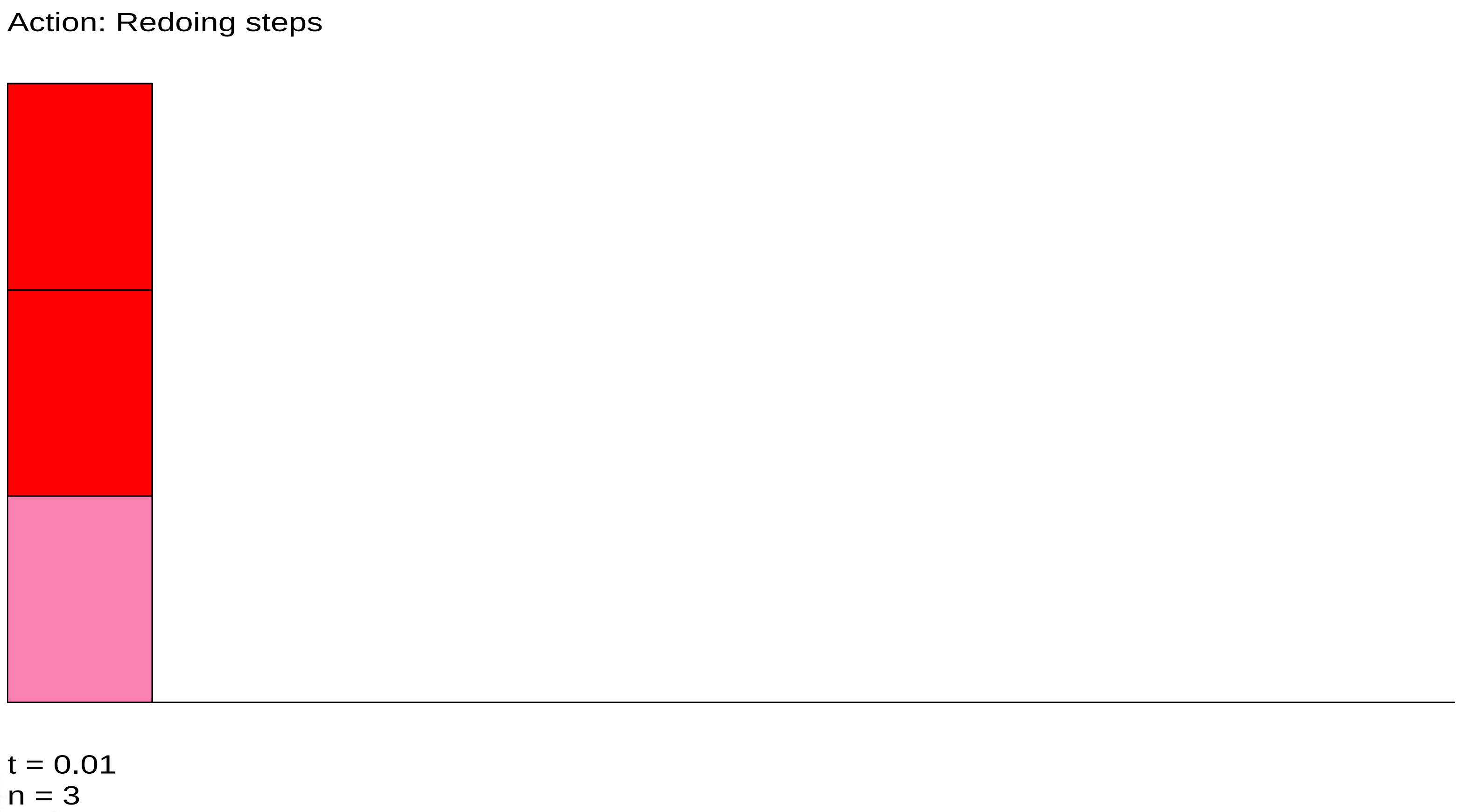

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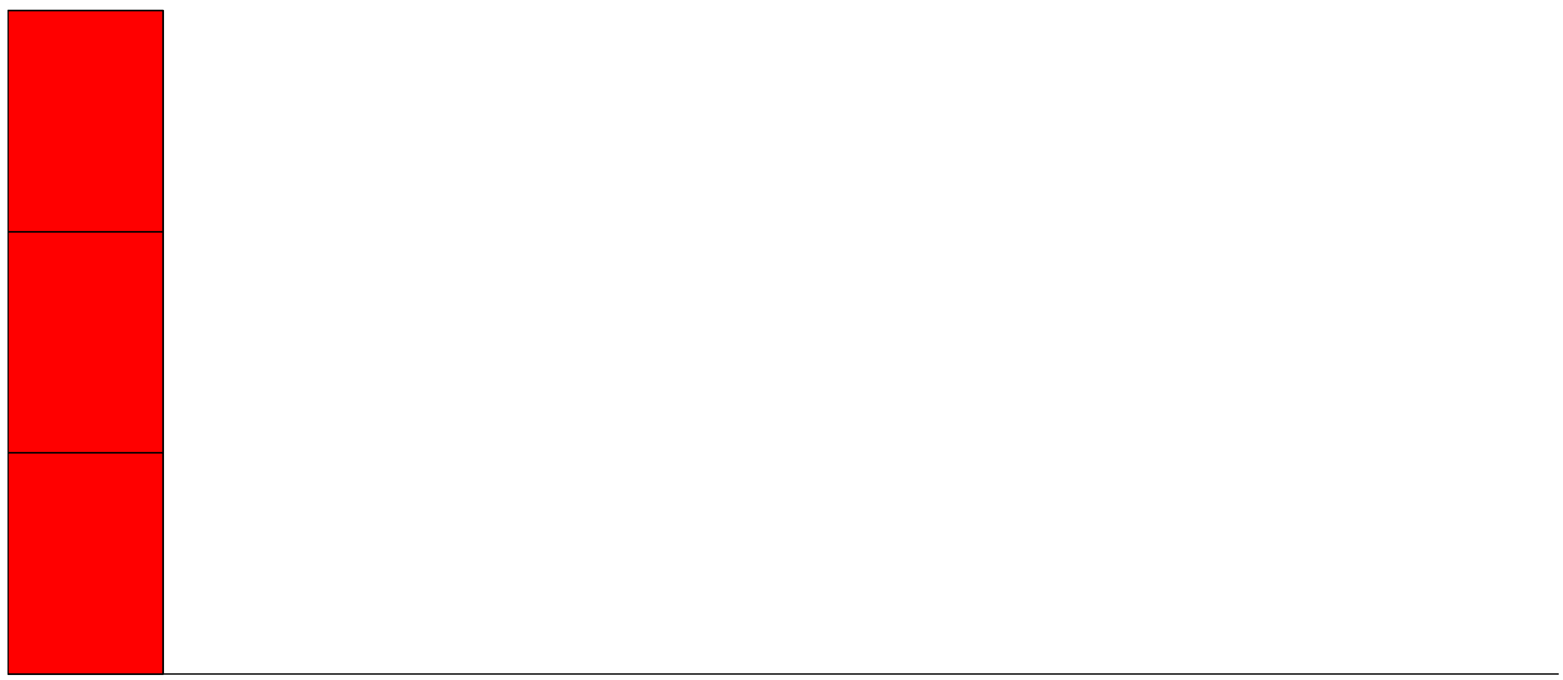
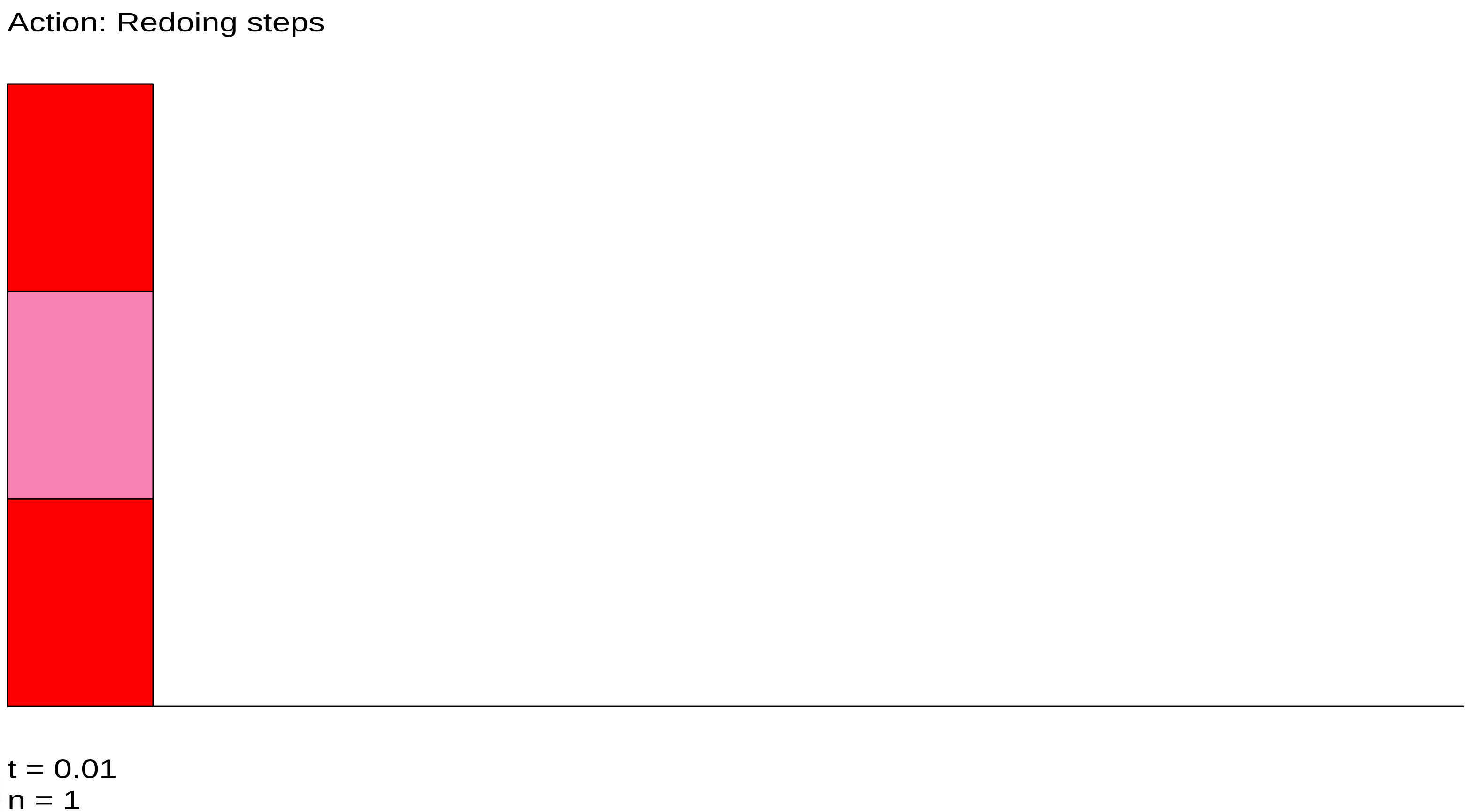
Action: Redoing steps



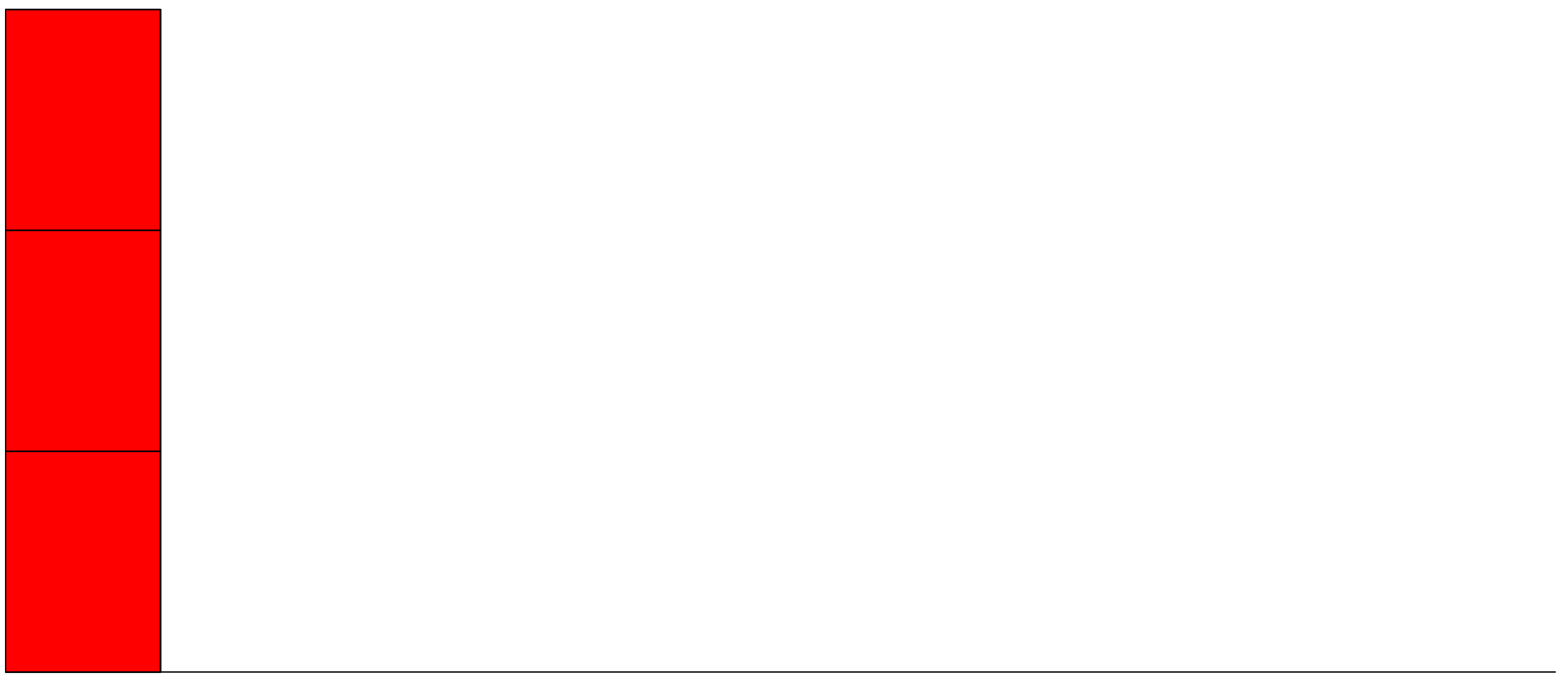
t = 0.01
n = 1

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

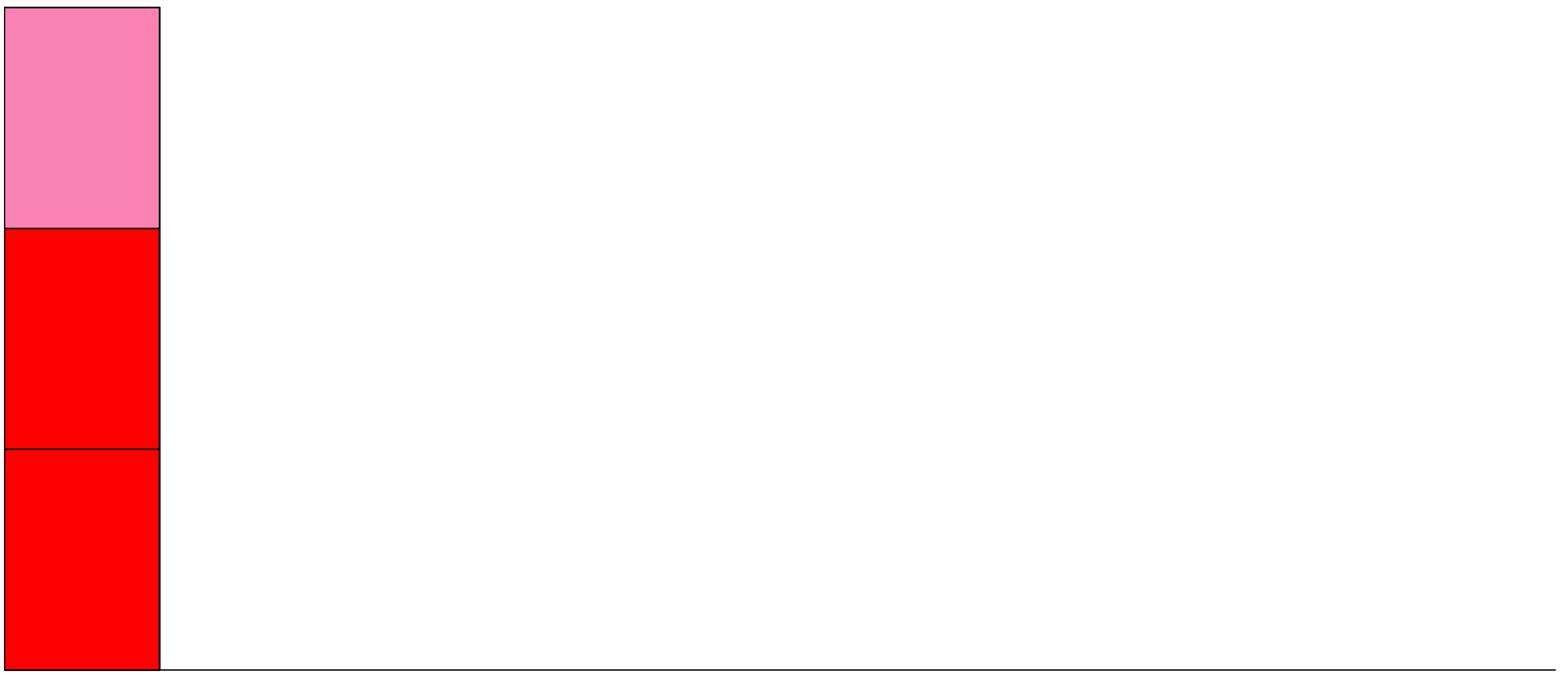
Action: Redoing steps



t = 0.01
n = 1

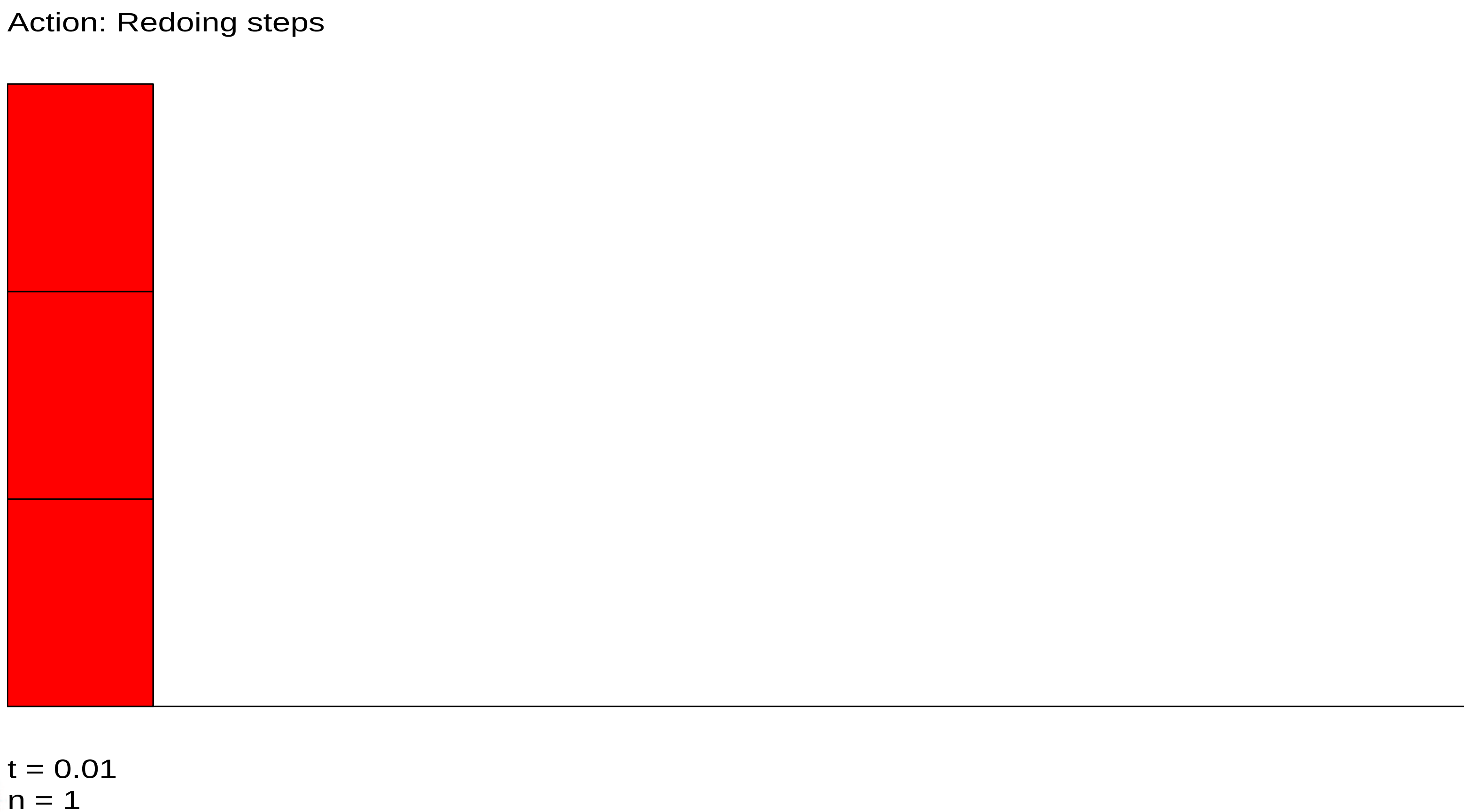
Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.01
n = 1

Multi-Adaptive Time-Stepping



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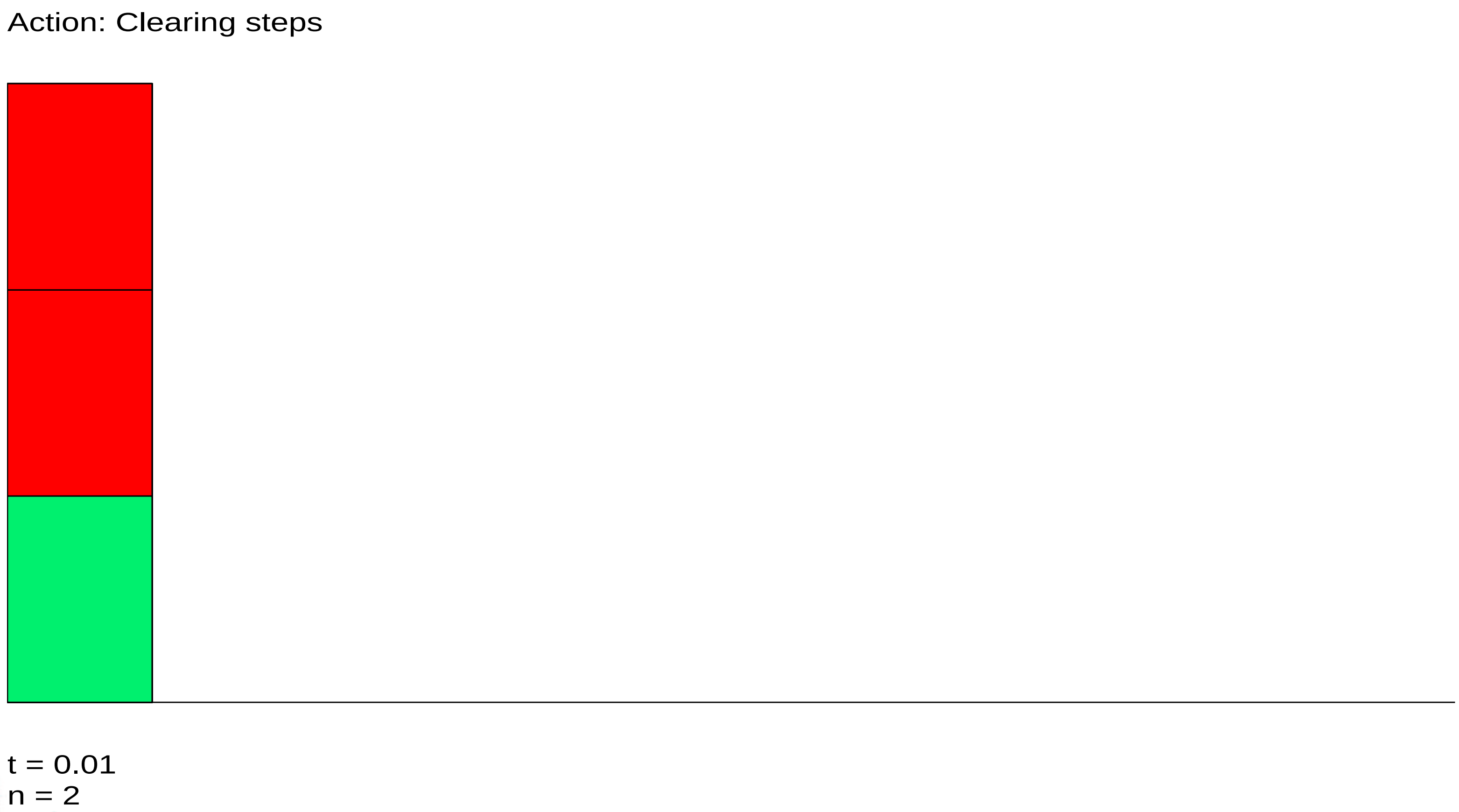
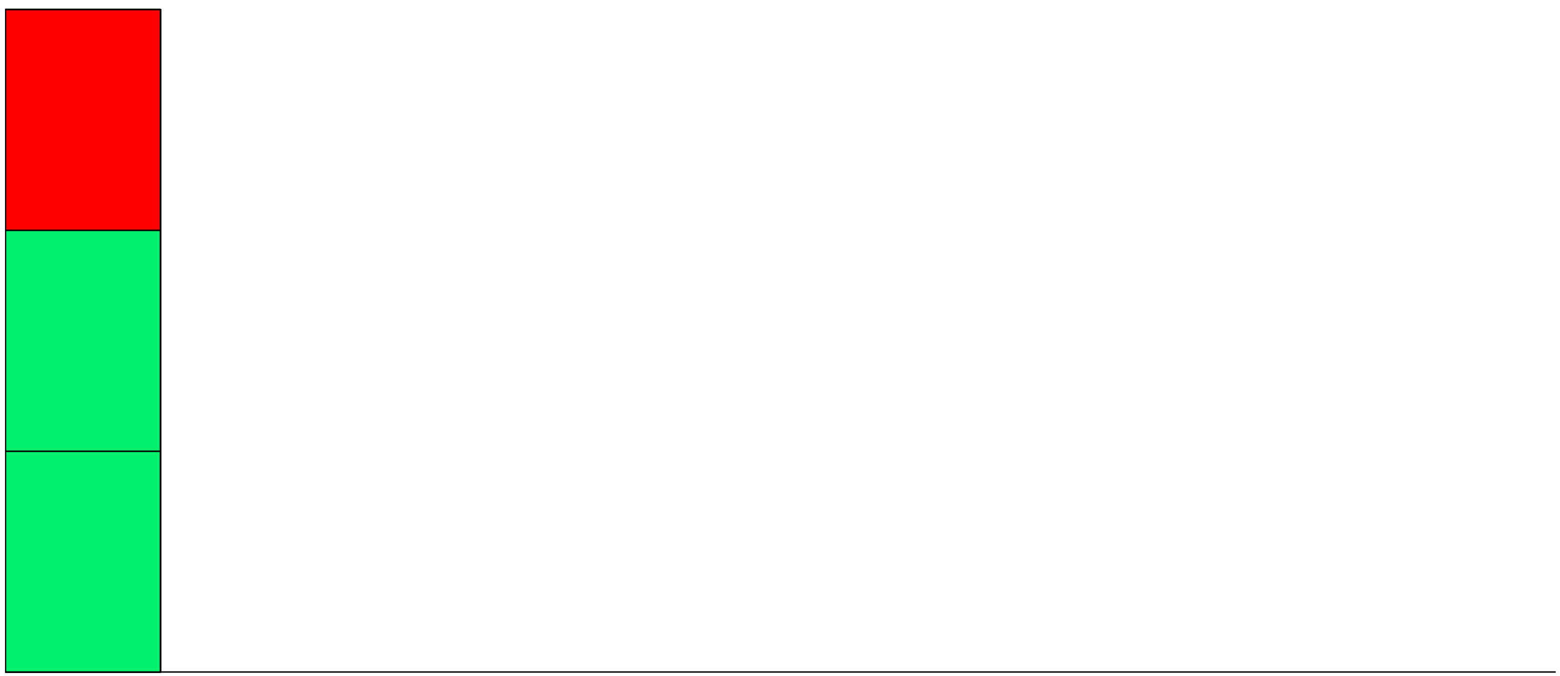


Multi-Adaptive Time-Stepping

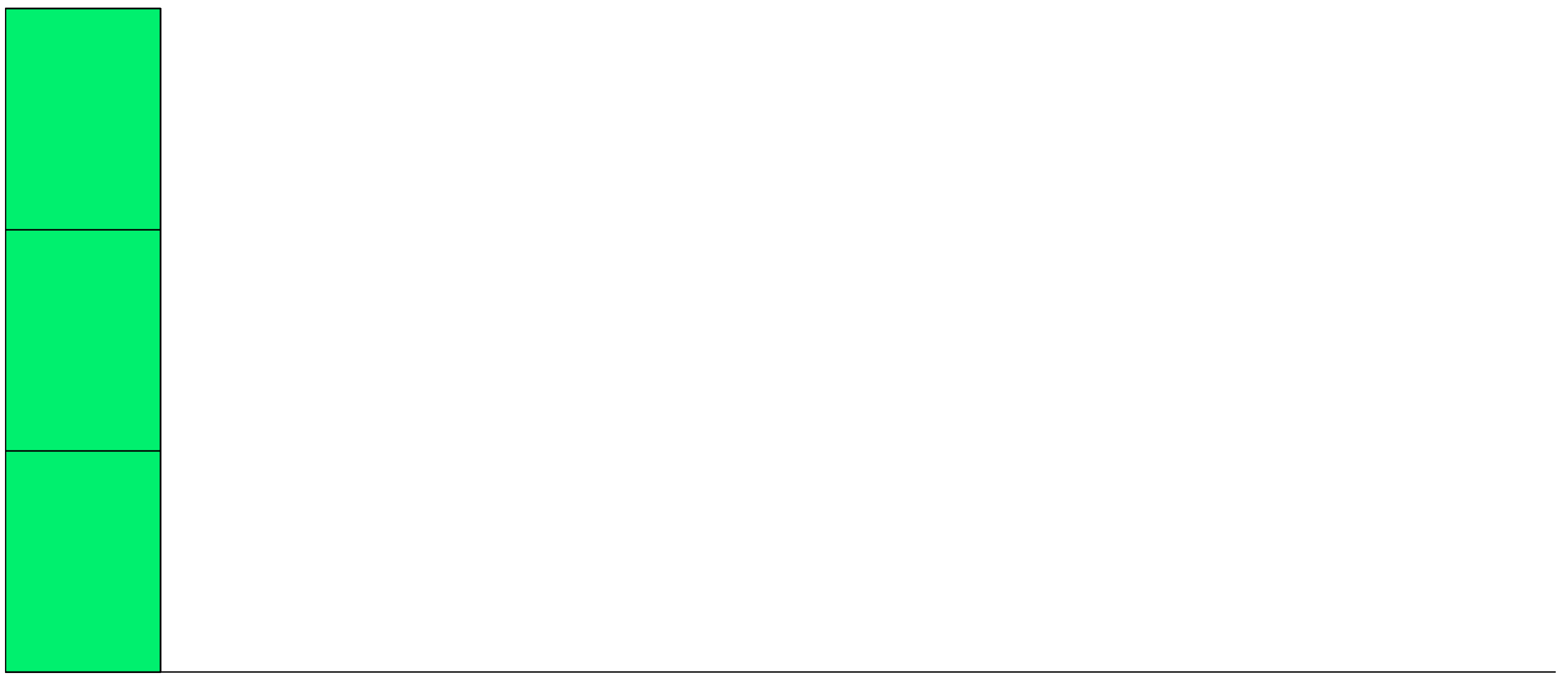
Action: Clearing steps



t = 0.01
n = 2

Multi-Adaptive Time-Stepping

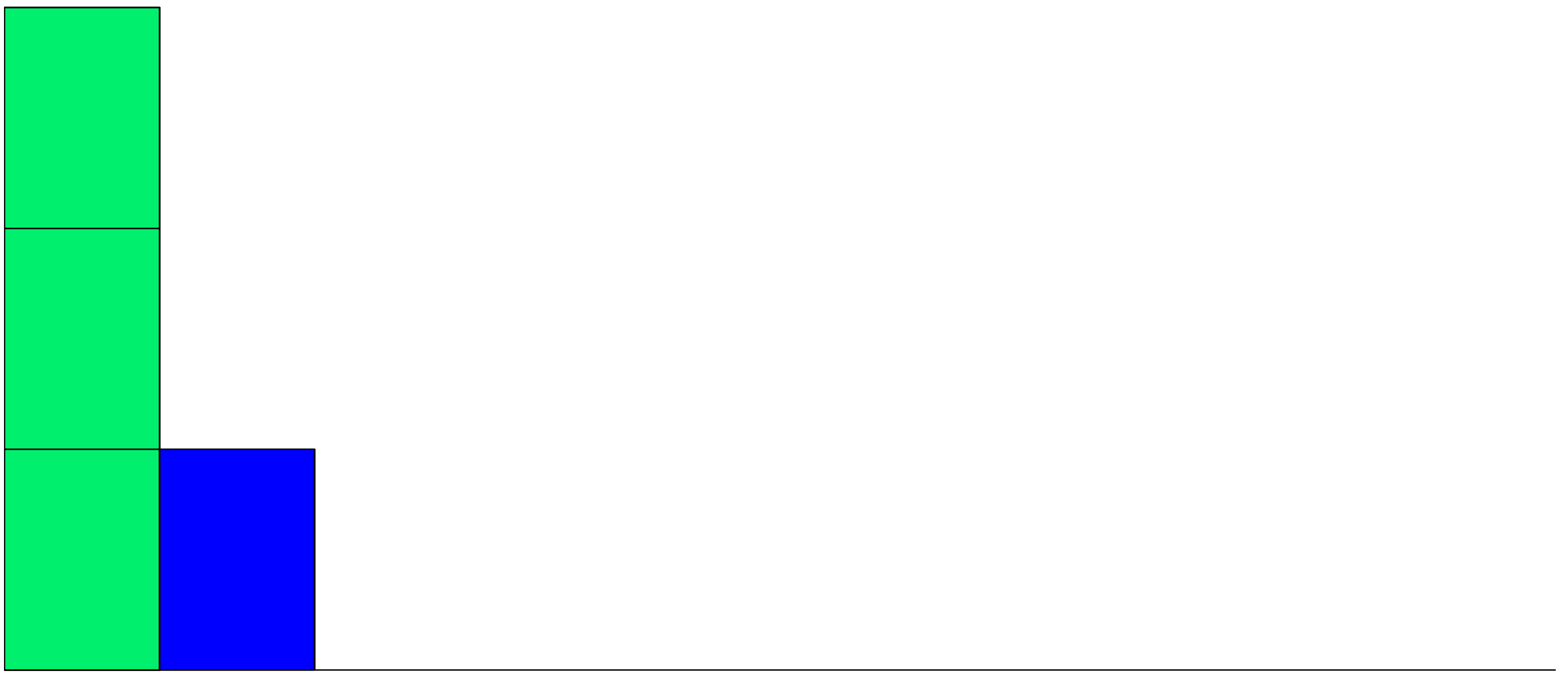
Action: Clearing steps



t = 0.01
n = 2

Multi-Adaptive Time-Stepping

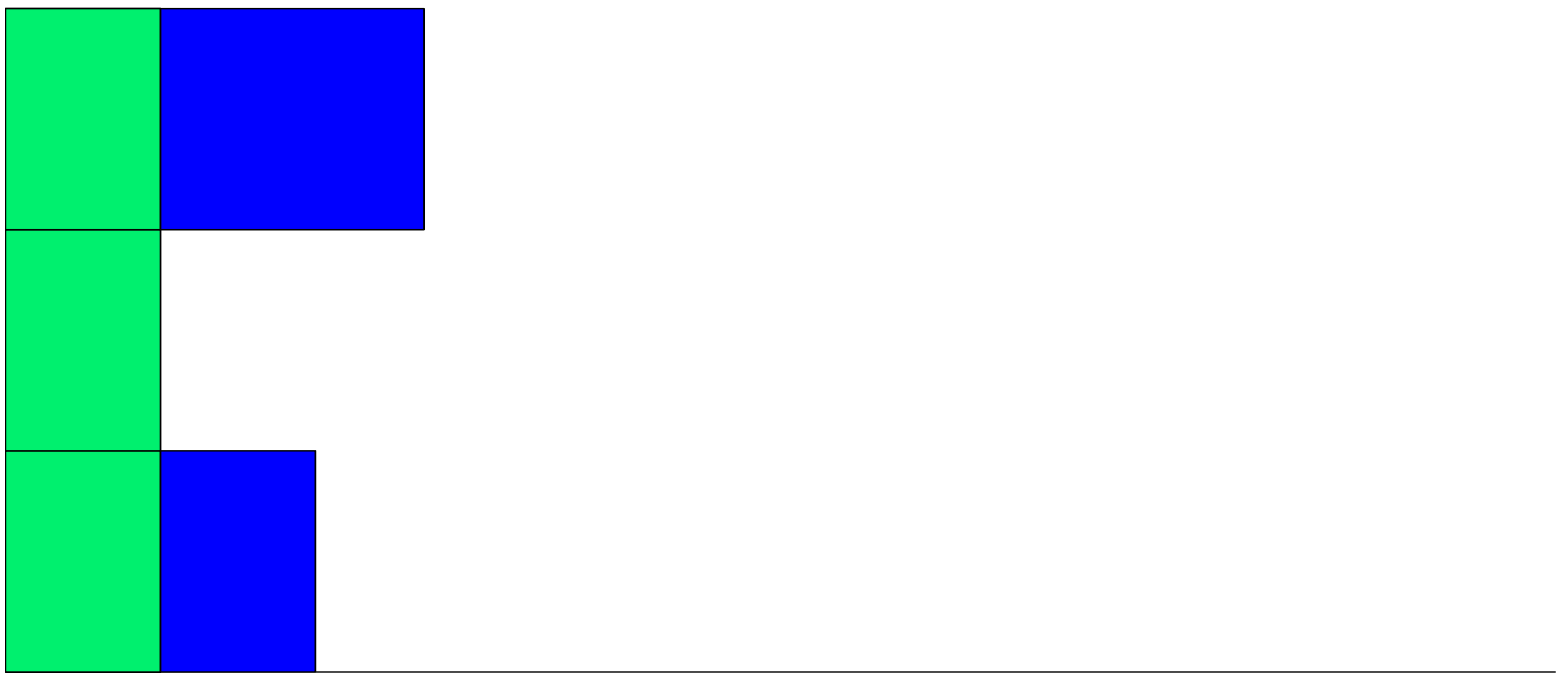
Action: Covering



t = 0.02
n = 2

Multi-Adaptive Time-Stepping

Action: Covering



t = 0.027
n = 2

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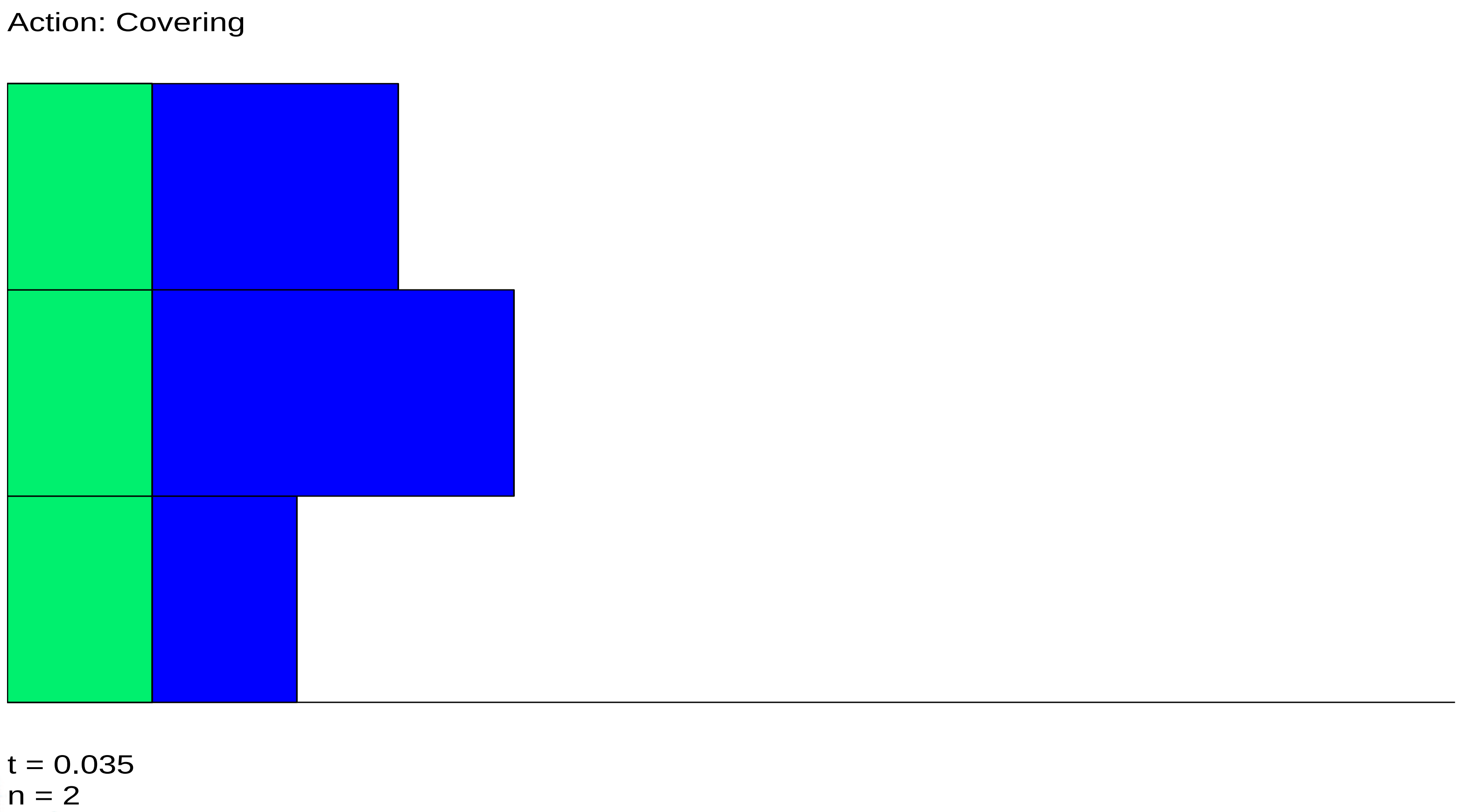
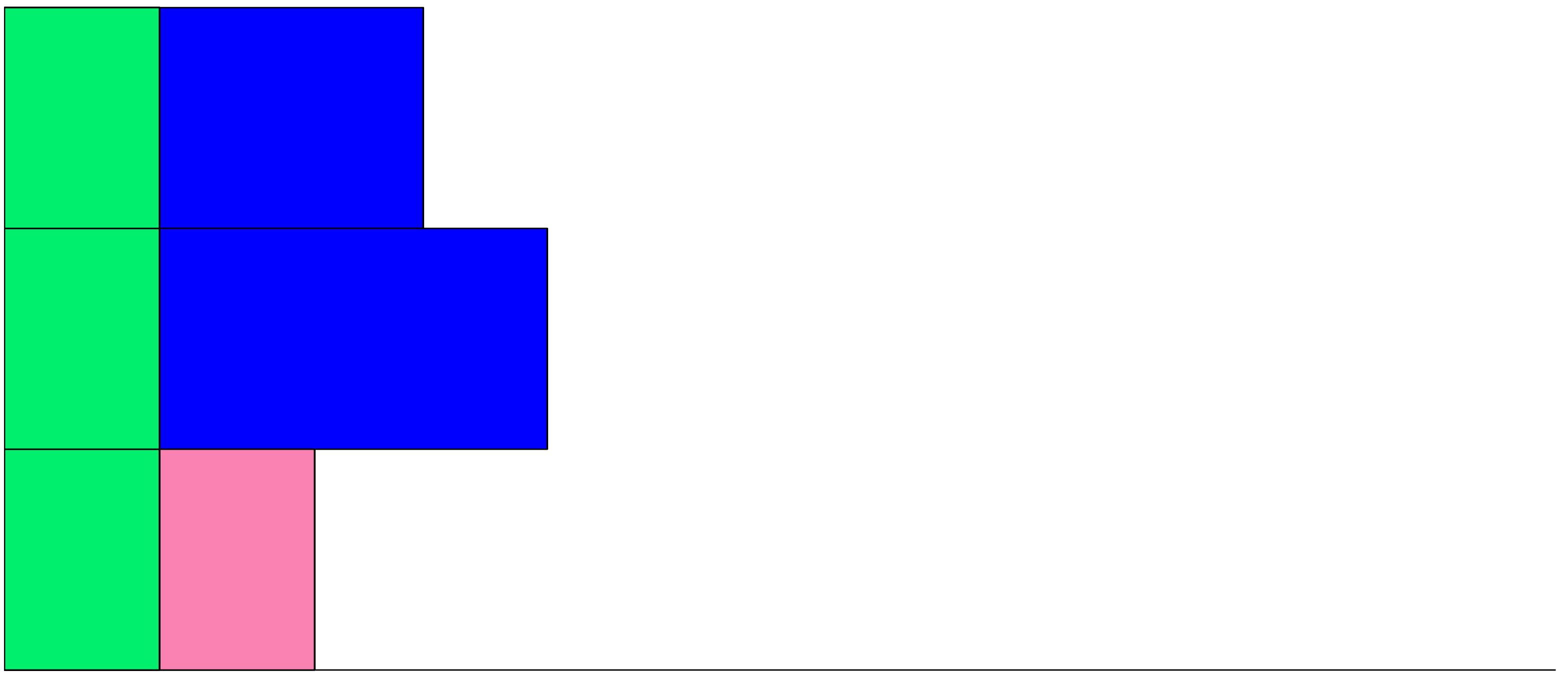
Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

Action: Redoing steps



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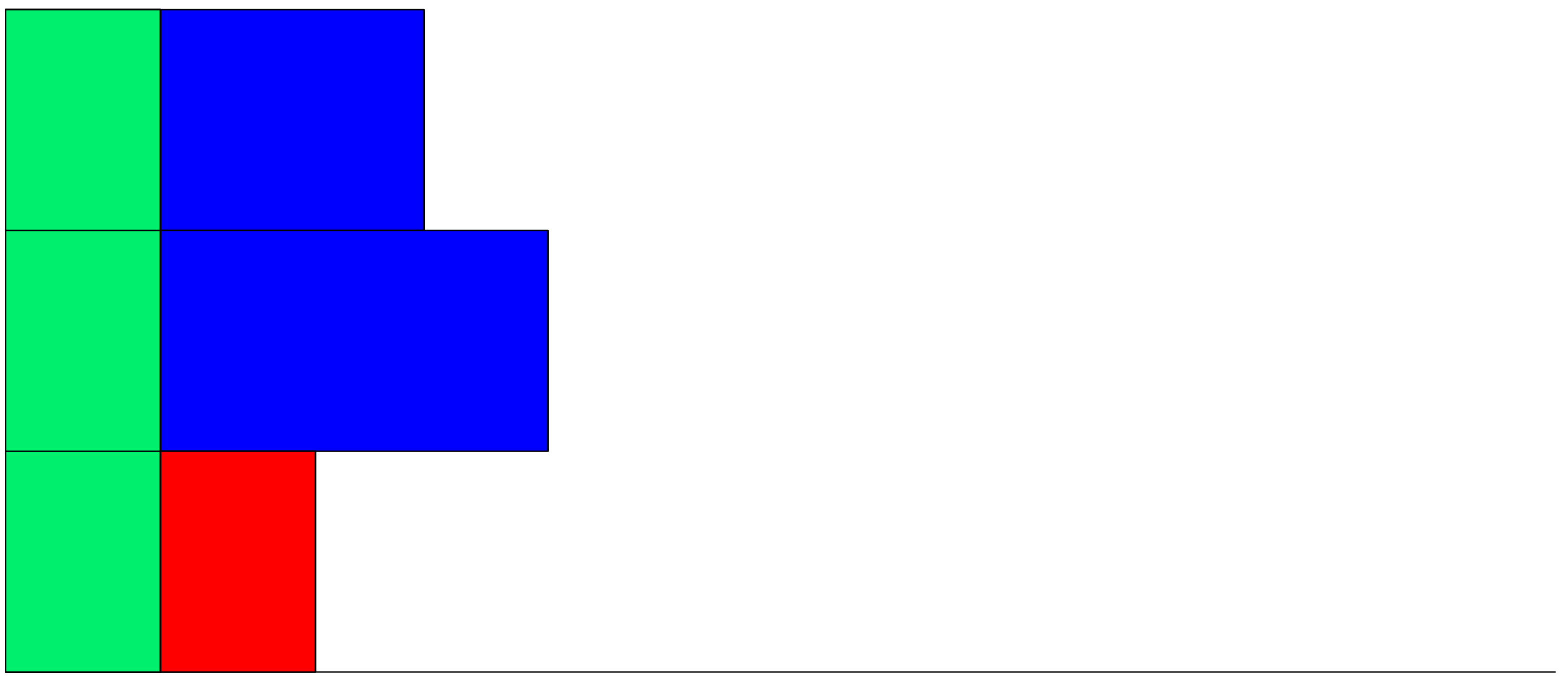
n = 2

Multi-Adaptive Time-Stepping

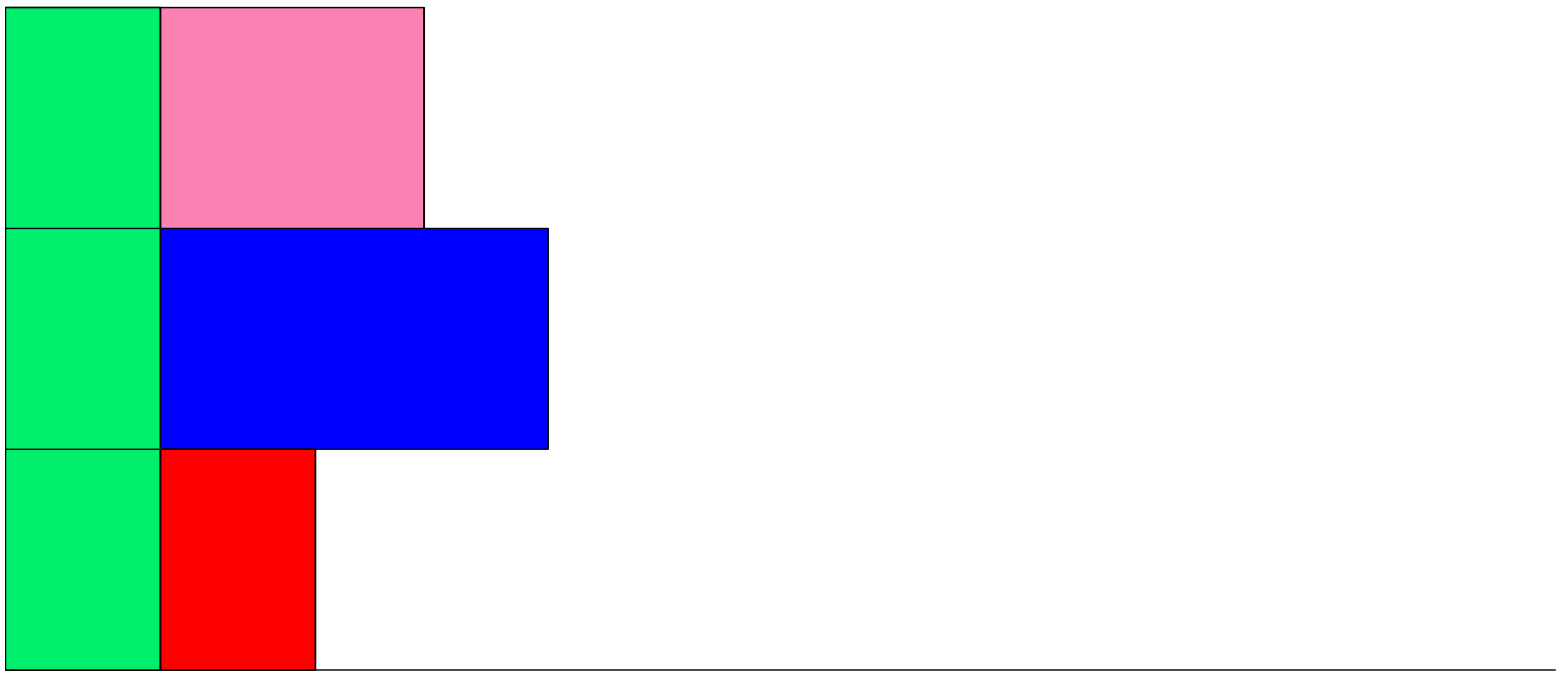
Action: Redoing steps



t = 0.02
n = 1

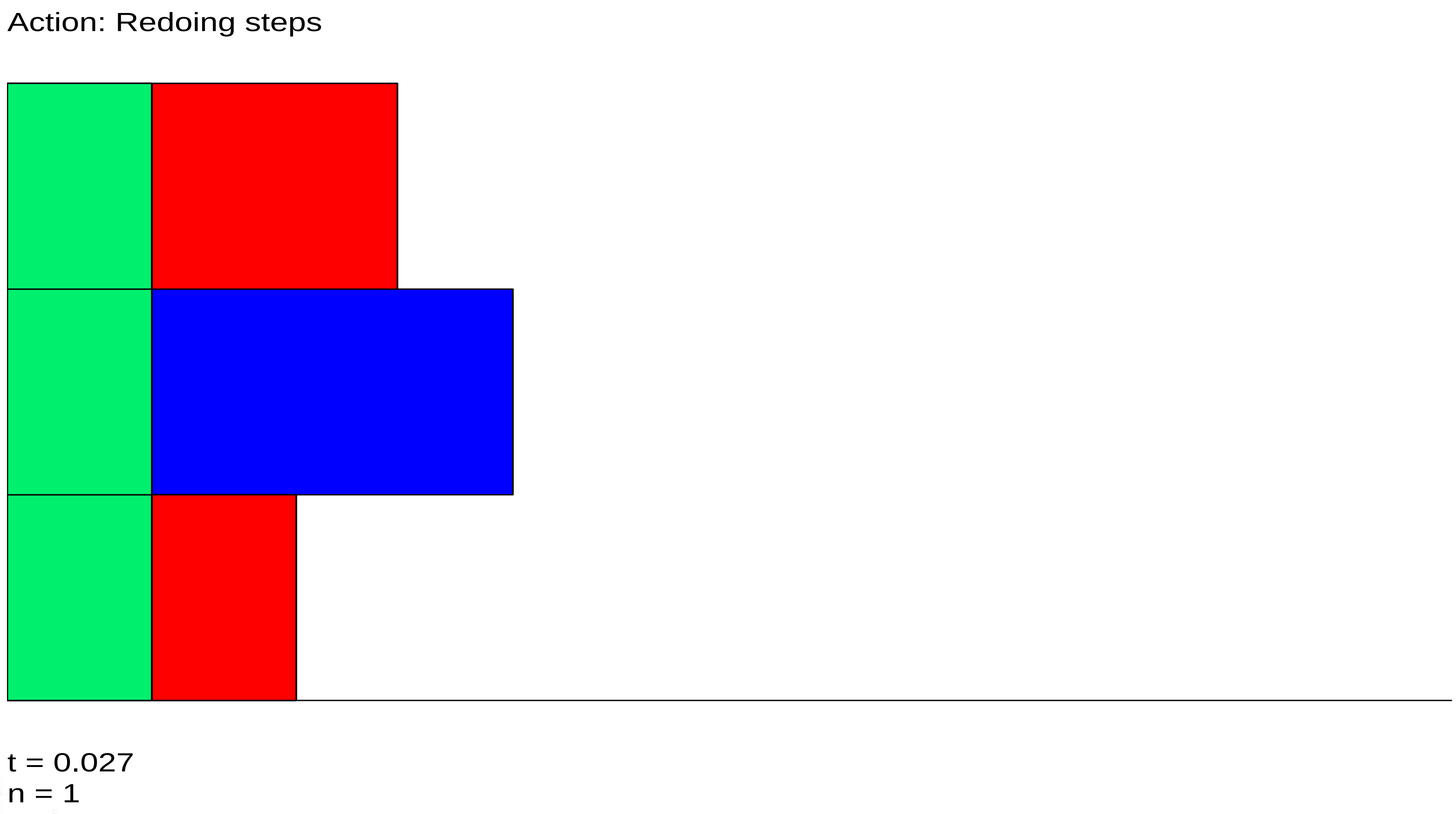
Multi-Adaptive Time-Stepping

Action: Redoing steps



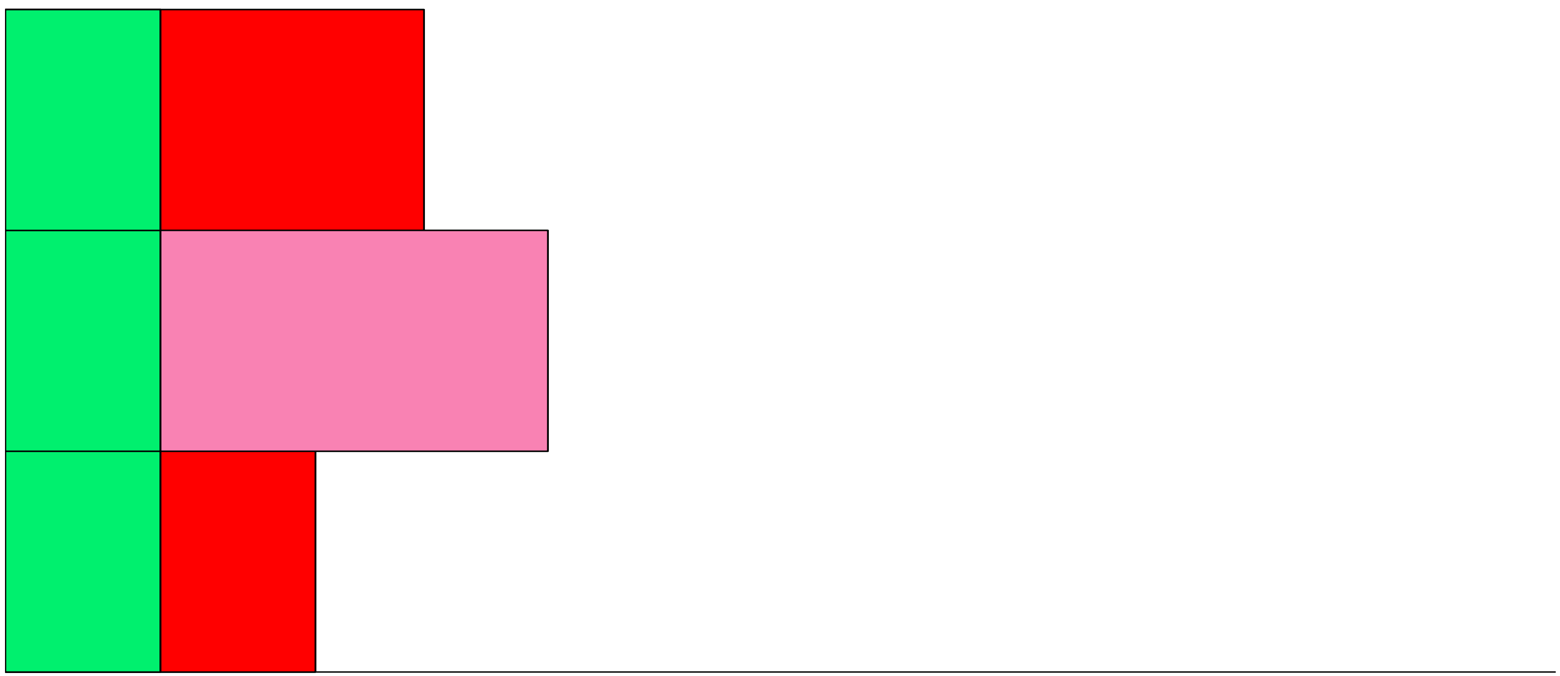
t = 0.02
n = 1

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

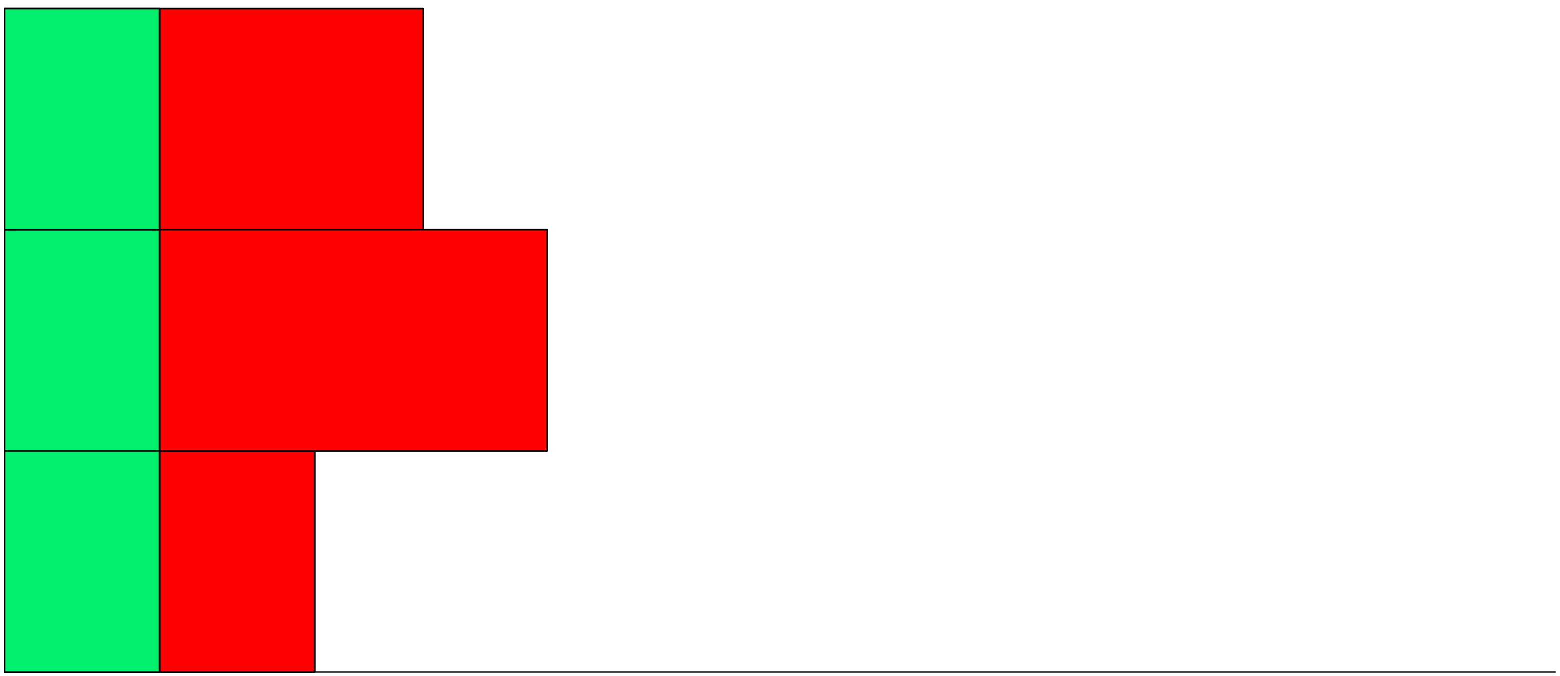
Action: Redoing steps



t = 0.027
n = 1

Multi-Adaptive Time-Stepping

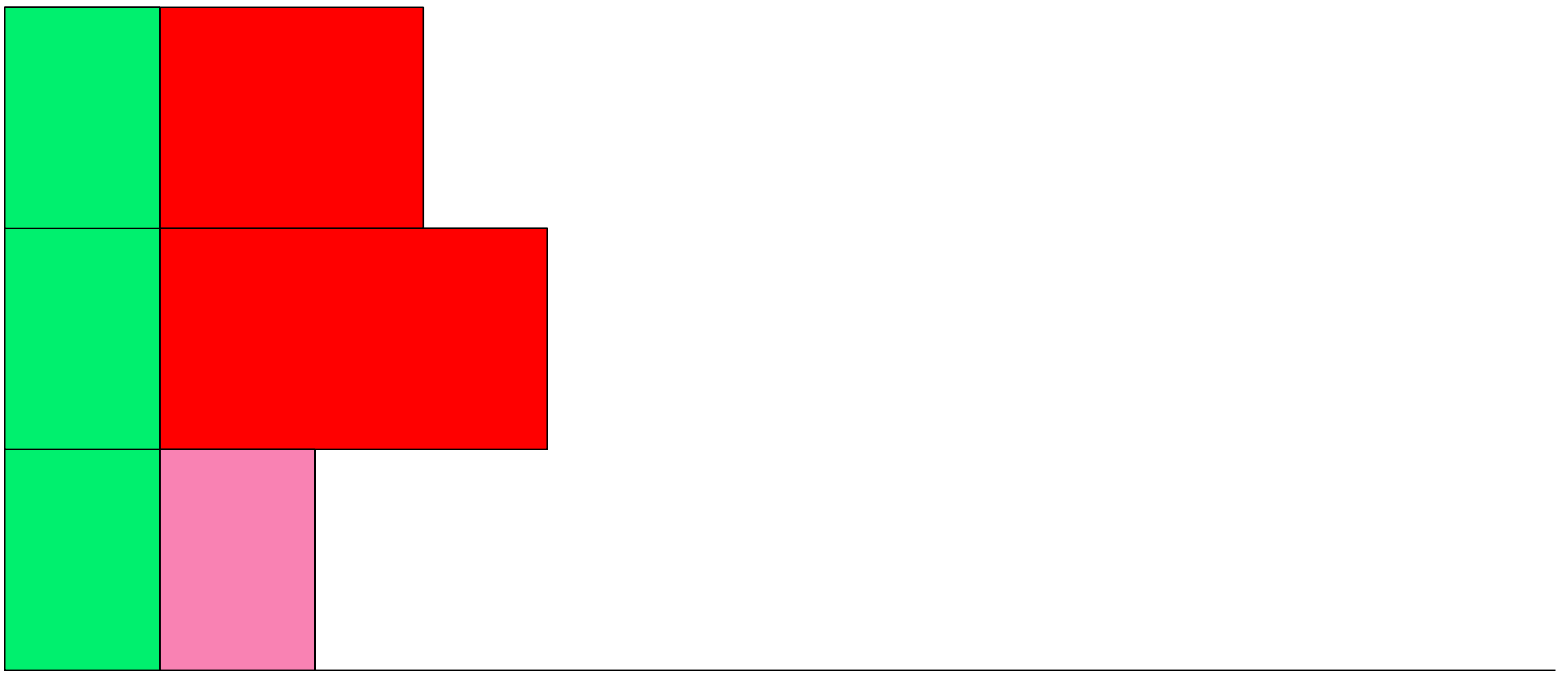
Action: Redoing steps



t = 0.035
n = 1

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.035
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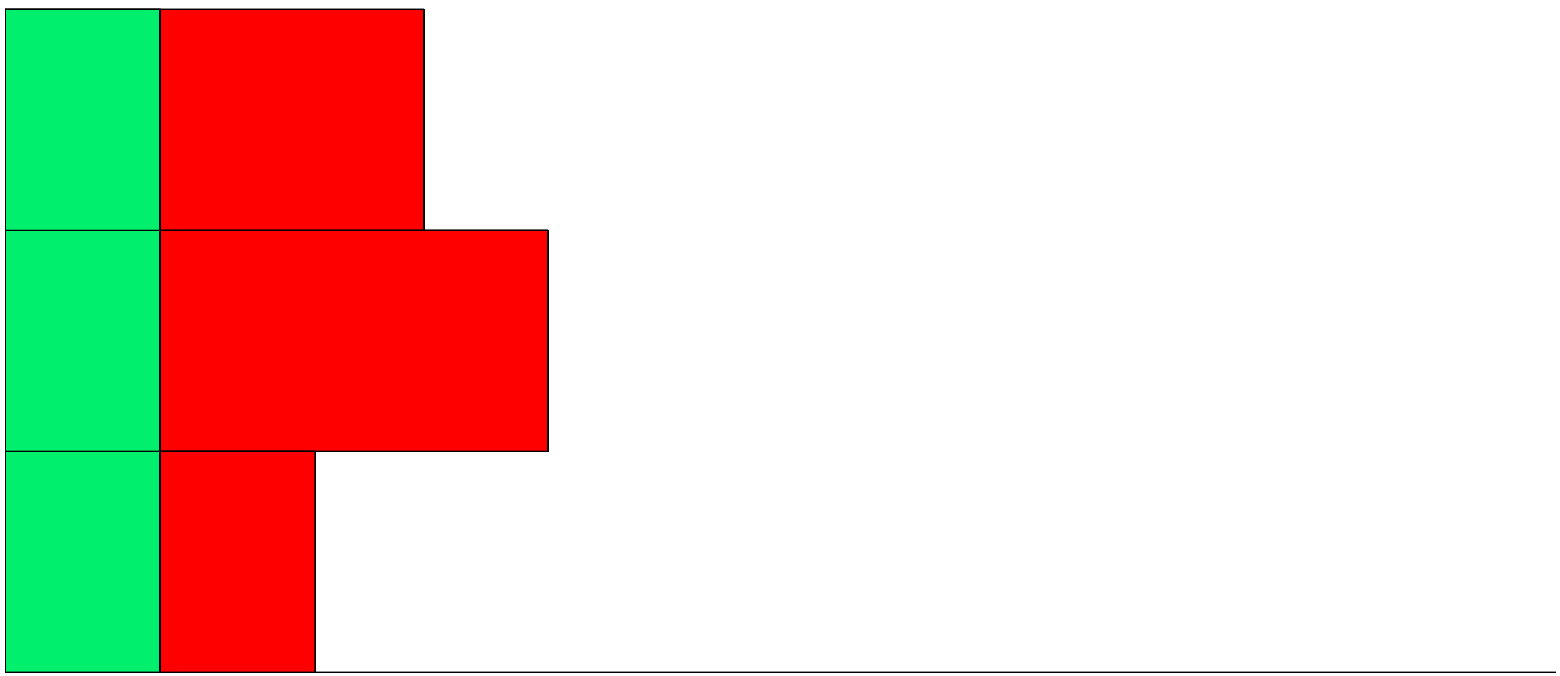
Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.02
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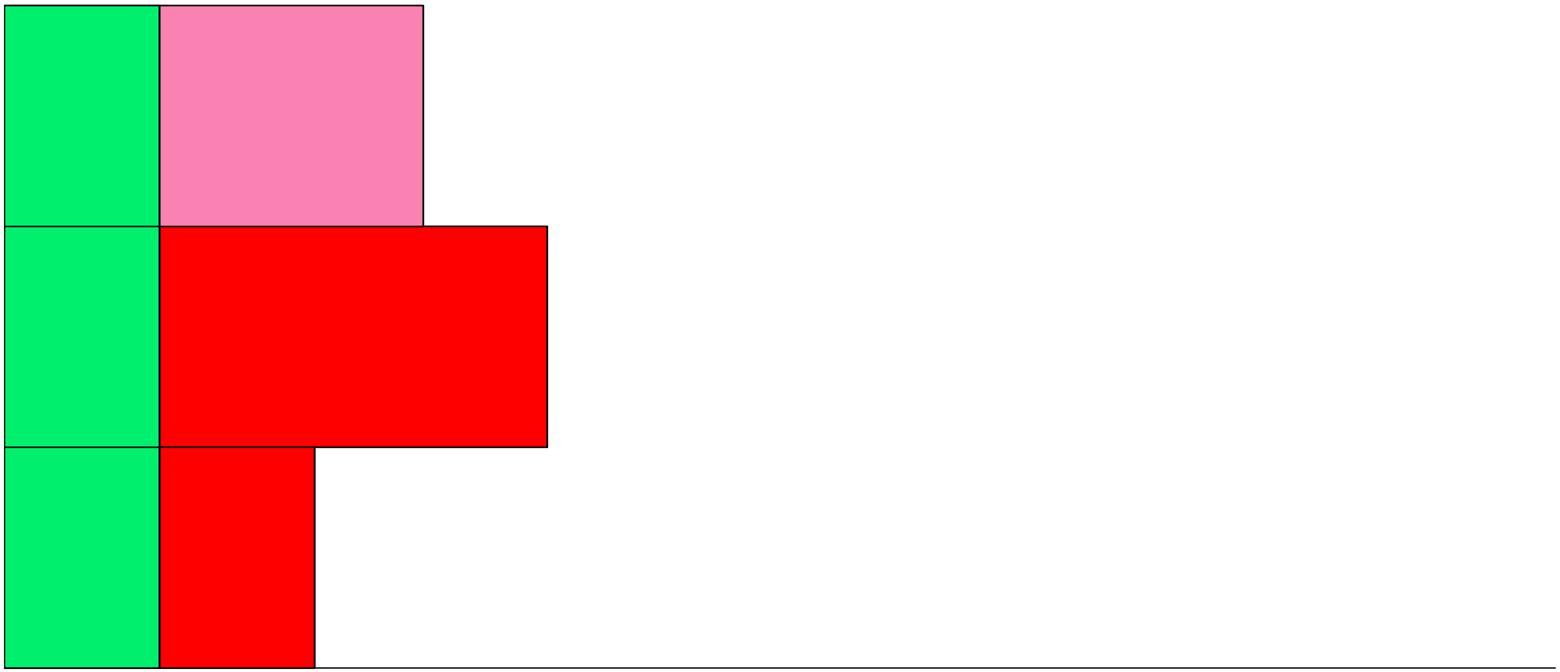

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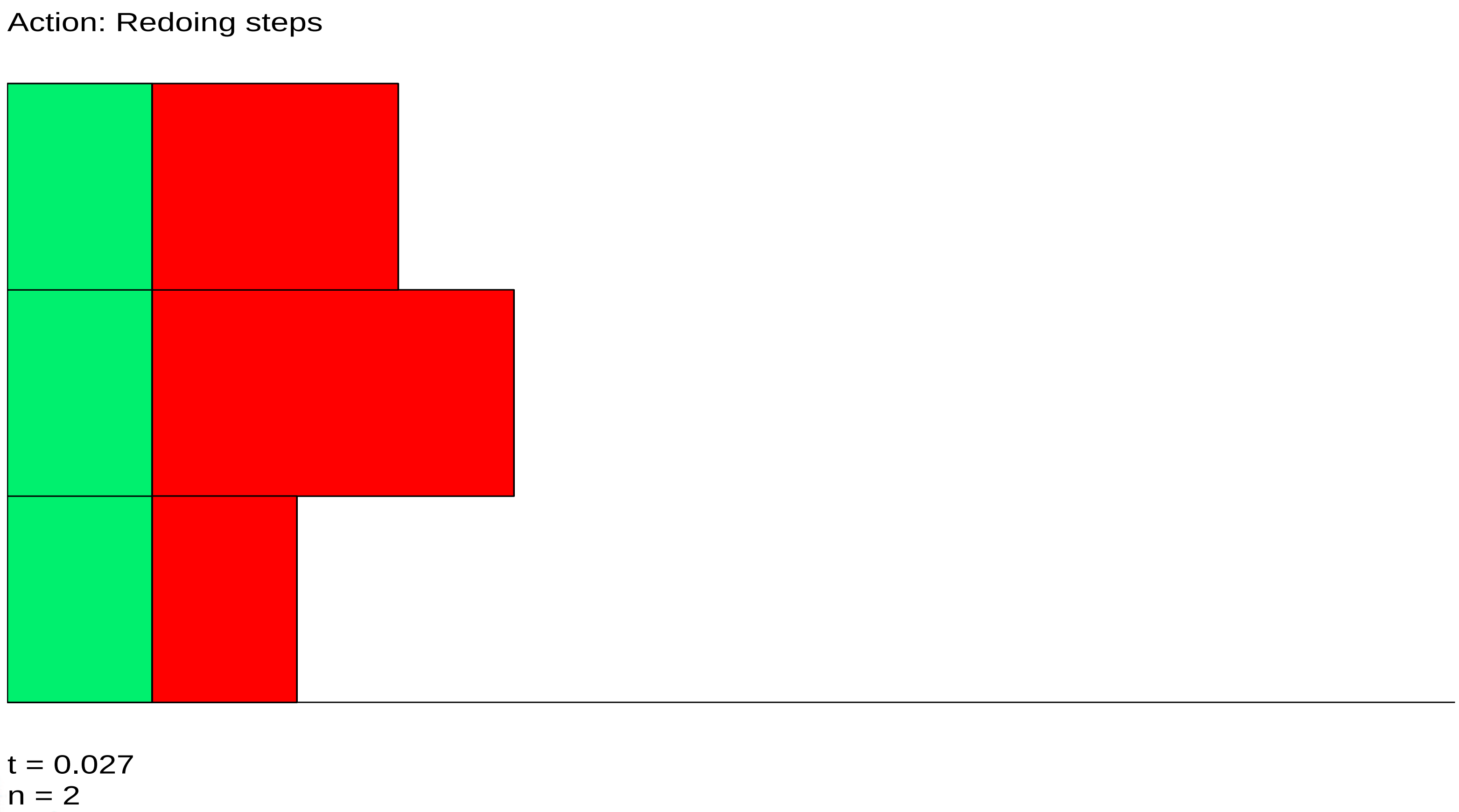

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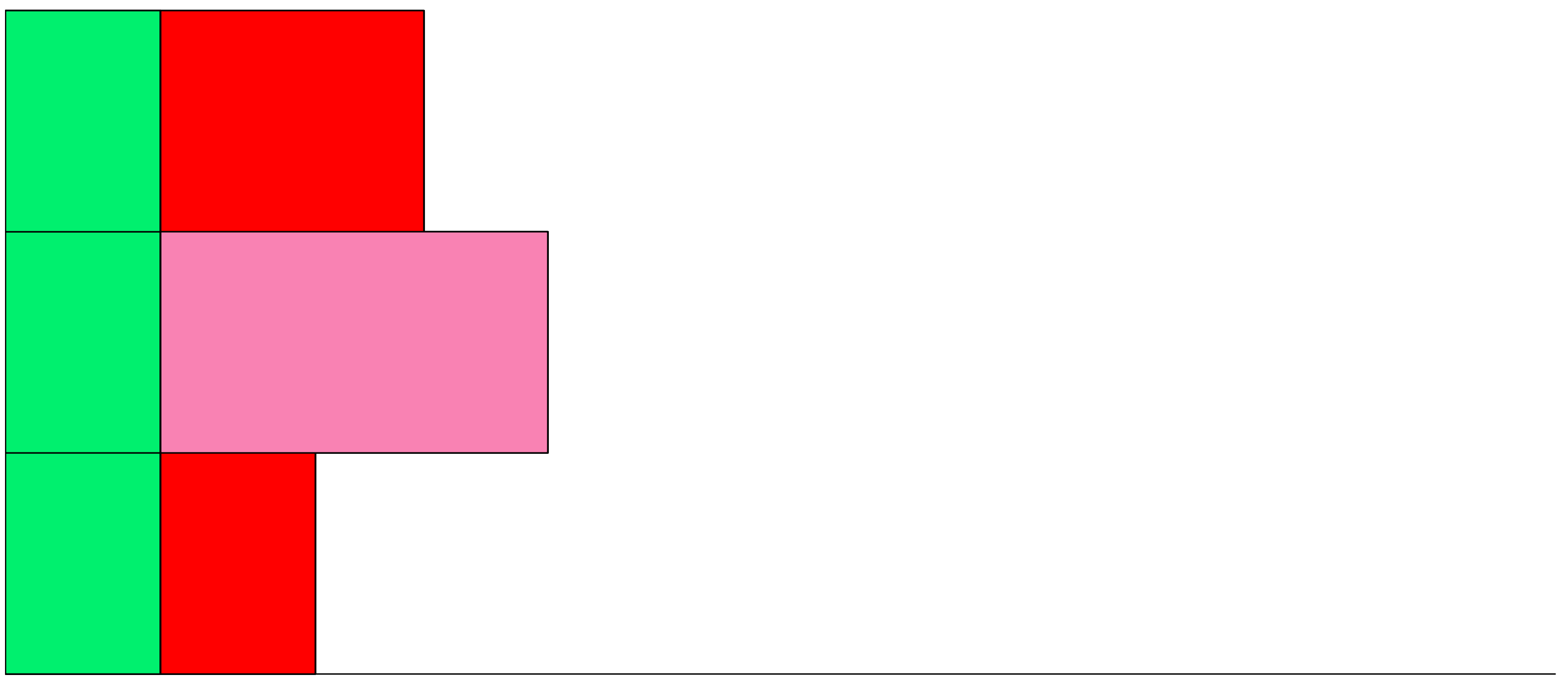

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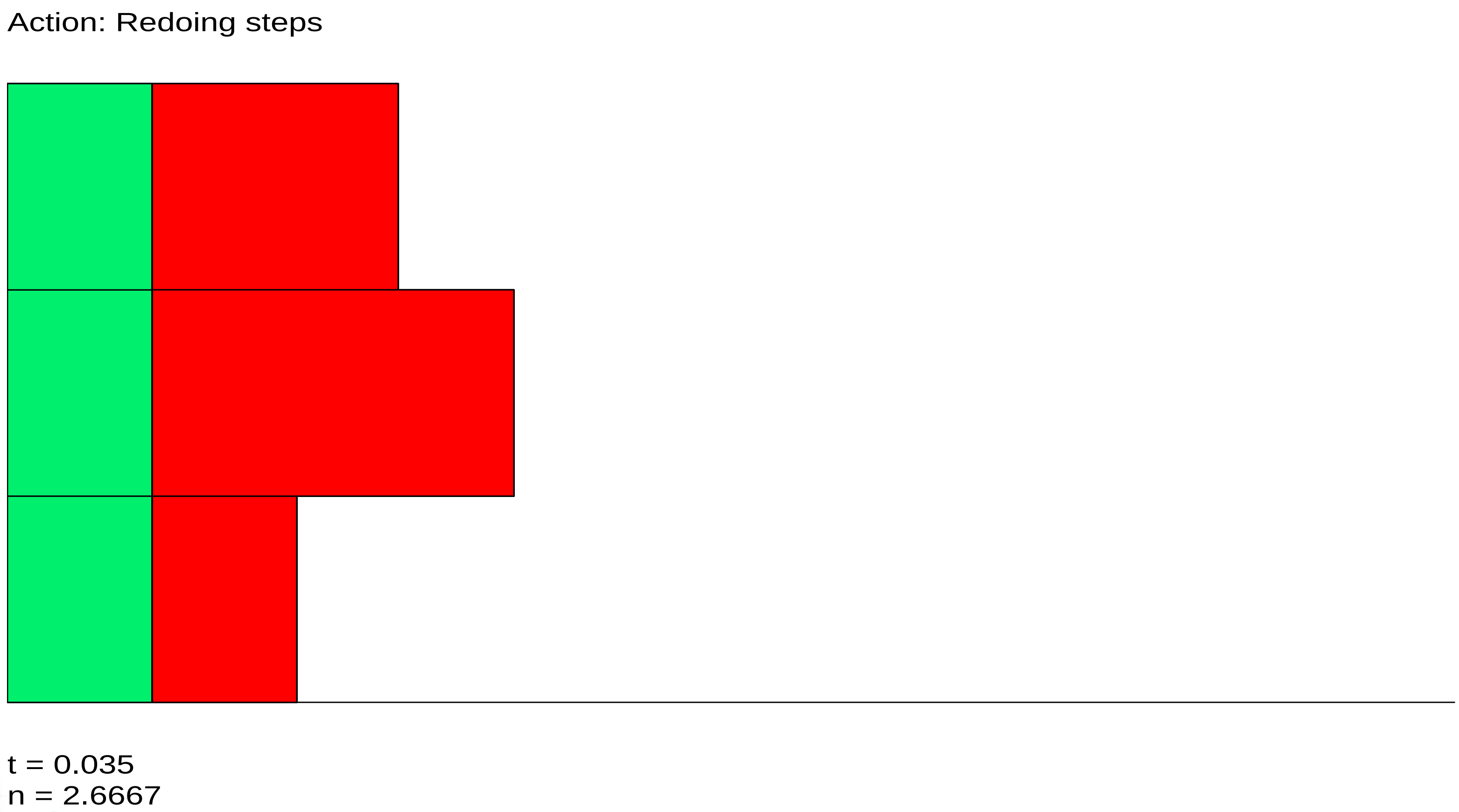
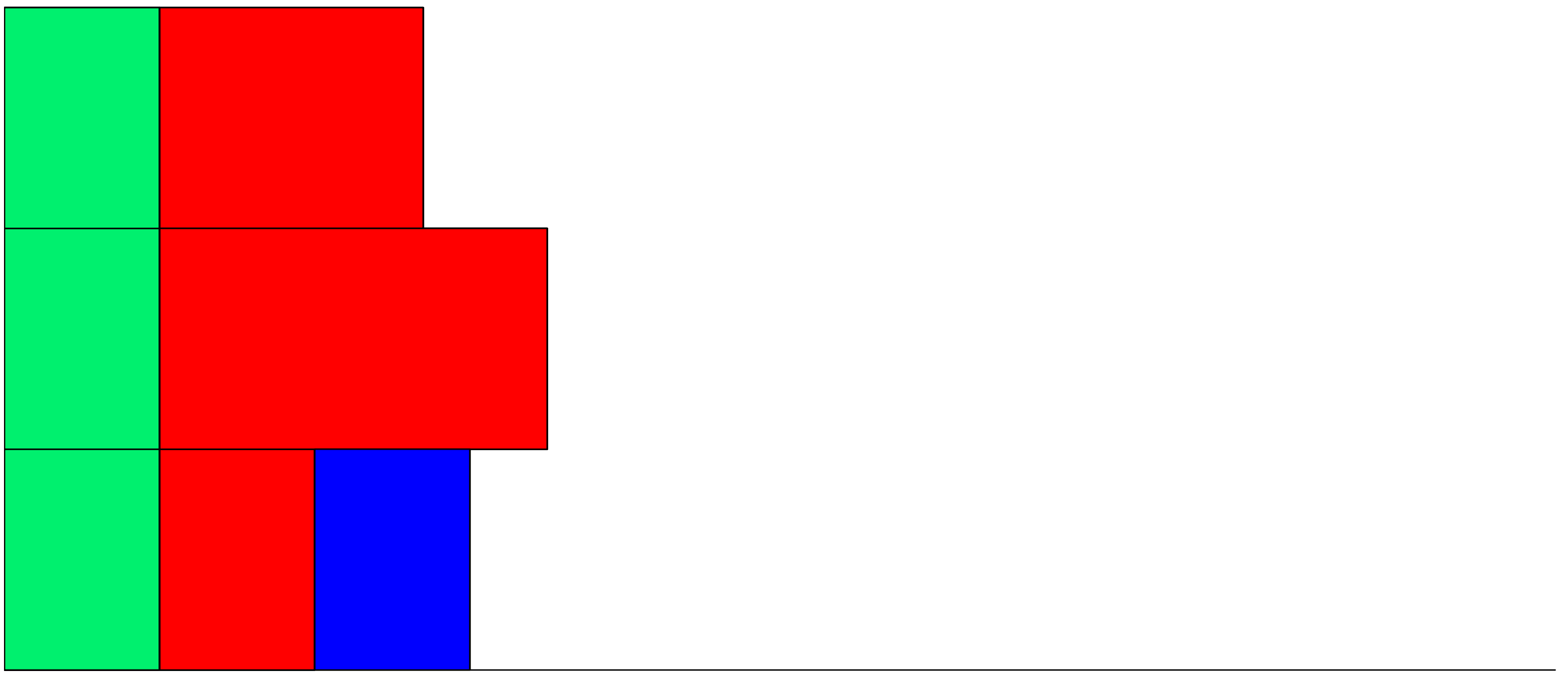
Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

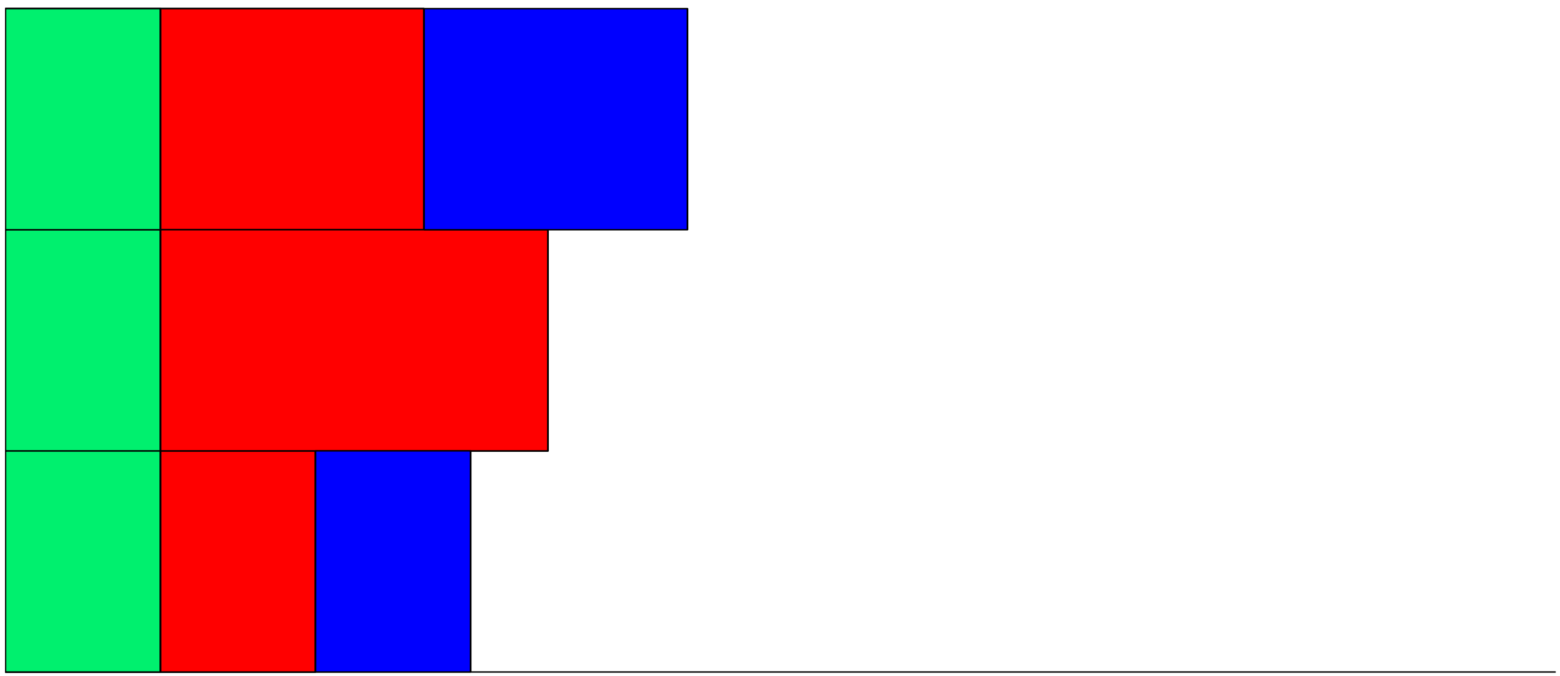
Action: Covering



t = 0.03
n = 3

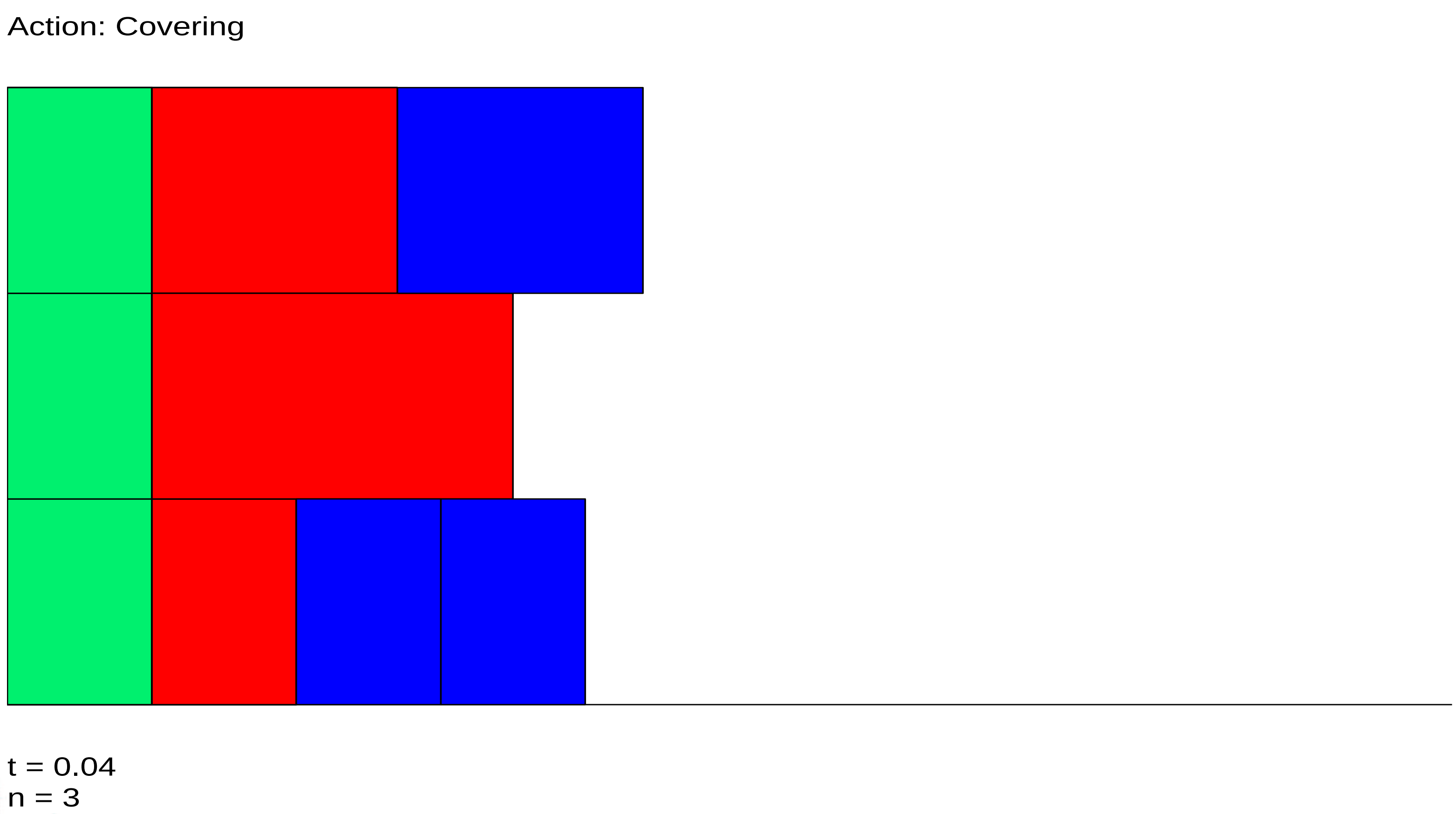
Multi-Adaptive Time-Stepping

Action: Covering



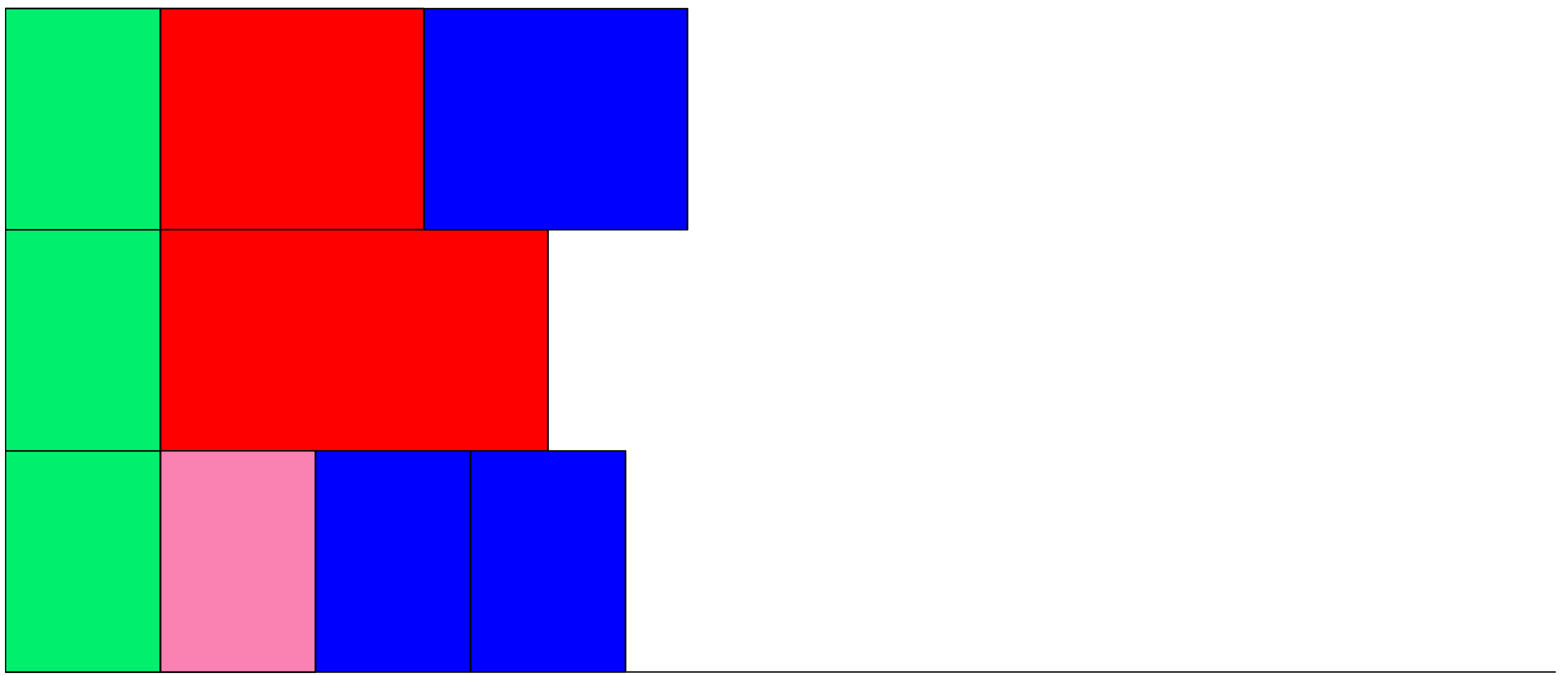
t = 0.044
n = 3

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

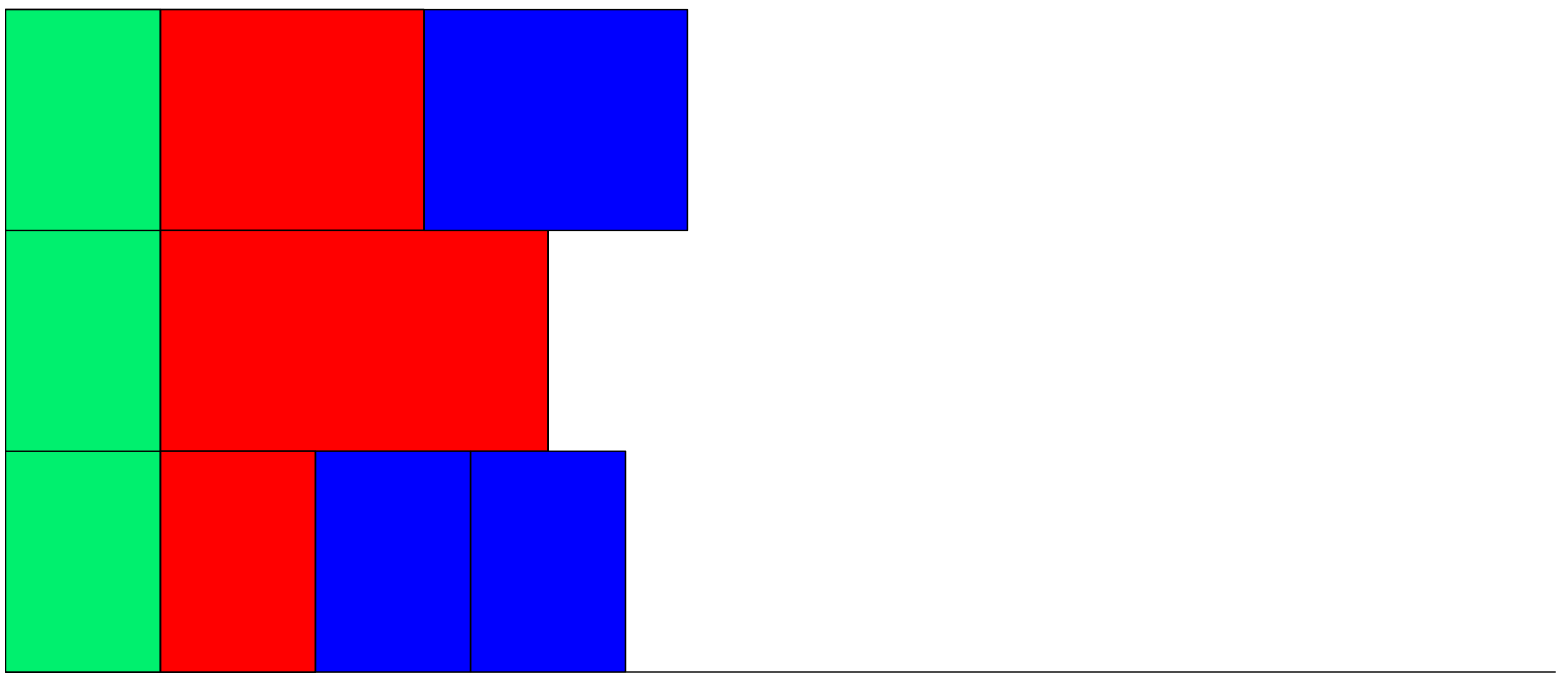
Action: Redoing steps



t = 0.04
n = 3

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.02
n = 1

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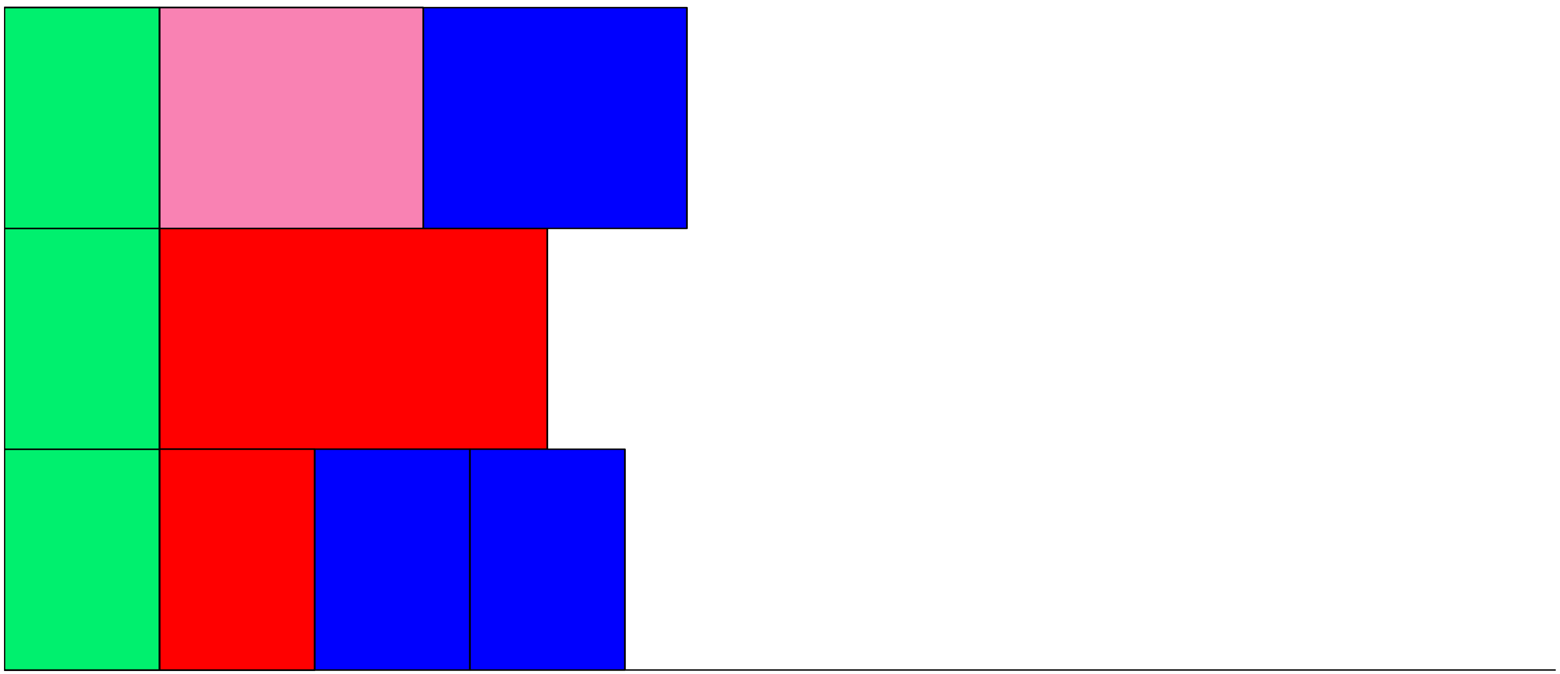
Multi-Adaptive Time-Stepping

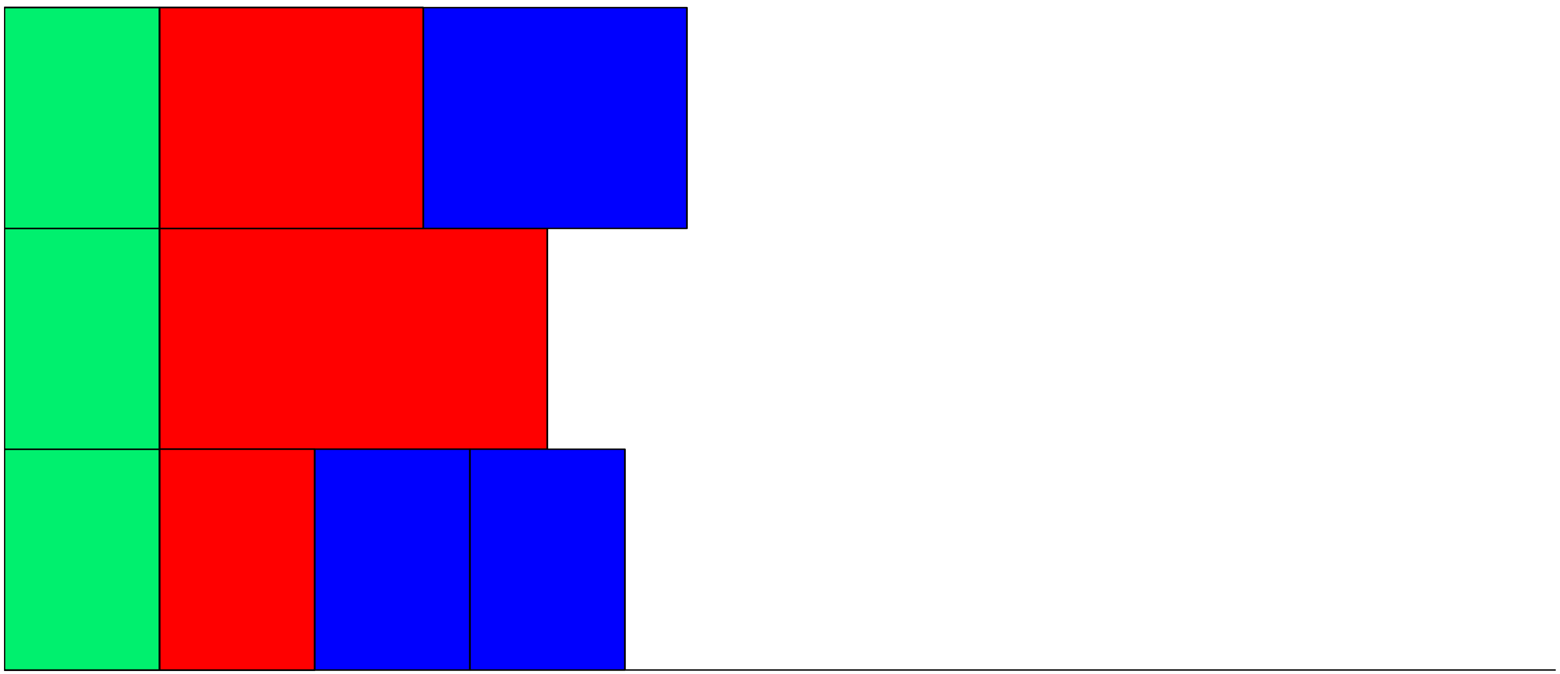
Action: Redoing steps



t = 0.02
n = 1

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.027
n = 1

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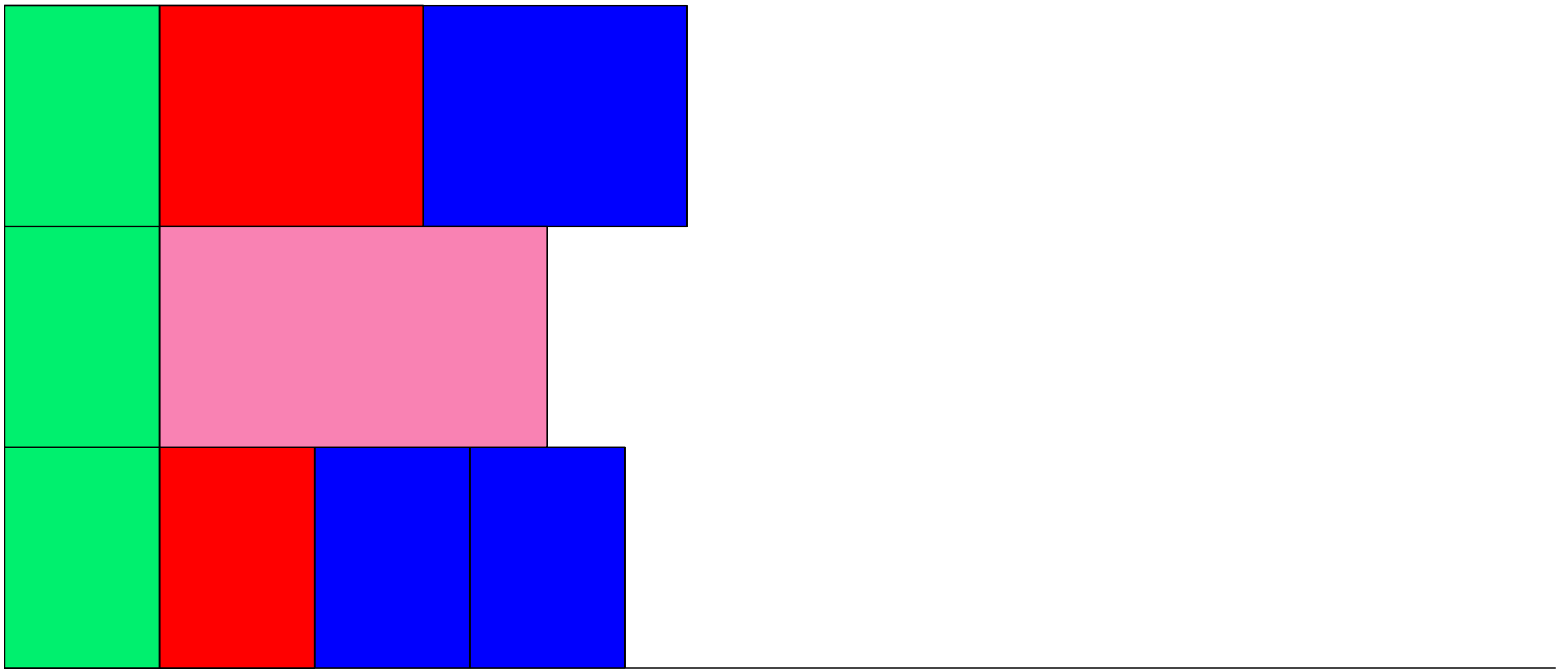
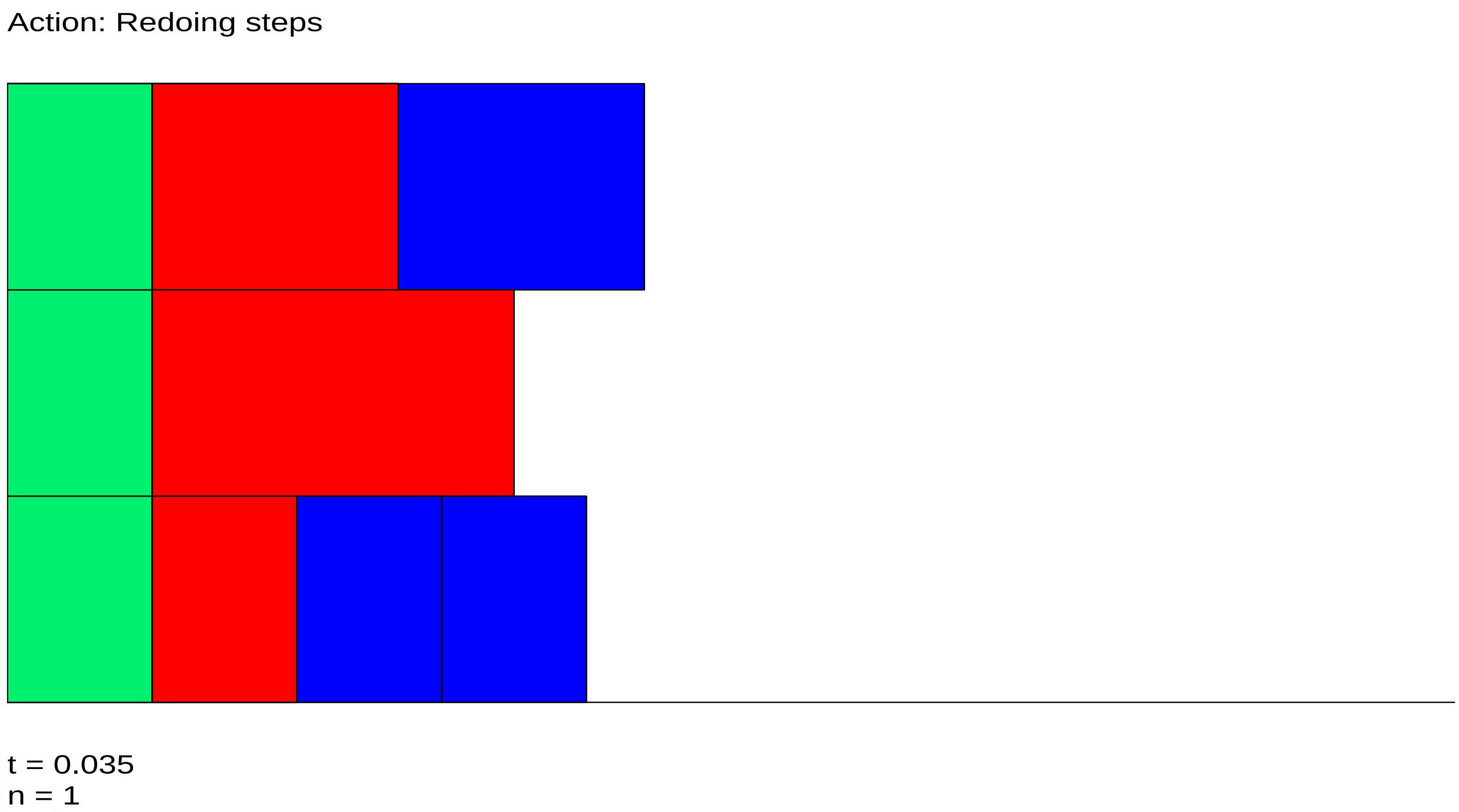
Multi-Adaptive Time-Stepping

Action: Redoing steps



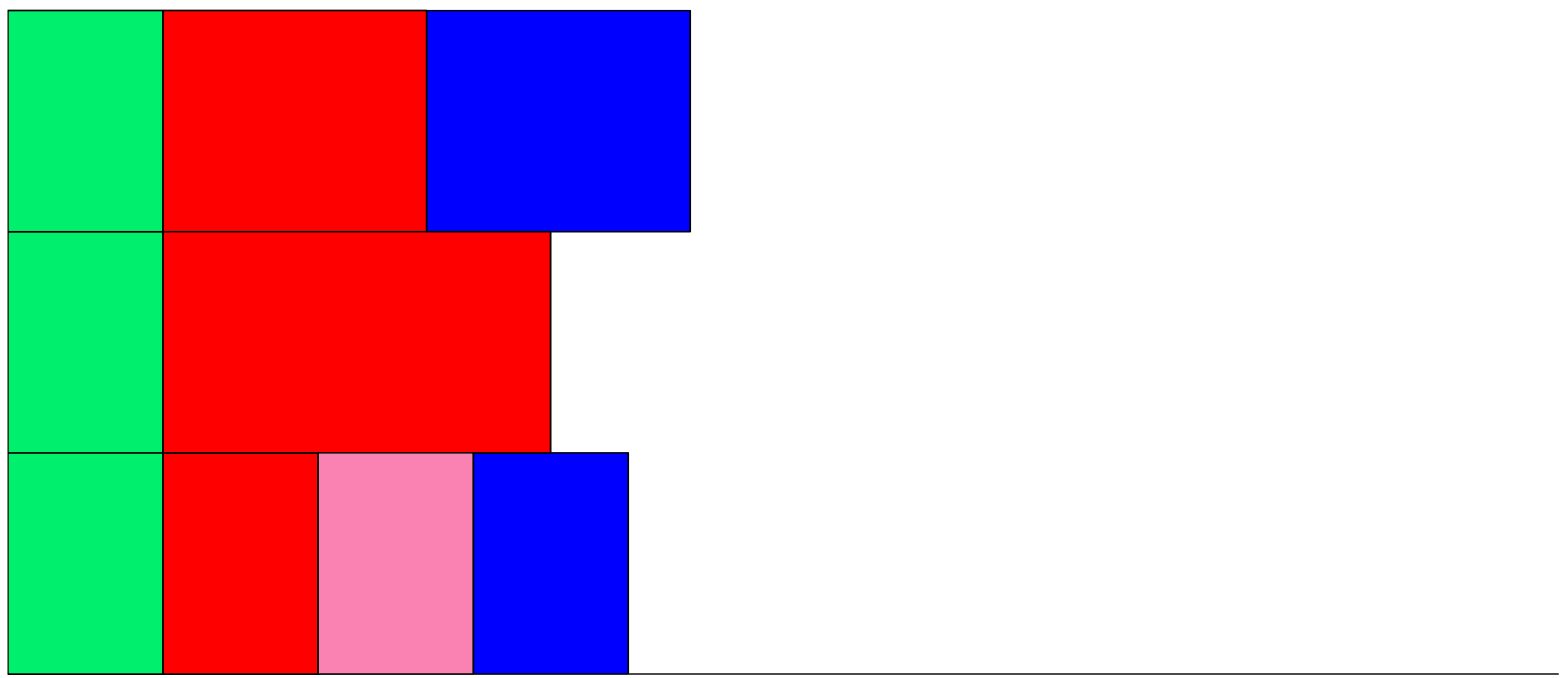
t = 0.027
n = 1

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

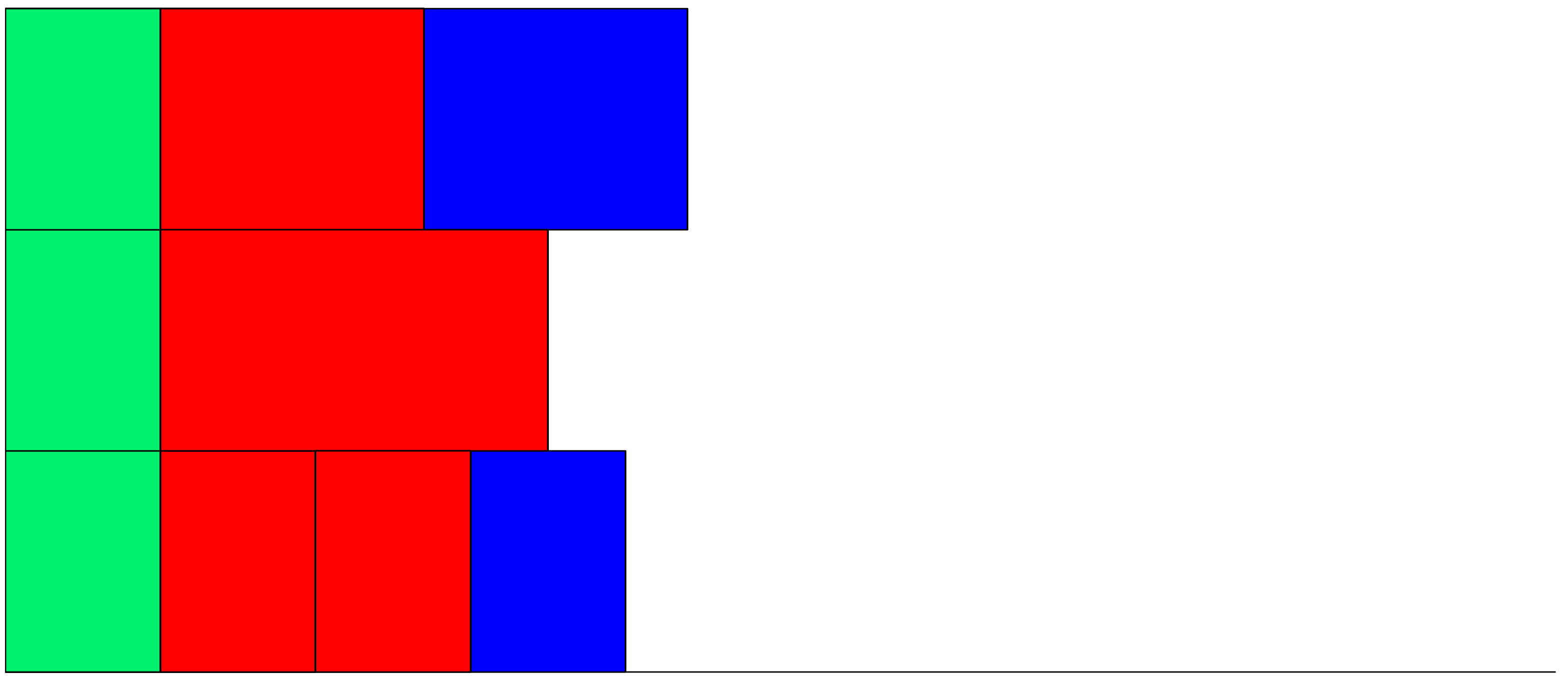
Action: Redoing steps



t = 0.035
n = 1

Multi-Adaptive Time-Stepping

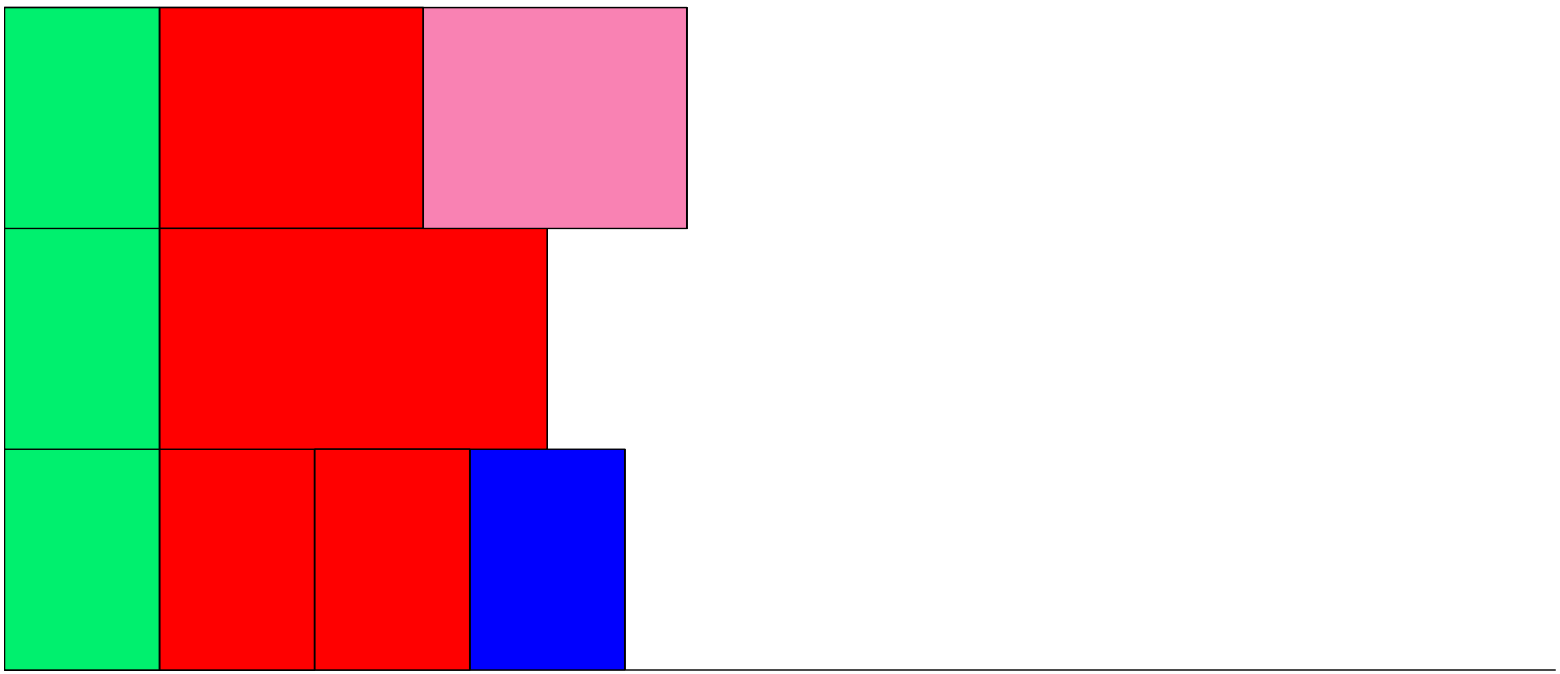
Action: Redoing steps



t = 0.03
n = 1

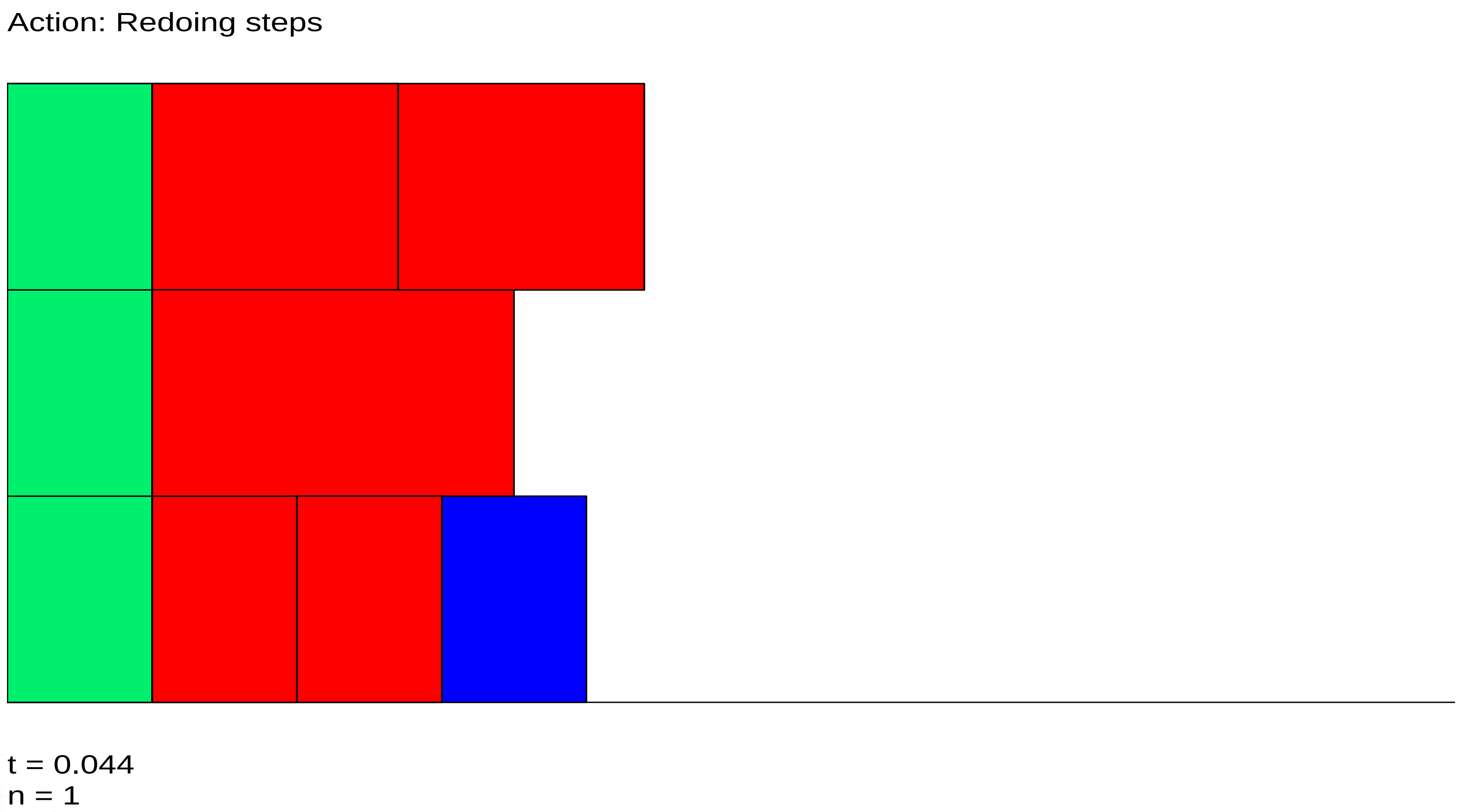
Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.03
n = 1

Multi-Adaptive Time-Stepping



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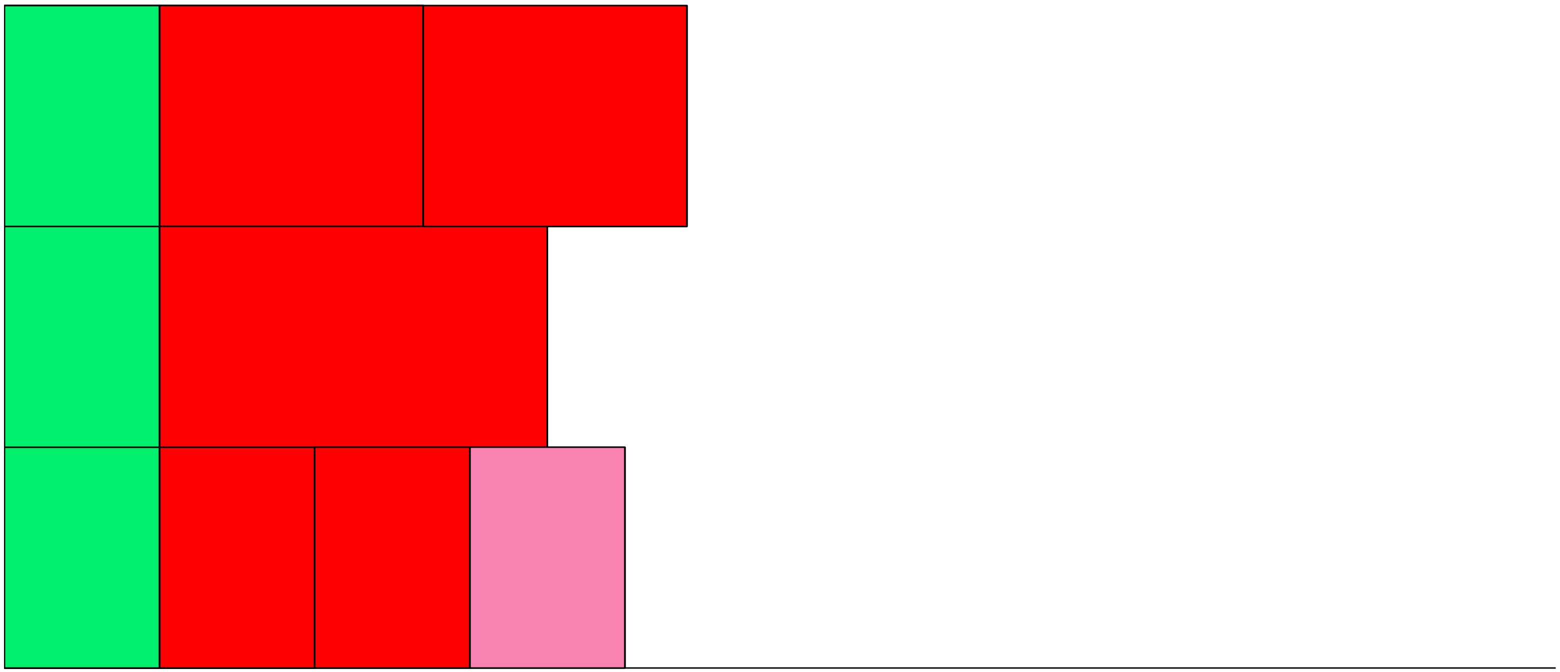
Multi-Adaptive Time-Stepping

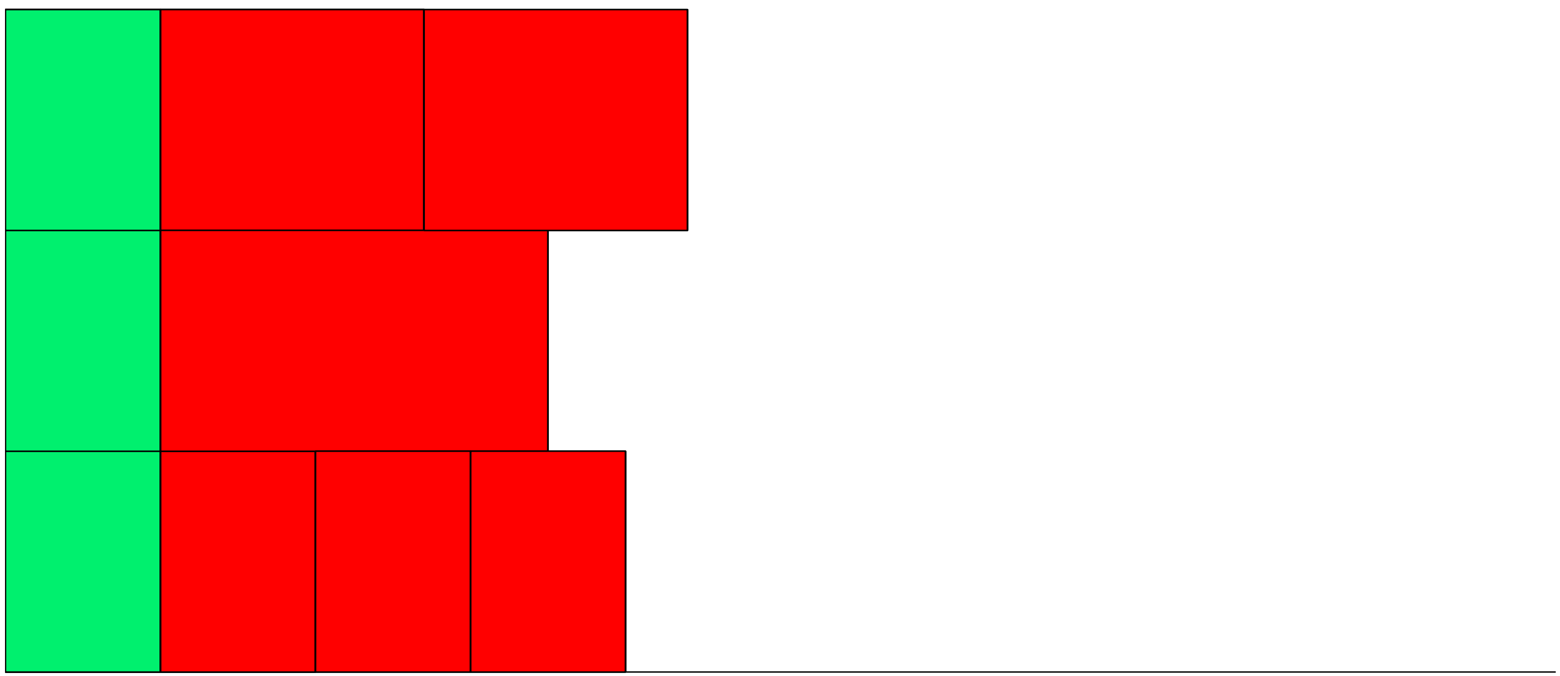
Action: Redoing steps



t = 0.044
n = 1

Multi-Adaptive Time-Stepping

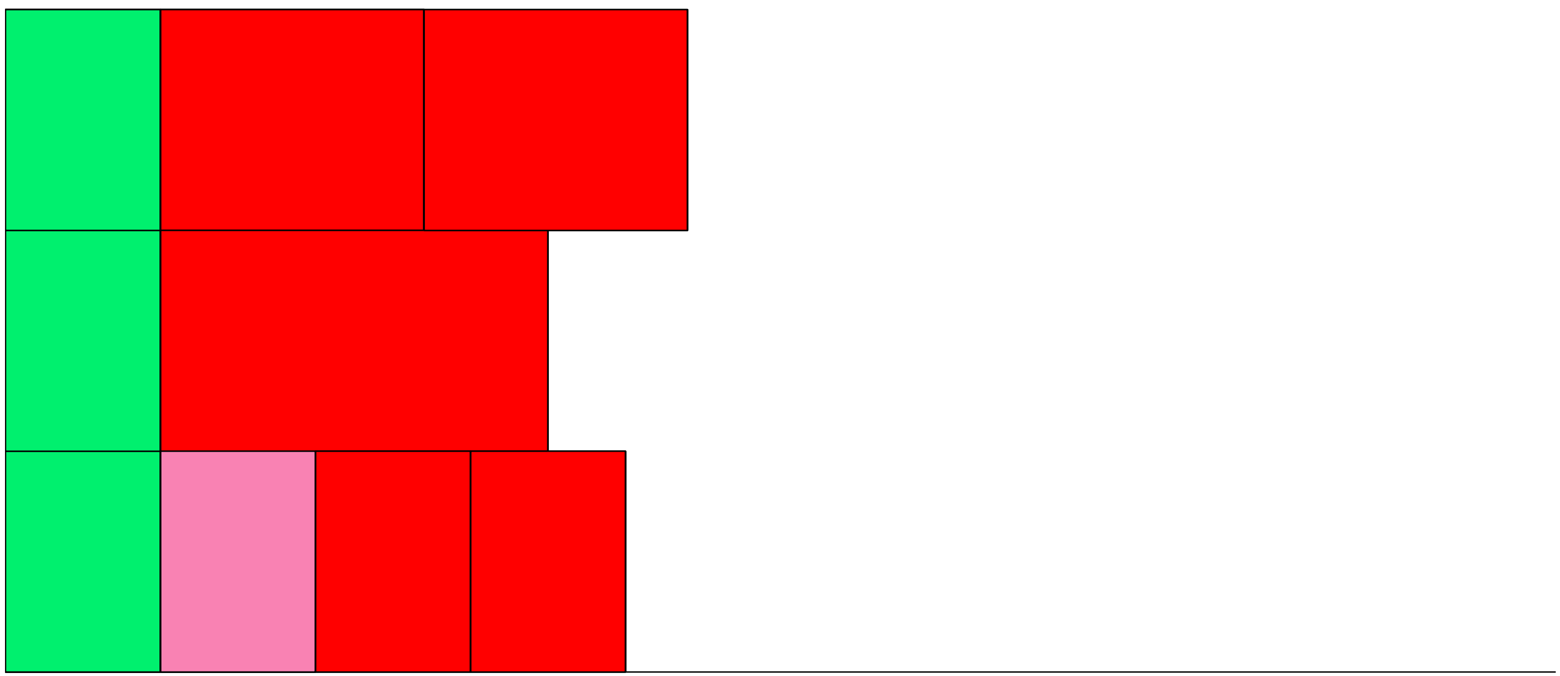
Action: Redoing steps



t = 0.04
n = 1

Multi-Adaptive Time-Stepping

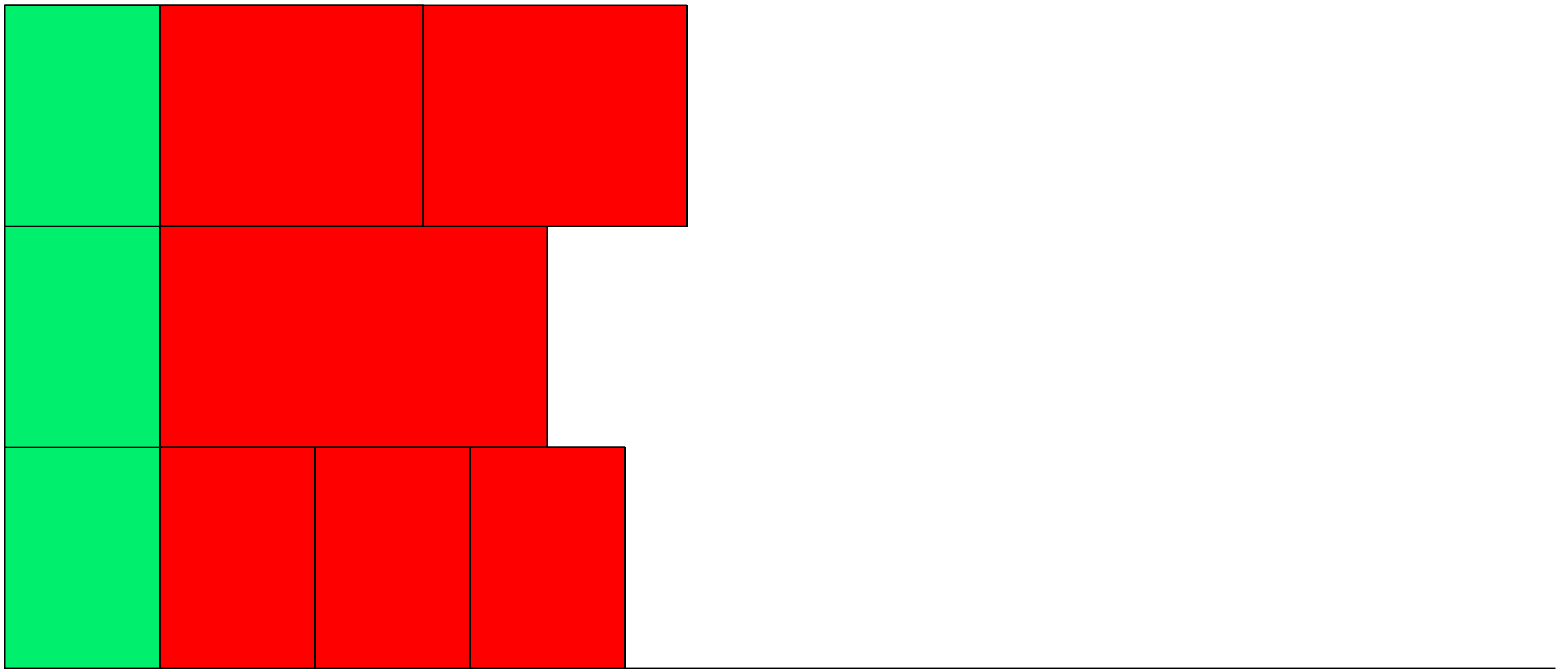
Action: Redoing steps



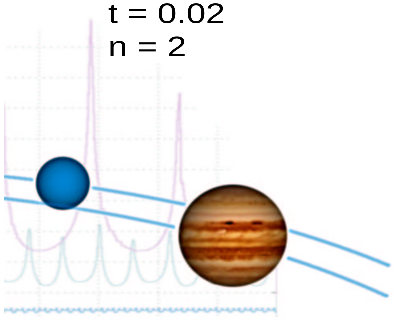
t = 0.04
n = 1

Multi-Adaptive Time-Stepping

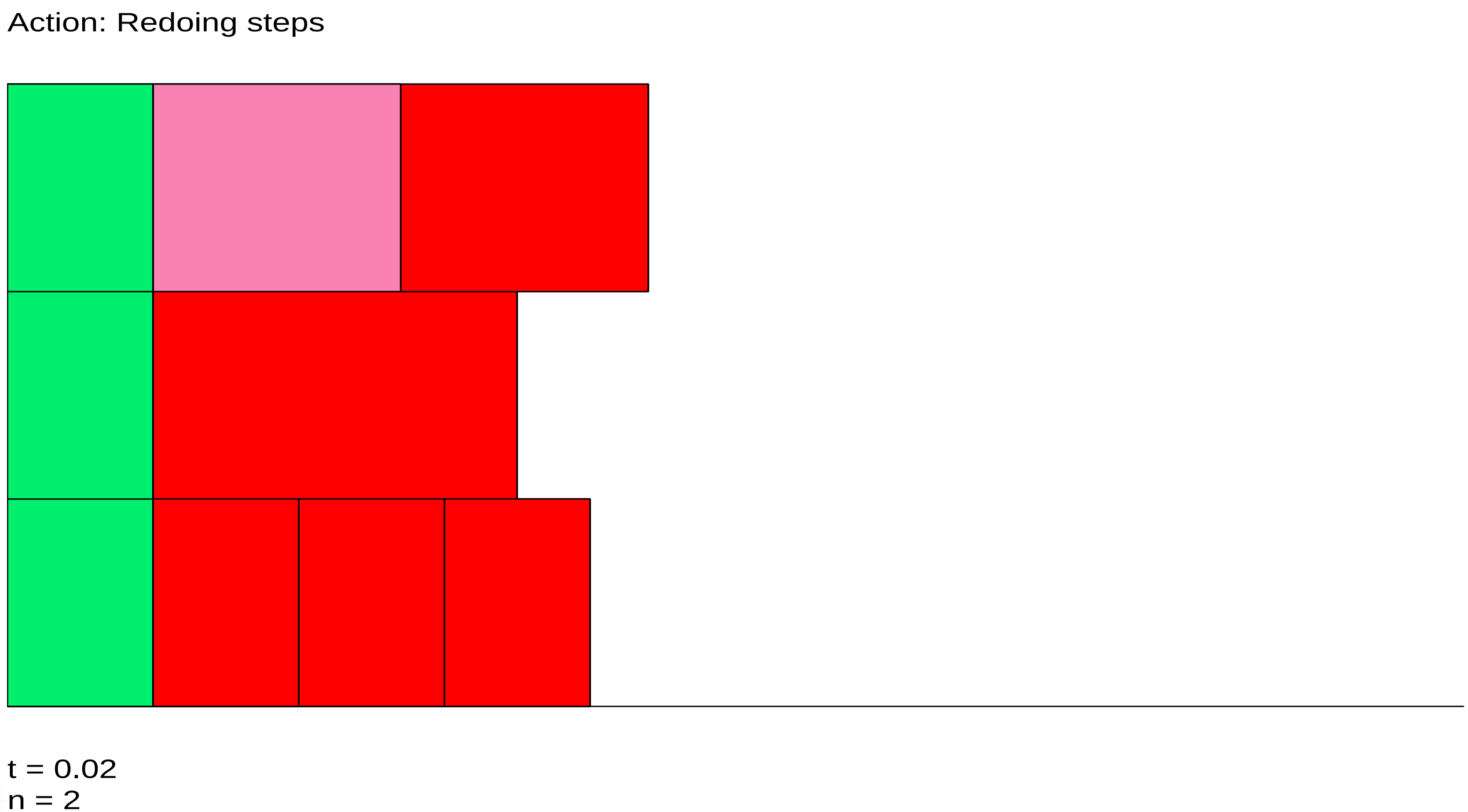
Action: Redoing steps



t = 0.02
n = 2

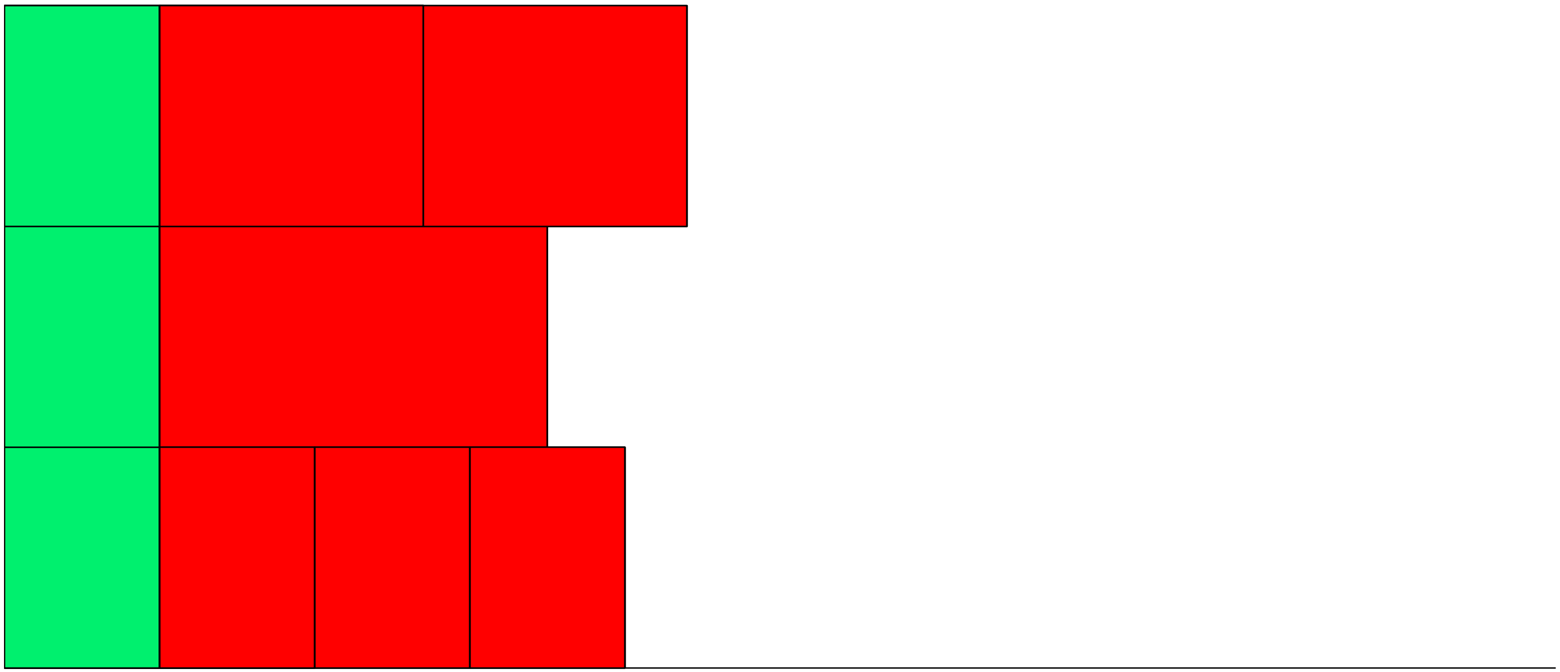


Multi-Adaptive Time-Stepping

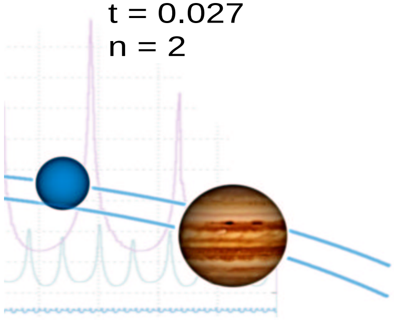


Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.027
n = 2



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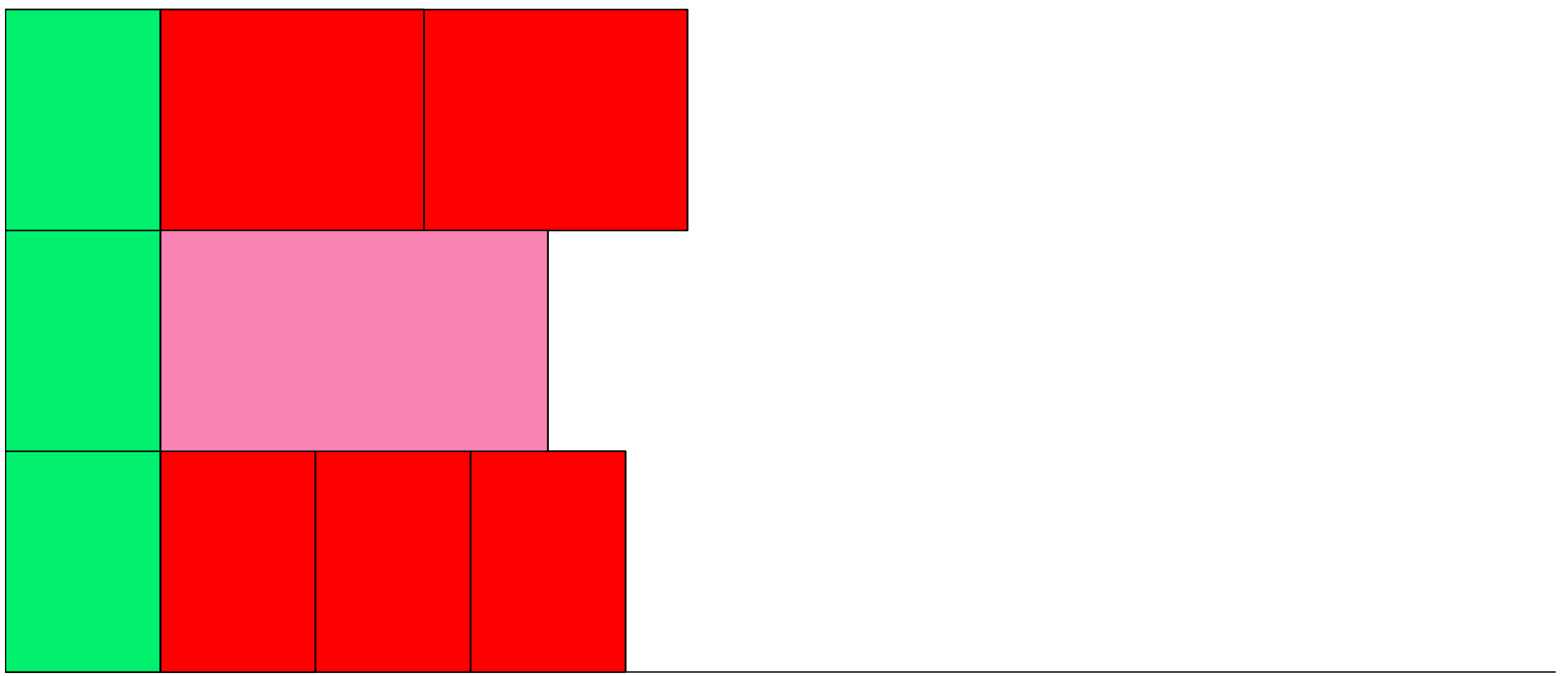
Multi-Adaptive Time-Stepping

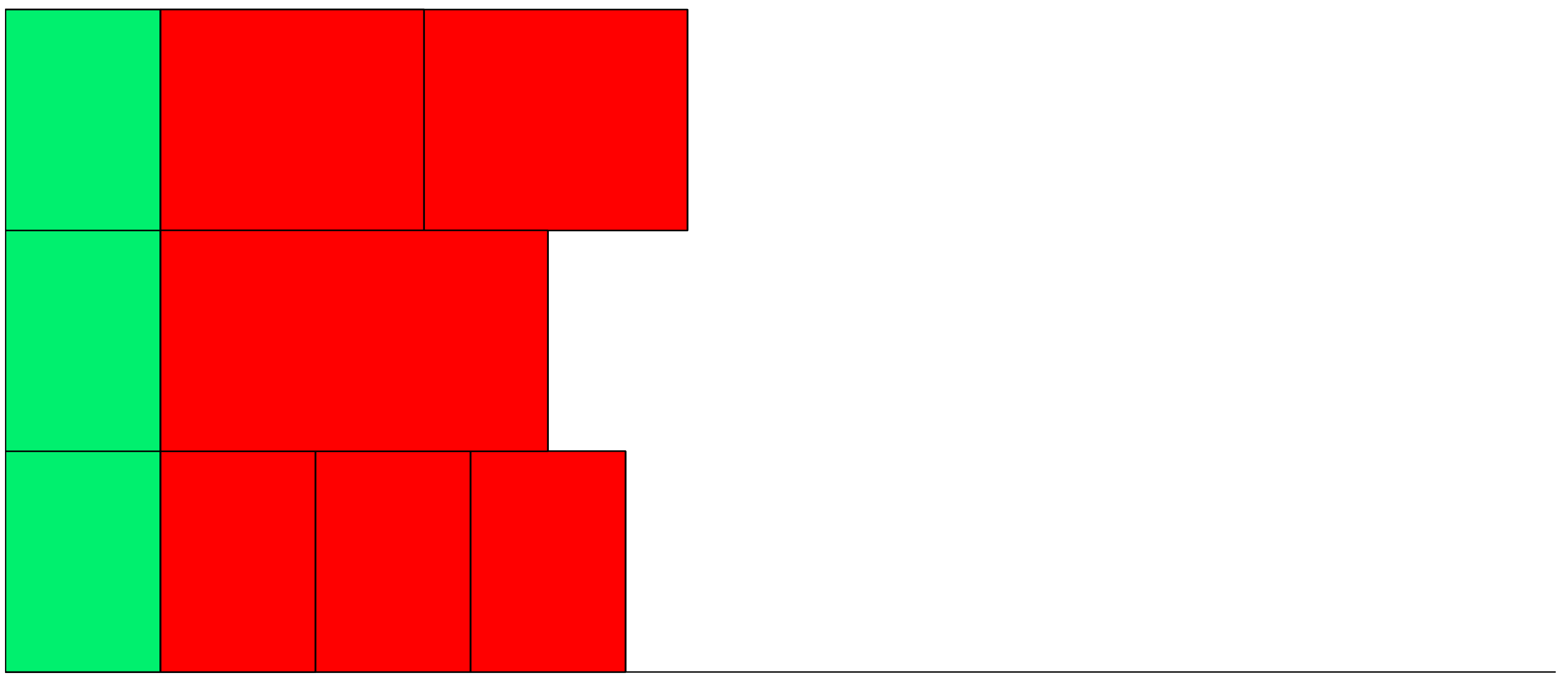
Action: Redoing steps



t = 0.027
n = 2

Multi-Adaptive Time-Stepping

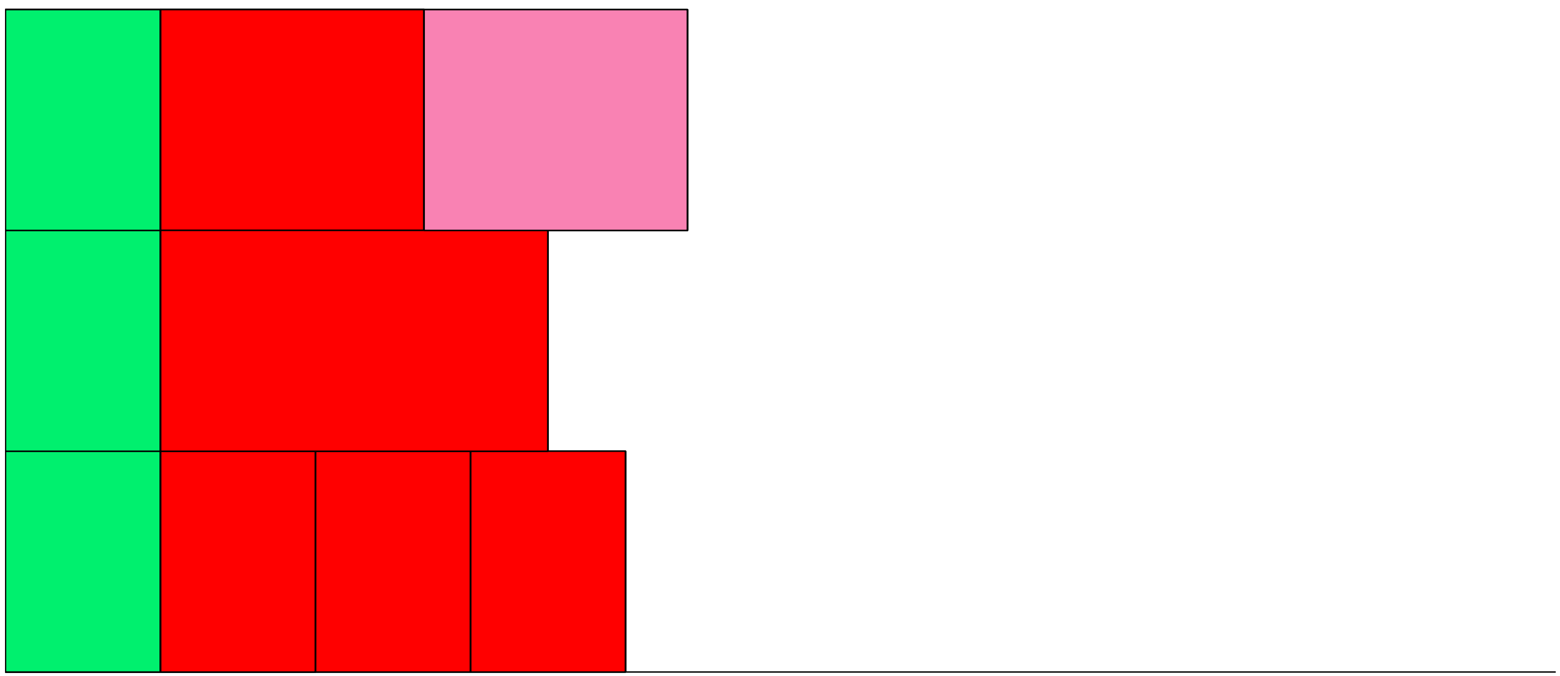
Action: Redoing steps



t = 0.035
n = 2

Multi-Adaptive Time-Stepping

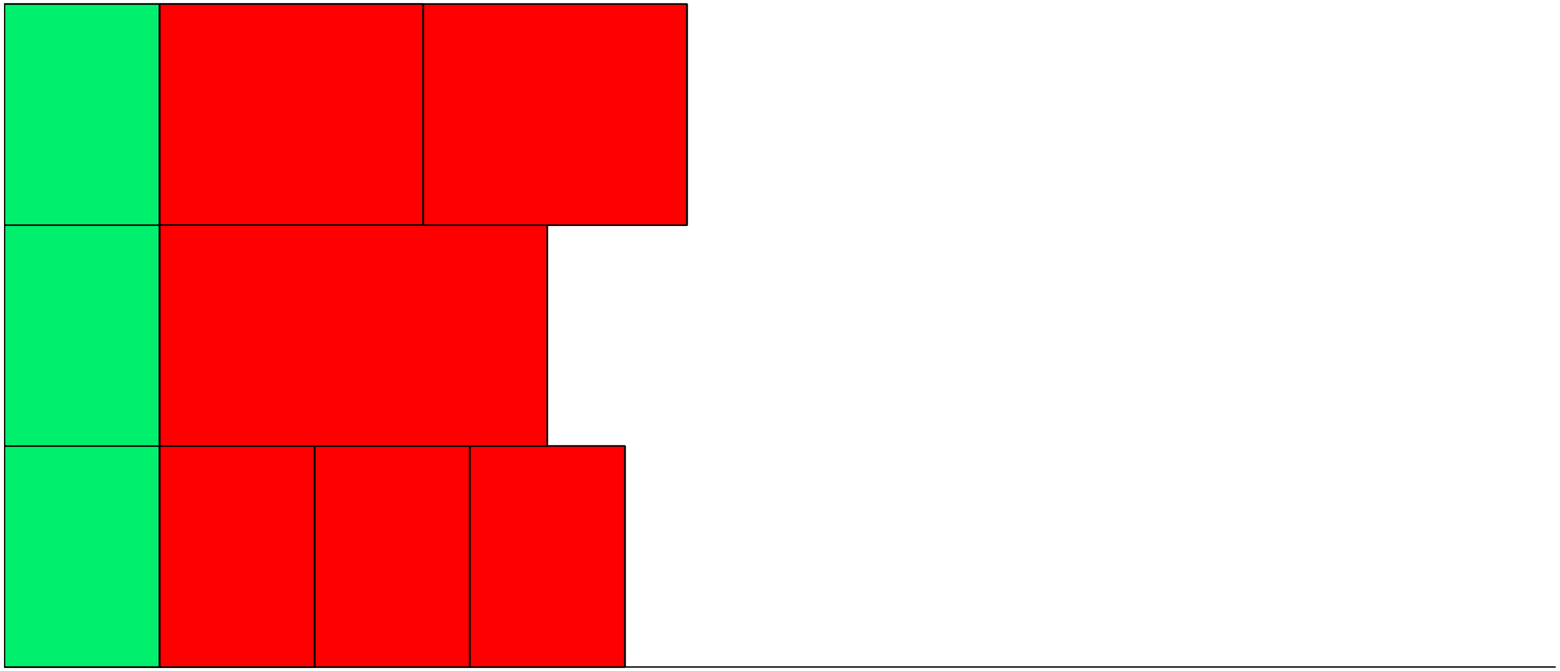
Action: Redoing steps



t = 0.035
n = 2

Multi-Adaptive Time-Stepping

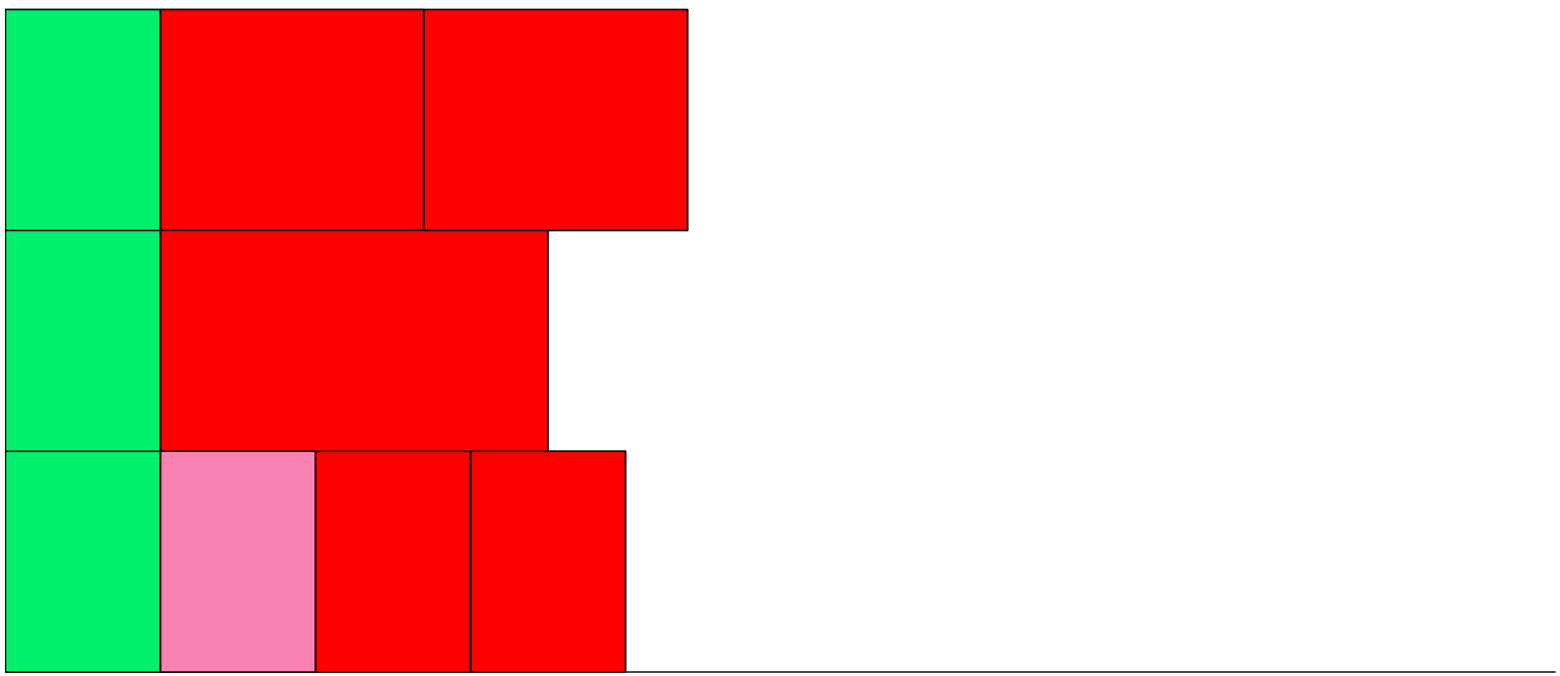
Action: Redoing steps



t = 0.044
n = 2

Multi-Adaptive Time-Stepping

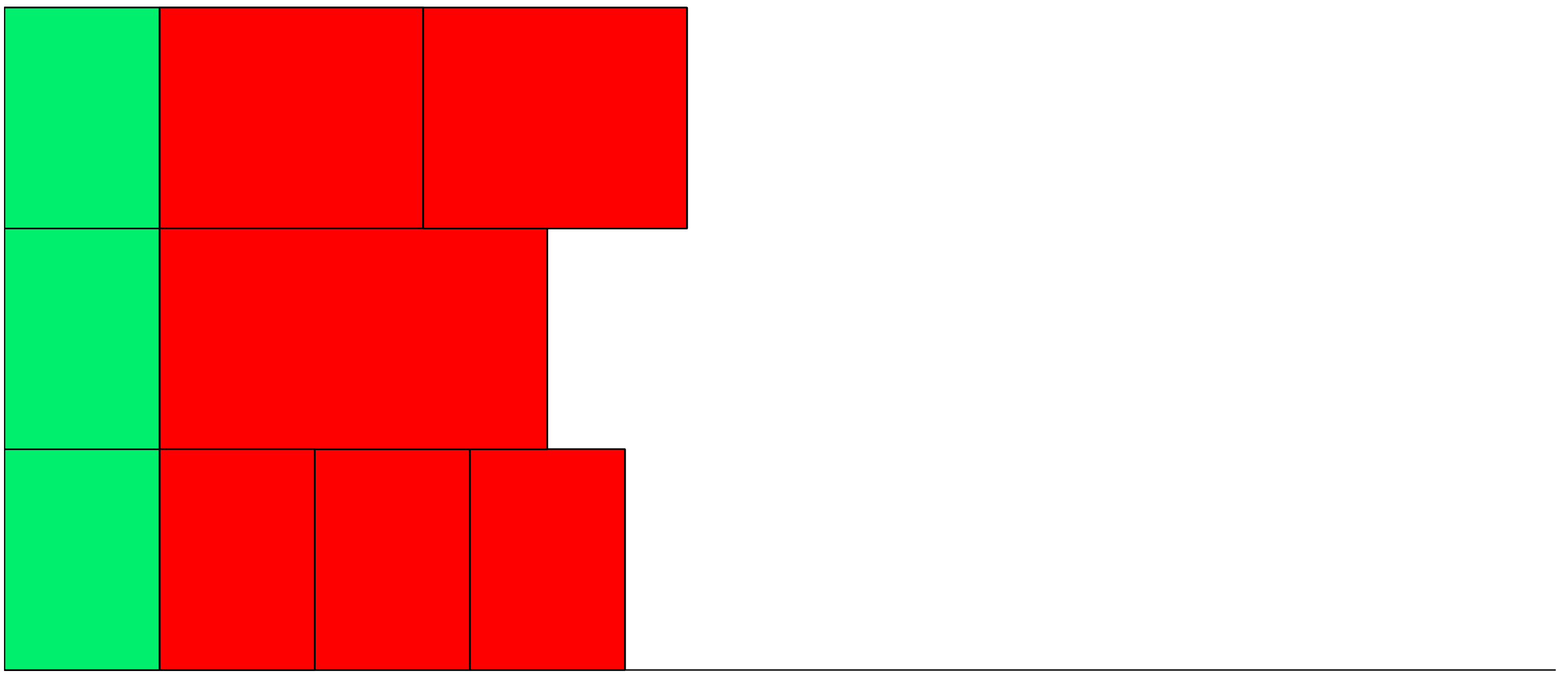
Action: Redoing steps



t = 0.044
n = 2

Multi-Adaptive Time-Stepping

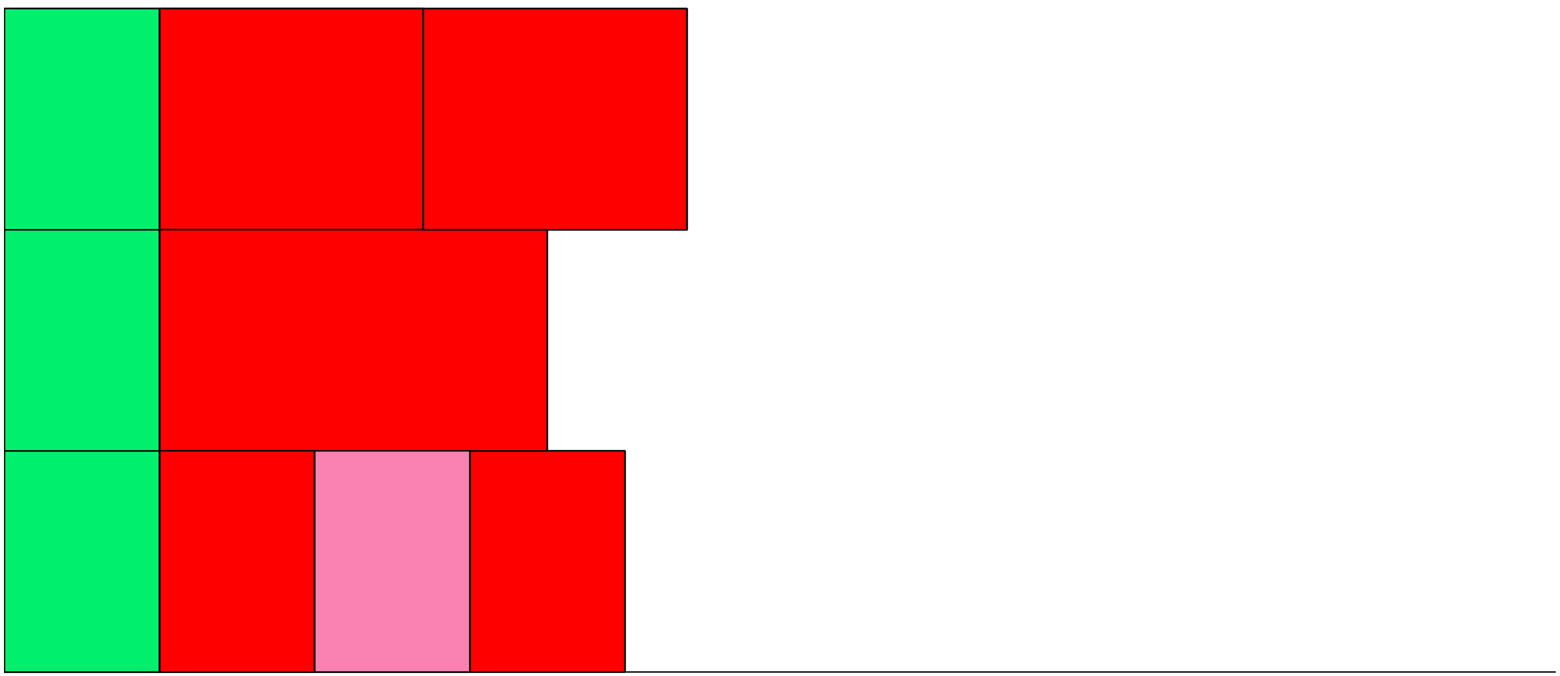
Action: Redoing steps



t = 0.02
n = 2.6667

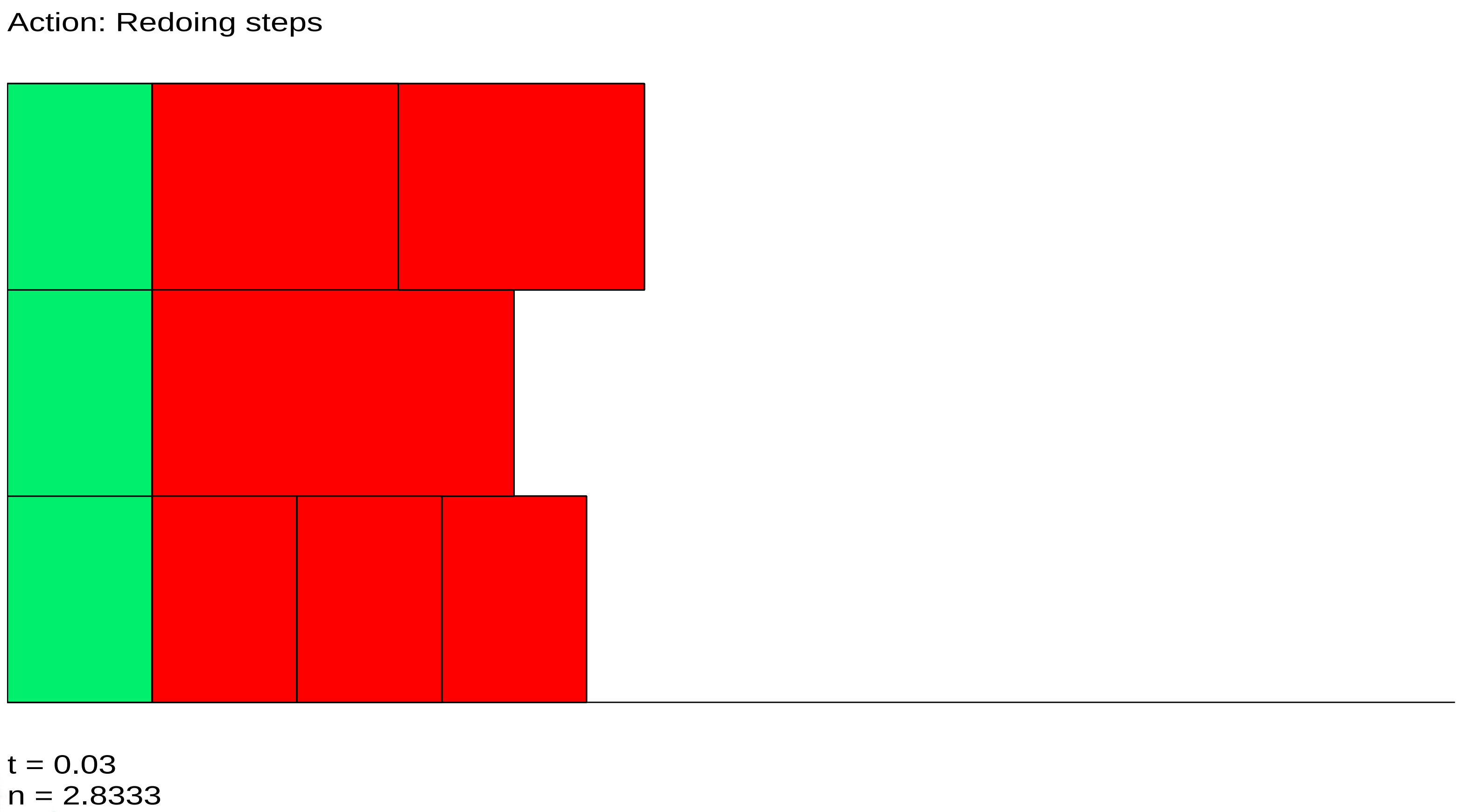
Multi-Adaptive Time-Stepping

Action: Redoing steps

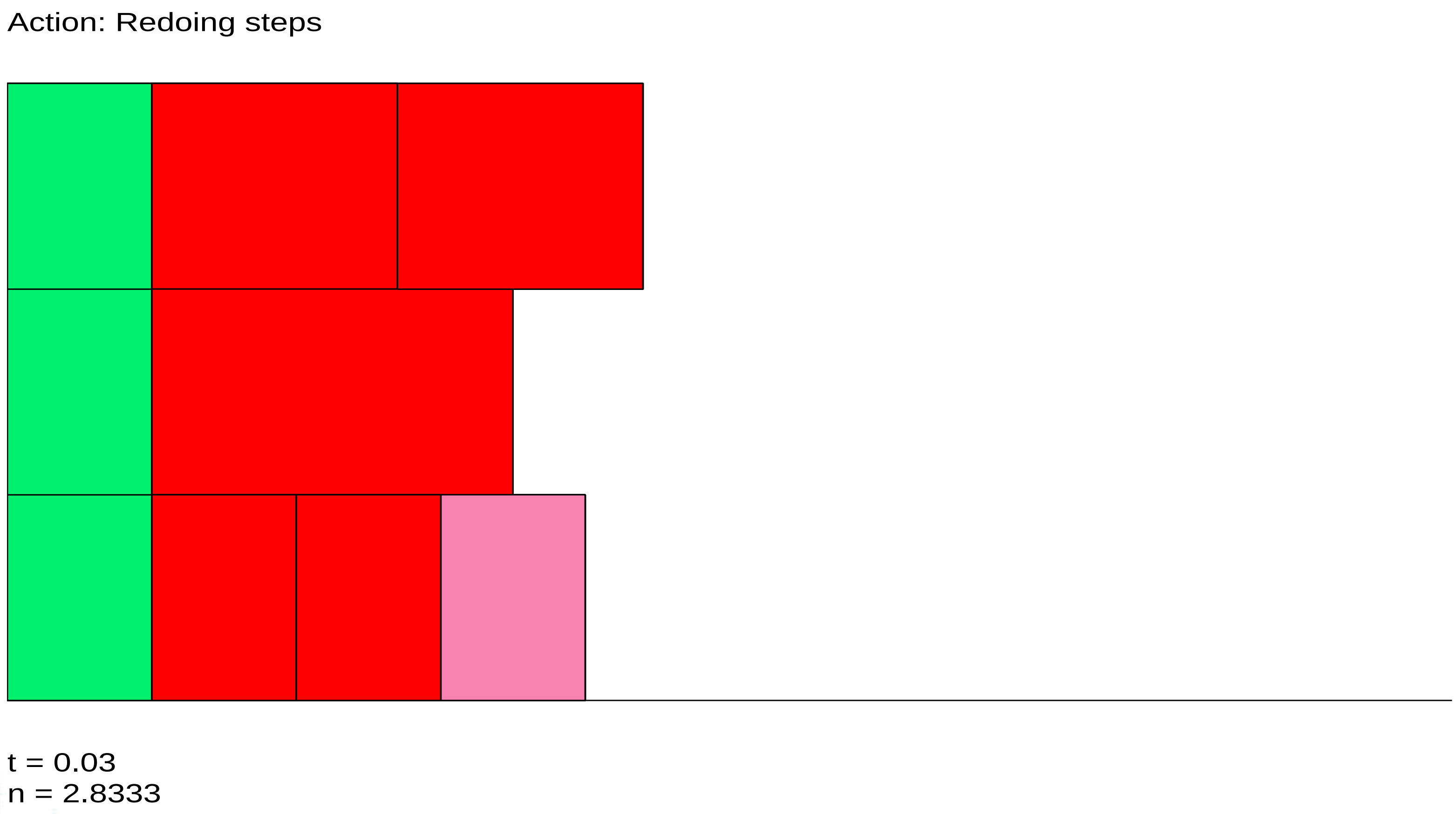


t = 0.02
n = 2.6667

Multi-Adaptive Time-Stepping

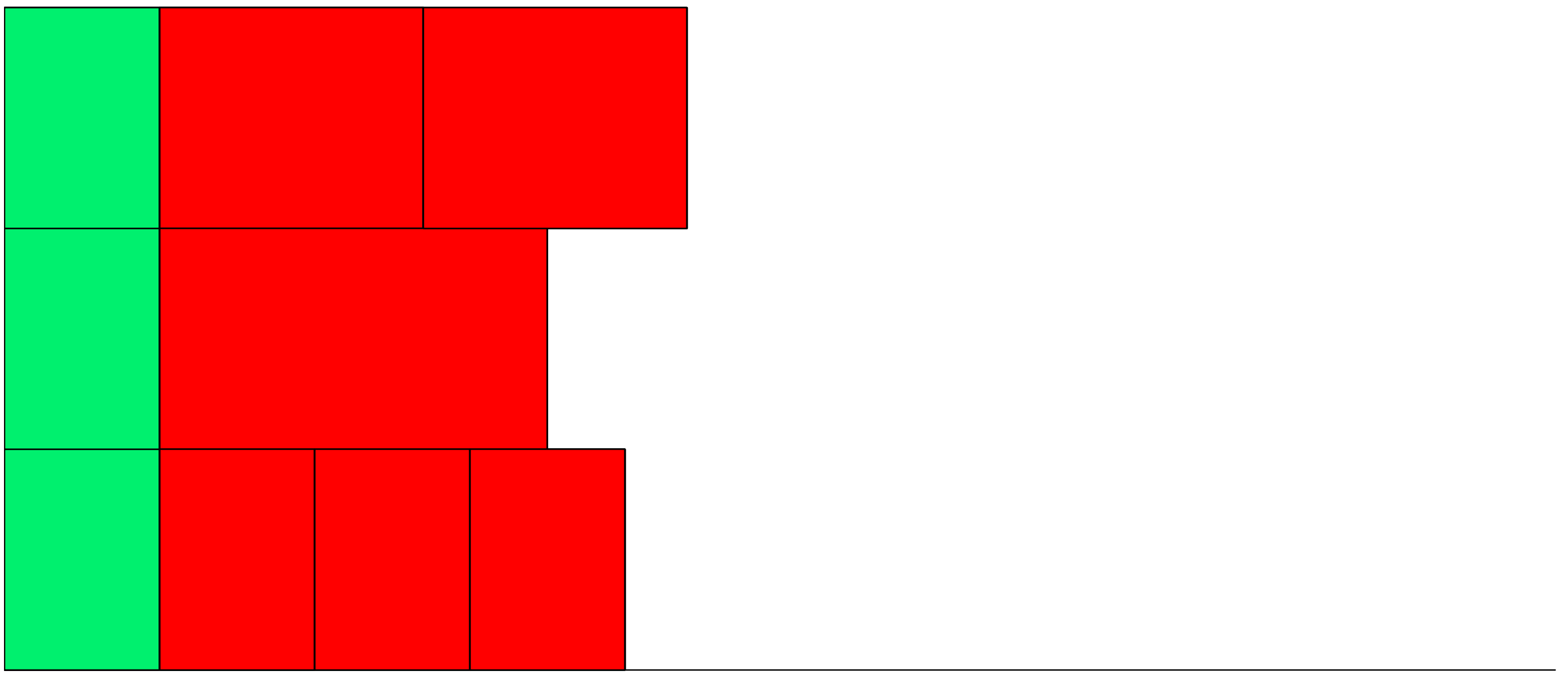


Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

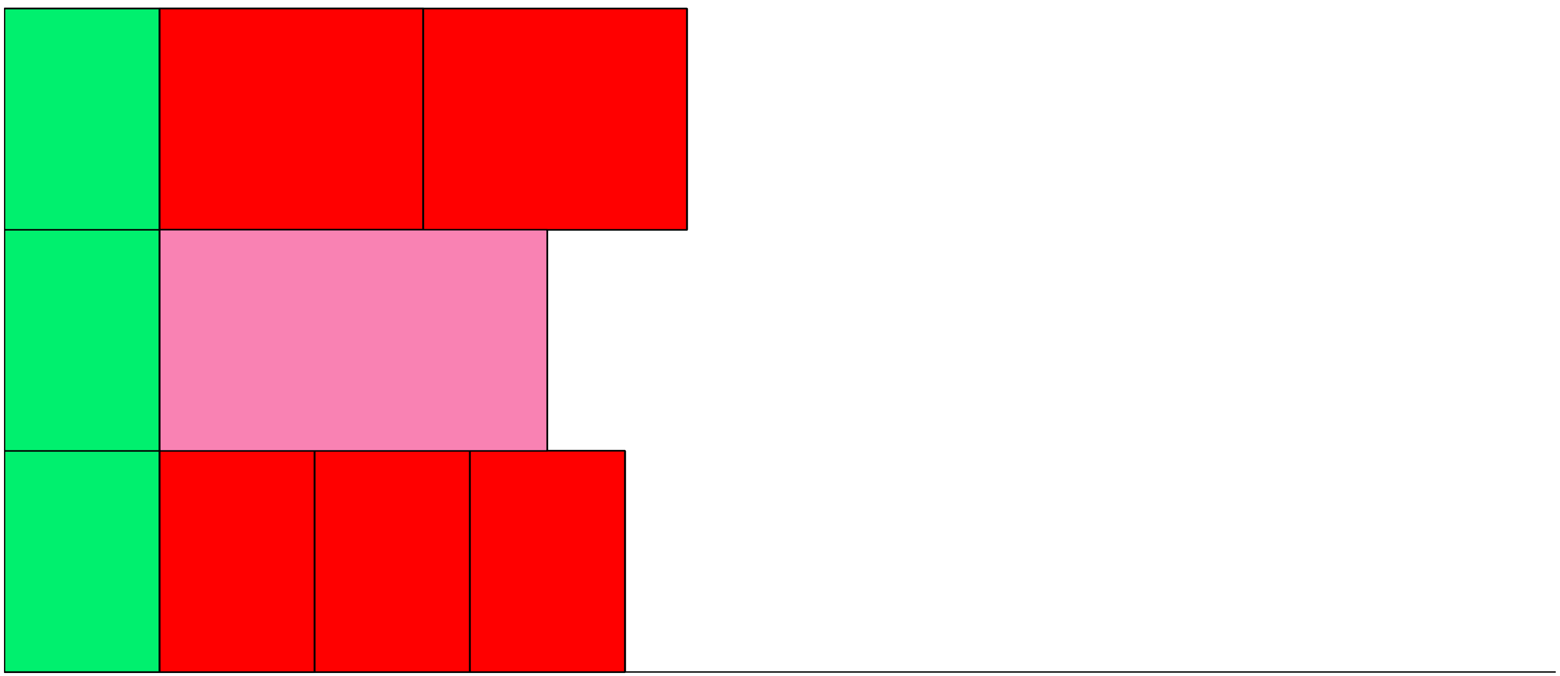
Action: Redoing steps



t = 0.04
n = 2.8333

Multi-Adaptive Time-Stepping

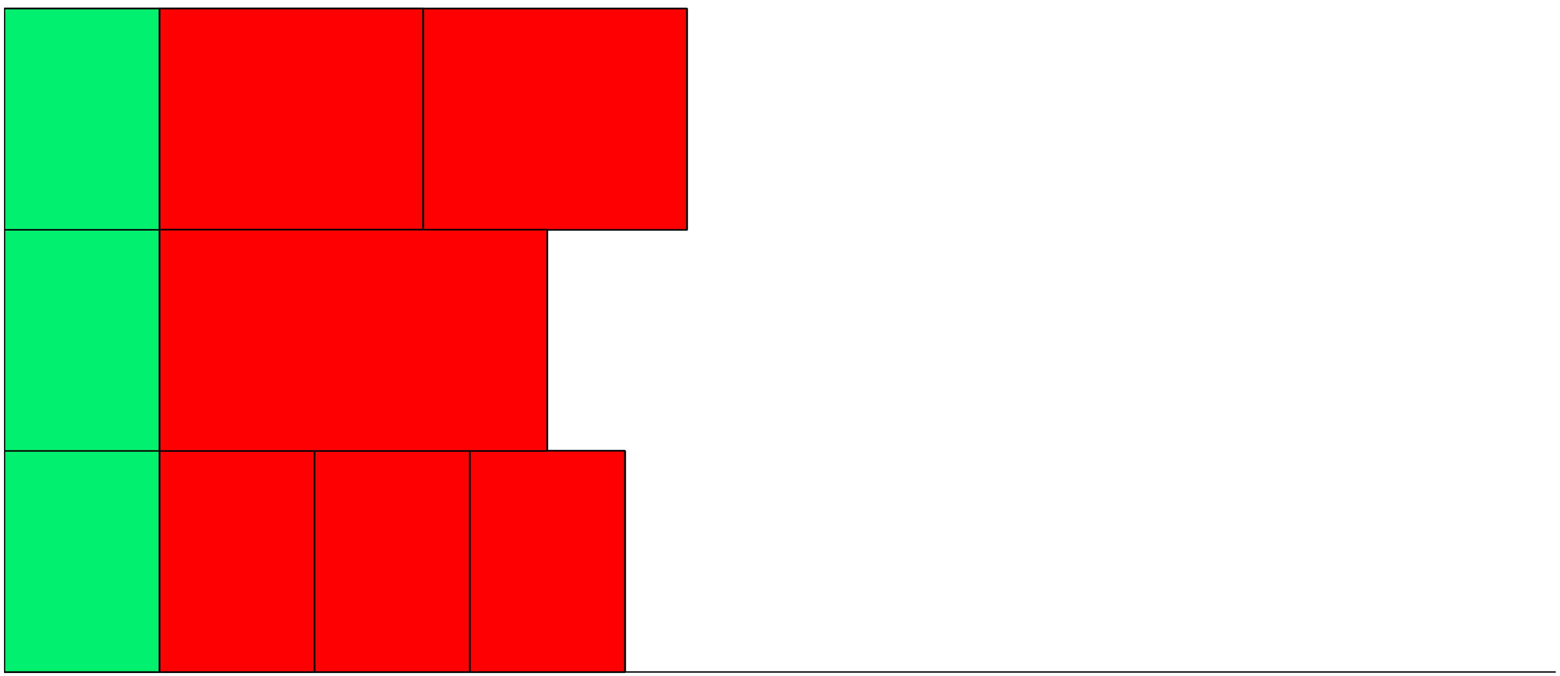
Action: Redoing steps



t = 0.04
n = 2.8333

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.035
n = 3.1667

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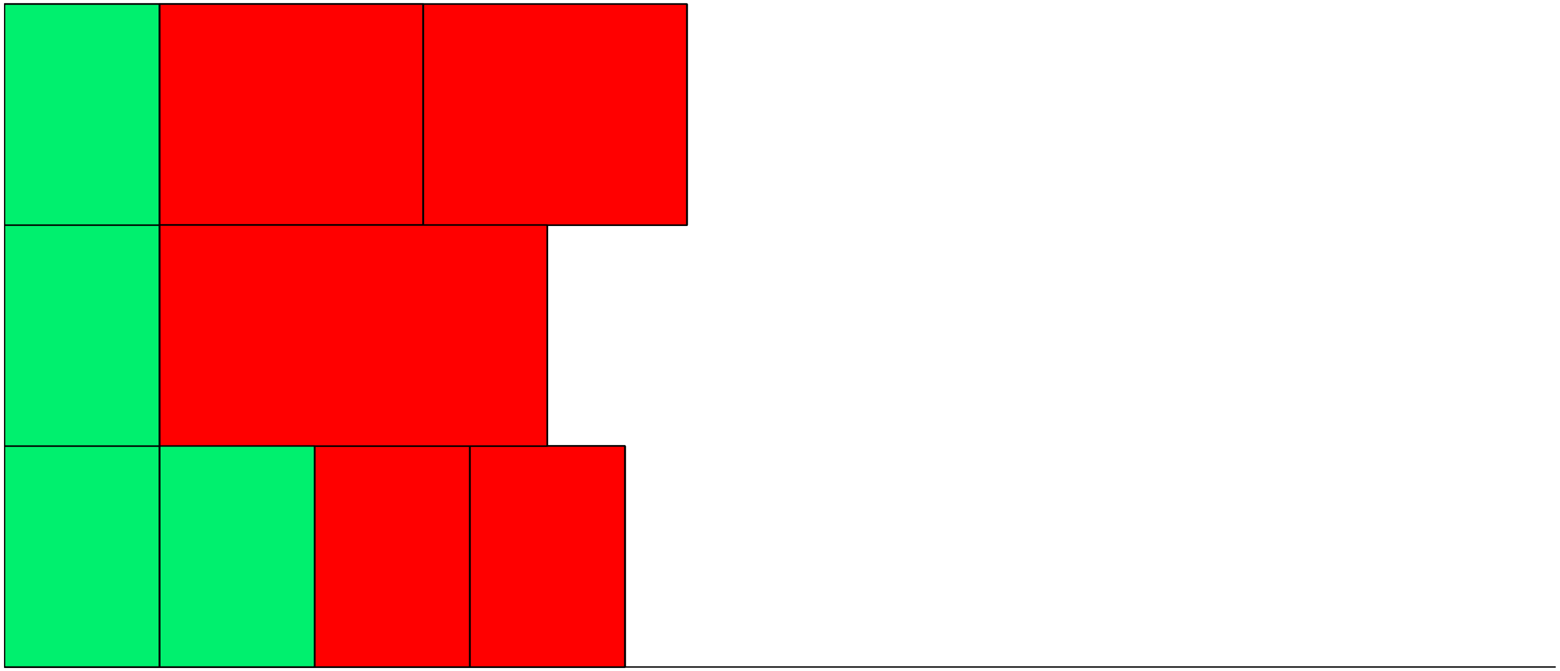
Multi-Adaptive Time-Stepping

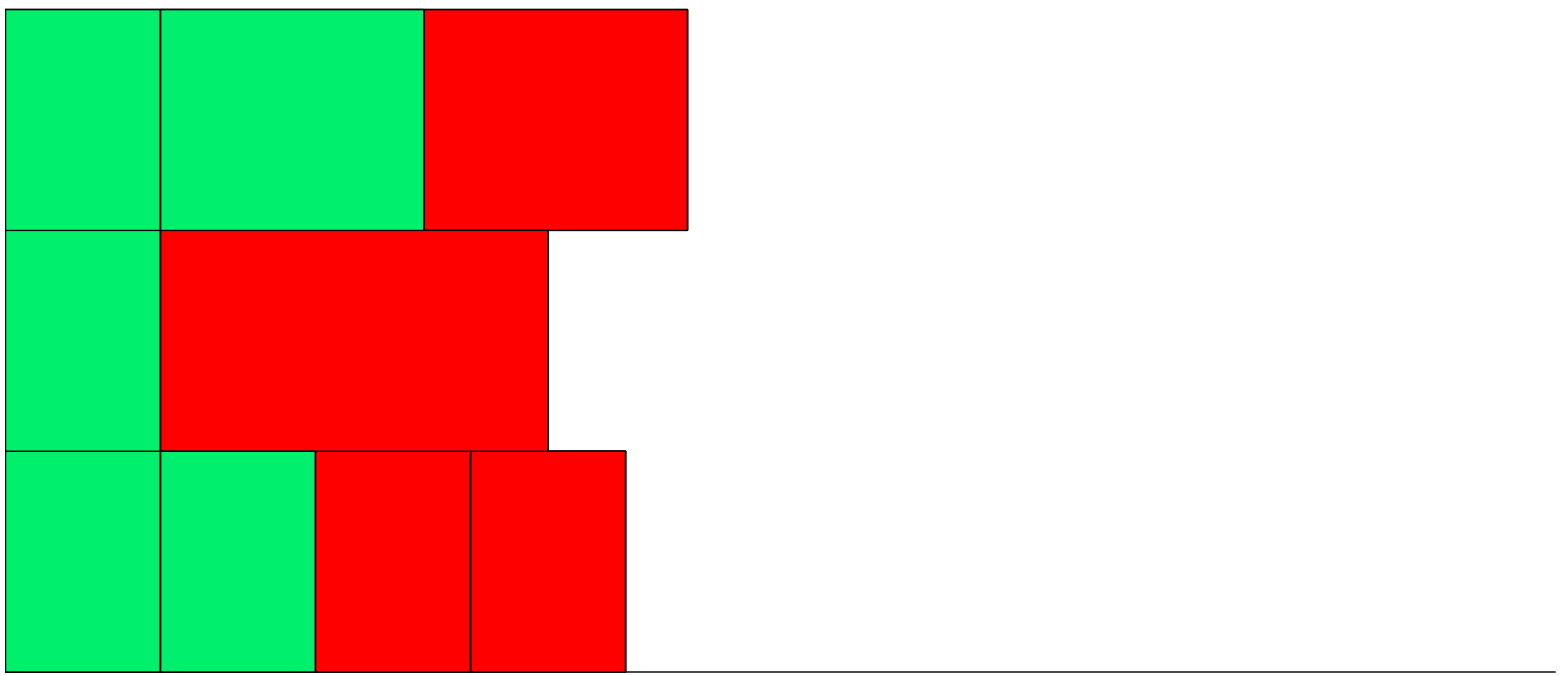
Action: Clearing steps



t = 0.02
n = 3.3333

Multi-Adaptive Time-Stepping

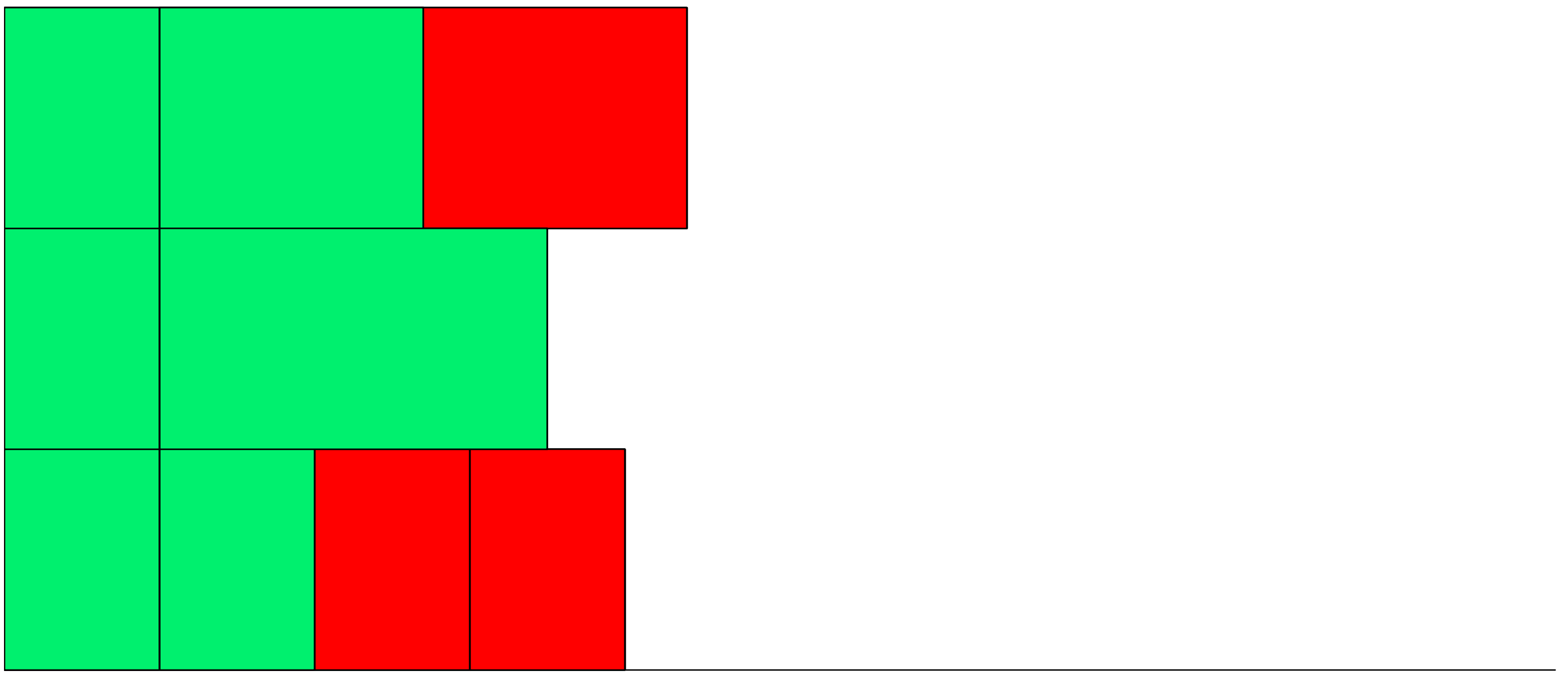
Action: Clearing steps



t = 0.027
n = 3.3333

Multi-Adaptive Time-Stepping

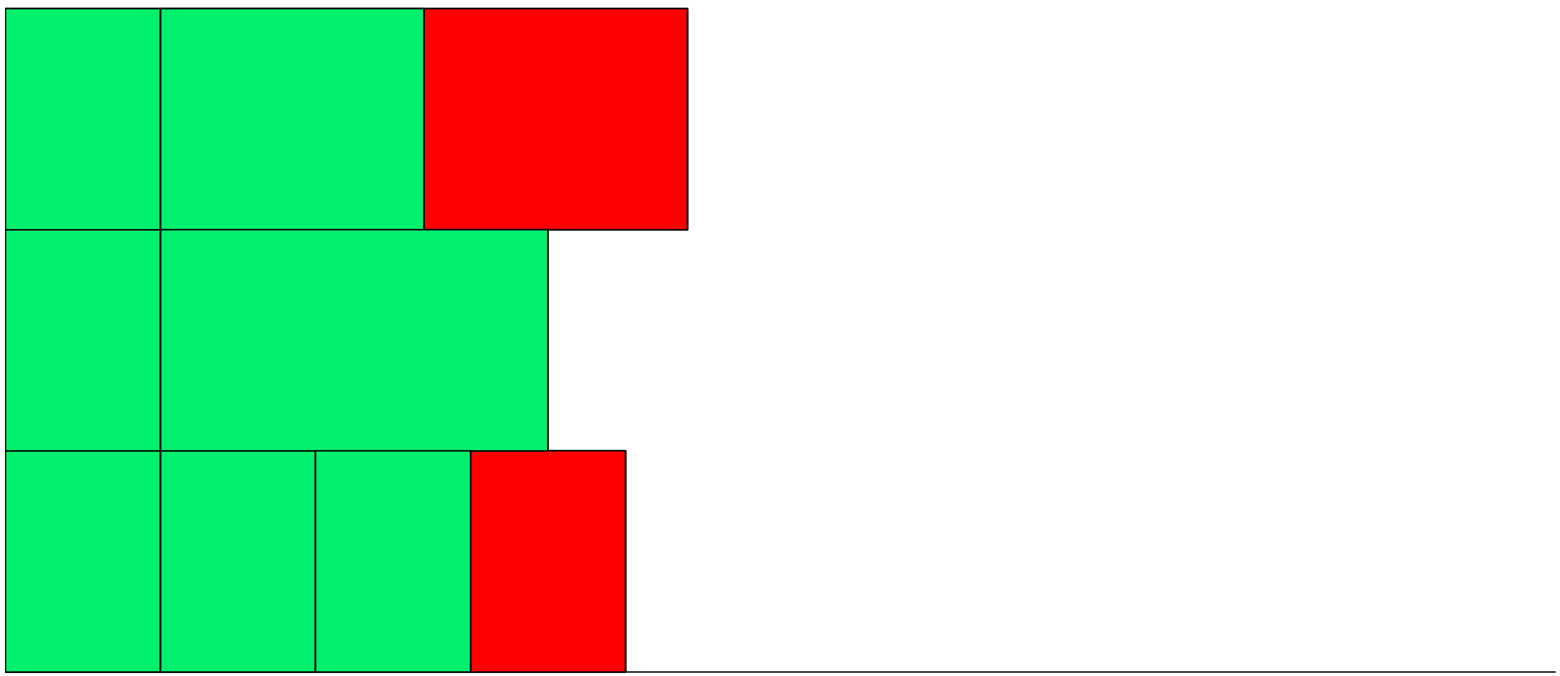
Action: Clearing steps



t = 0.035
n = 3.3333

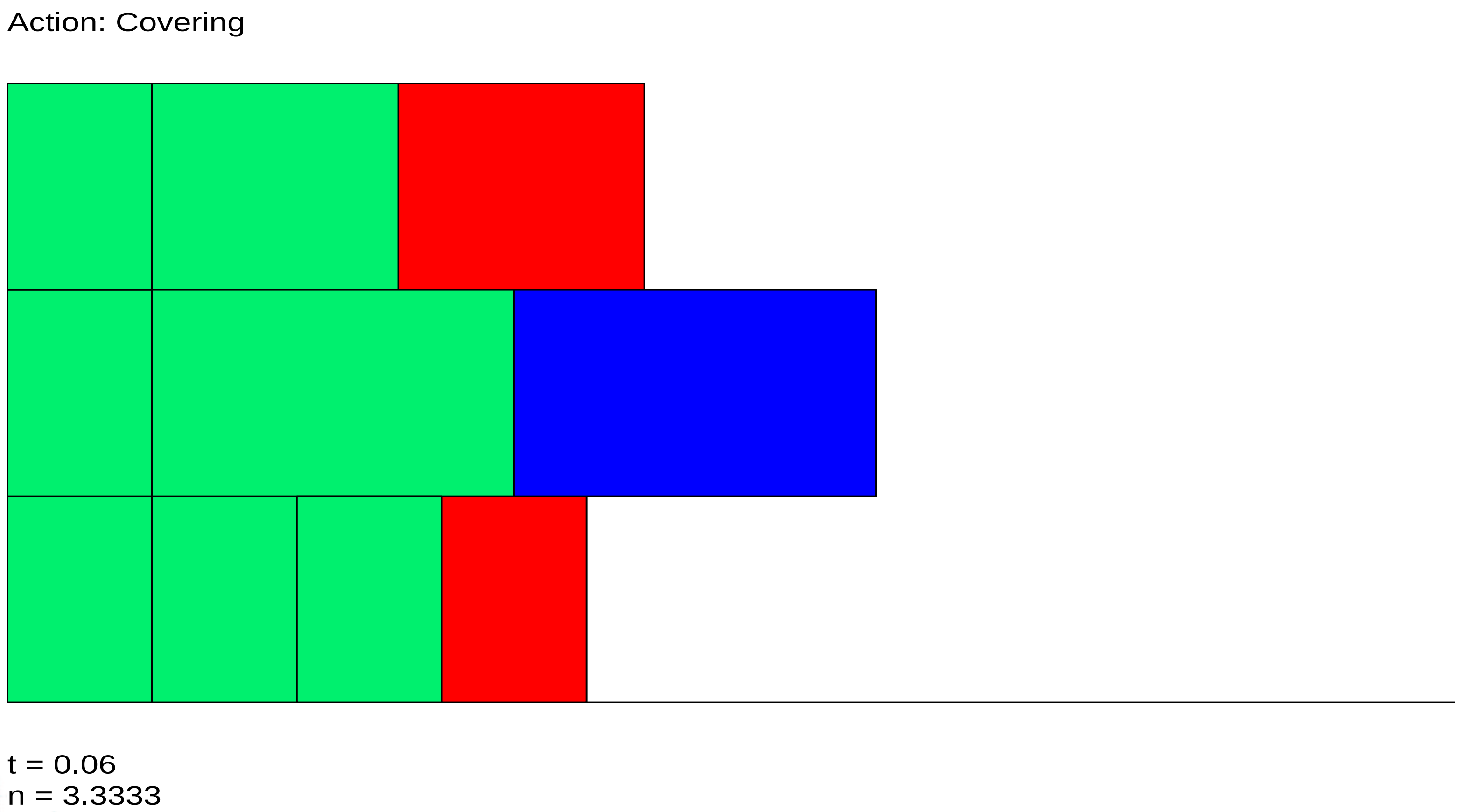
Multi-Adaptive Time-Stepping

Action: Clearing steps

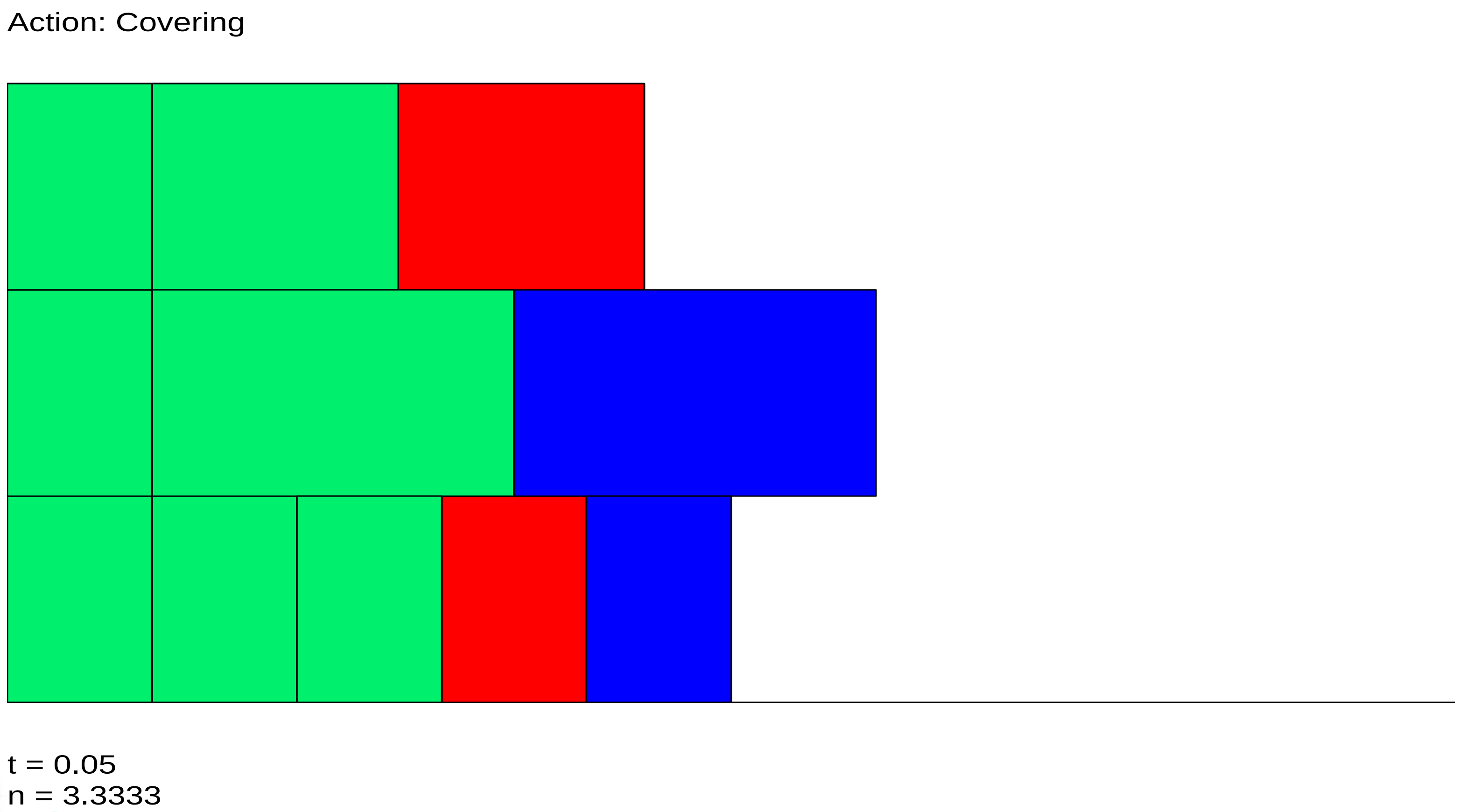


t = 0.03
n = 3.3333

Multi-Adaptive Time-Stepping

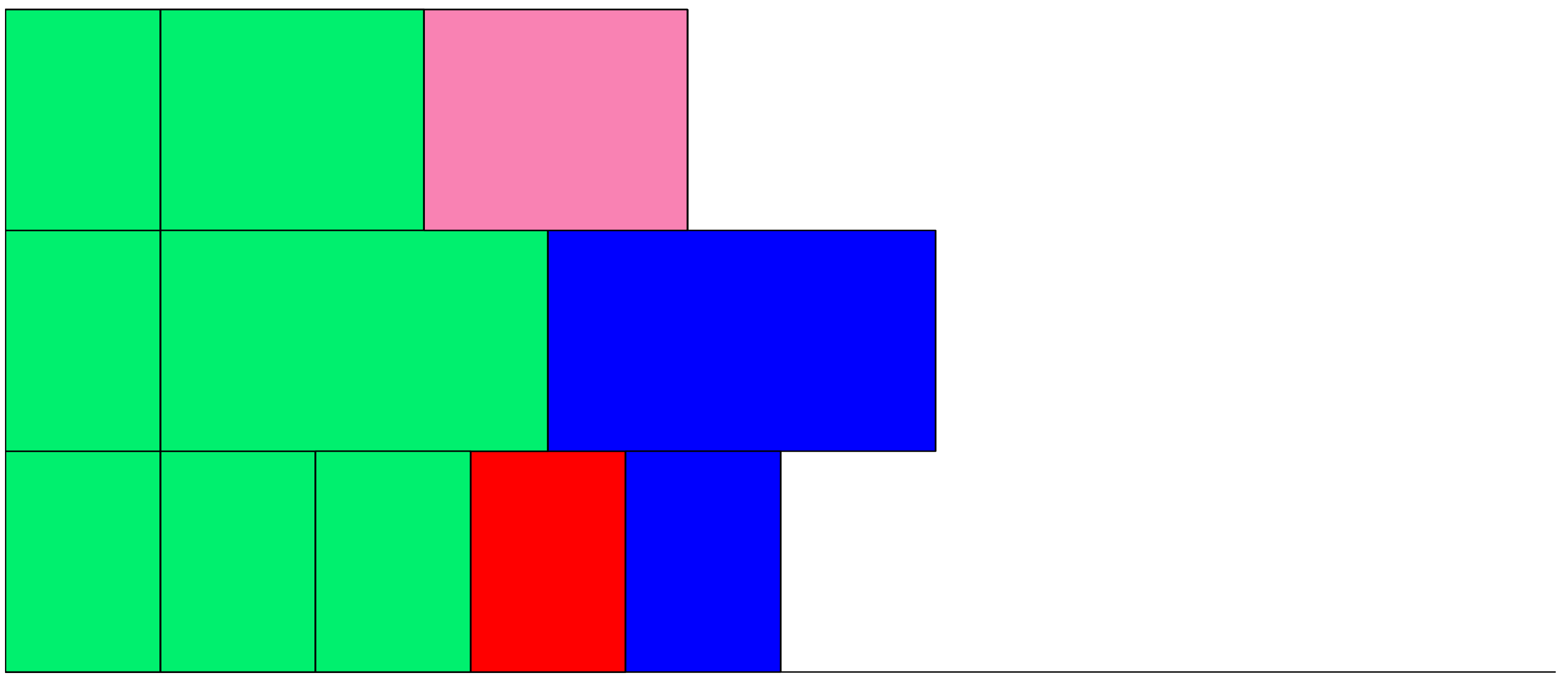


Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

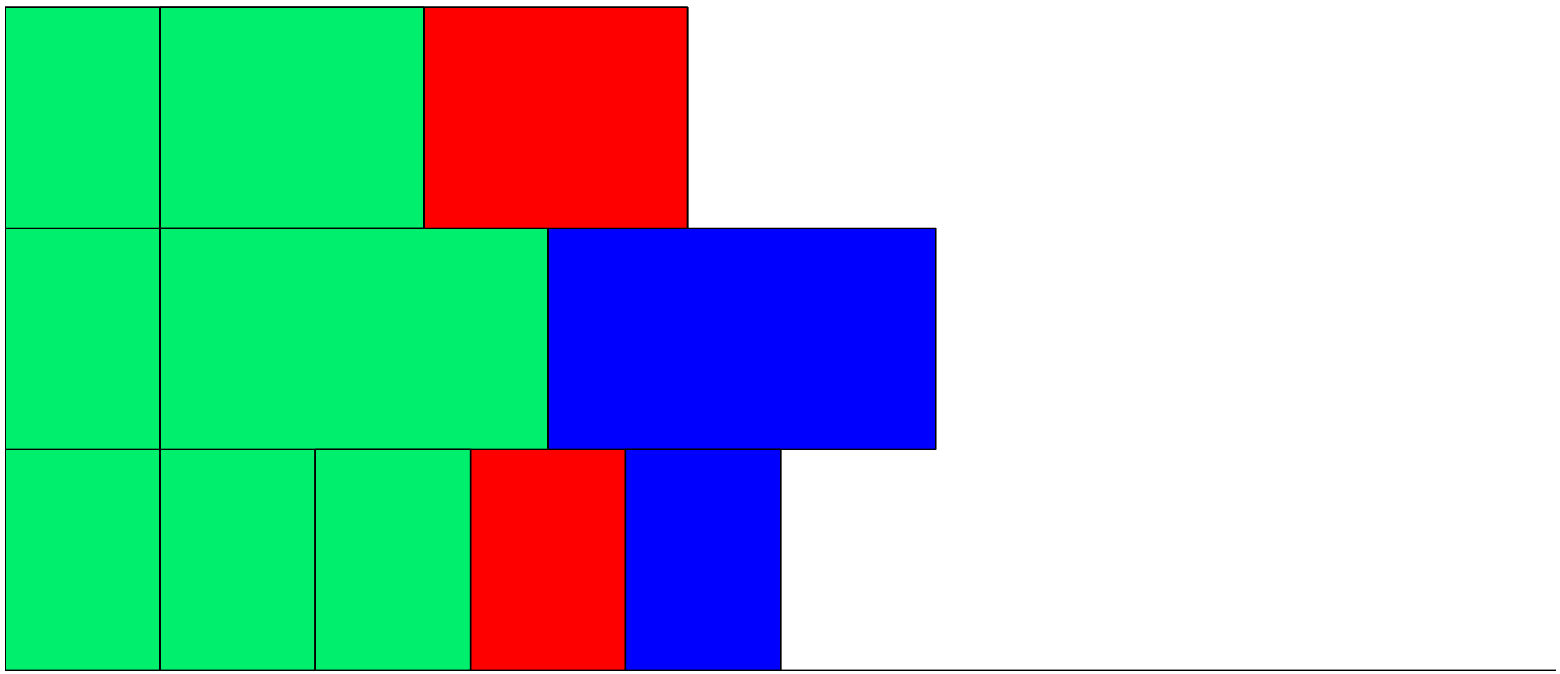
Action: Redoing steps



t = 0.05
n = 3.3333

Multi-Adaptive Time-Stepping

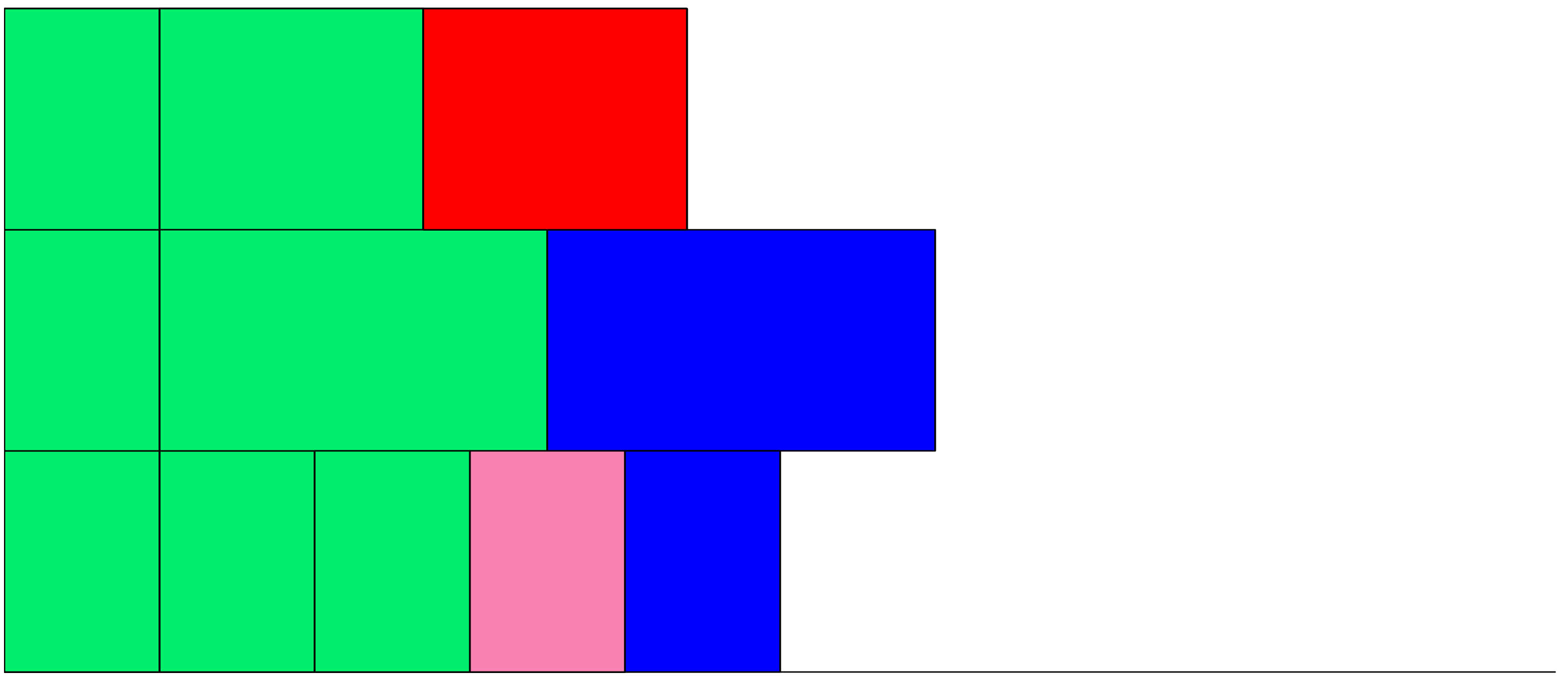
Action: Redoing steps



t = 0.044
n = 1

Multi-Adaptive Time-Stepping

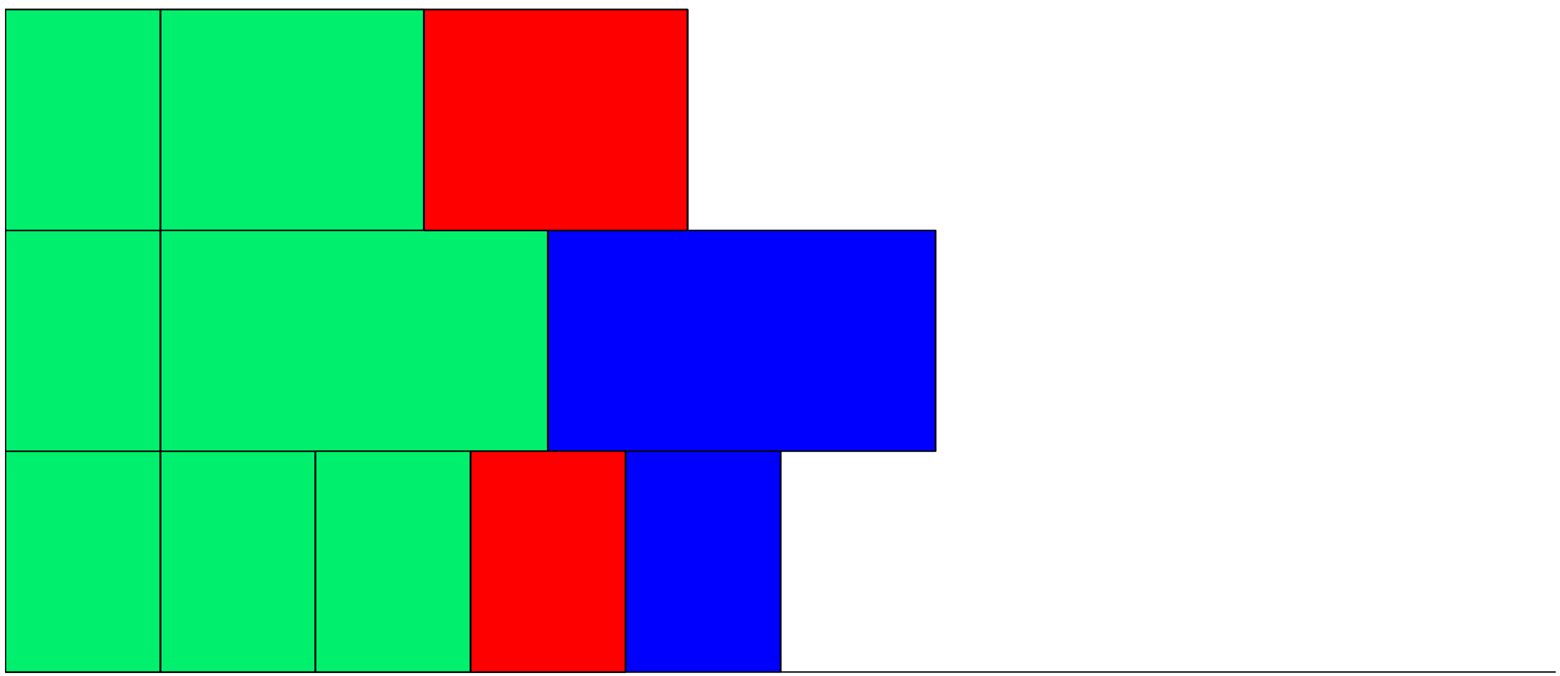
Action: Redoing steps



t = 0.044
n = 1

Multi-Adaptive Time-Stepping

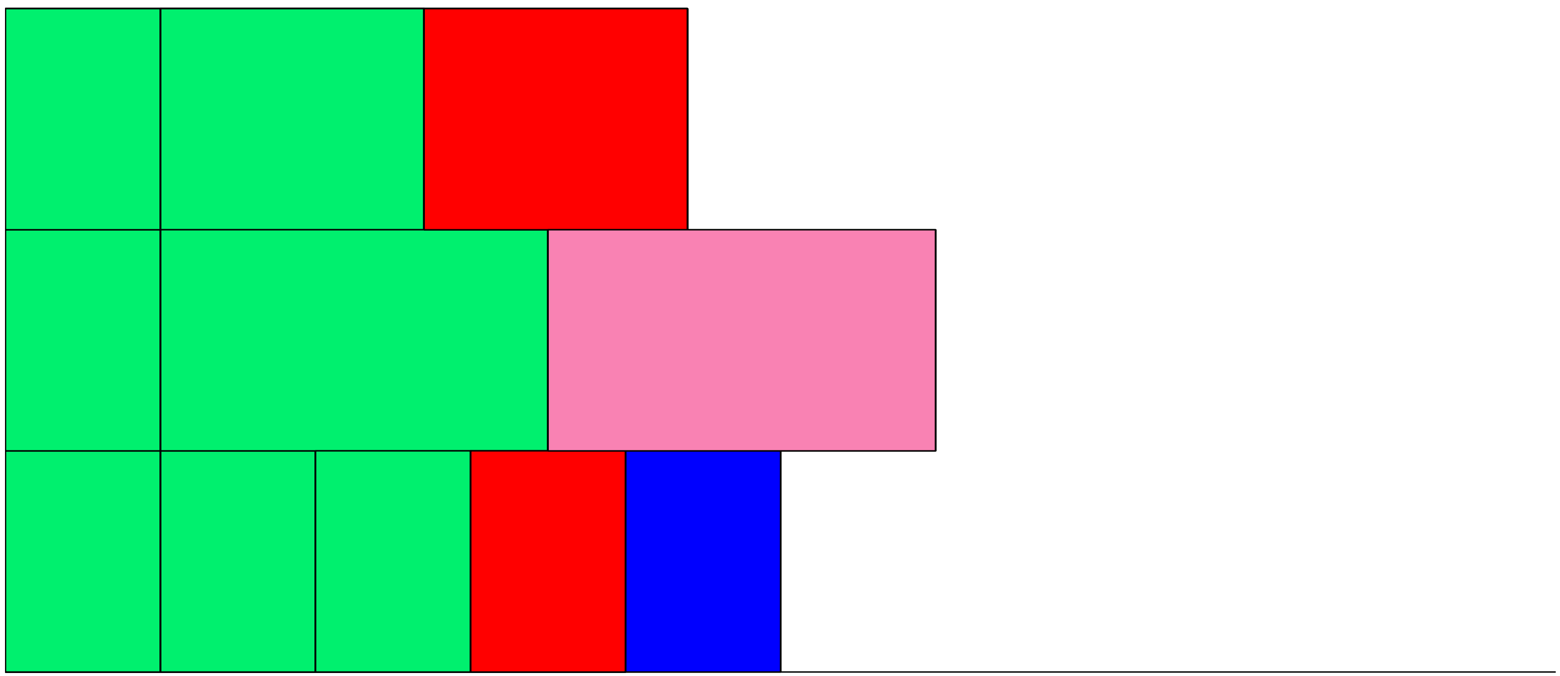
Action: Redoing steps



t = 0.04
n = 1

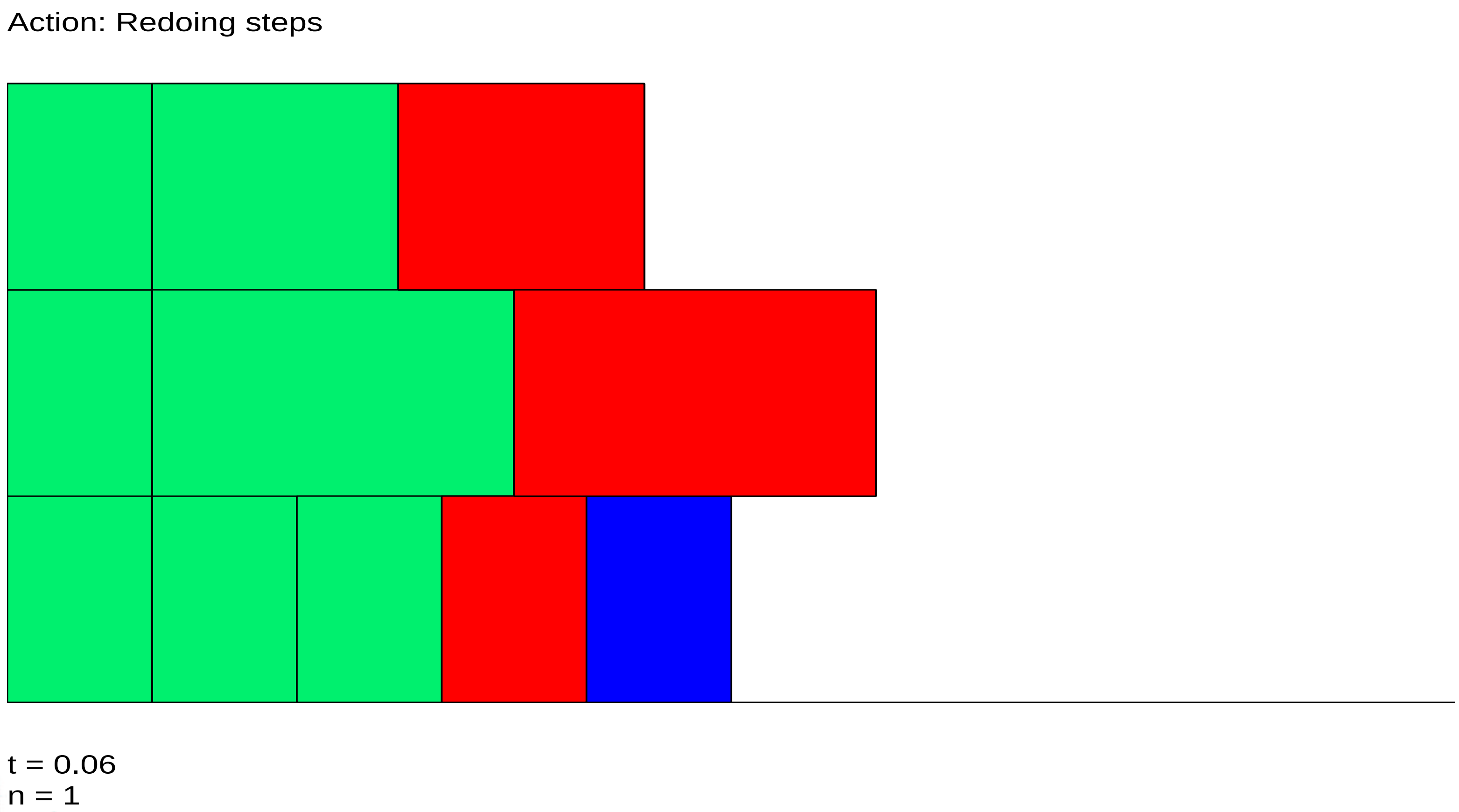
Multi-Adaptive Time-Stepping

Action: Redoing steps

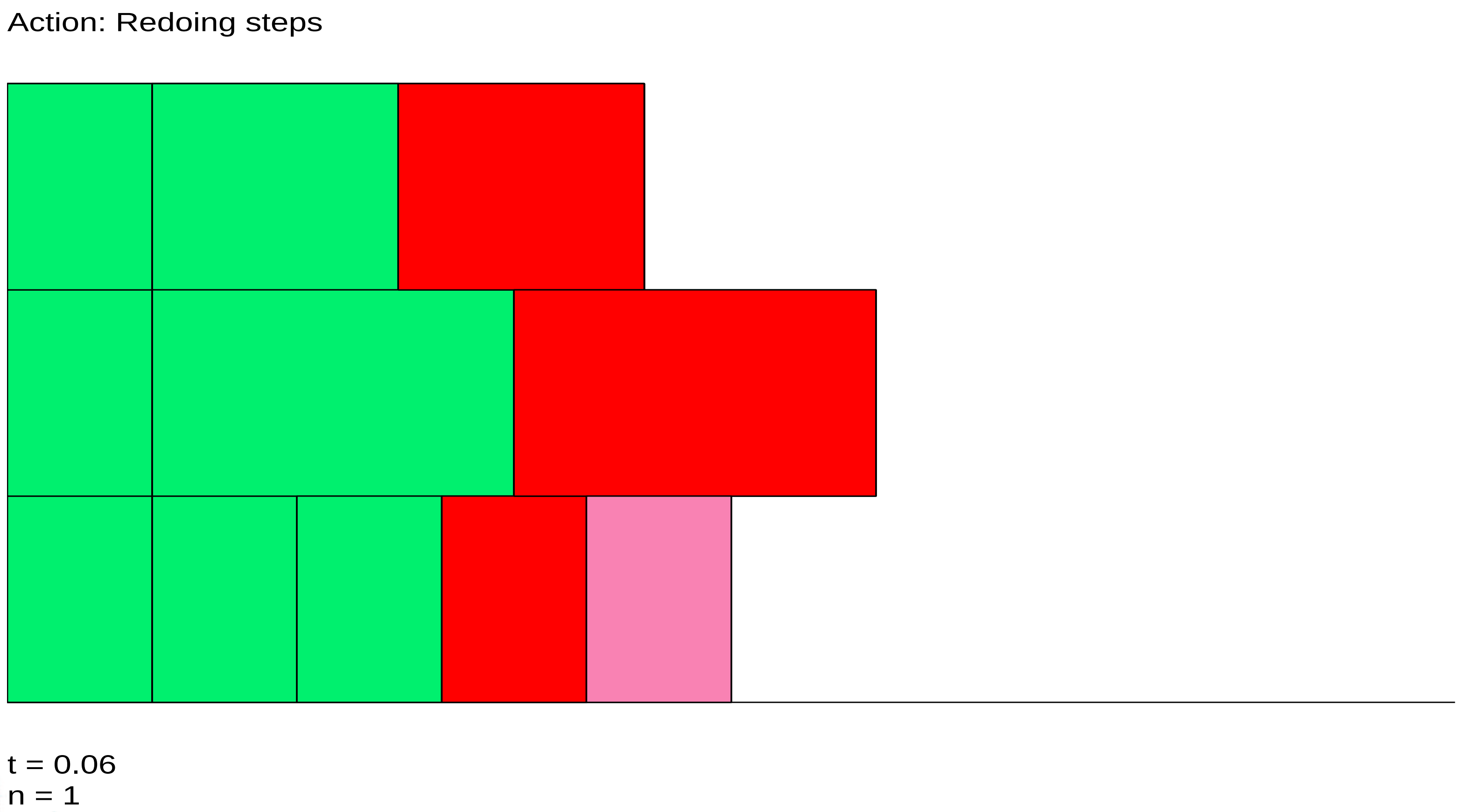


t = 0.04
n = 1

Multi-Adaptive Time-Stepping

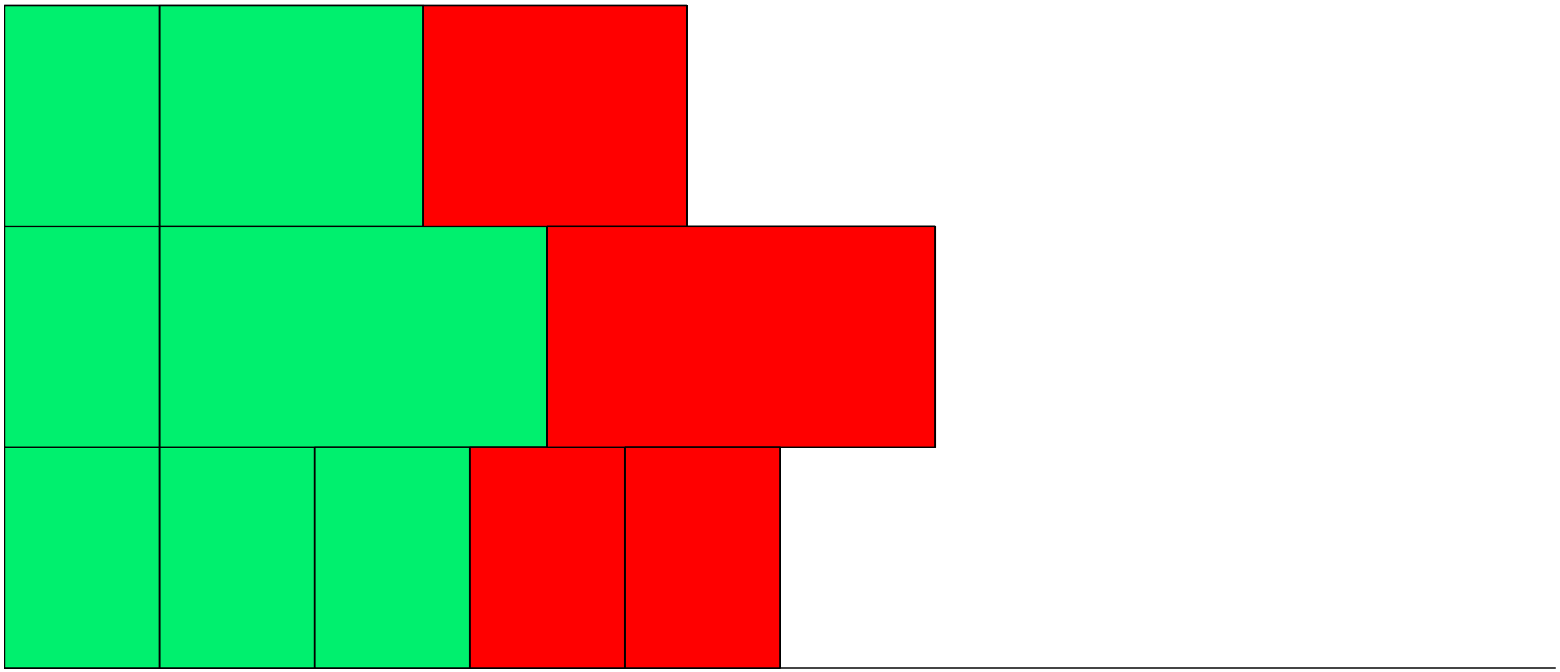


Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

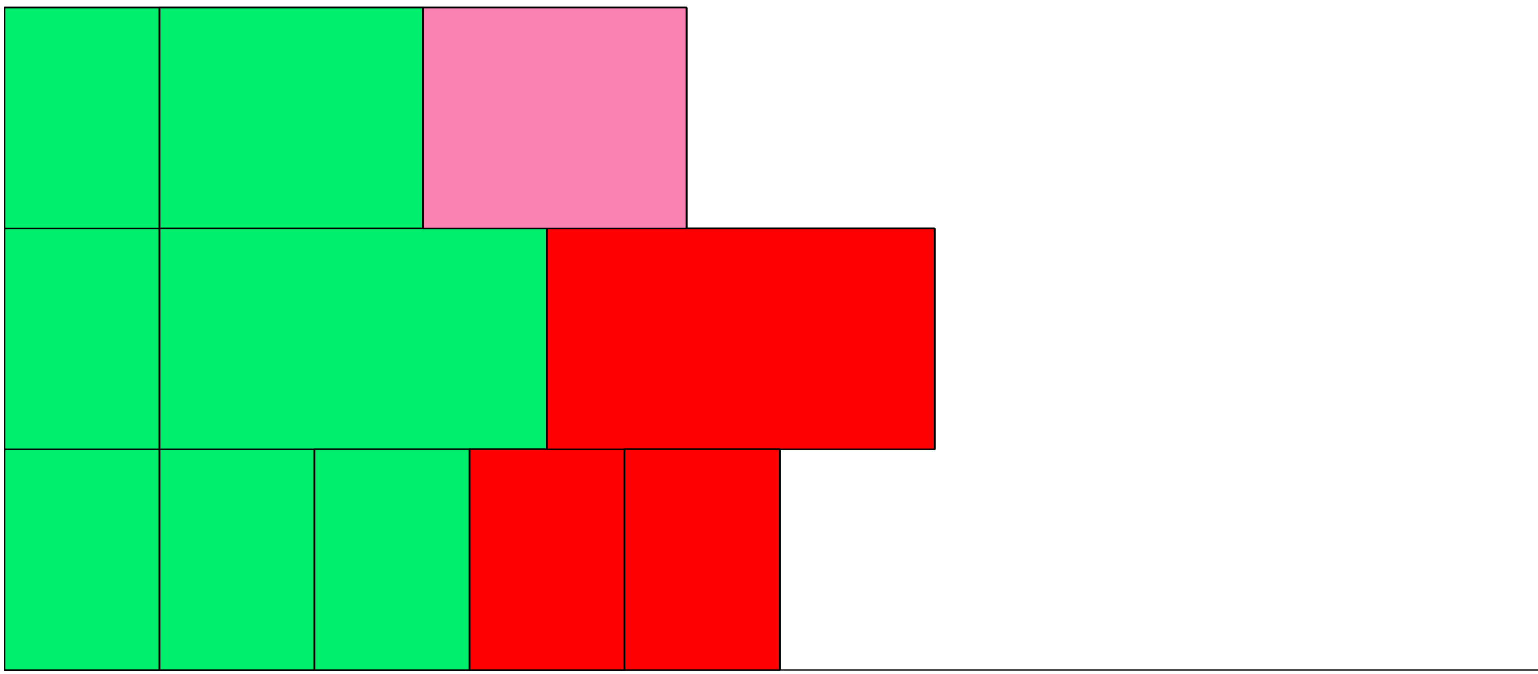
Action: Redoing steps



t = 0.05
n = 1

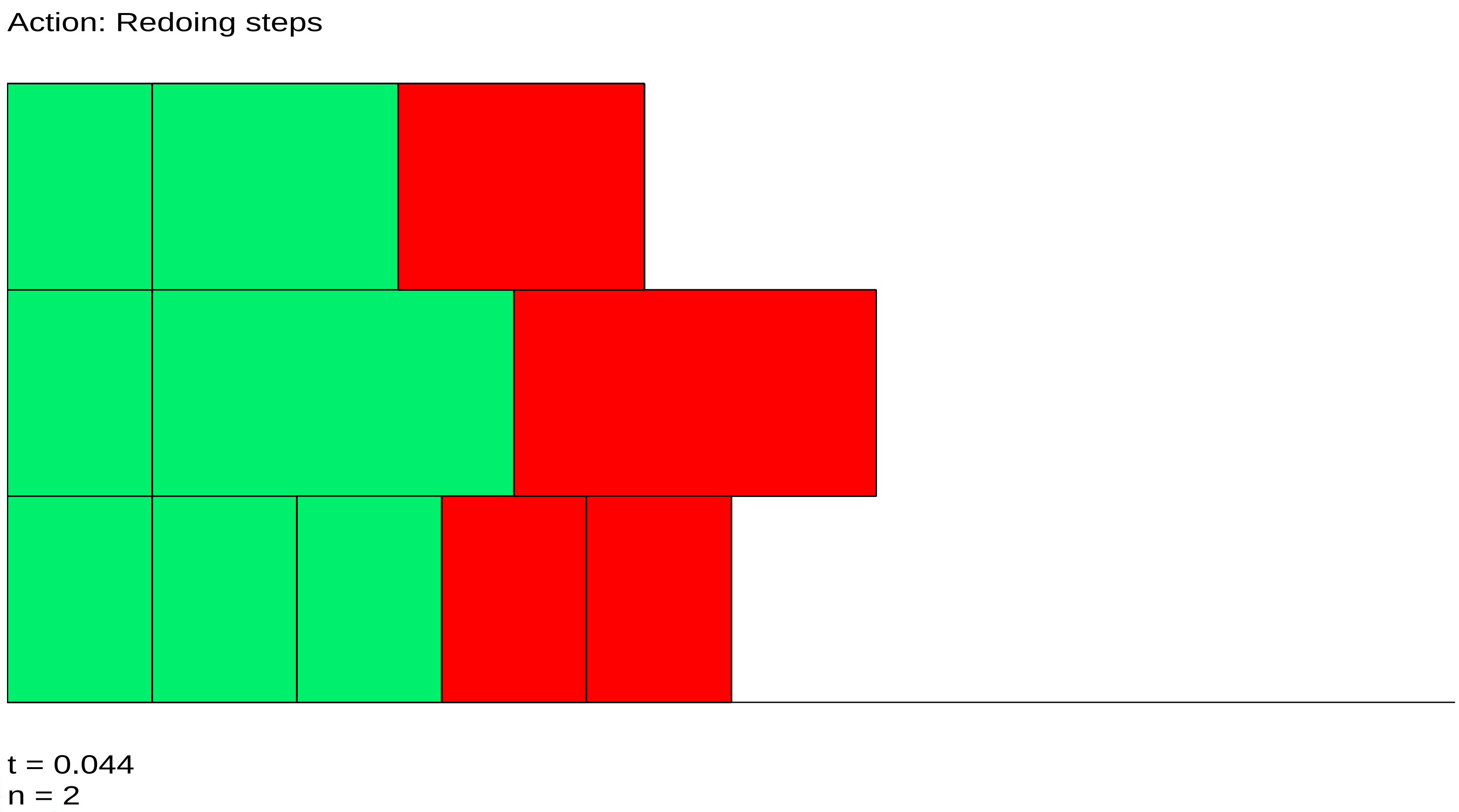
Multi-Adaptive Time-Stepping

Action: Redoing steps

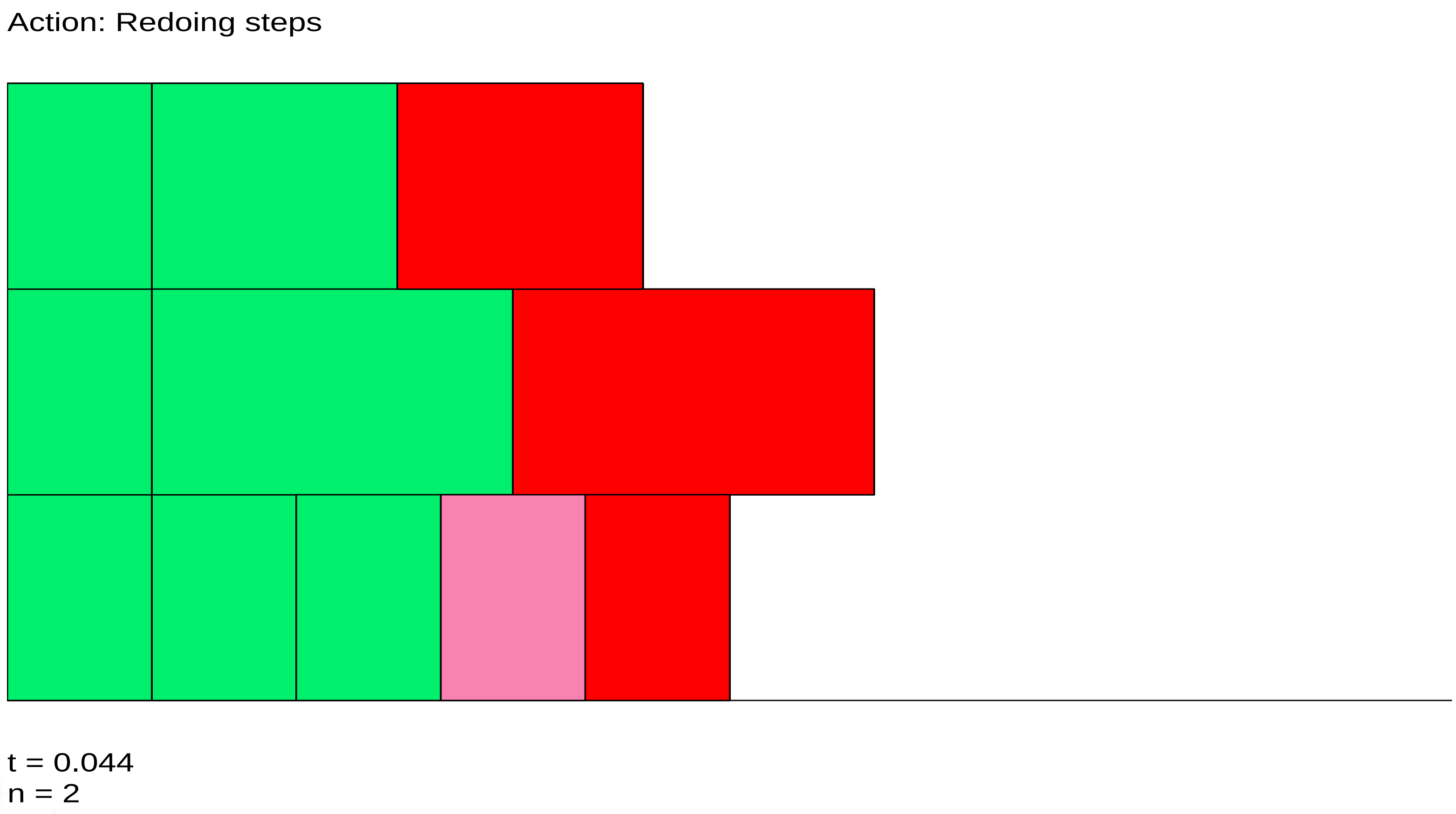


t = 0.05
n = 1

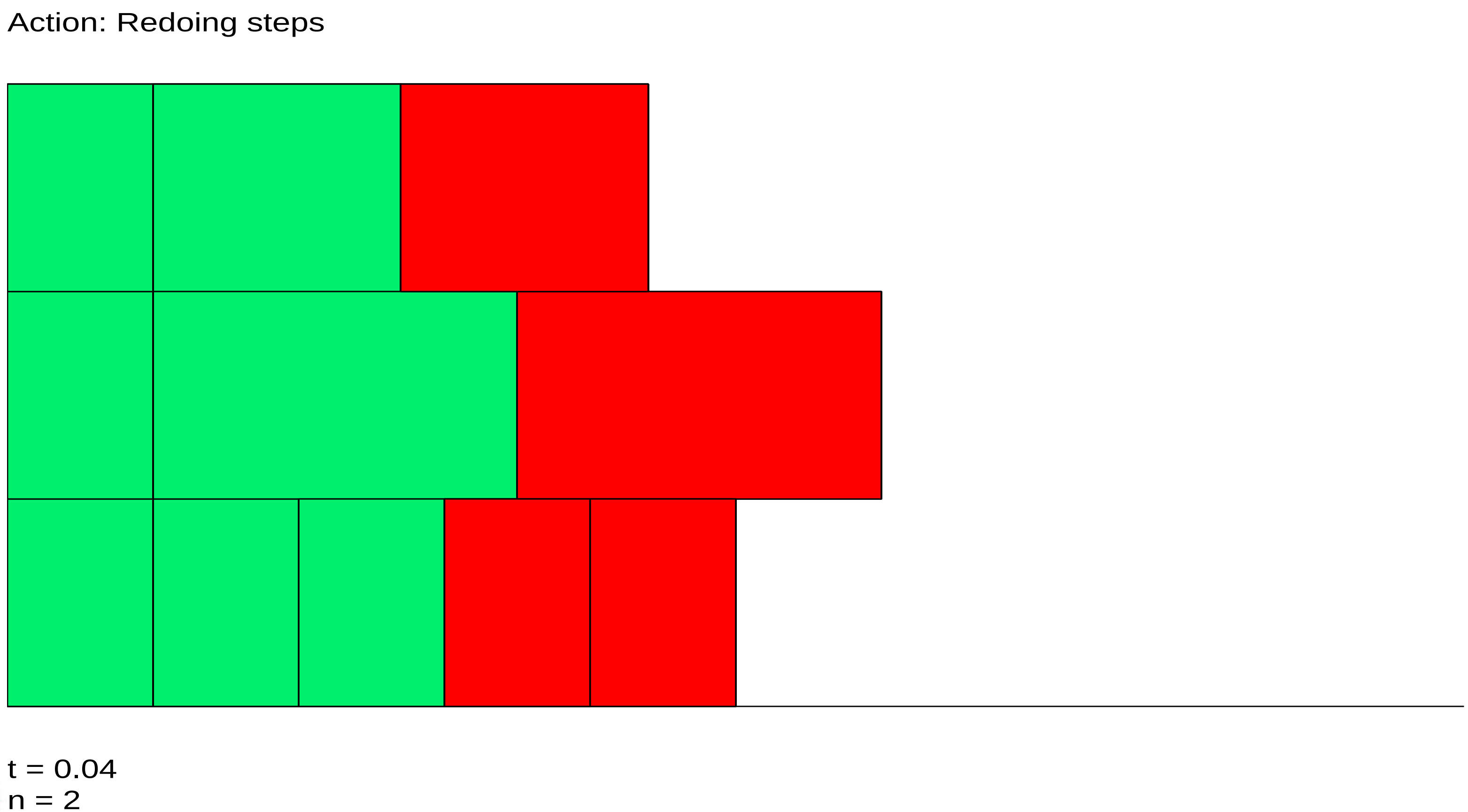
Multi-Adaptive Time-Stepping



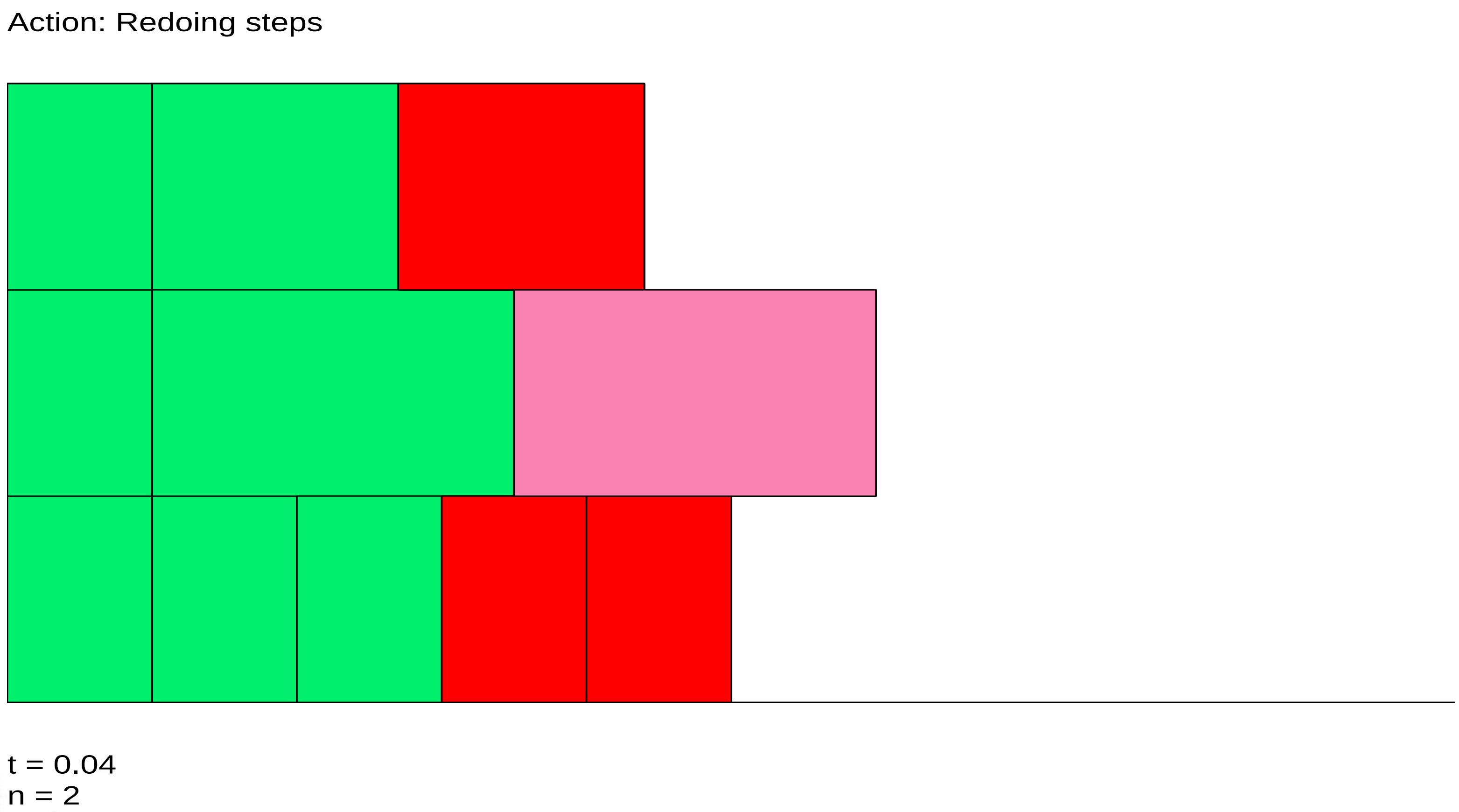
Multi-Adaptive Time-Stepping



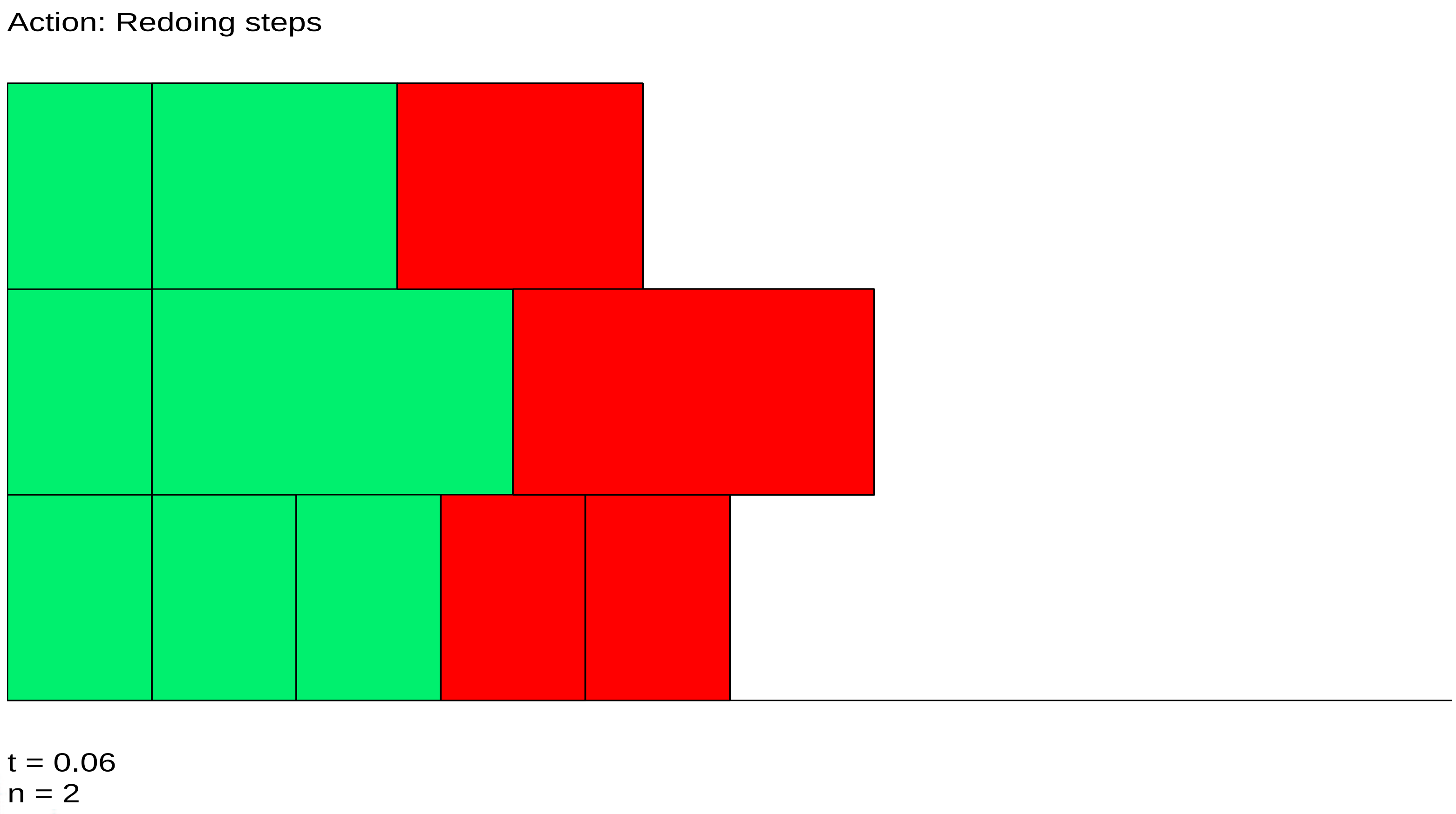
Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

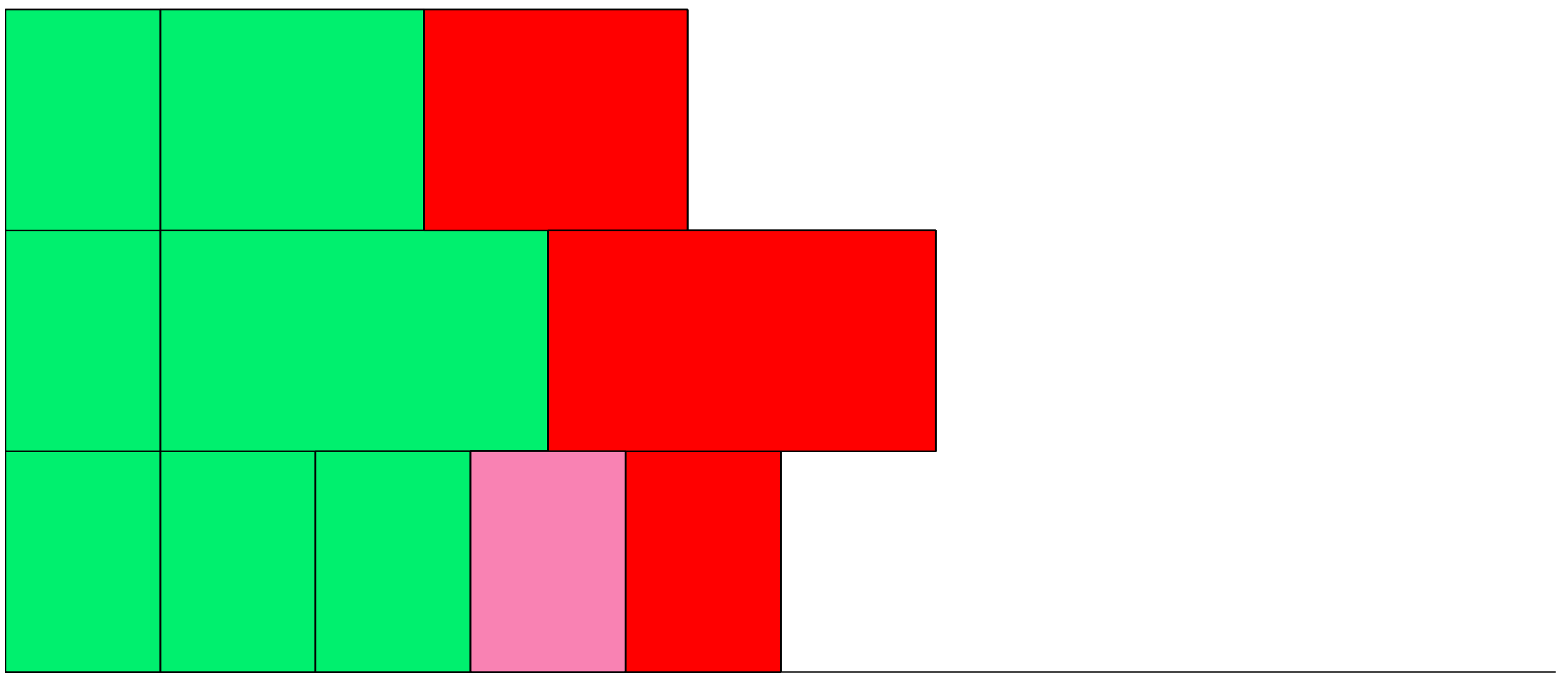


Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

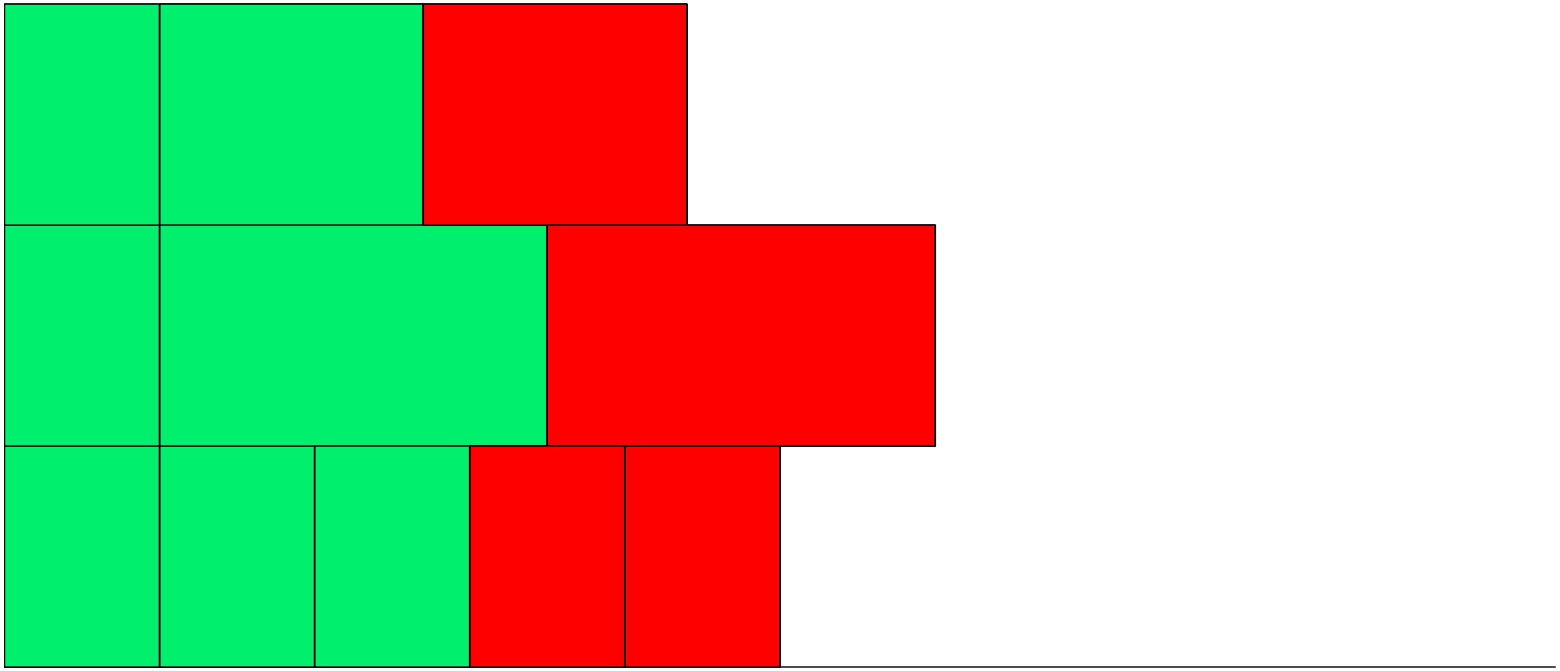
Action: Redoing steps



t = 0.06
n = 2

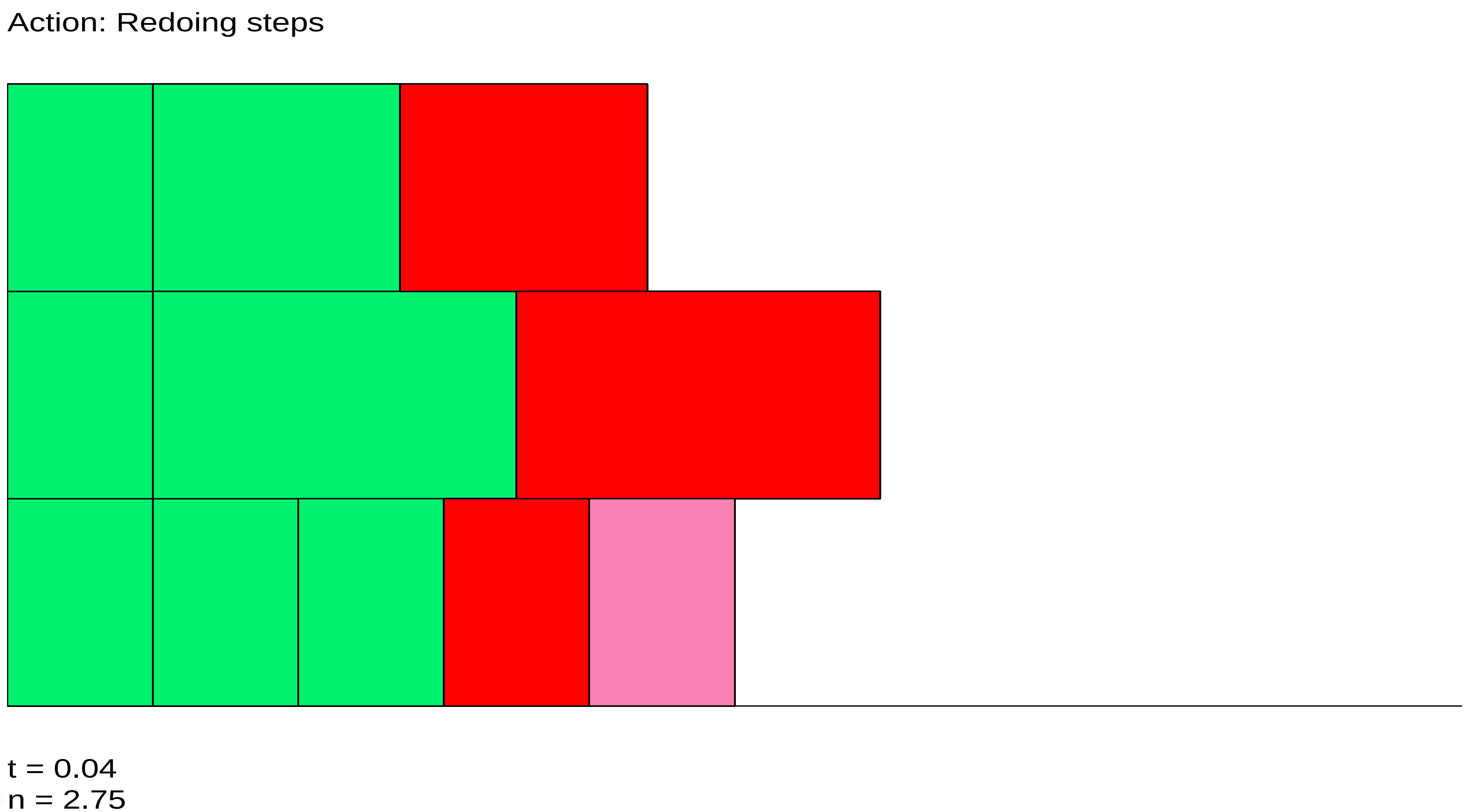
Multi-Adaptive Time-Stepping

Action: Redoing steps



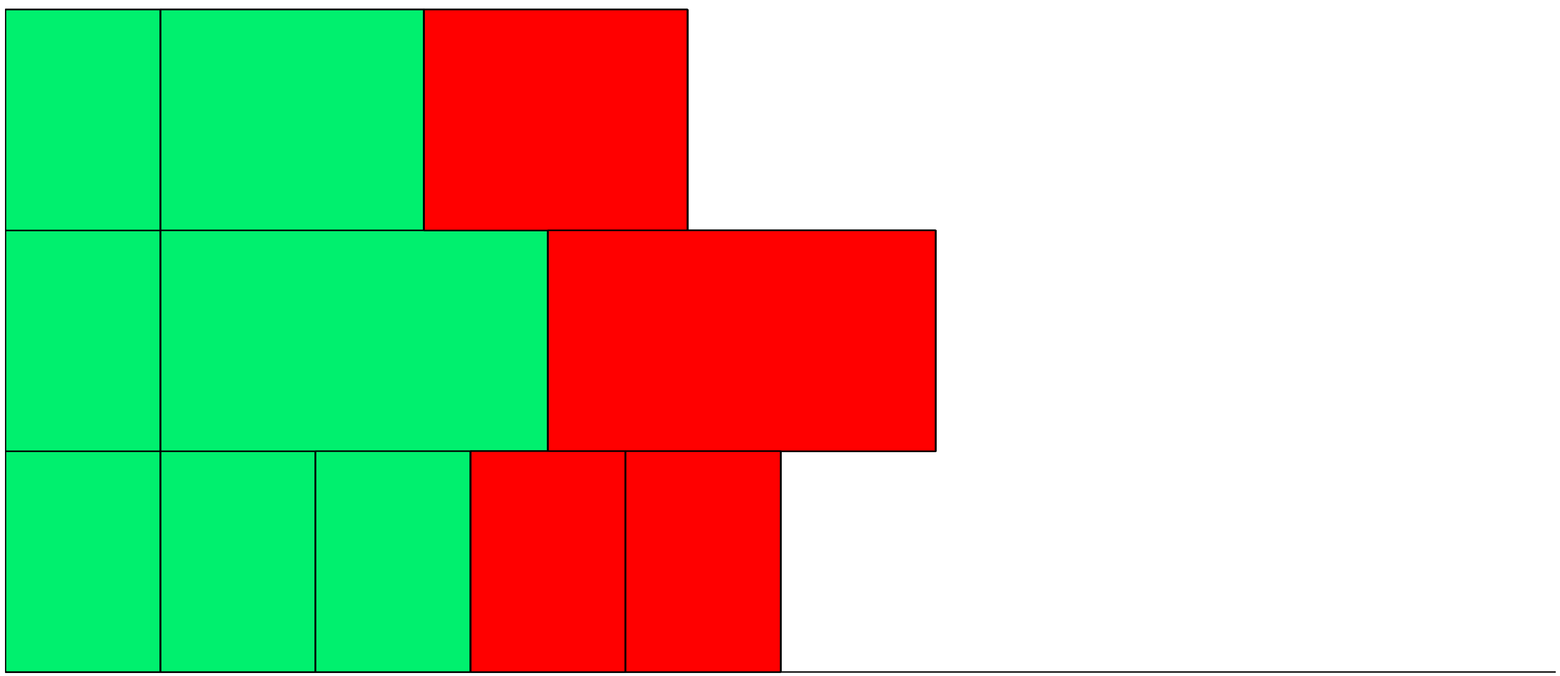
t = 0.04
n = 2.75

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

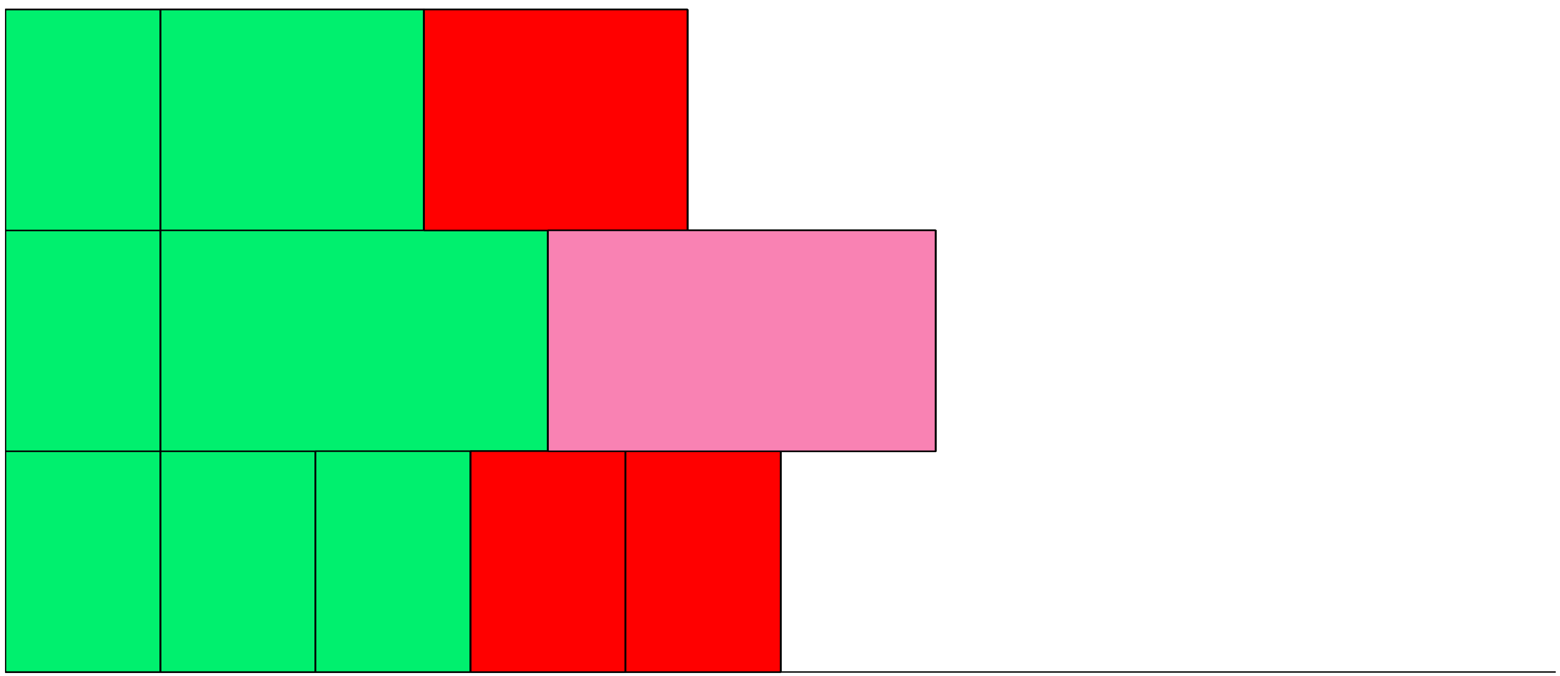
Action: Redoing steps



t = 0.05
n = 3

Multi-Adaptive Time-Stepping

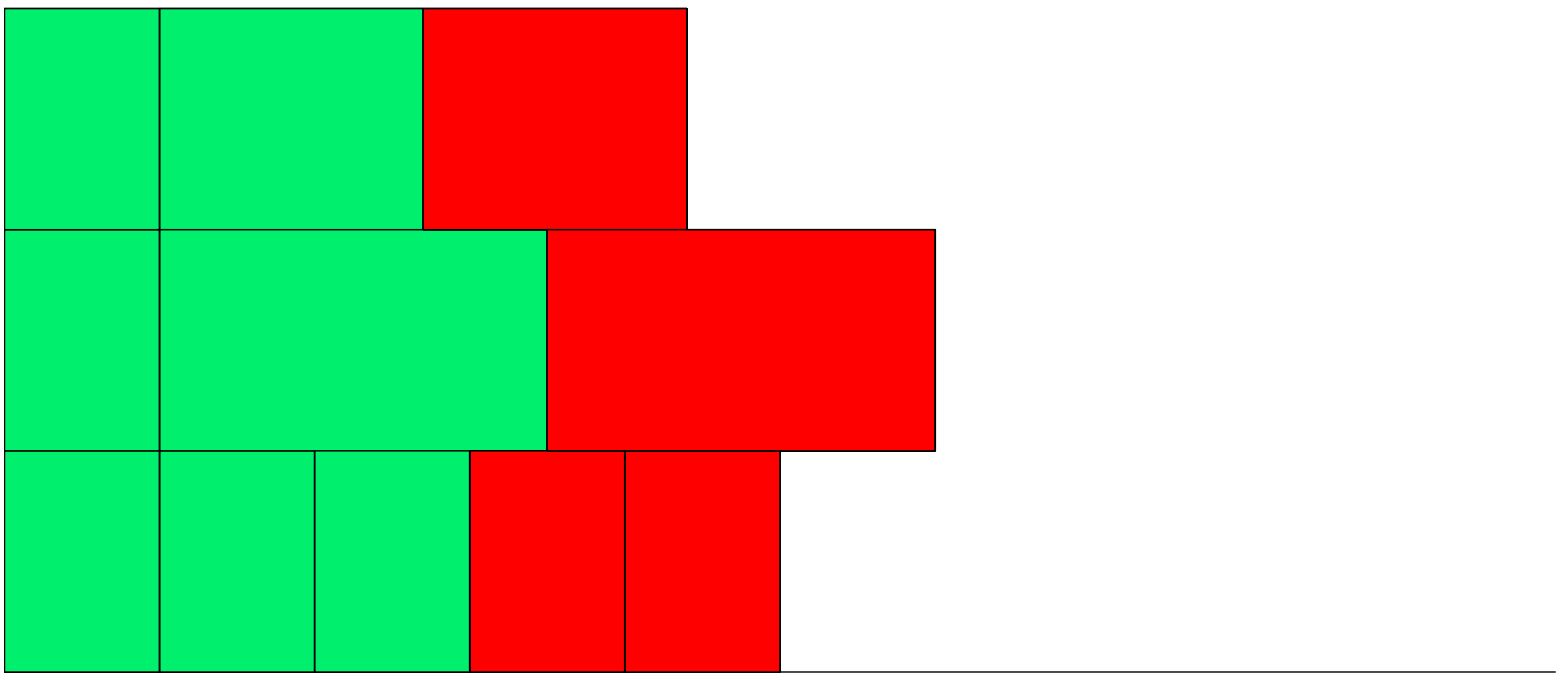
Action: Redoing steps



t = 0.05
n = 3

Multi-Adaptive Time-Stepping

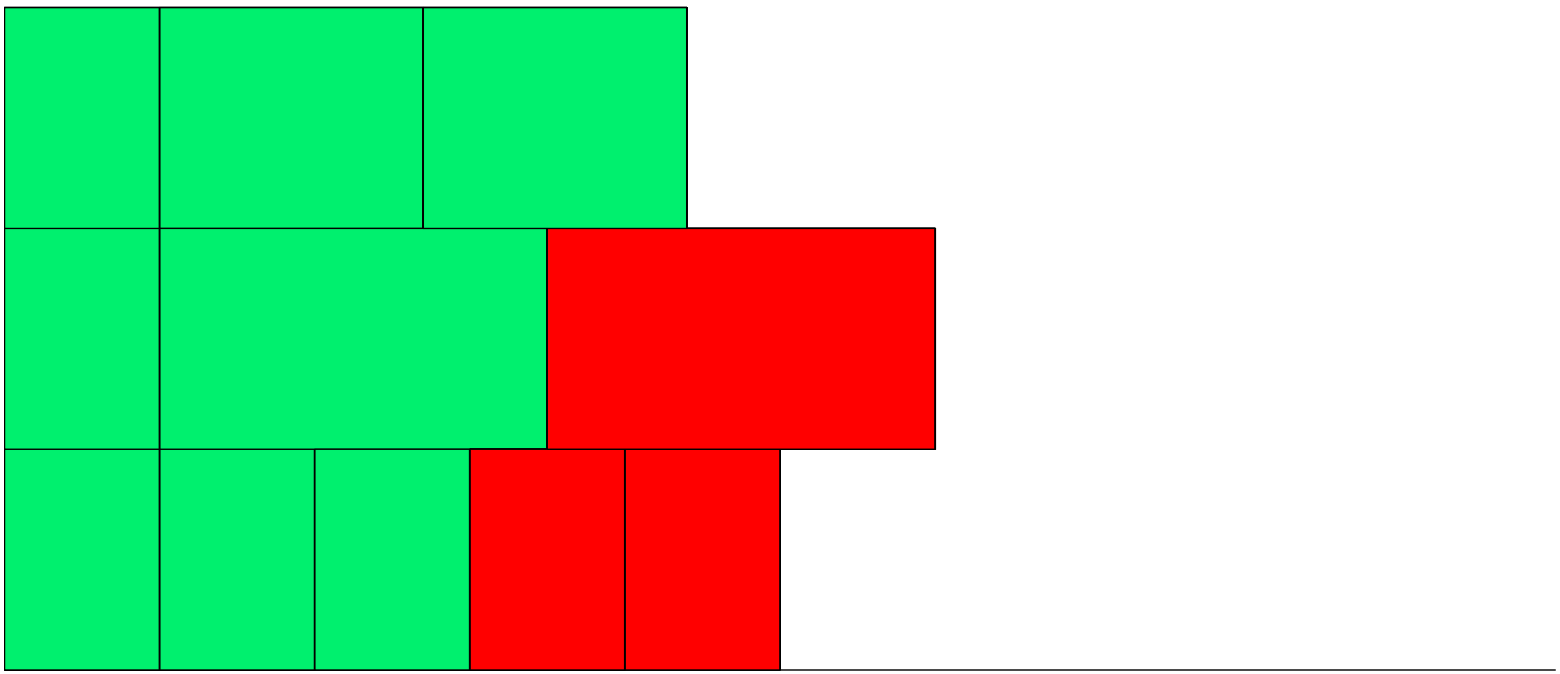
Action: Redoing steps



t = 0.06
n = 3.25

Multi-Adaptive Time-Stepping

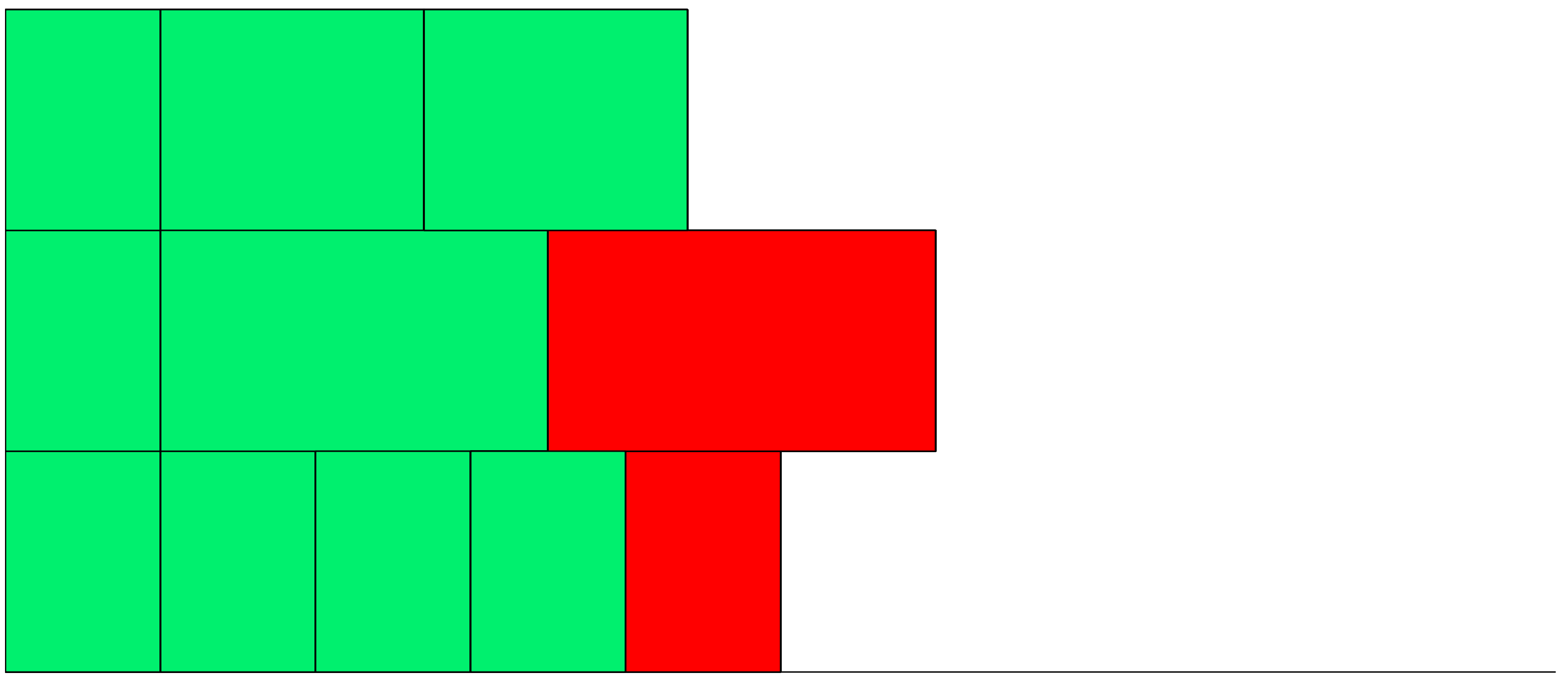
Action: Clearing steps



t = 0.044
n = 3.5

Multi-Adaptive Time-Stepping

Action: Clearing steps



t = 0.04
n = 3.5

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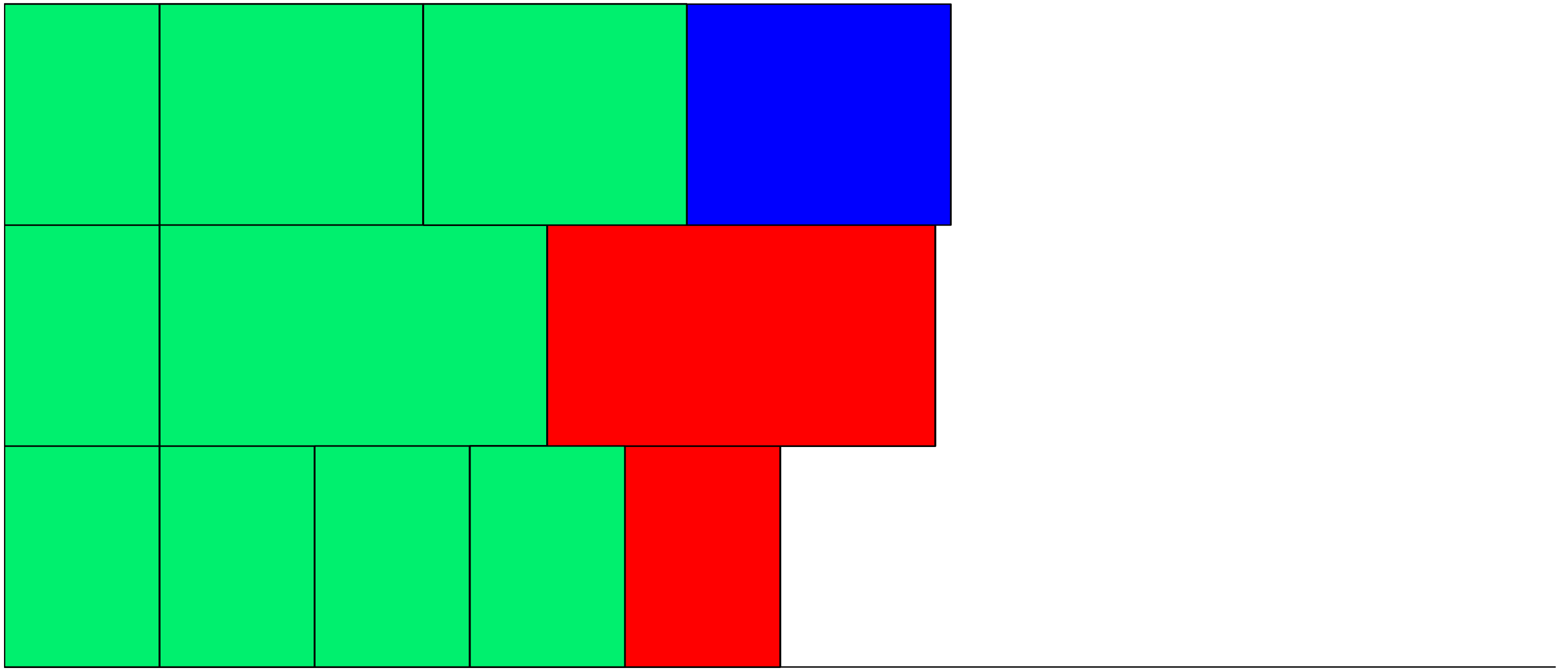
Multi-Adaptive Time-Stepping

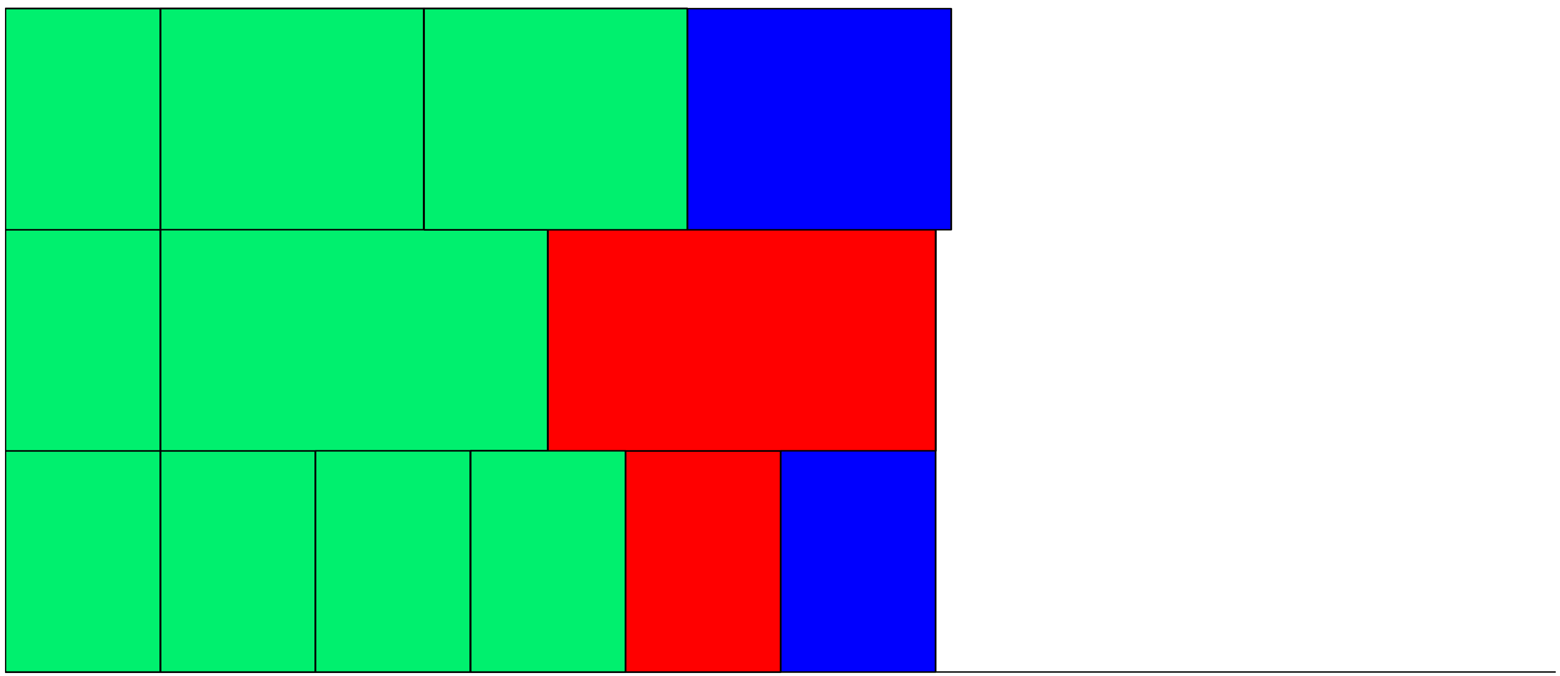
Action: Covering



t = 0.061
n = 3.5

Multi-Adaptive Time-Stepping

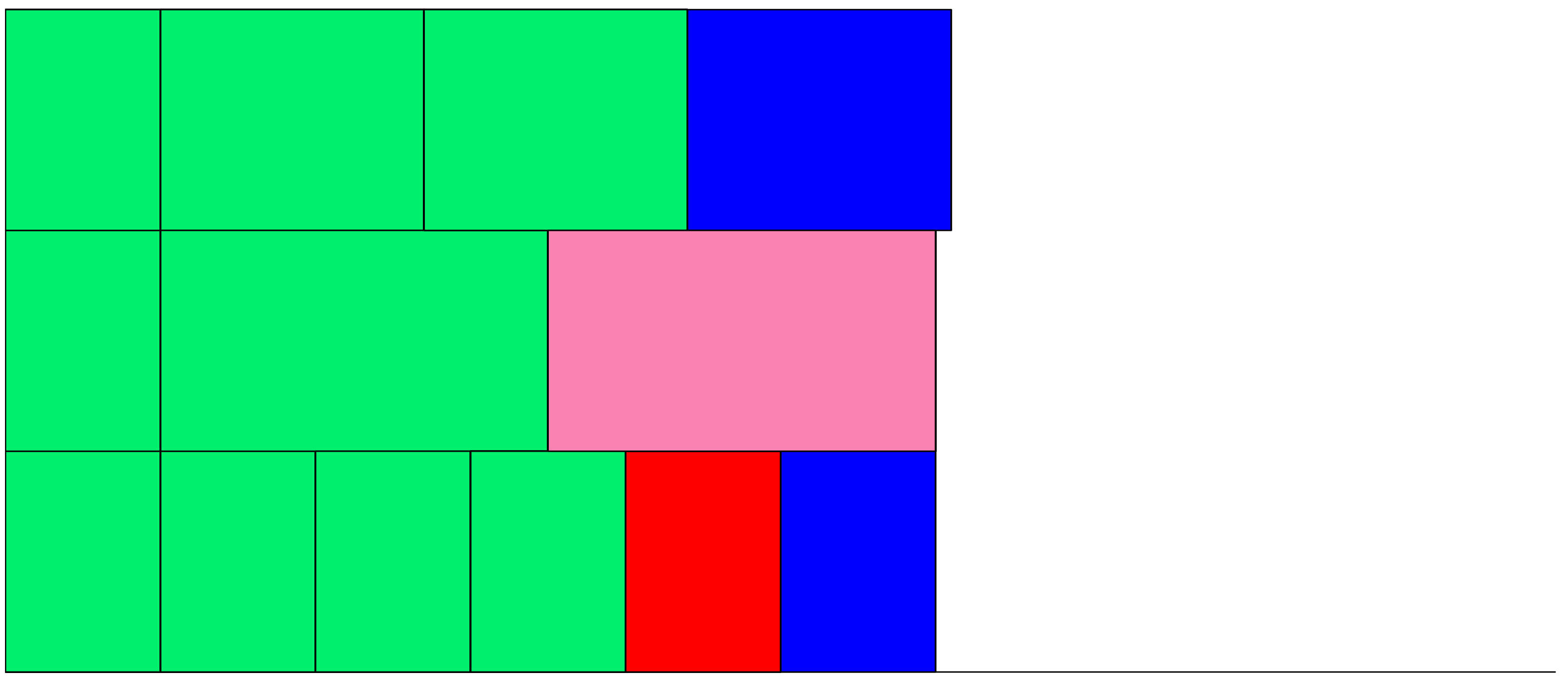
Action: Covering



t = 0.06
n = 3.5

Multi-Adaptive Time-Stepping

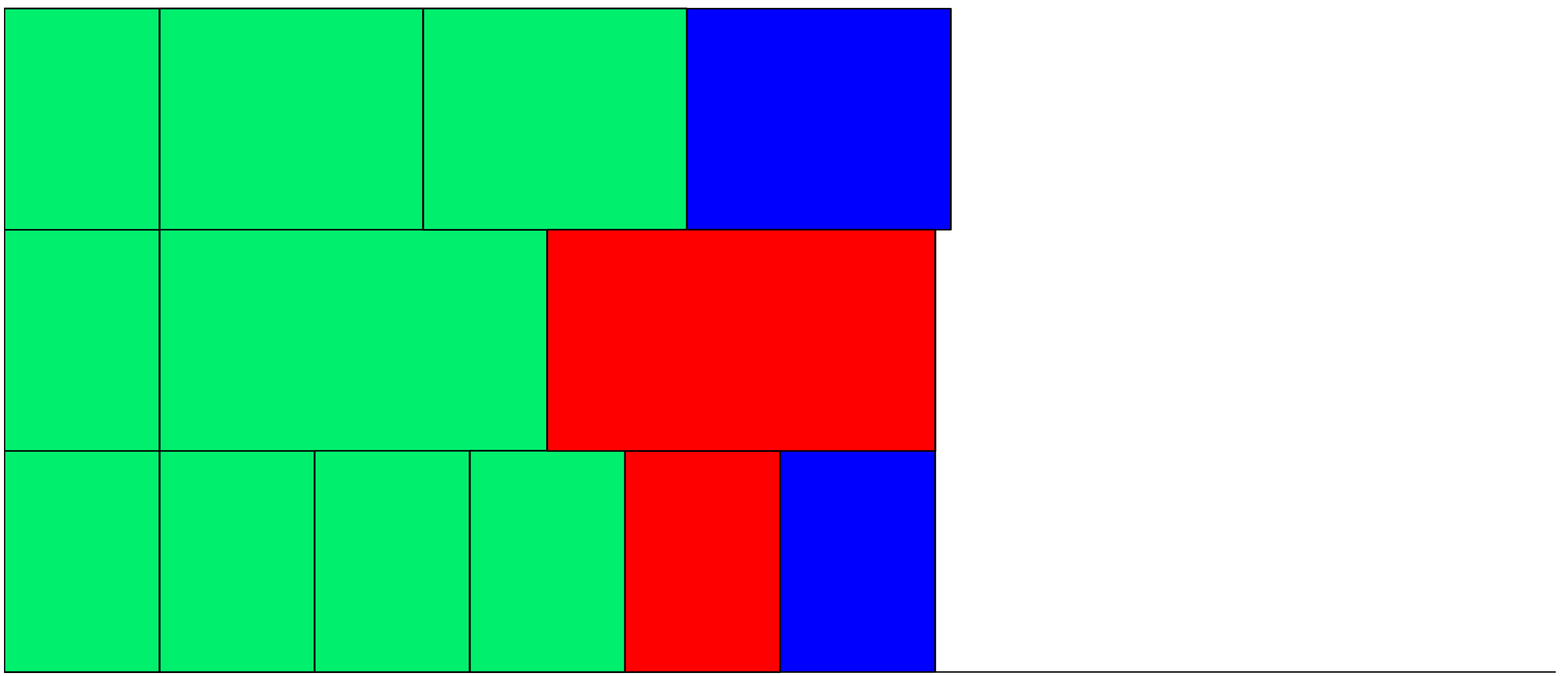
Action: Redoing steps



t = 0.06
n = 3.5

Multi-Adaptive Time-Stepping

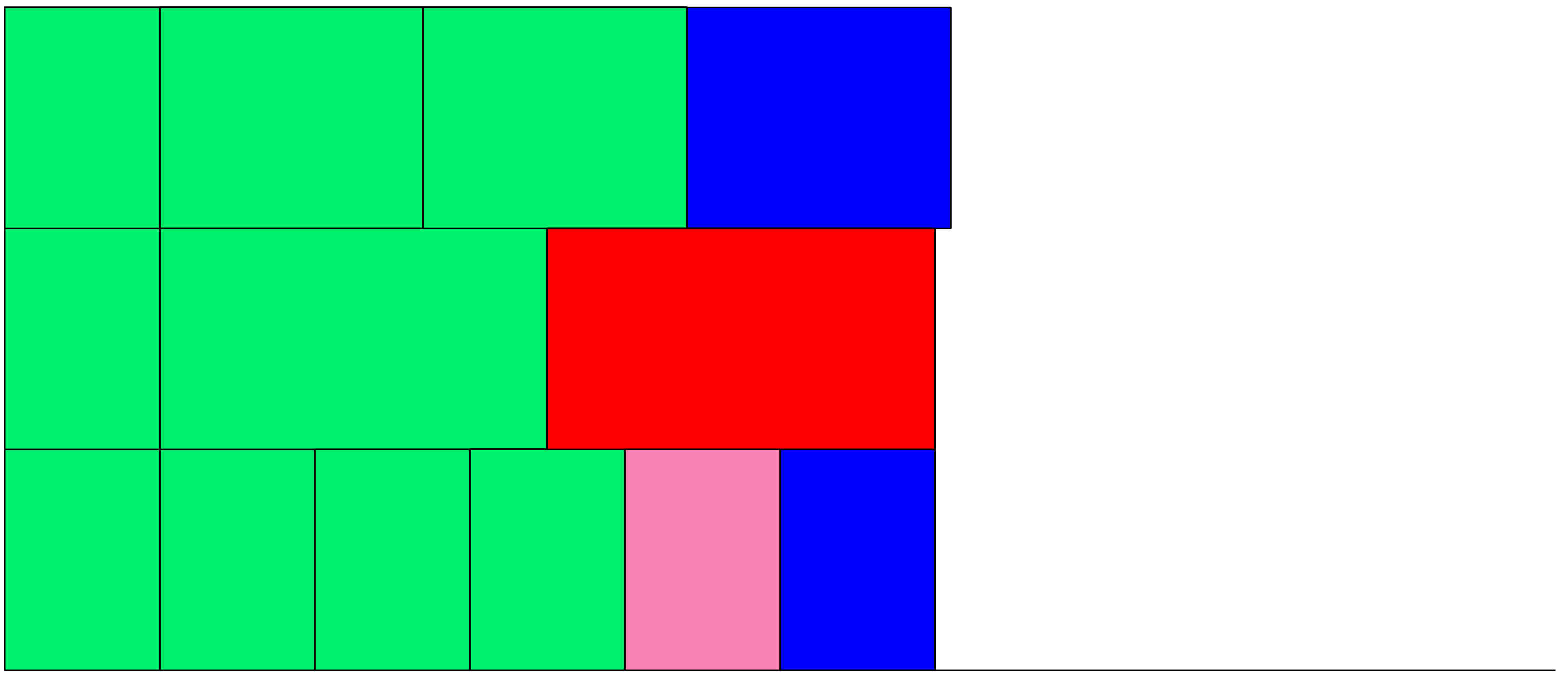
Action: Redoing steps



t = 0.06
n = 1

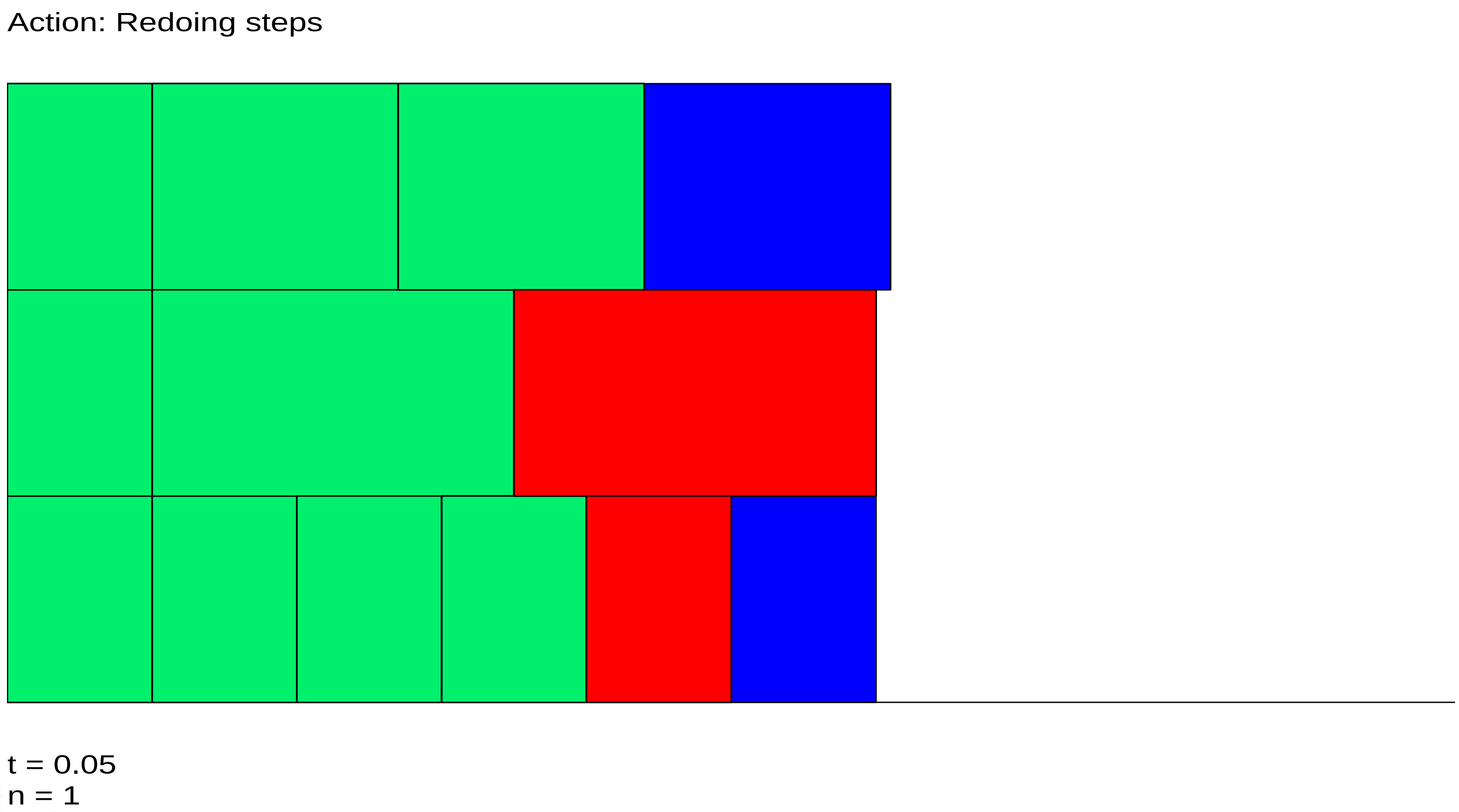
Multi-Adaptive Time-Stepping

Action: Redoing steps



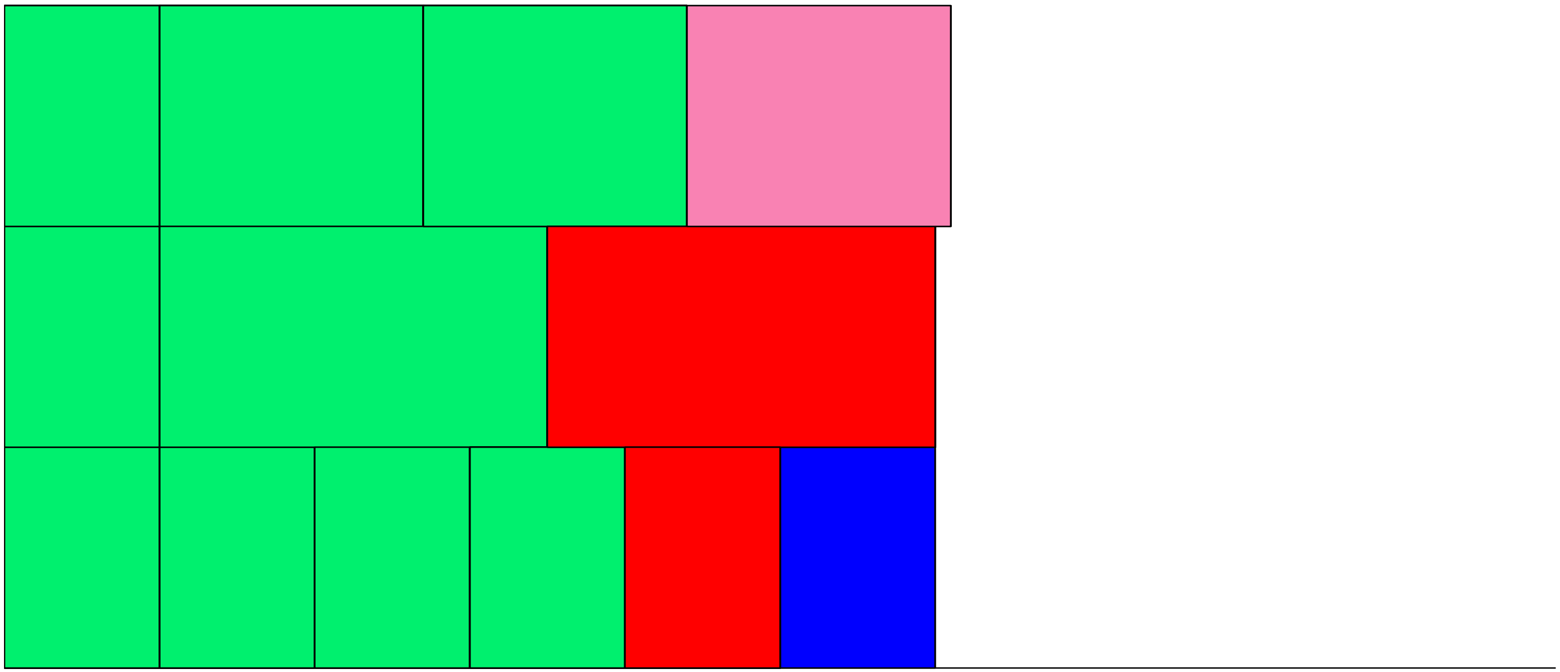
t = 0.06
n = 1

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.05
n = 1

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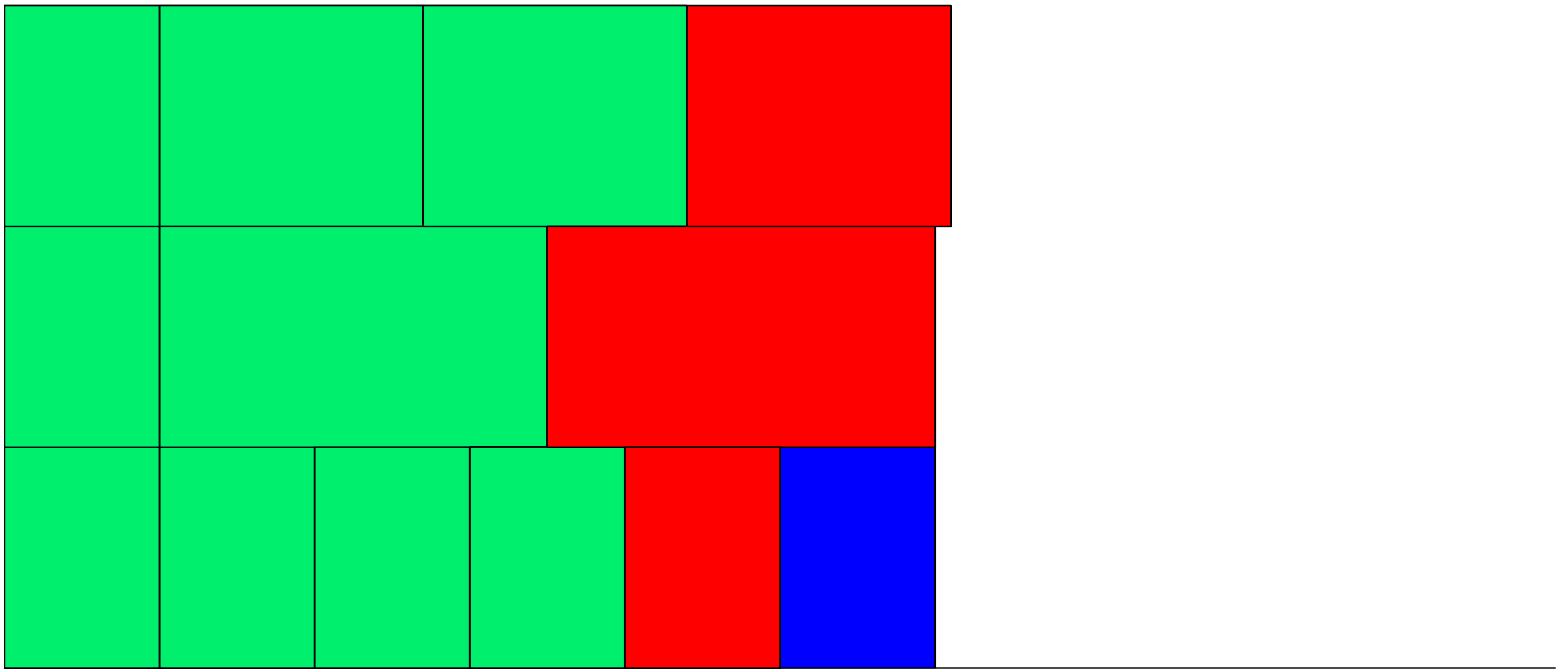
CHALMERS FINITE ELEMENT CENTER

Multi-Adaptive Time-Stepping

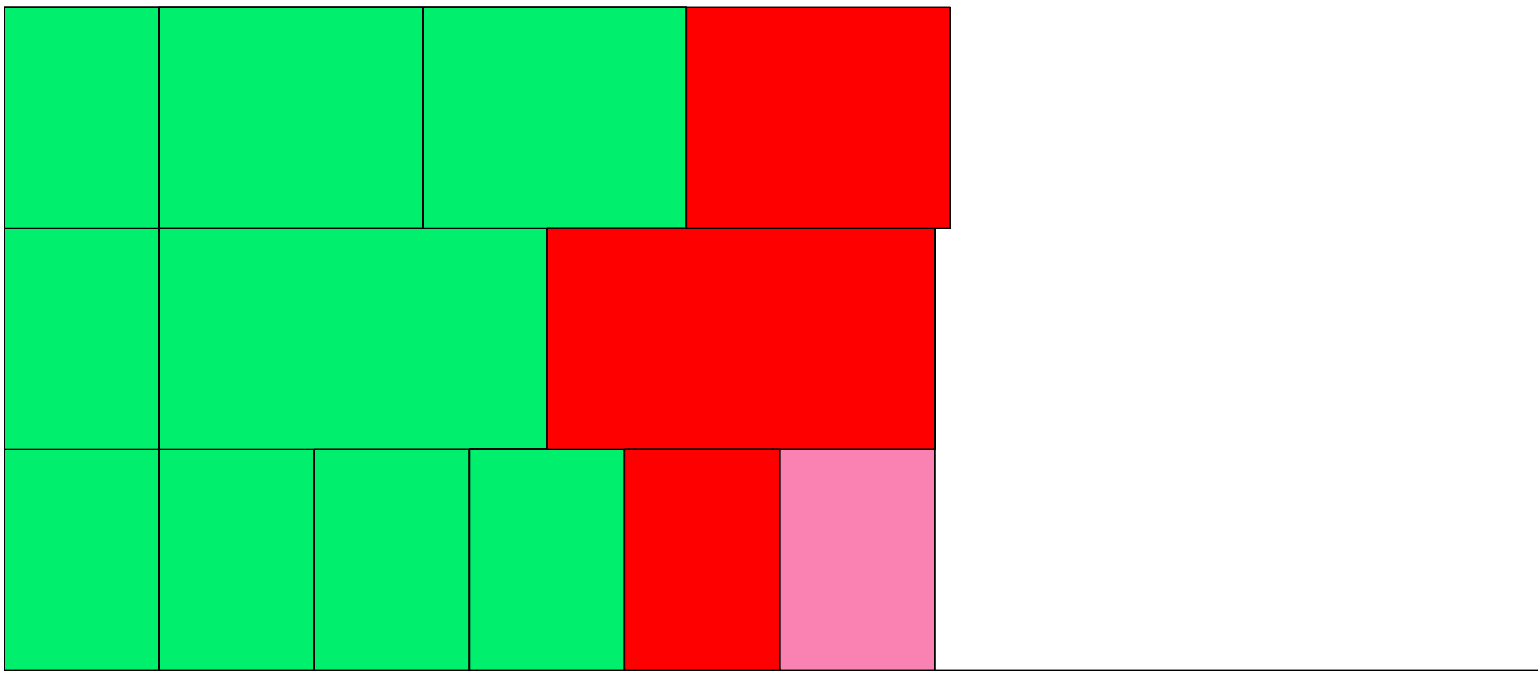
Action: Redoing steps



t = 0.061
n = 1

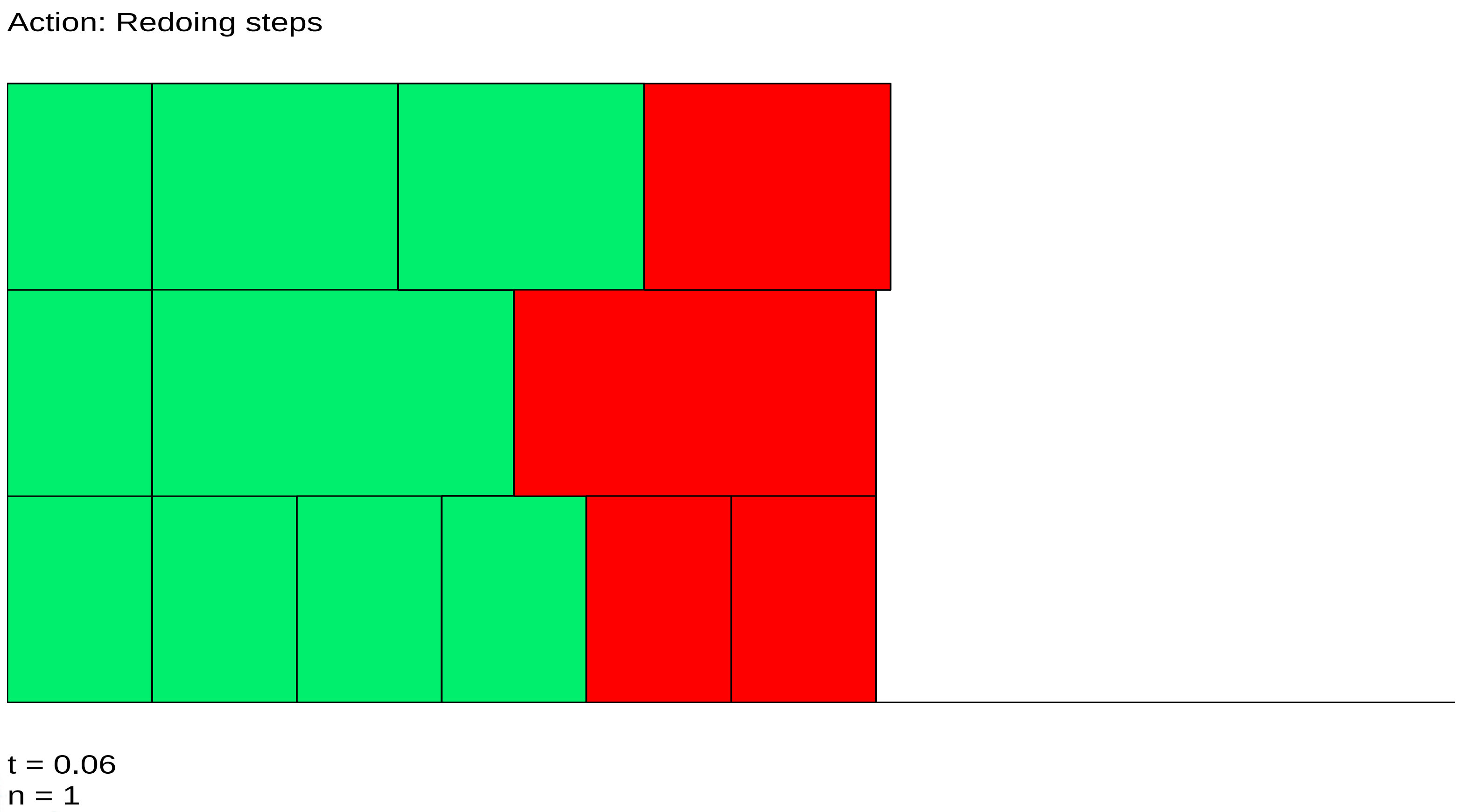
Multi-Adaptive Time-Stepping

Action: Redoing steps

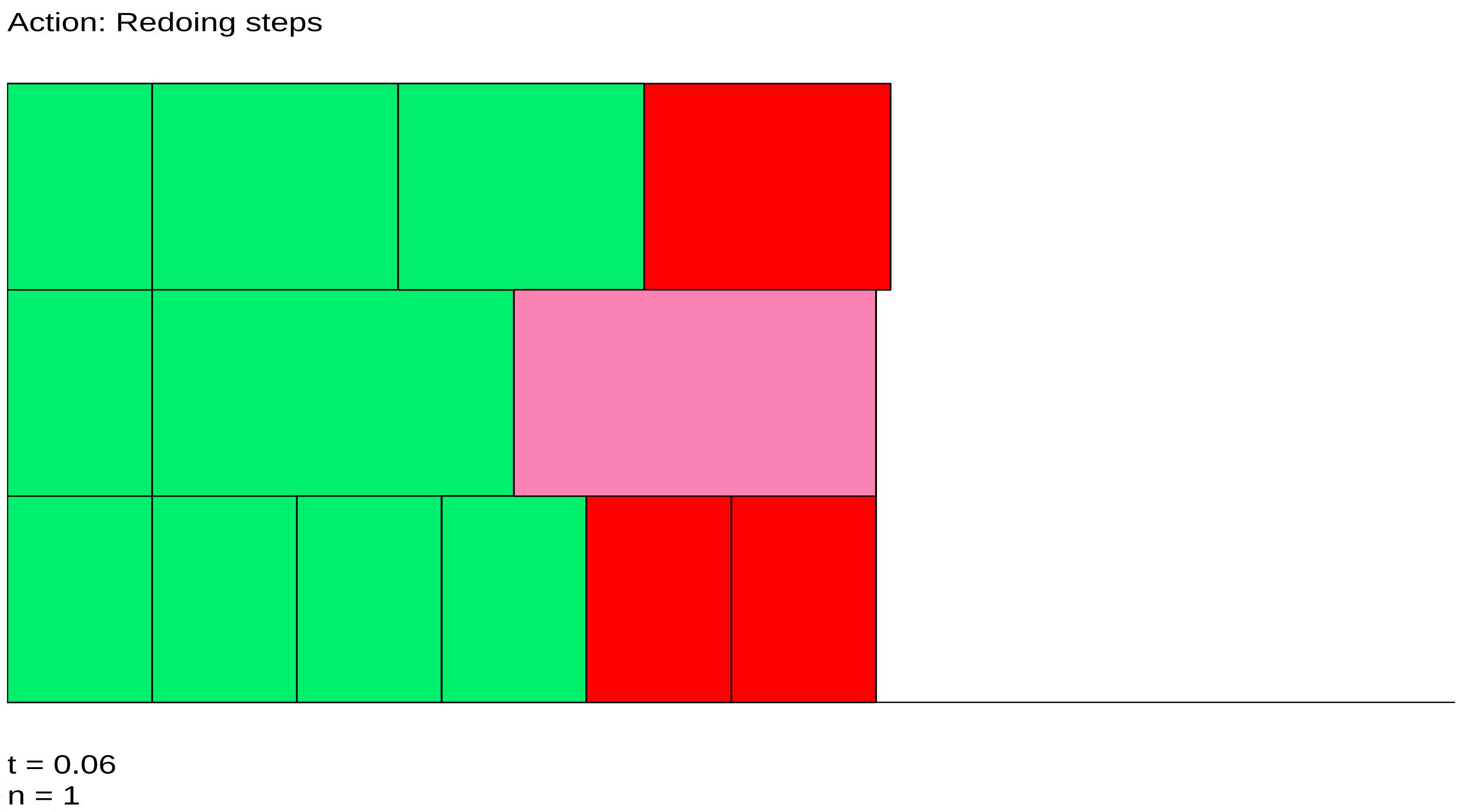


t = 0.061
n = 1

Multi-Adaptive Time-Stepping

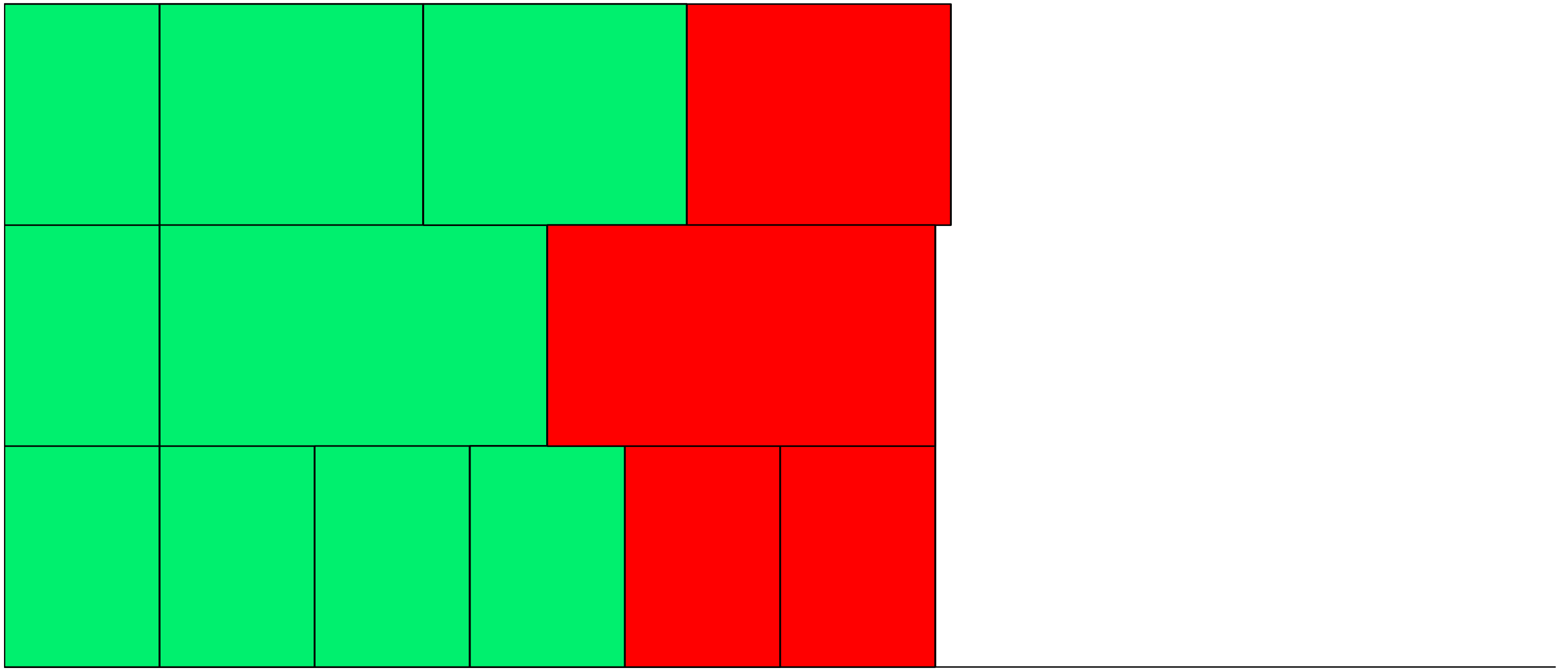


Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

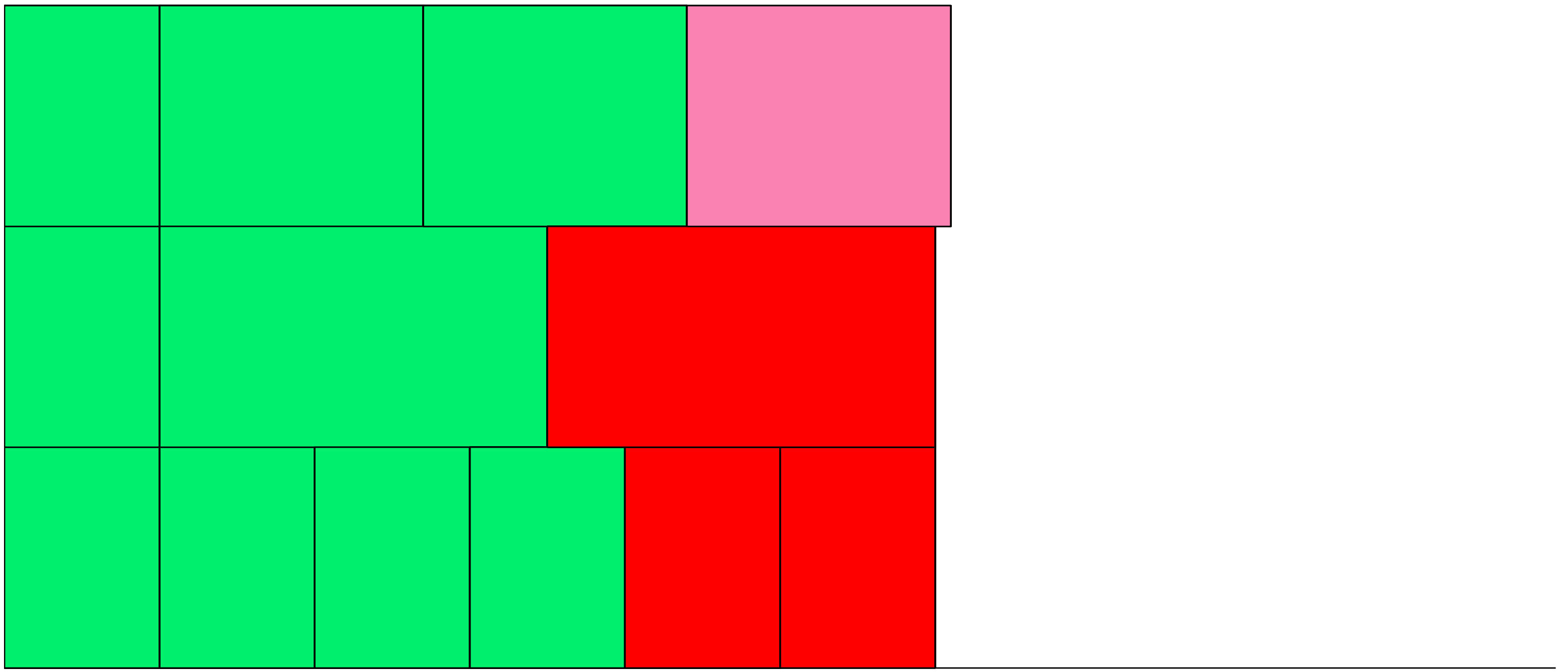
Action: Redoing steps



t = 0.06
n = 2

Multi-Adaptive Time-Stepping

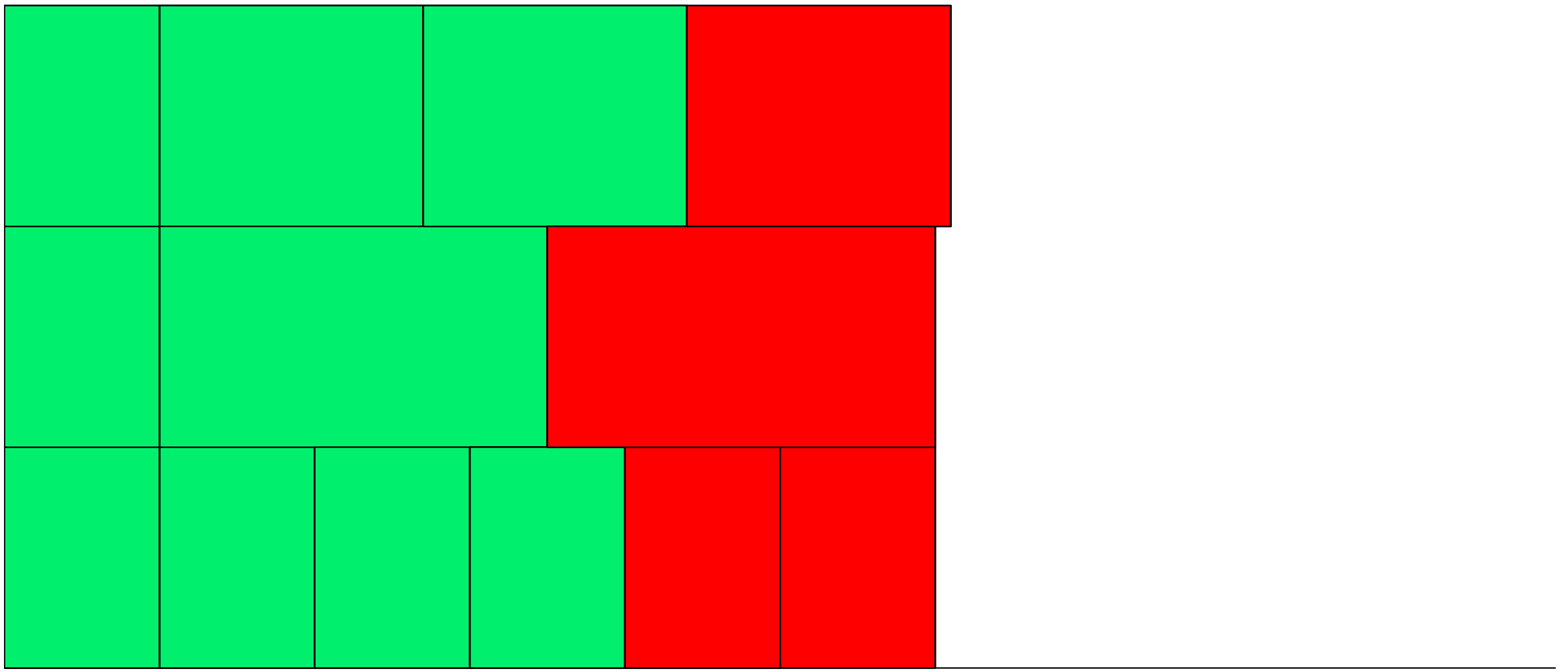
Action: Redoing steps



t = 0.06
n = 2

Multi-Adaptive Time-Stepping

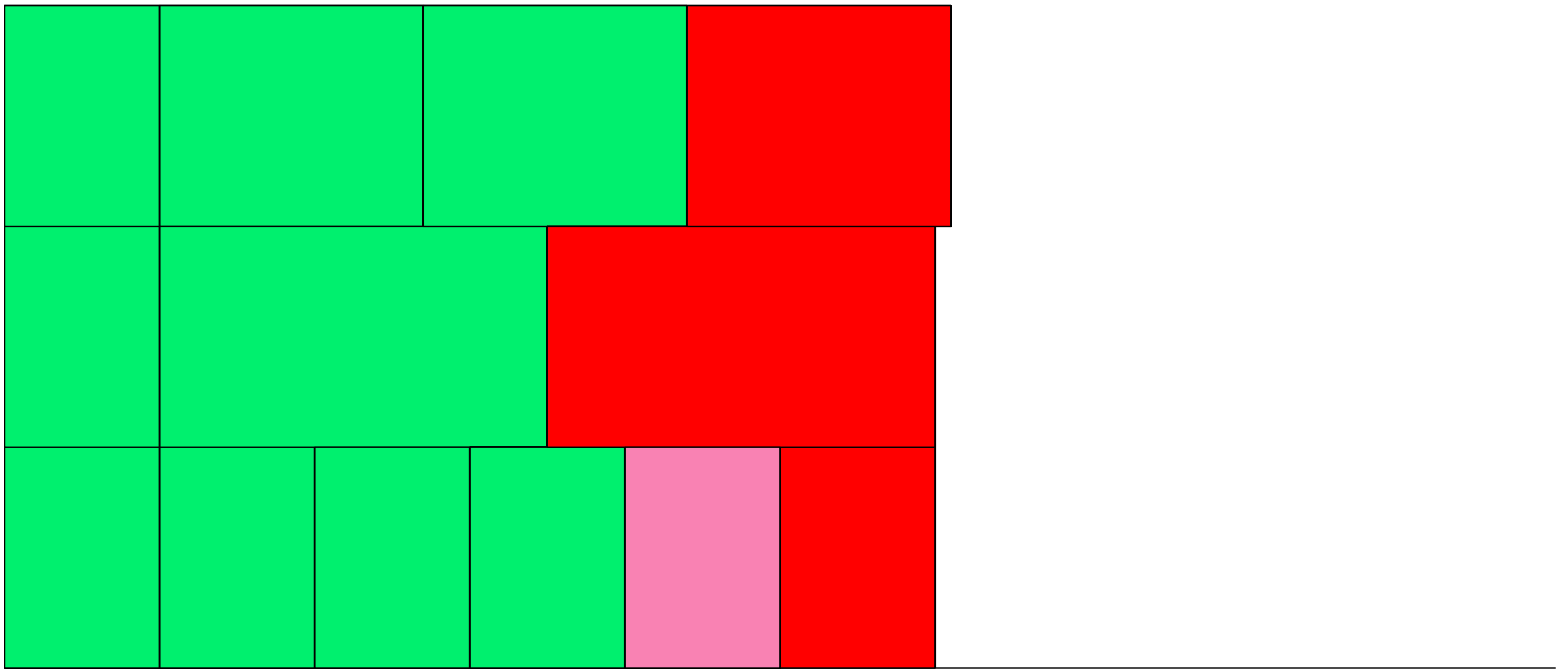
Action: Redoing steps



t = 0.061
n = 2

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.061
n = 2

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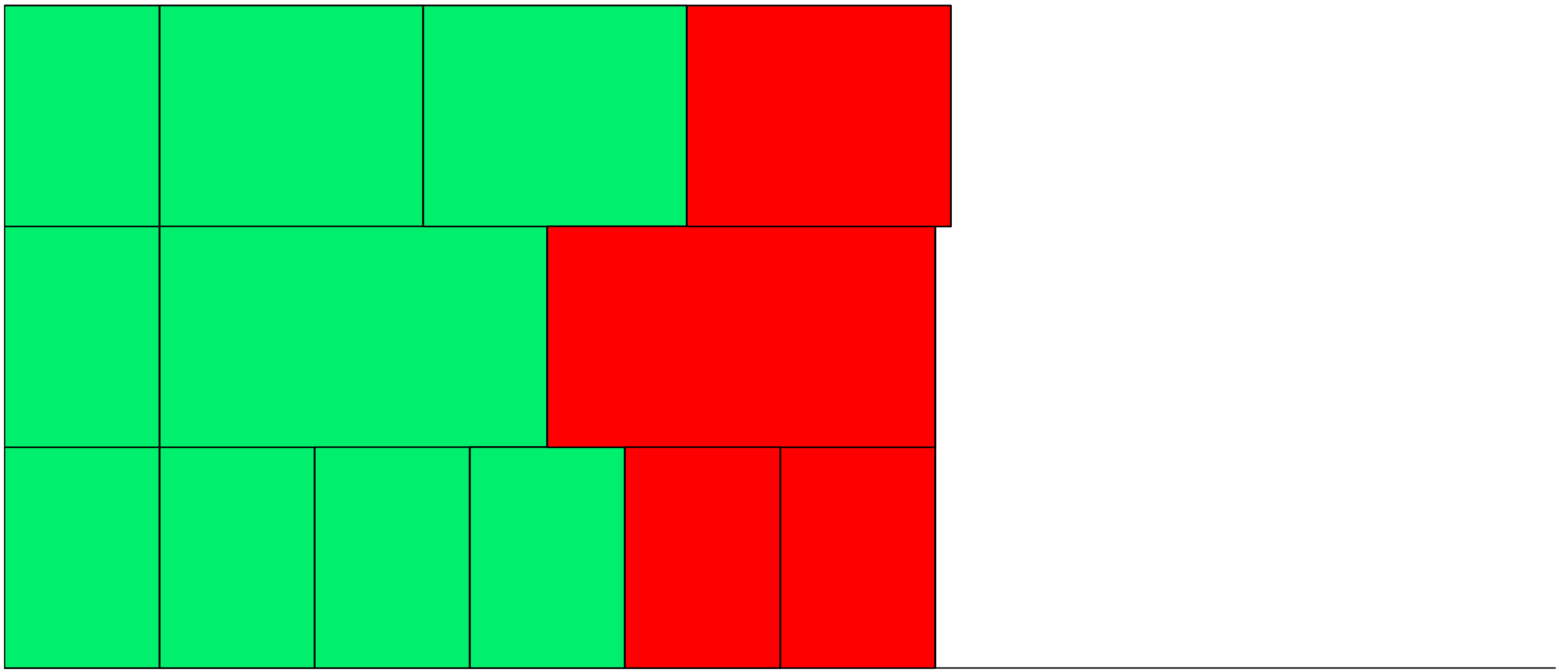
Multi-Adaptive Time-Stepping

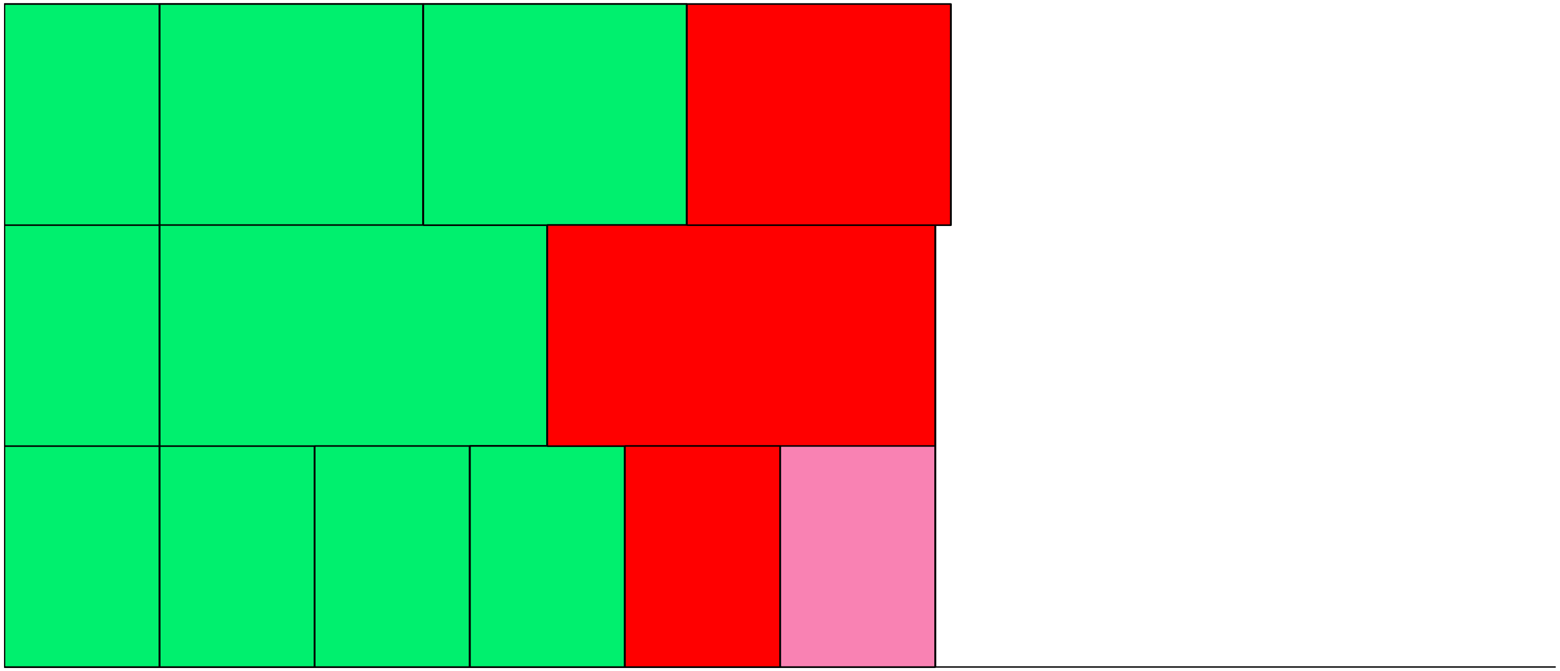
Action: Redoing steps



t = 0.05
n = 2.5

Multi-Adaptive Time-Stepping

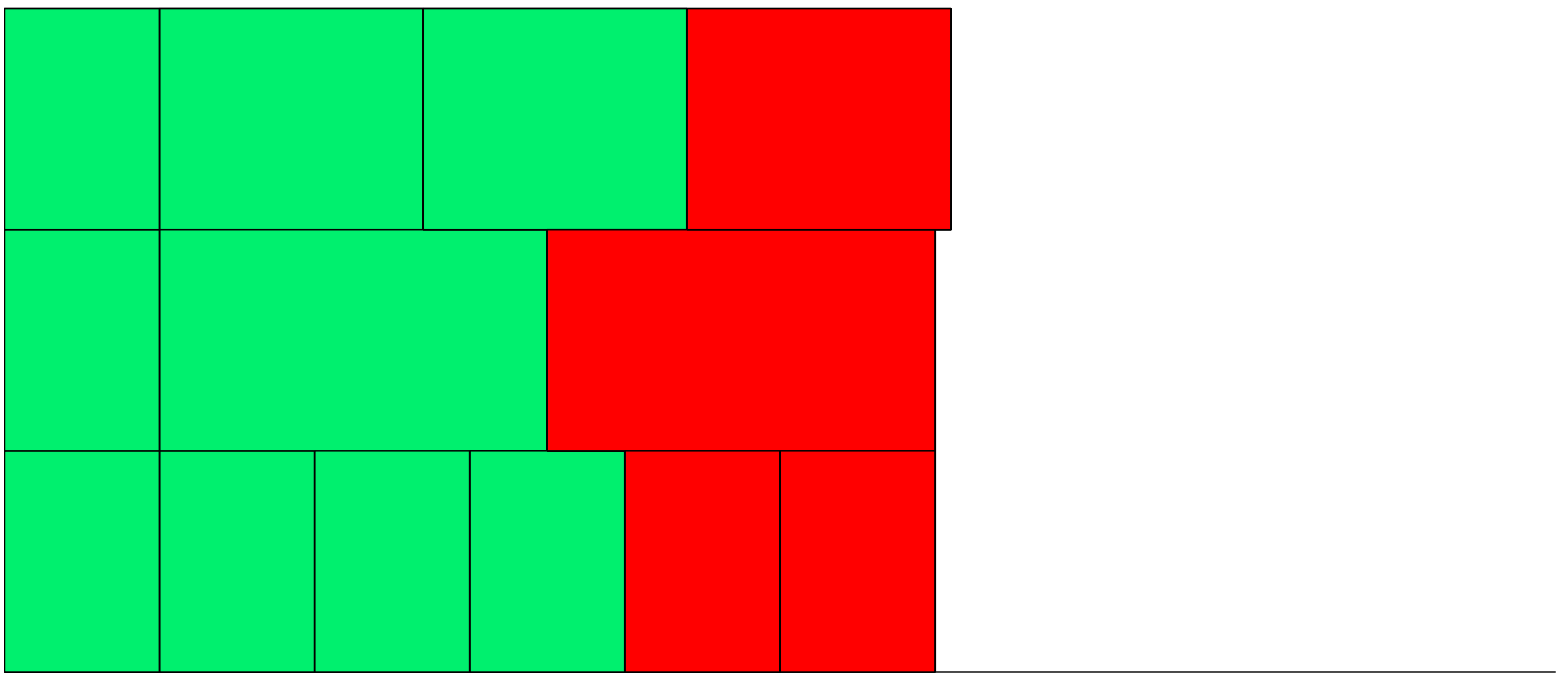
Action: Redoing steps



t = 0.05
n = 2.5

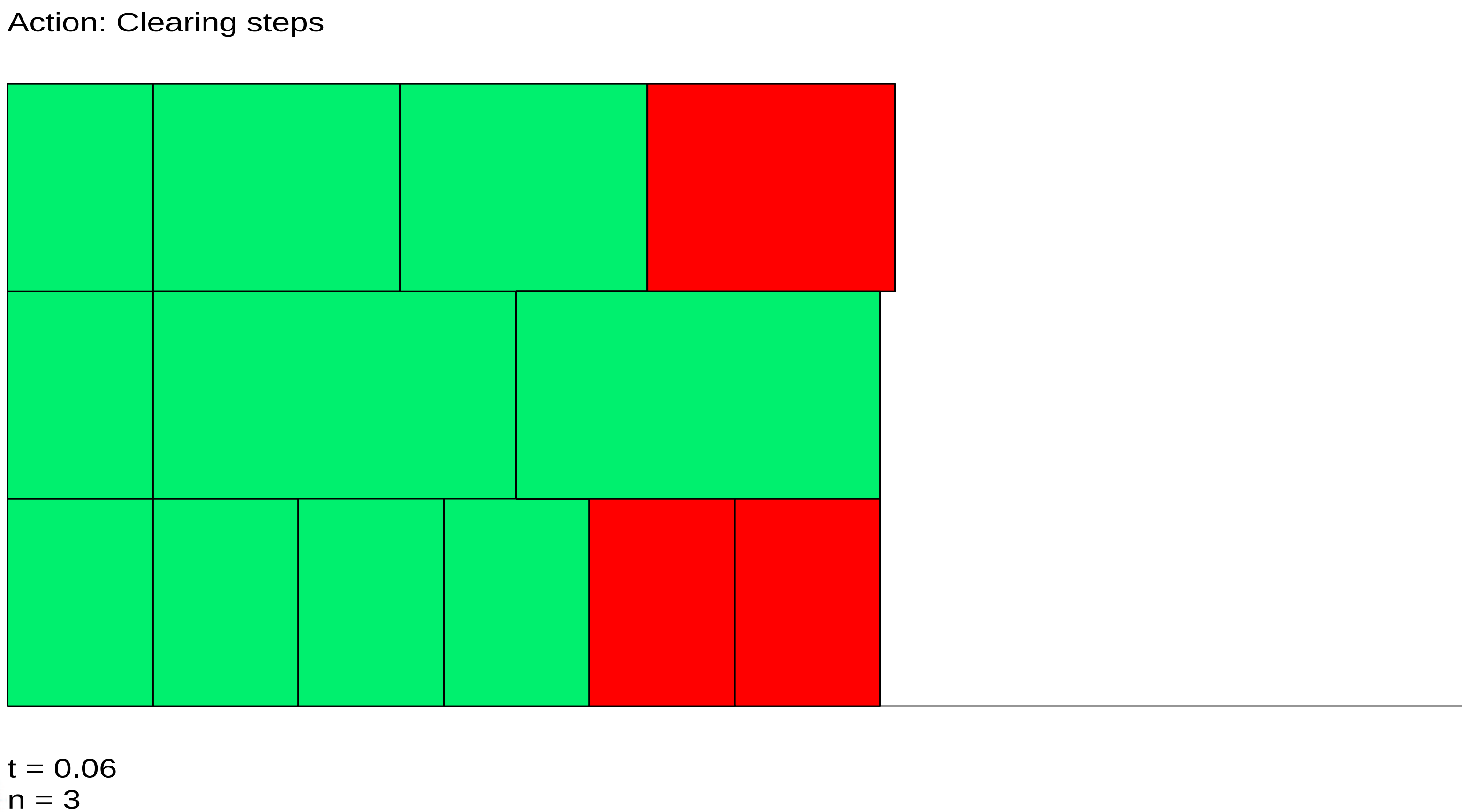
Multi-Adaptive Time-Stepping

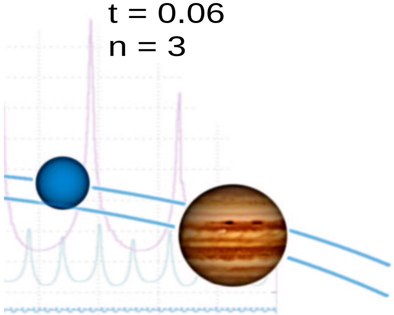
Action: Redoing steps



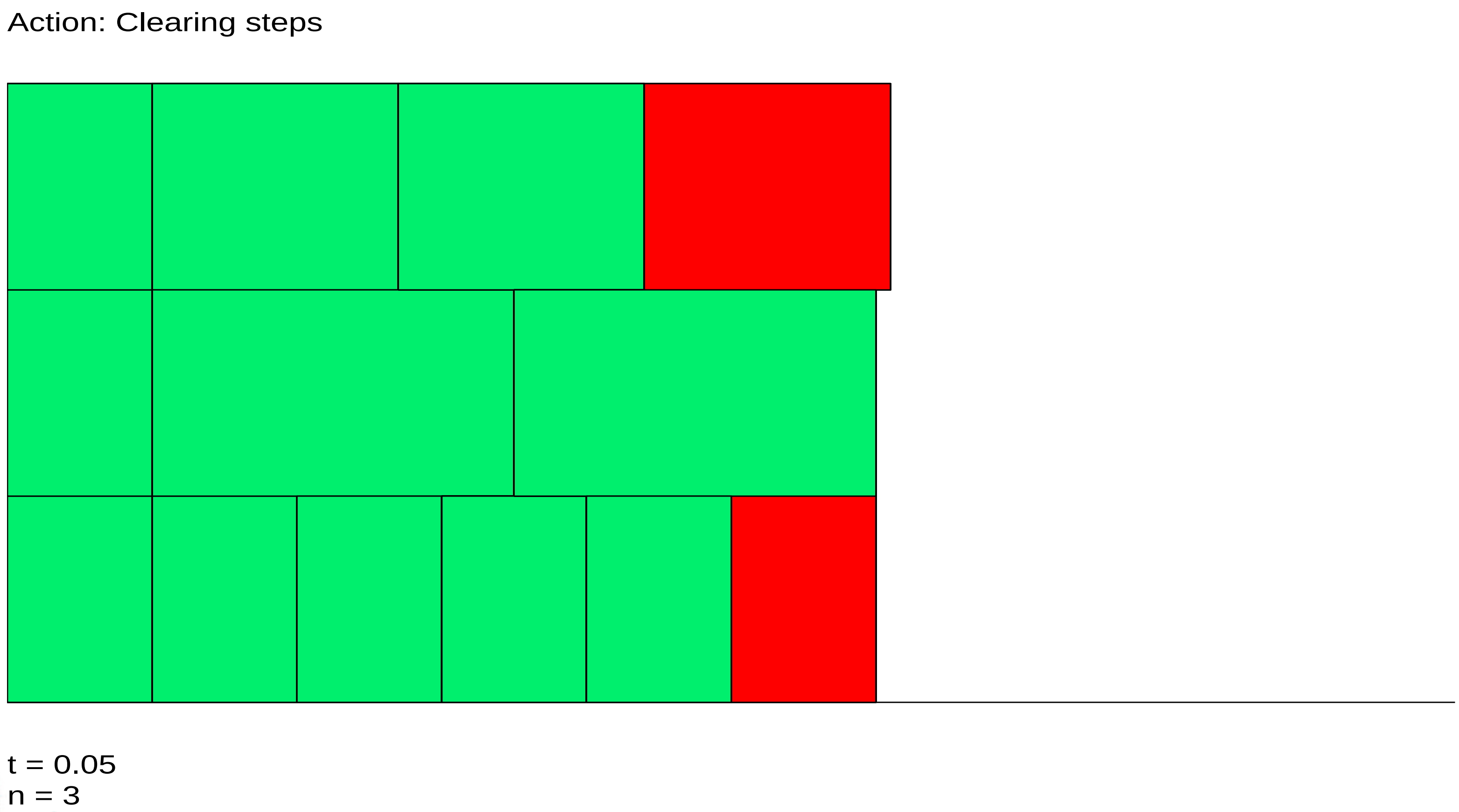
t = 0.06
n = 2.5

Multi-Adaptive Time-Stepping

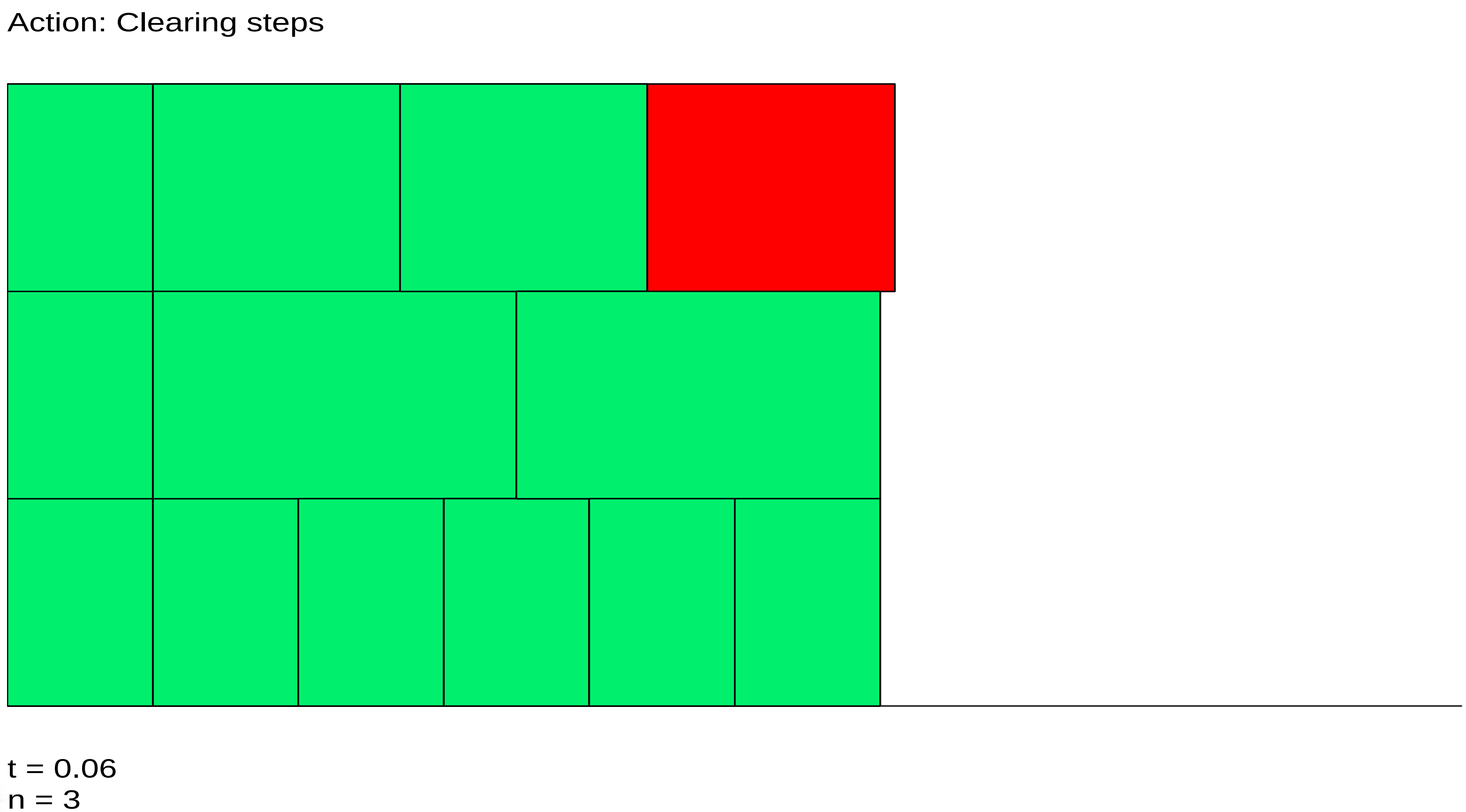




Multi-Adaptive Time-Stepping

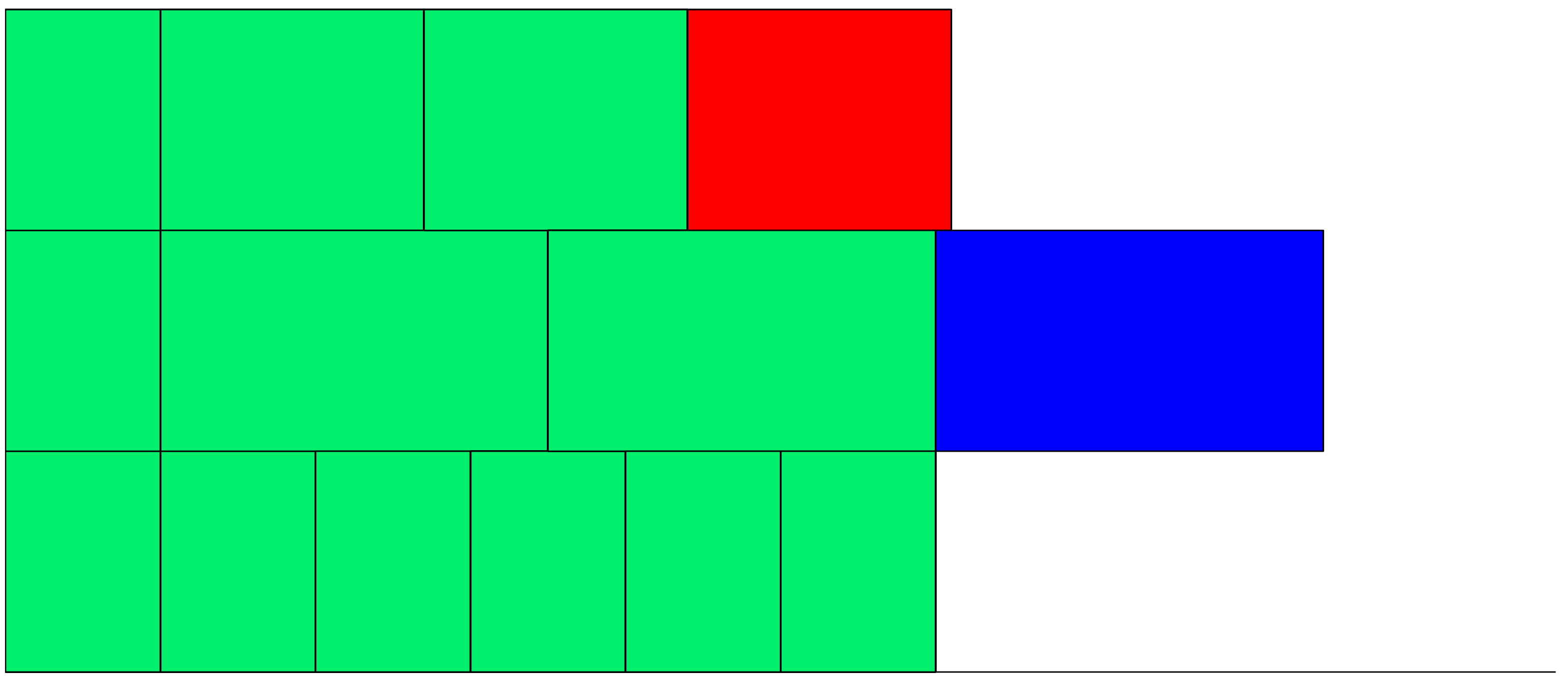


Multi-Adaptive Time-Stepping



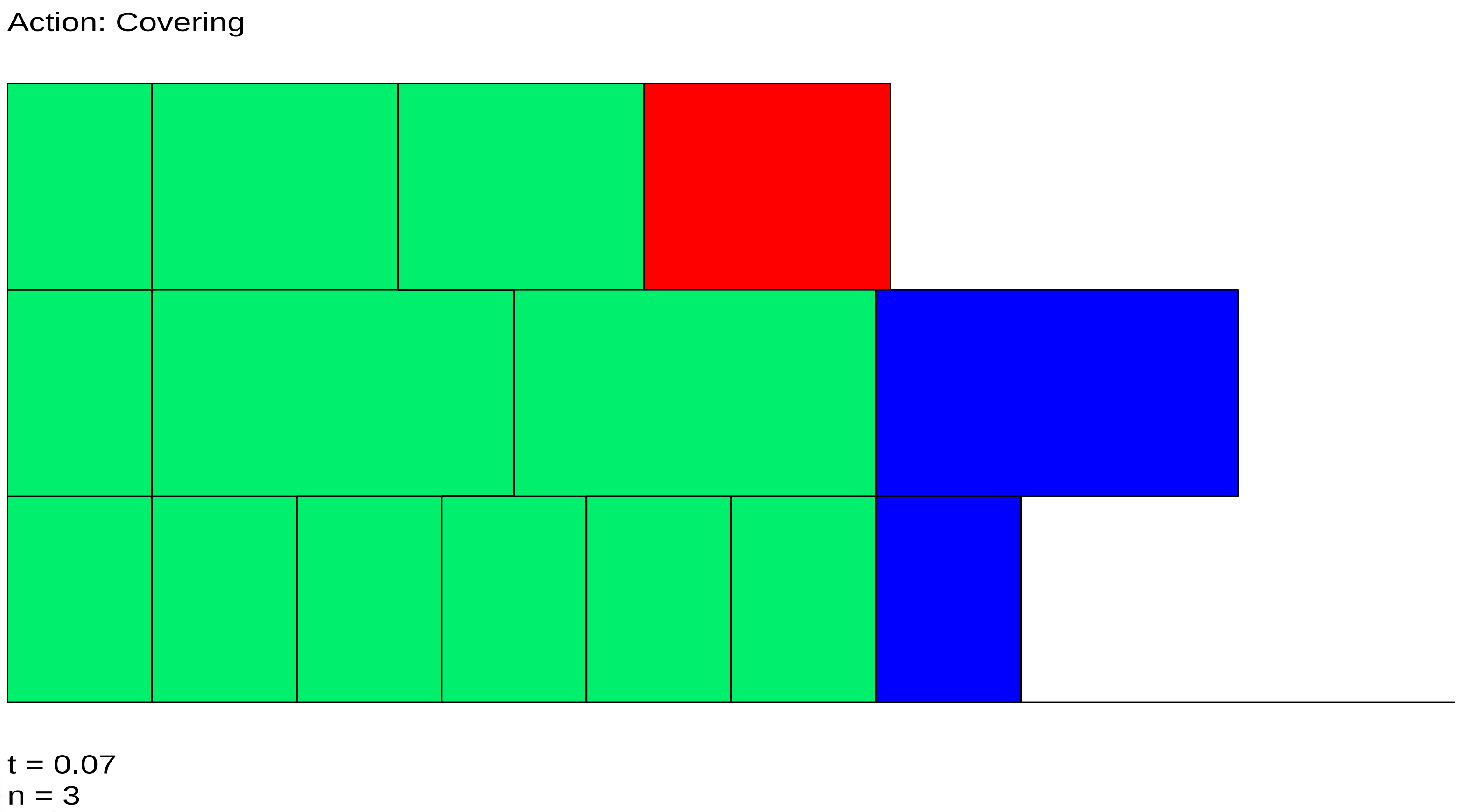
Multi-Adaptive Time-Stepping

Action: Covering



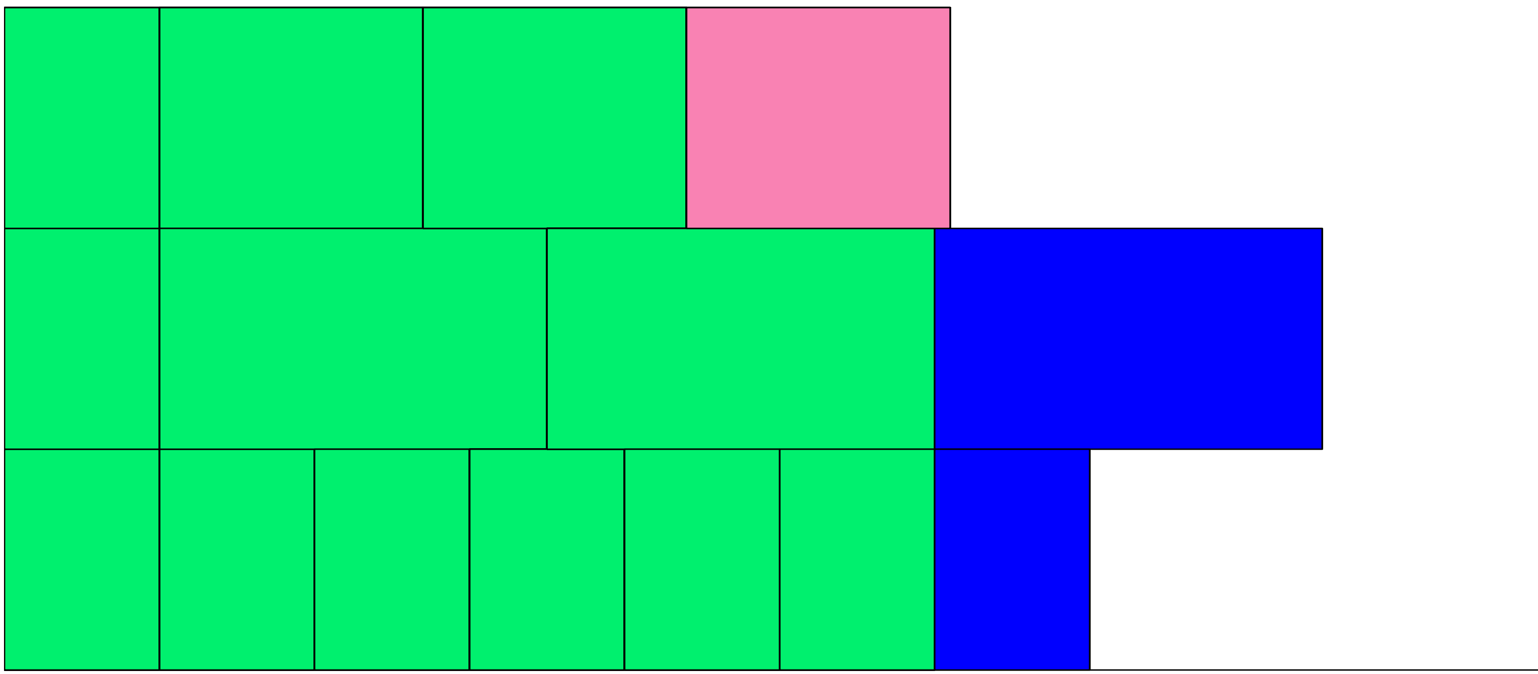
t = 0.085
n = 3

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.07
n = 3

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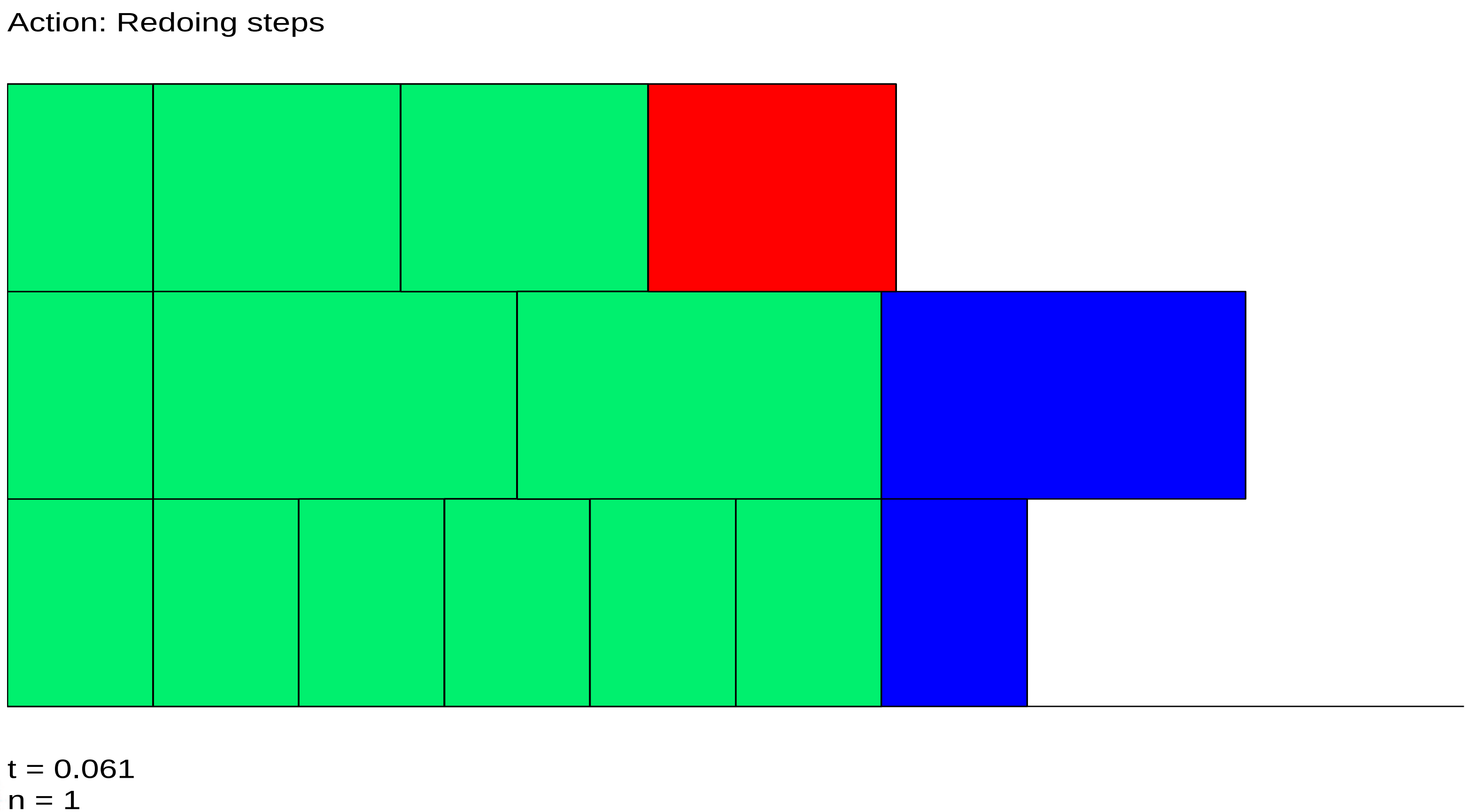
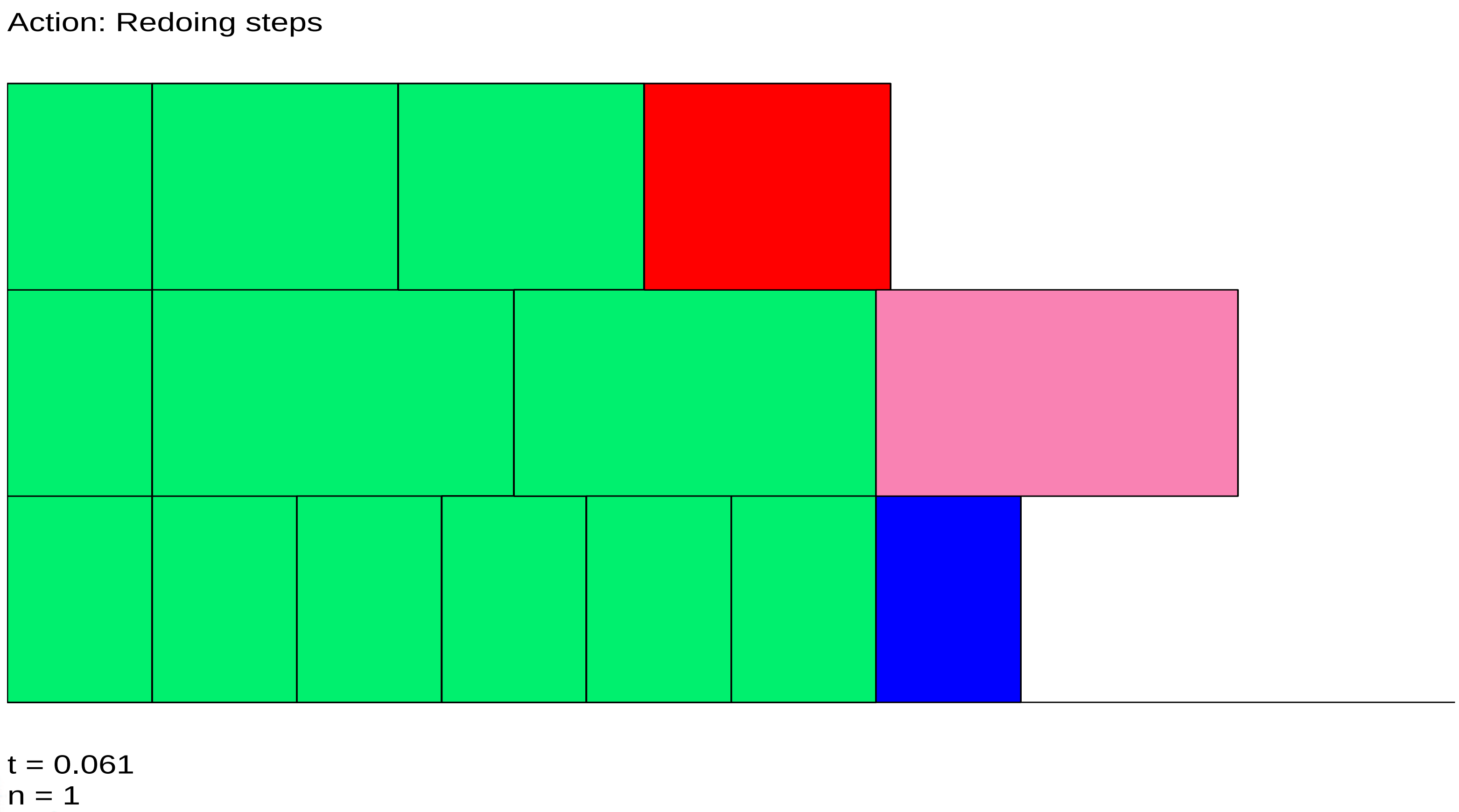
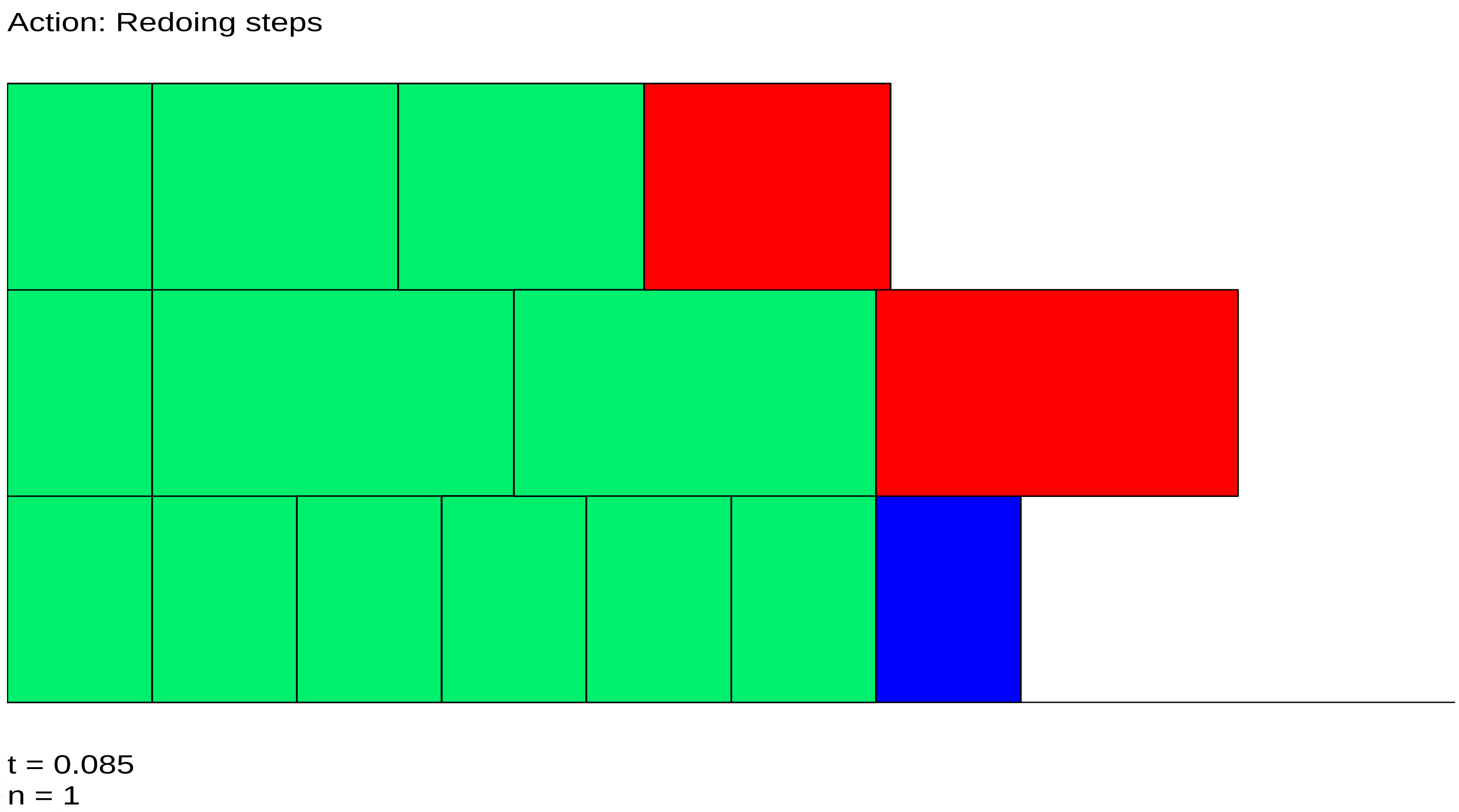
Multi-Adaptive Time-Stepping



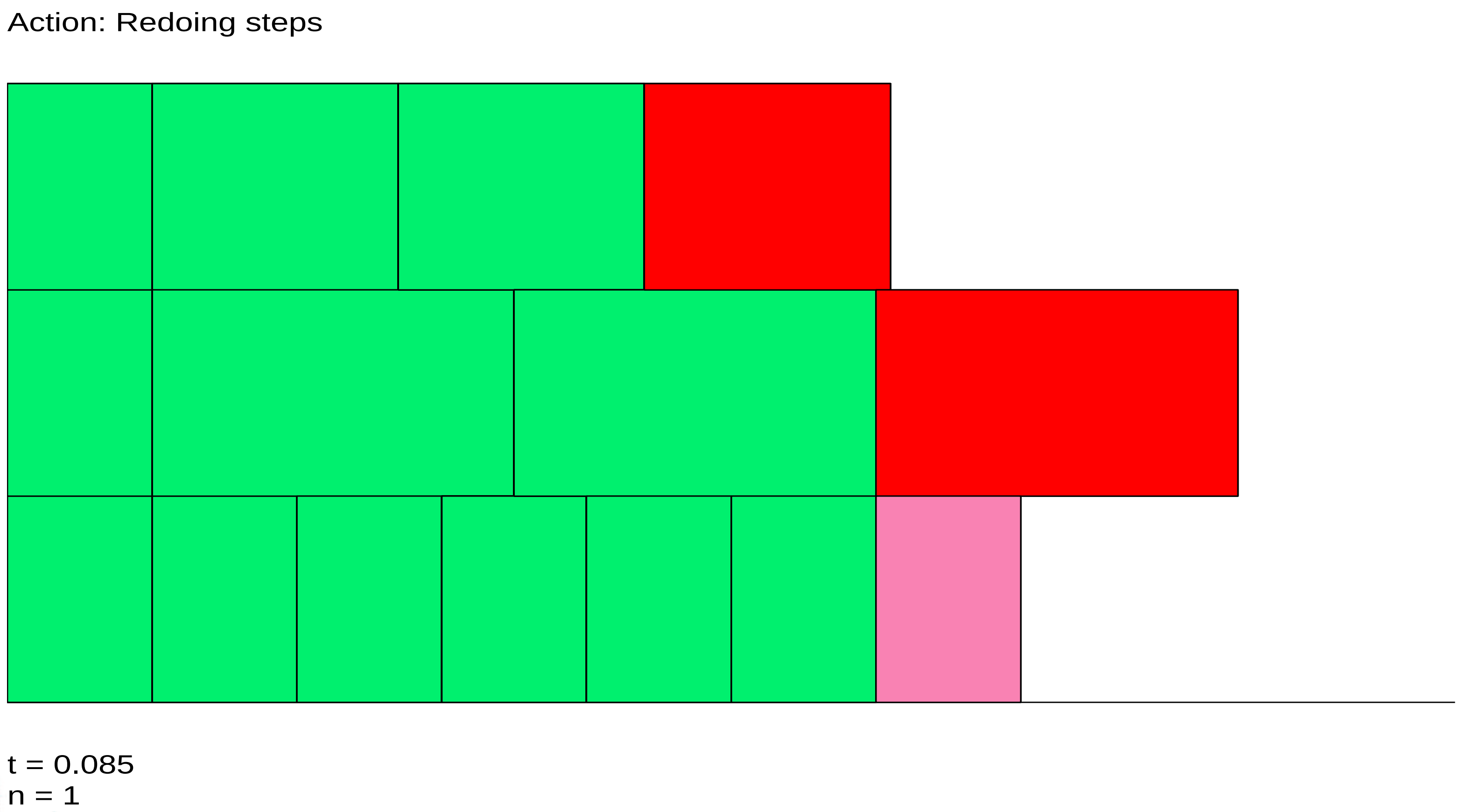
Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

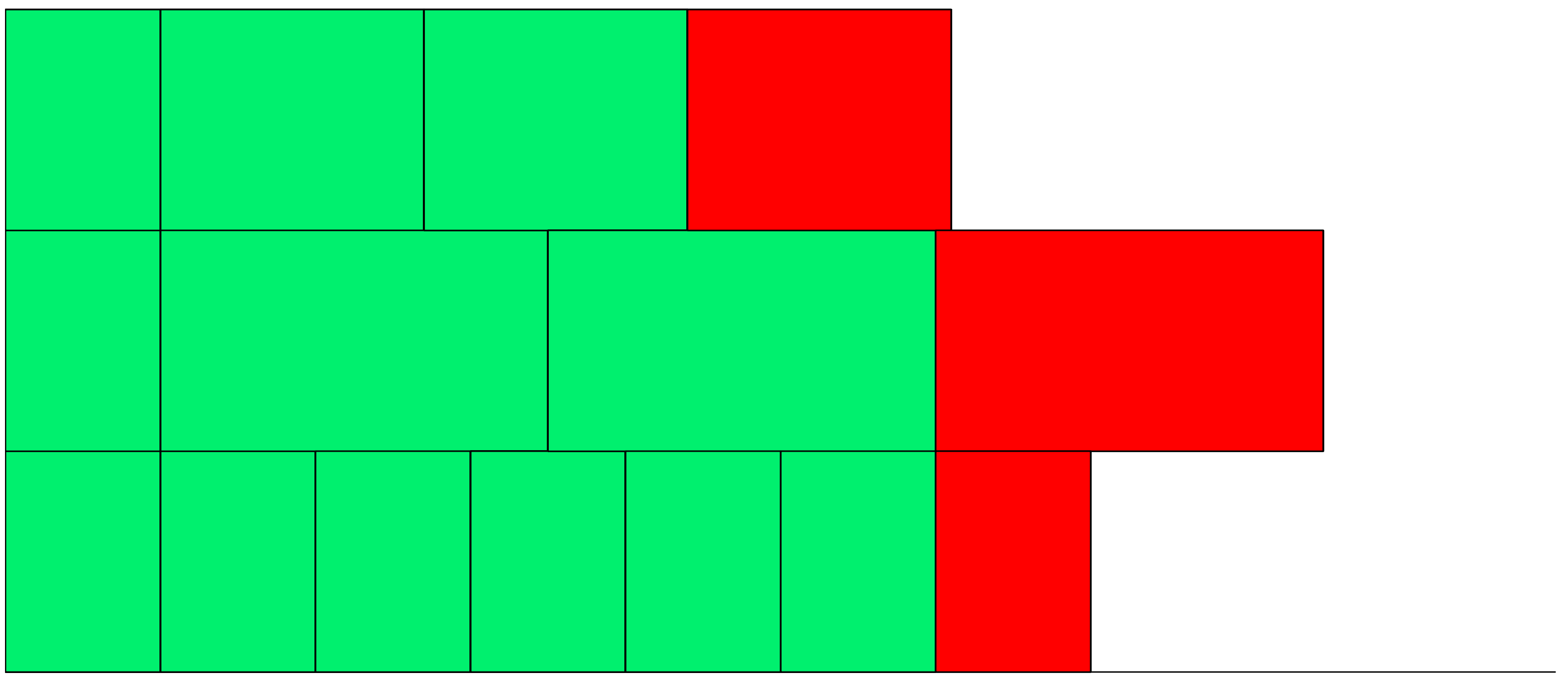


Multi-Adaptive Time-Stepping



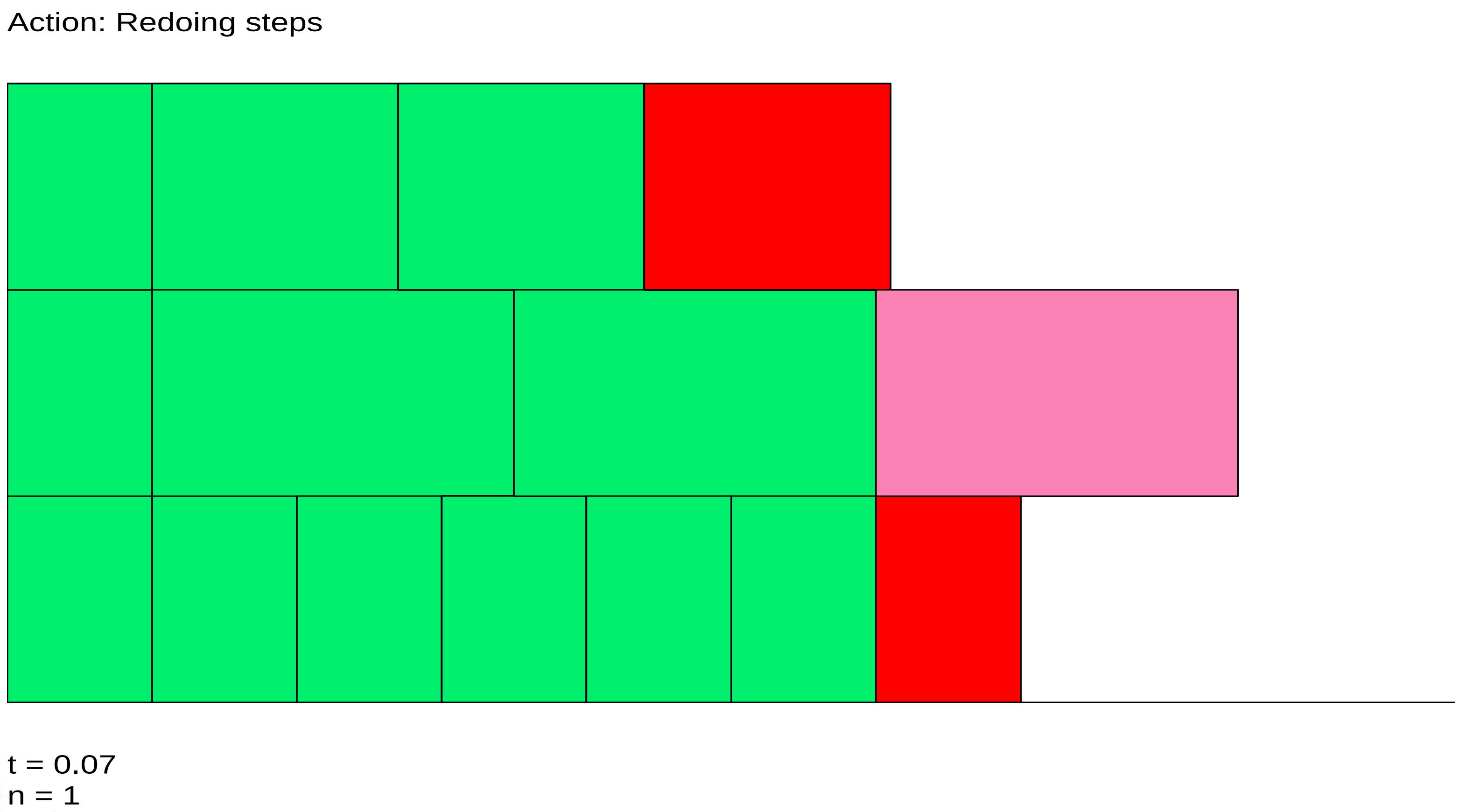
Multi-Adaptive Time-Stepping

Action: Redoing steps



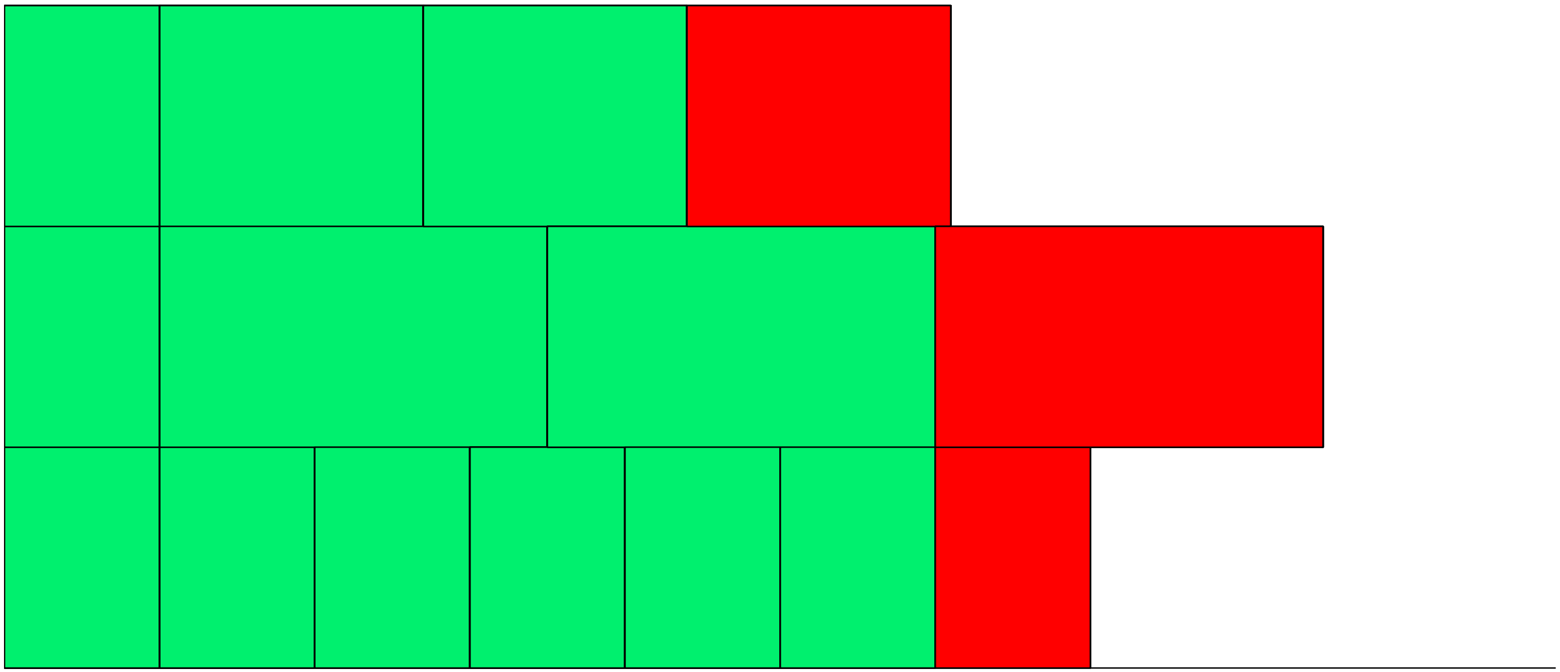
t = 0.07
n = 1

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

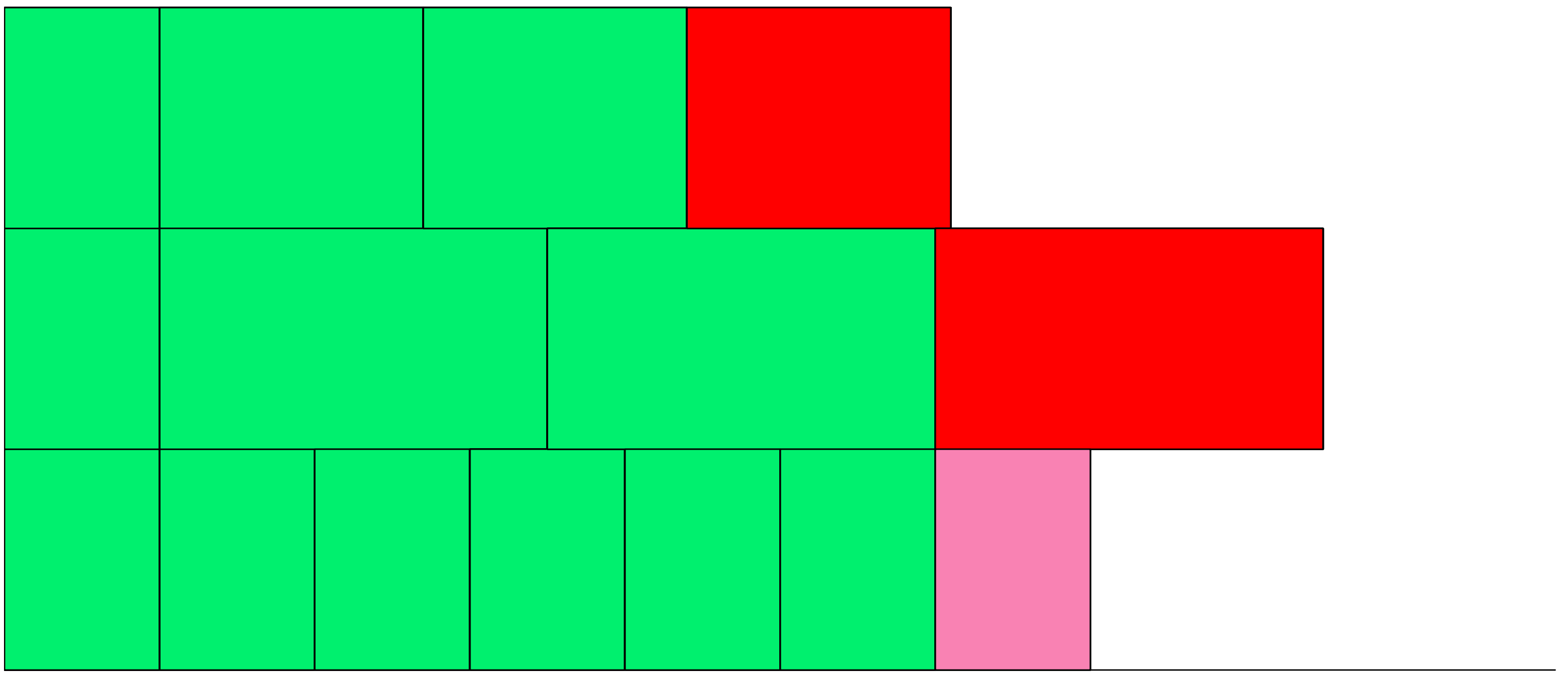
Action: Redoing steps



t = 0.085
n = 2

Multi-Adaptive Time-Stepping

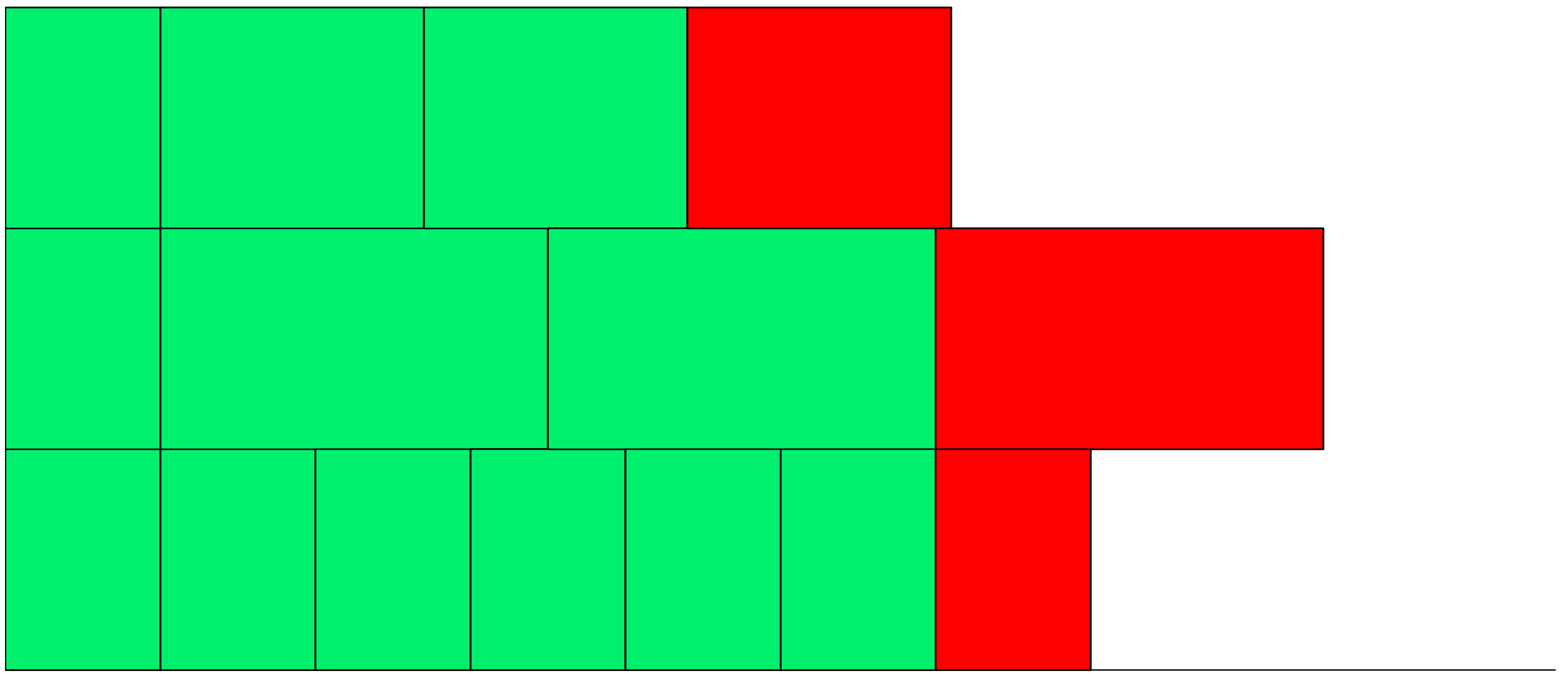
Action: Redoing steps



t = 0.085
n = 2

Multi-Adaptive Time-Stepping

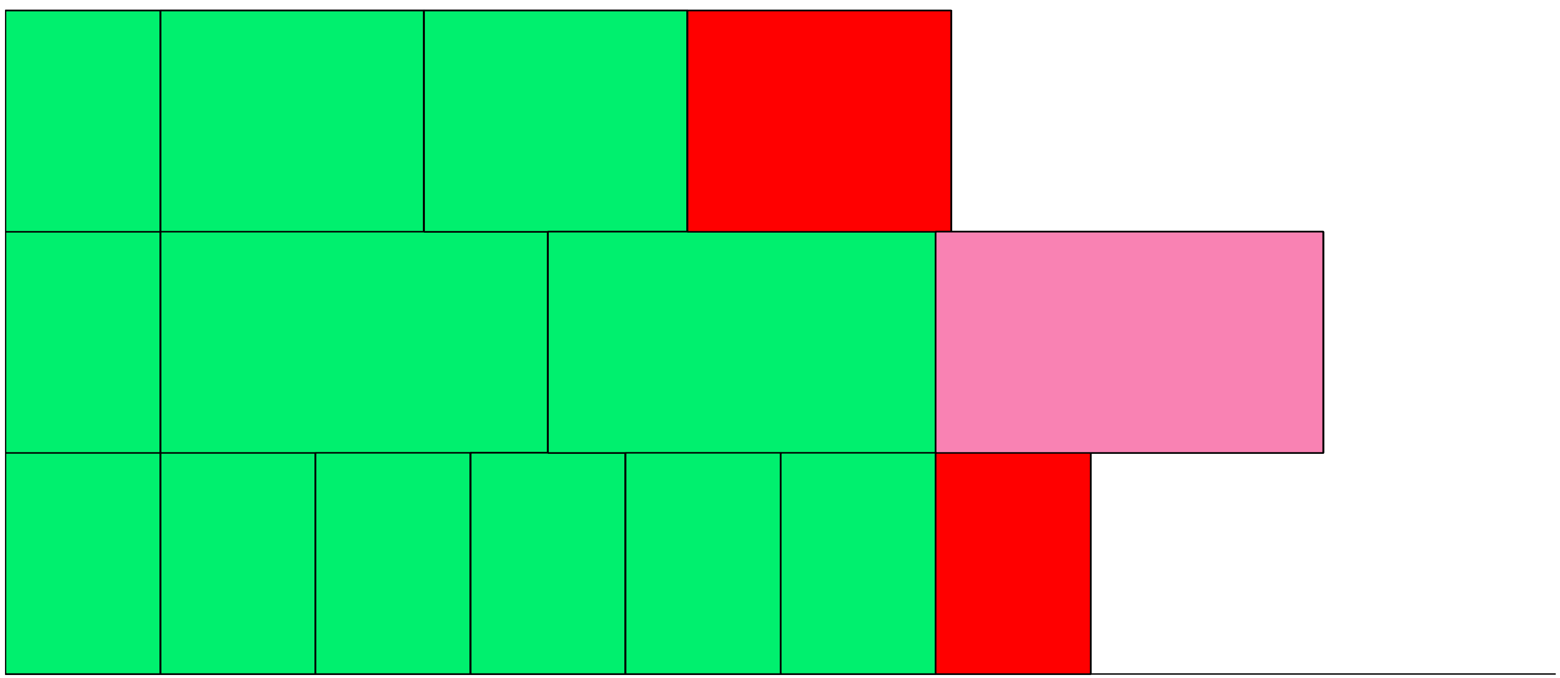
Action: Redoing steps



t = 0.07
n = 2.3333

Multi-Adaptive Time-Stepping

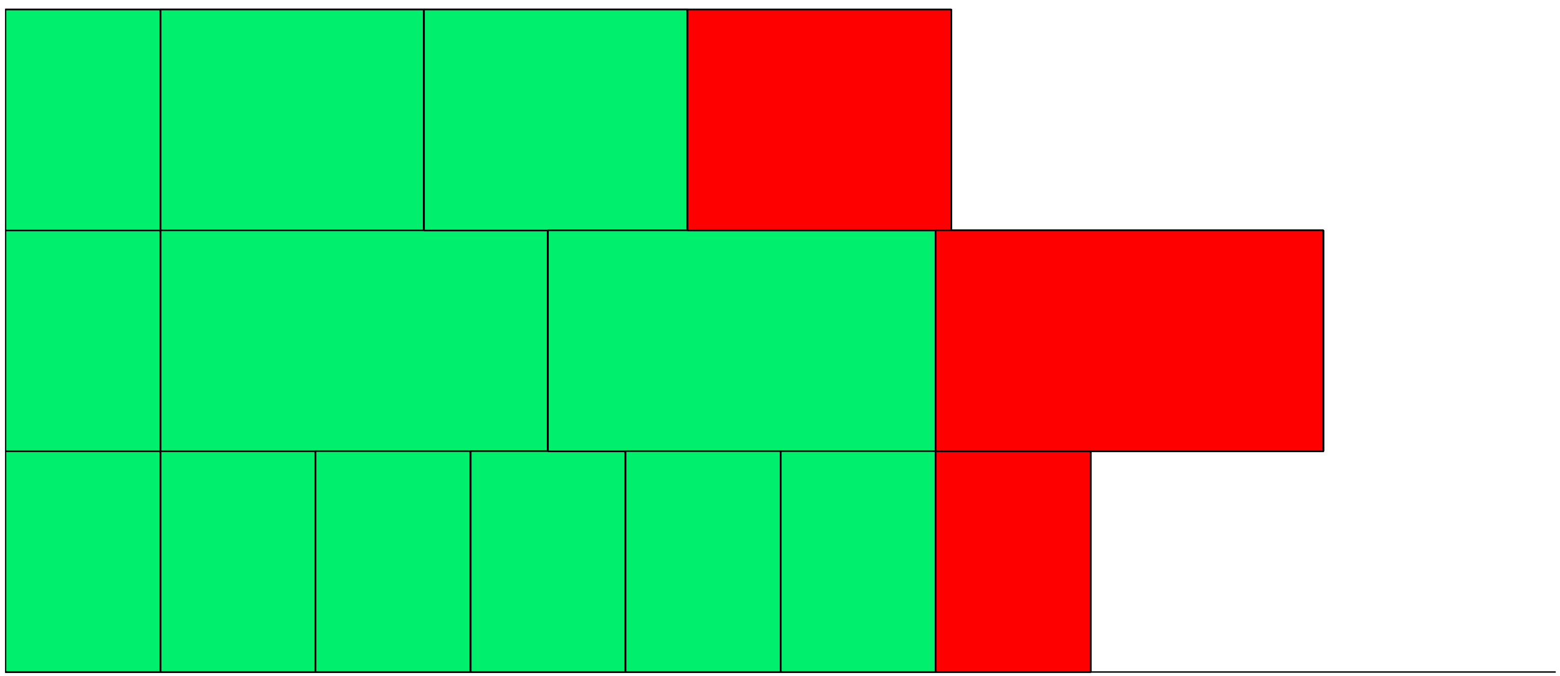
Action: Redoing steps



t = 0.07
n = 2.3333

Multi-Adaptive Time-Stepping

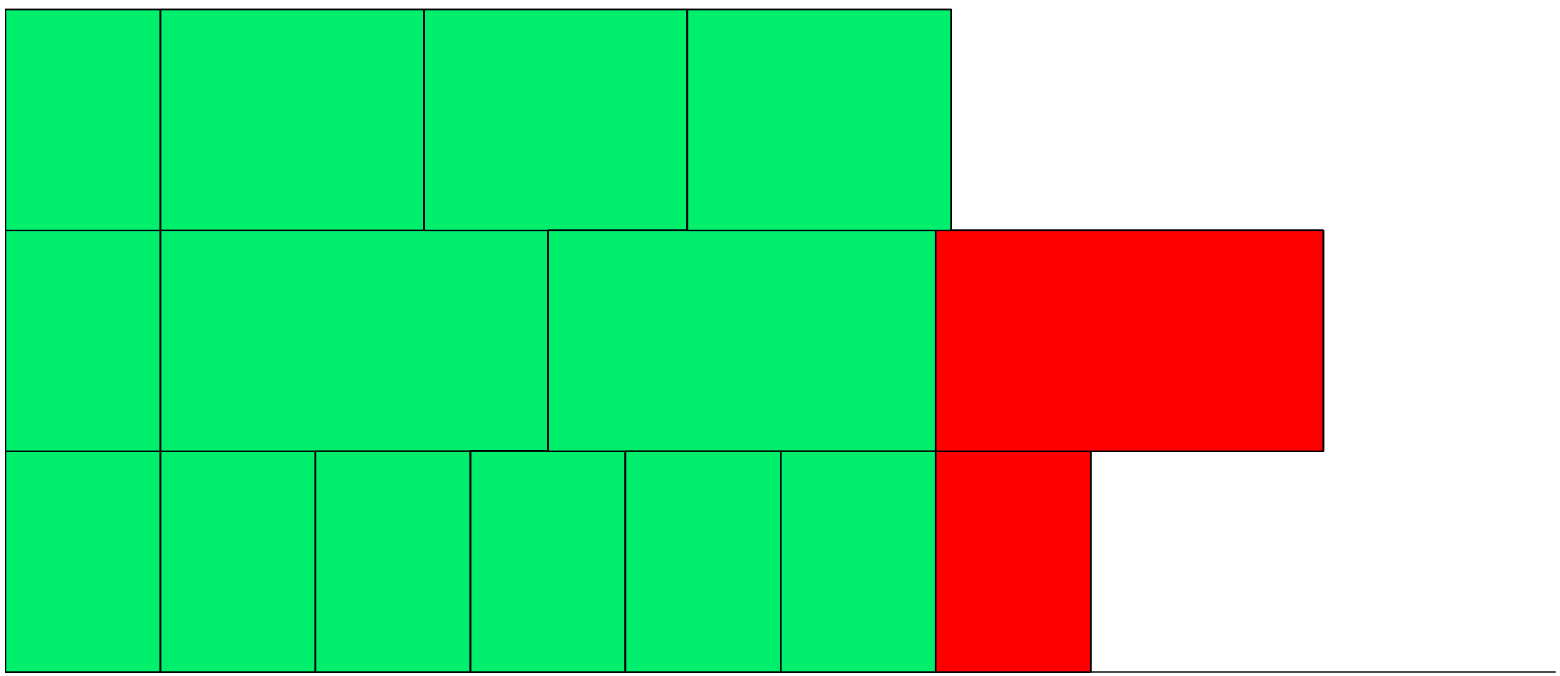
Action: Redoing steps



t = 0.085
n = 2.6667

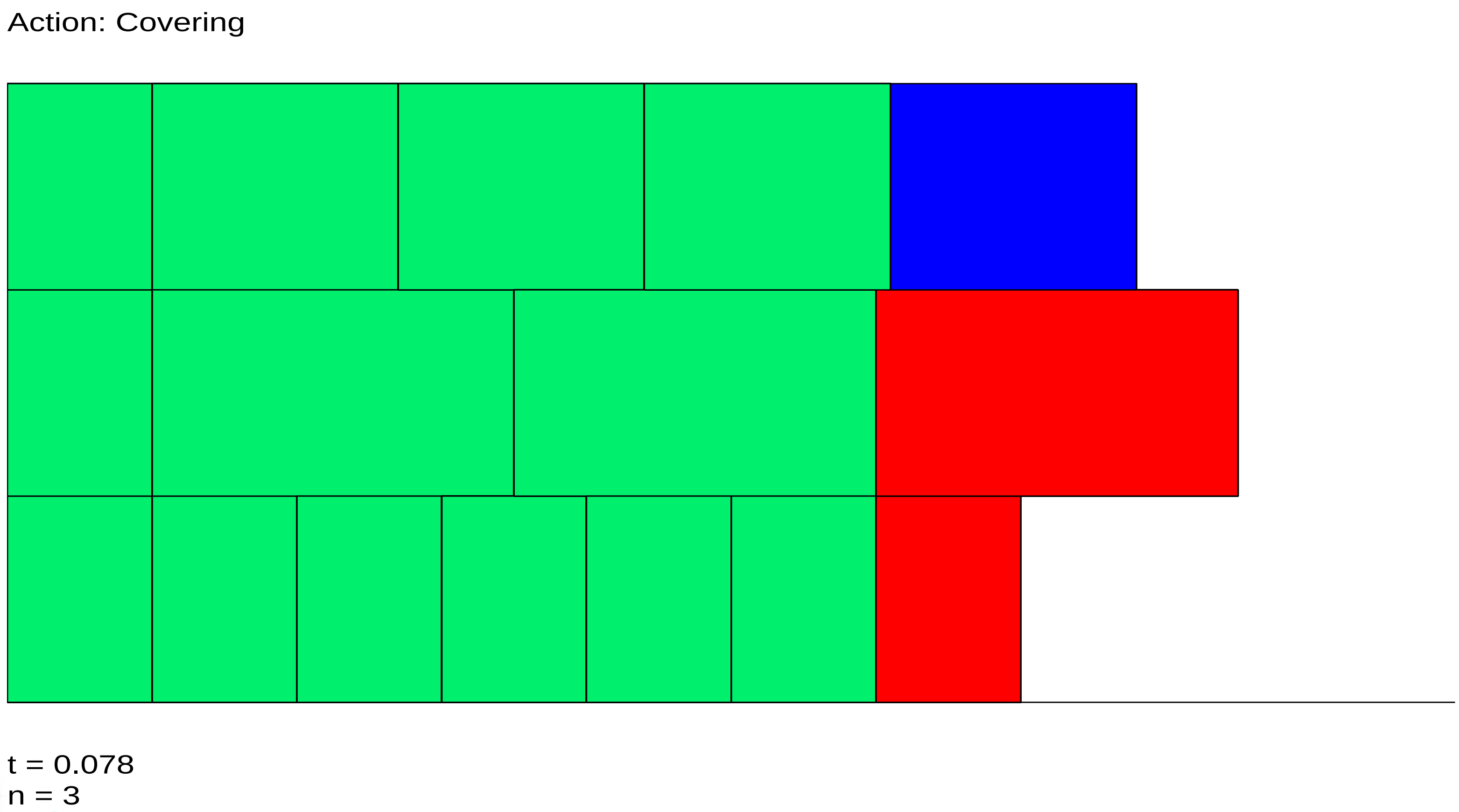
Multi-Adaptive Time-Stepping

Action: Clearing steps

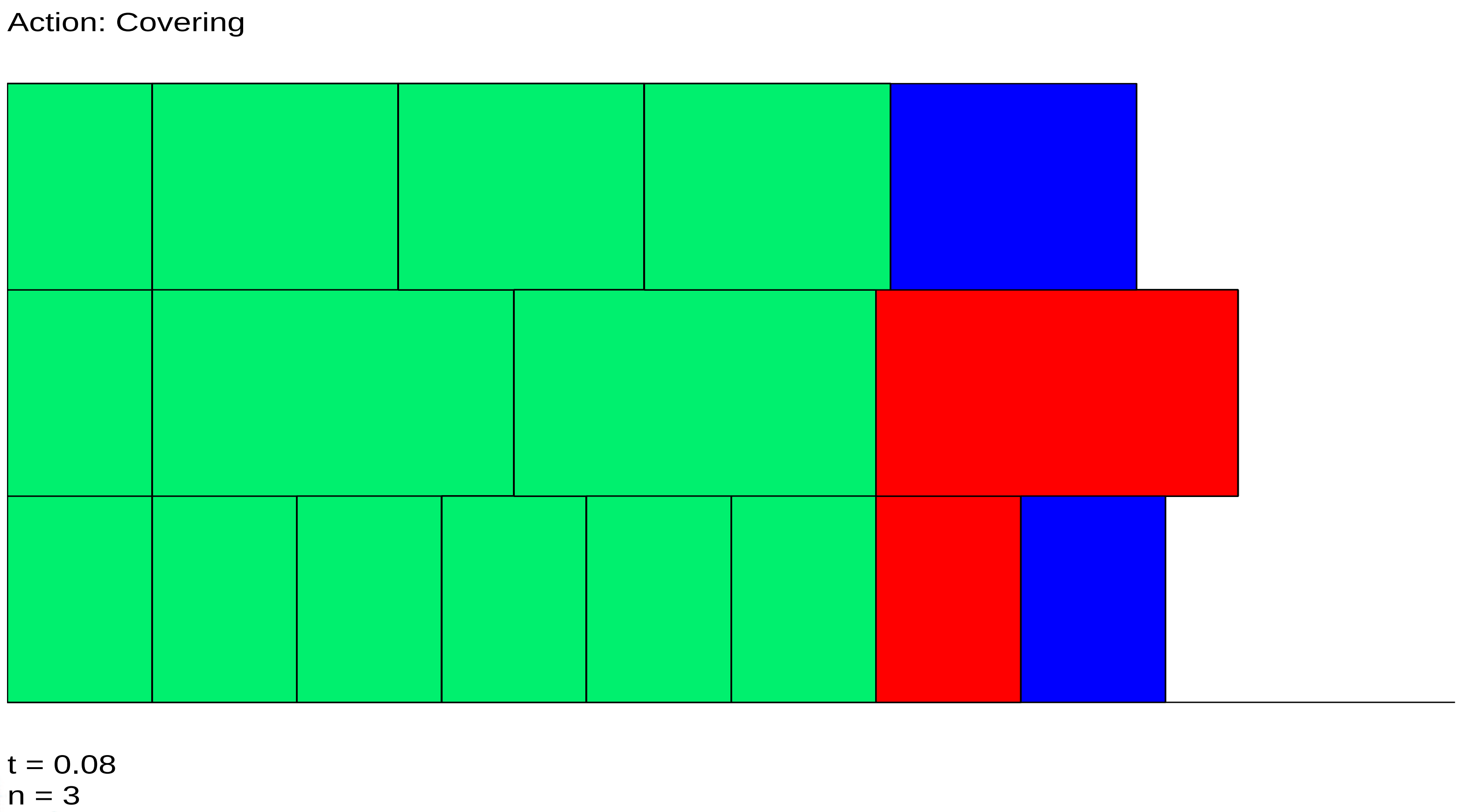


t = 0.061
n = 3

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping



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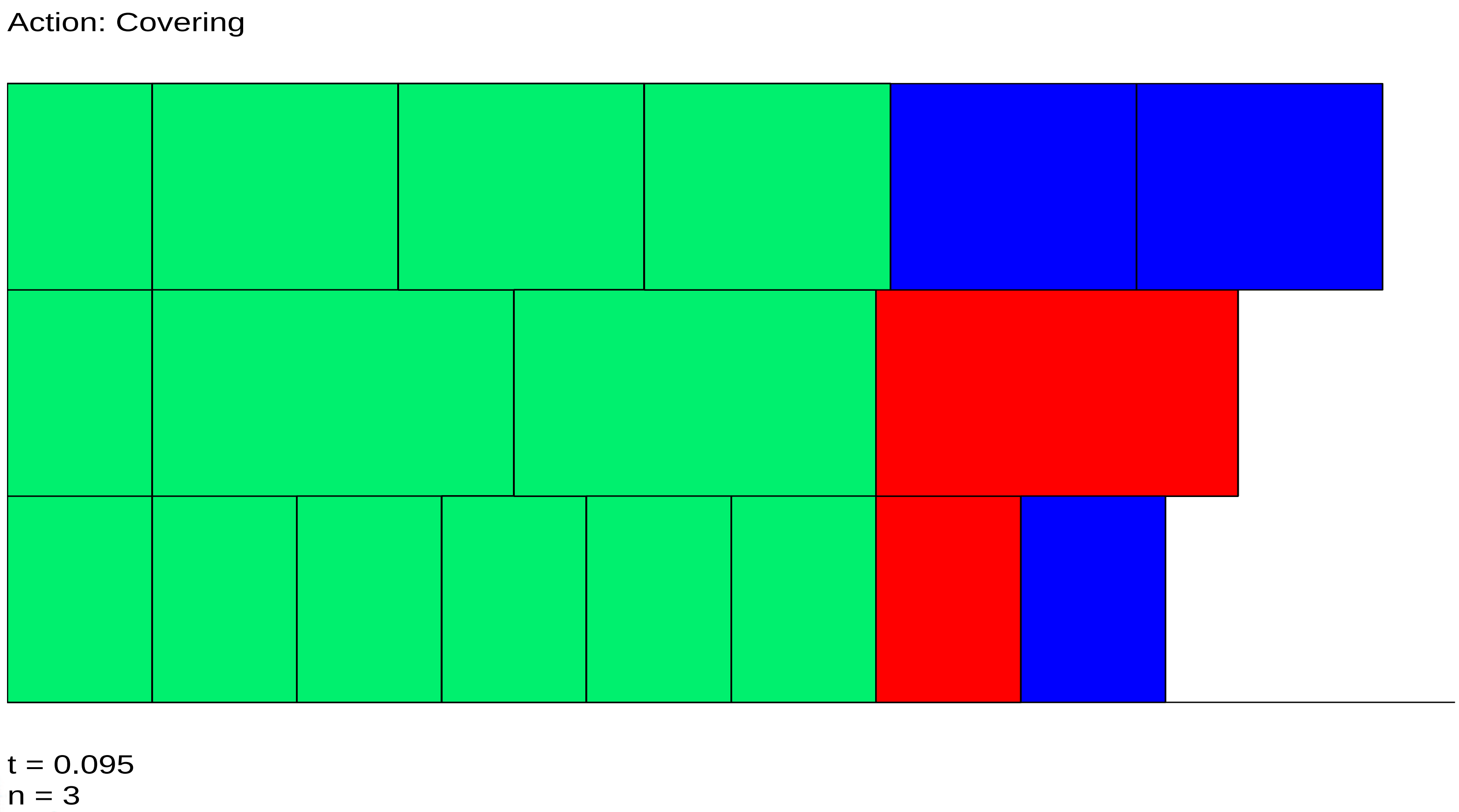
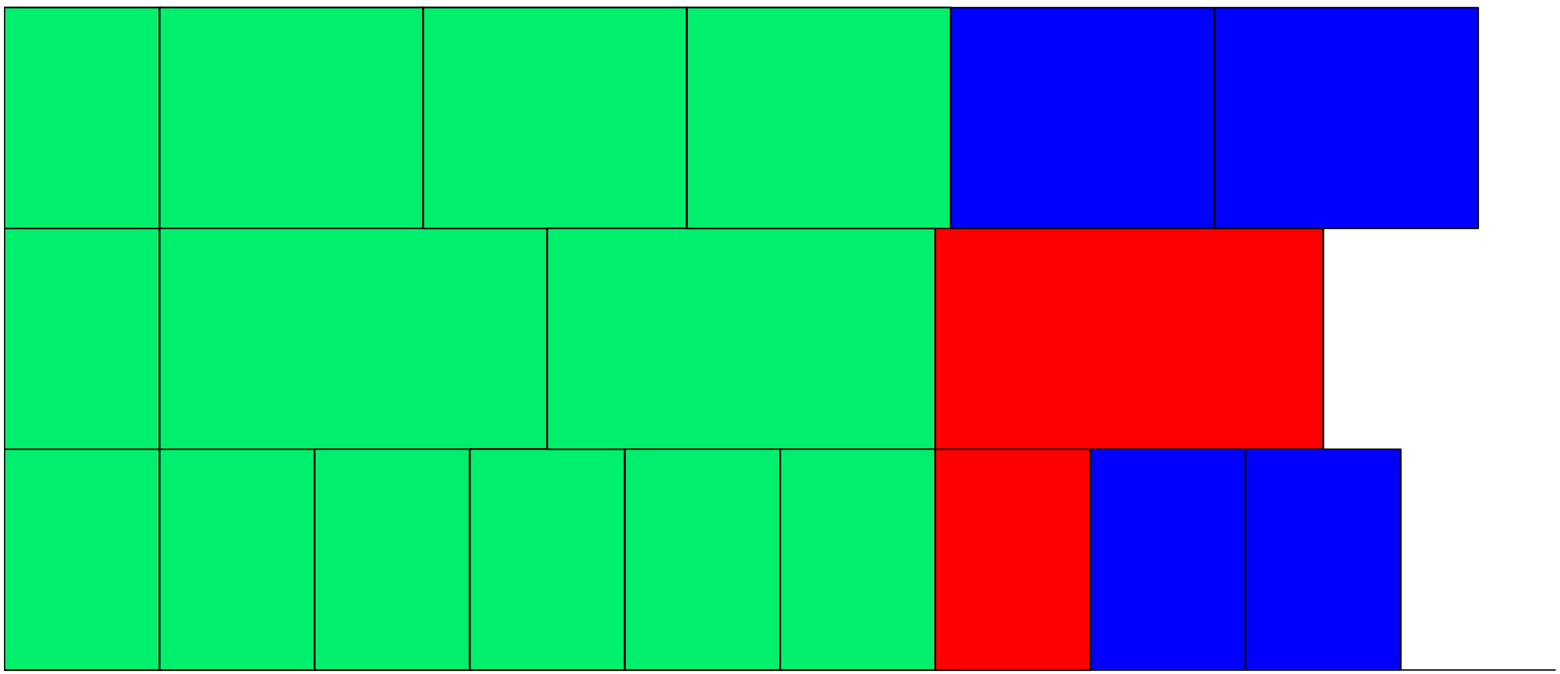
Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

Action: Covering



t = 0.09
n = 3

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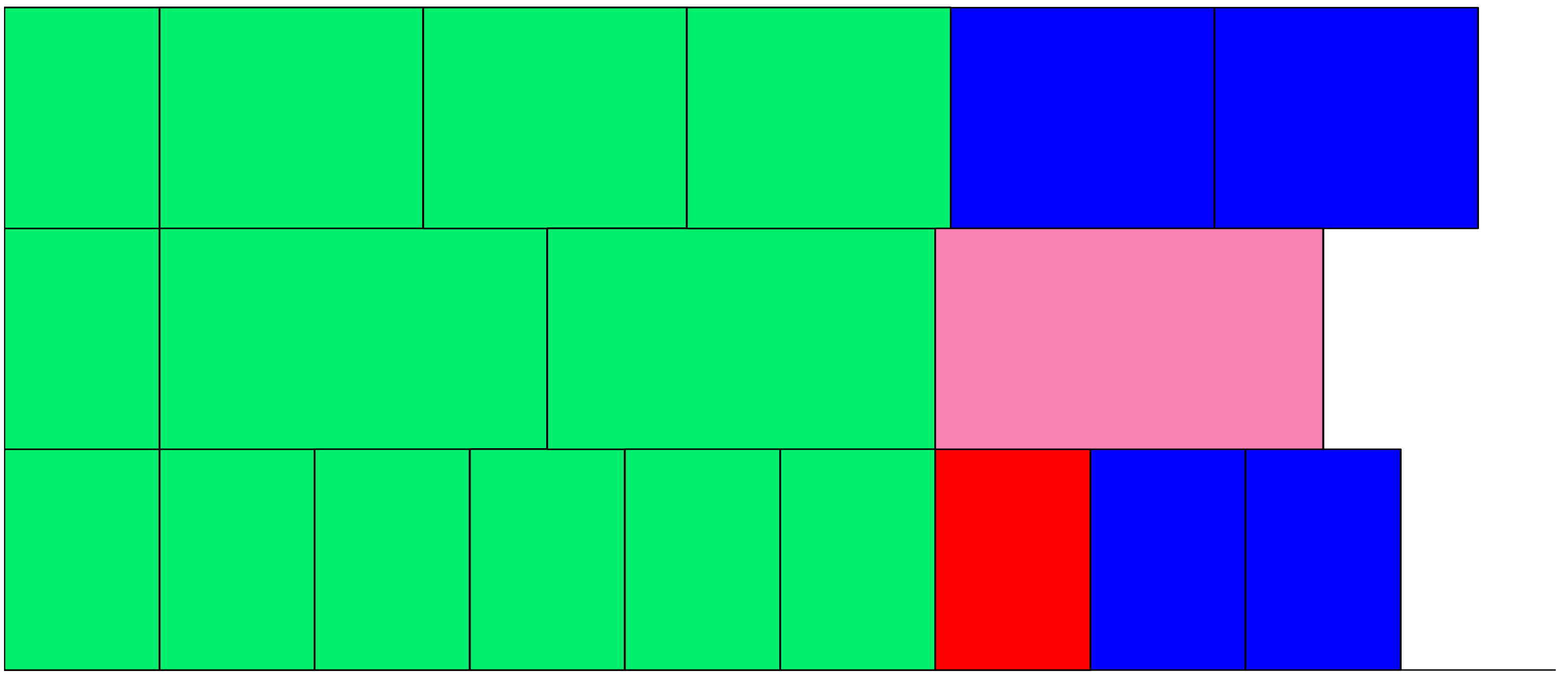
CHALMERS FINITE ELEMENT CENTER

Multi-Adaptive Time-Stepping

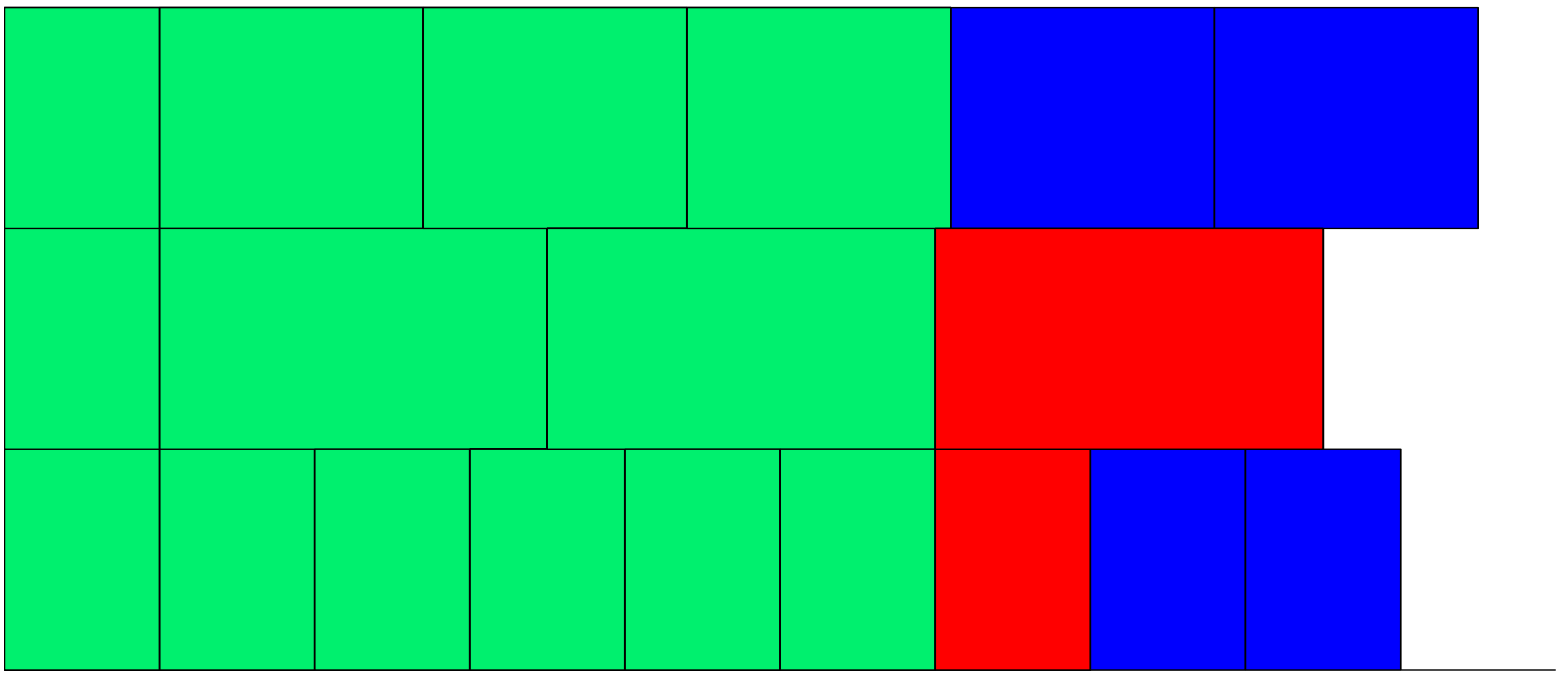
Action: Redoing steps



t = 0.09
n = 3

Multi-Adaptive Time-Stepping

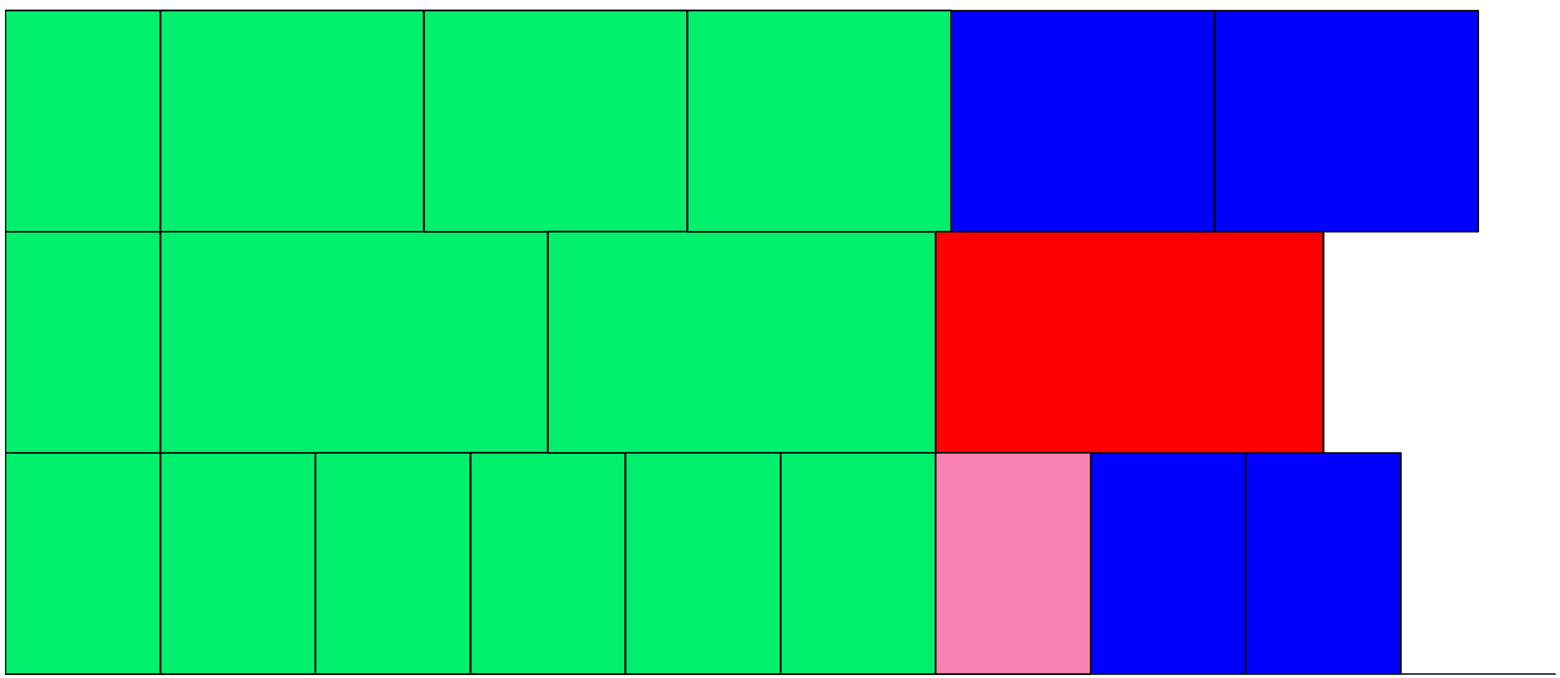
Action: Redoing steps



t = 0.085
n = 1

Multi-Adaptive Time-Stepping

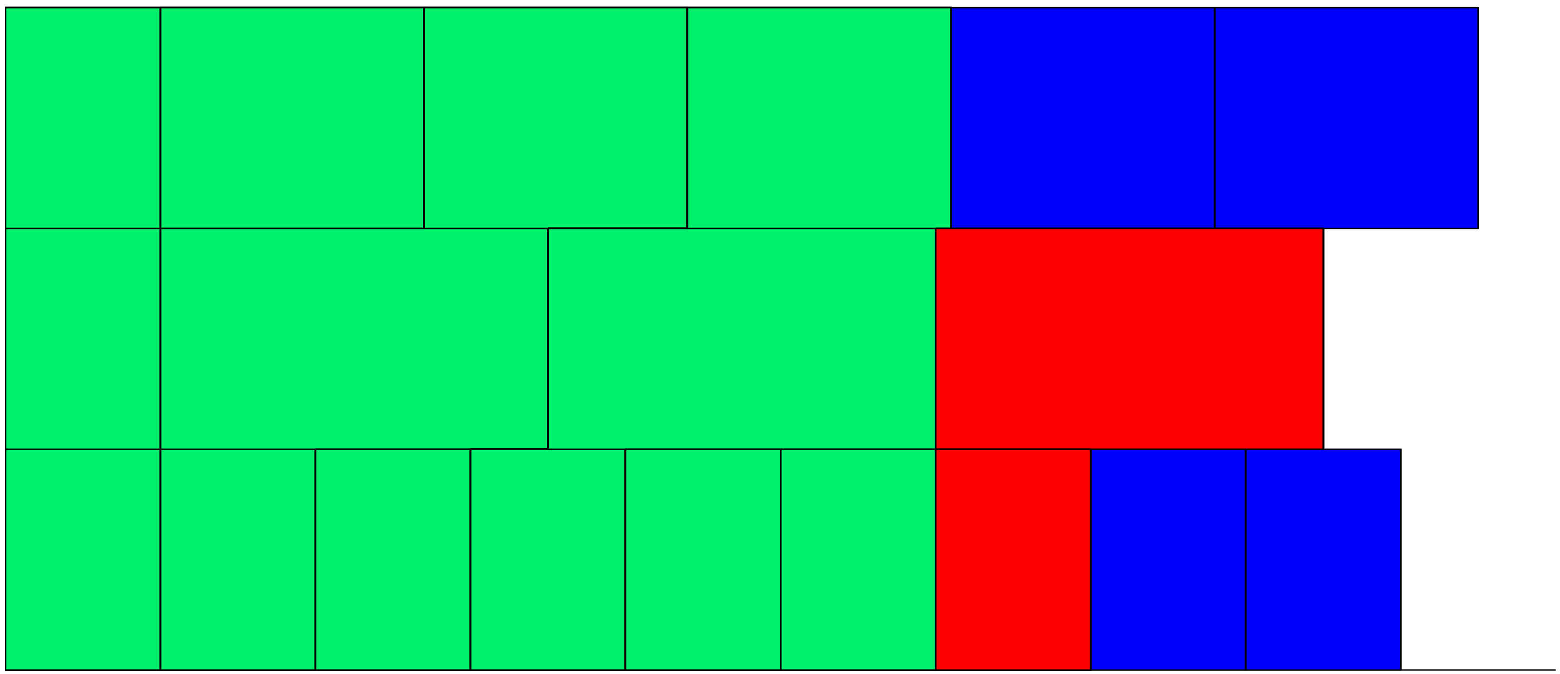
Action: Redoing steps



t = 0.085
n = 1

Multi-Adaptive Time-Stepping

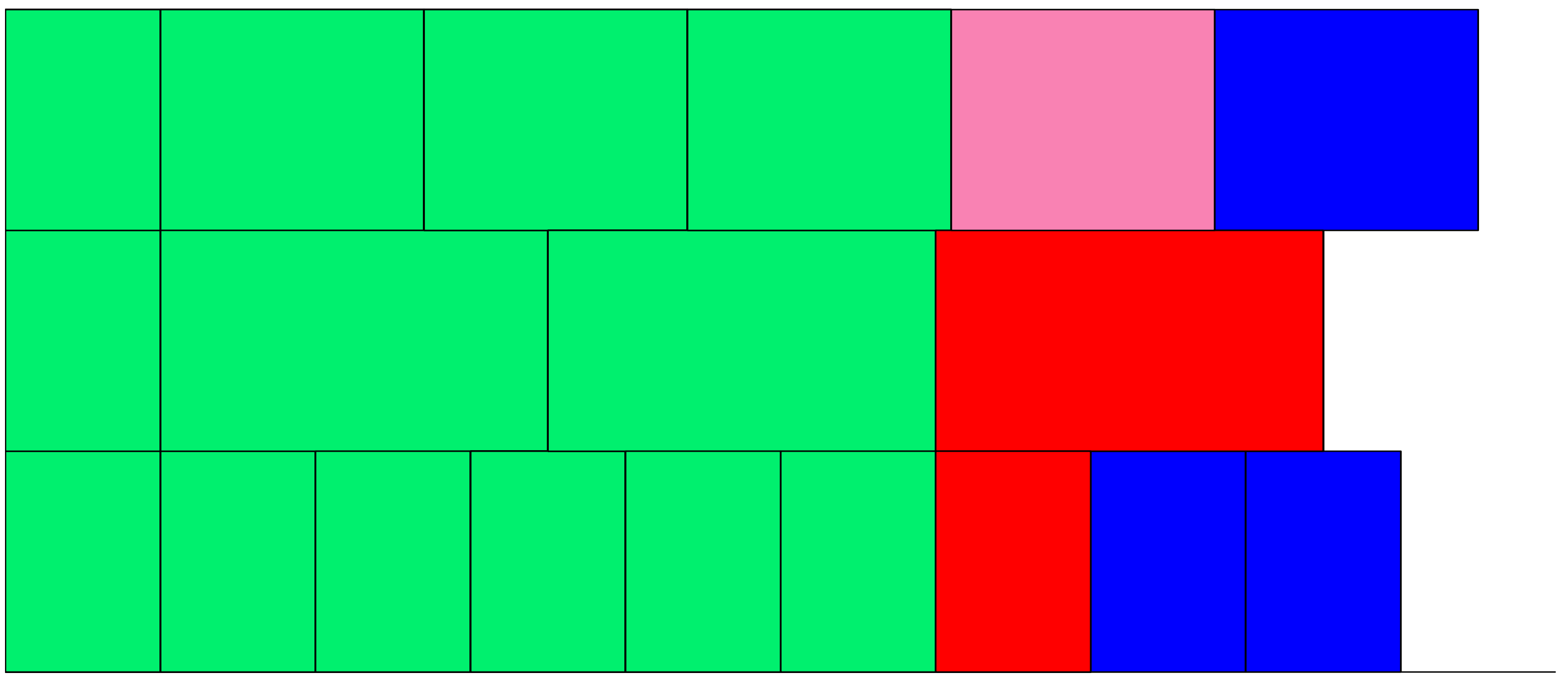
Action: Redoing steps



t = 0.07
n = 1

Multi-Adaptive Time-Stepping

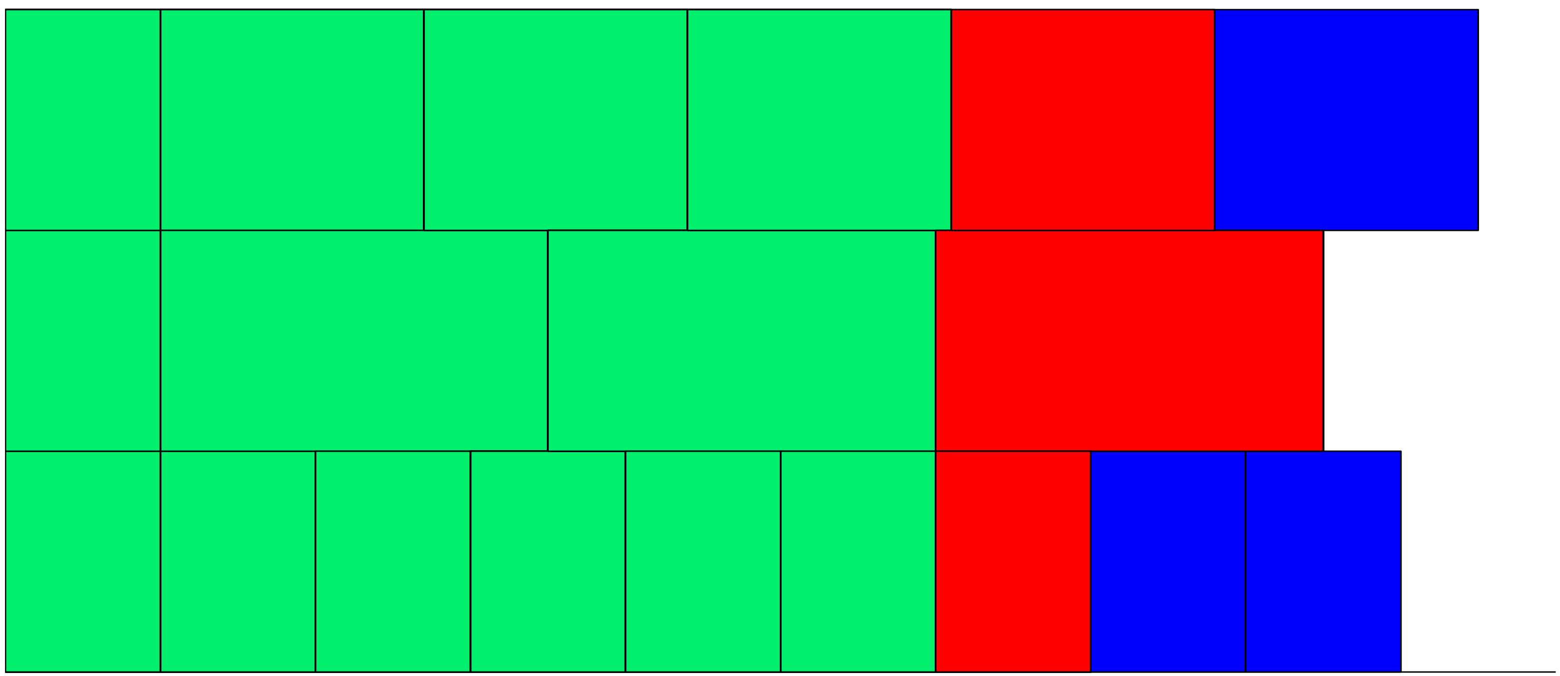
Action: Redoing steps



t = 0.07
n = 1

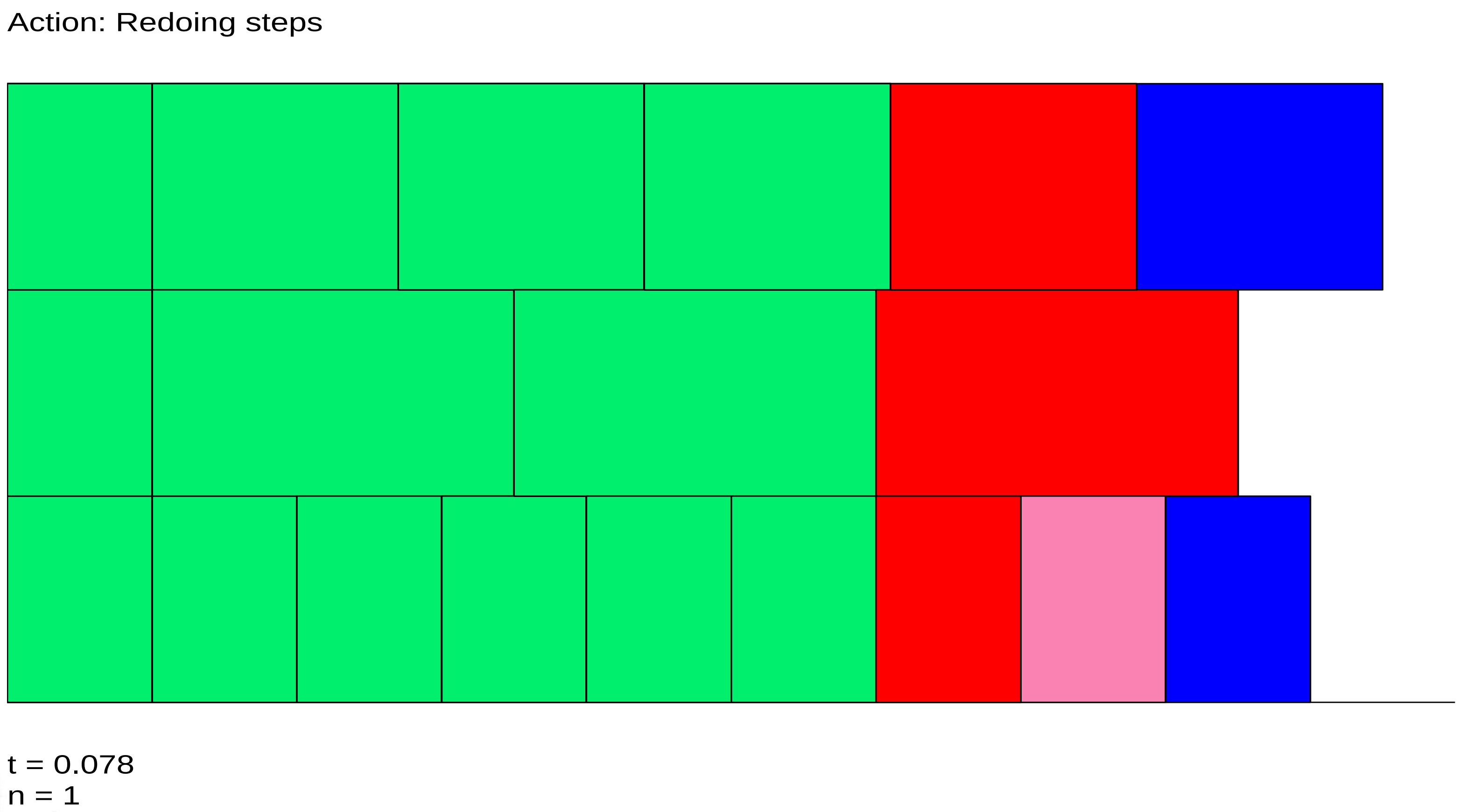
Multi-Adaptive Time-Stepping

Action: Redoing steps



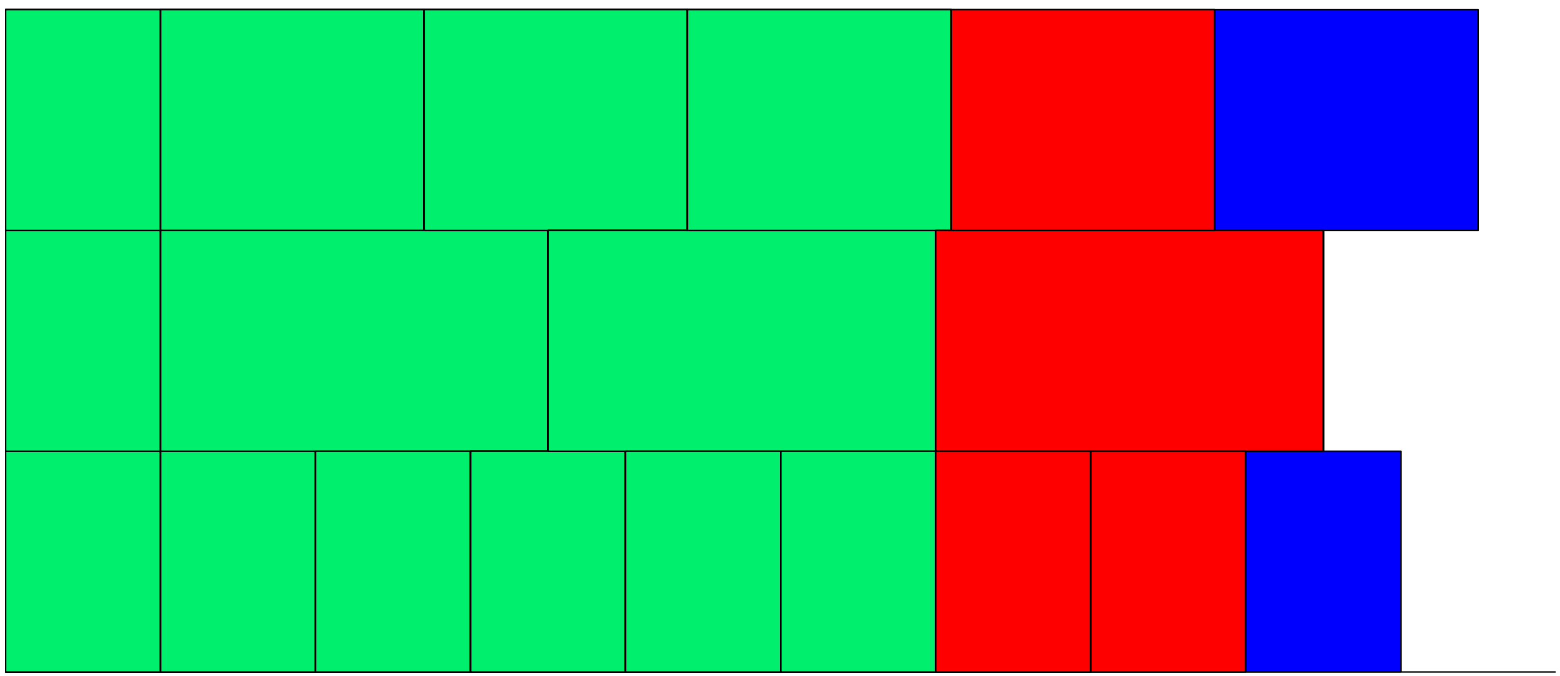
t = 0.078
n = 1

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

Action: Redoing steps



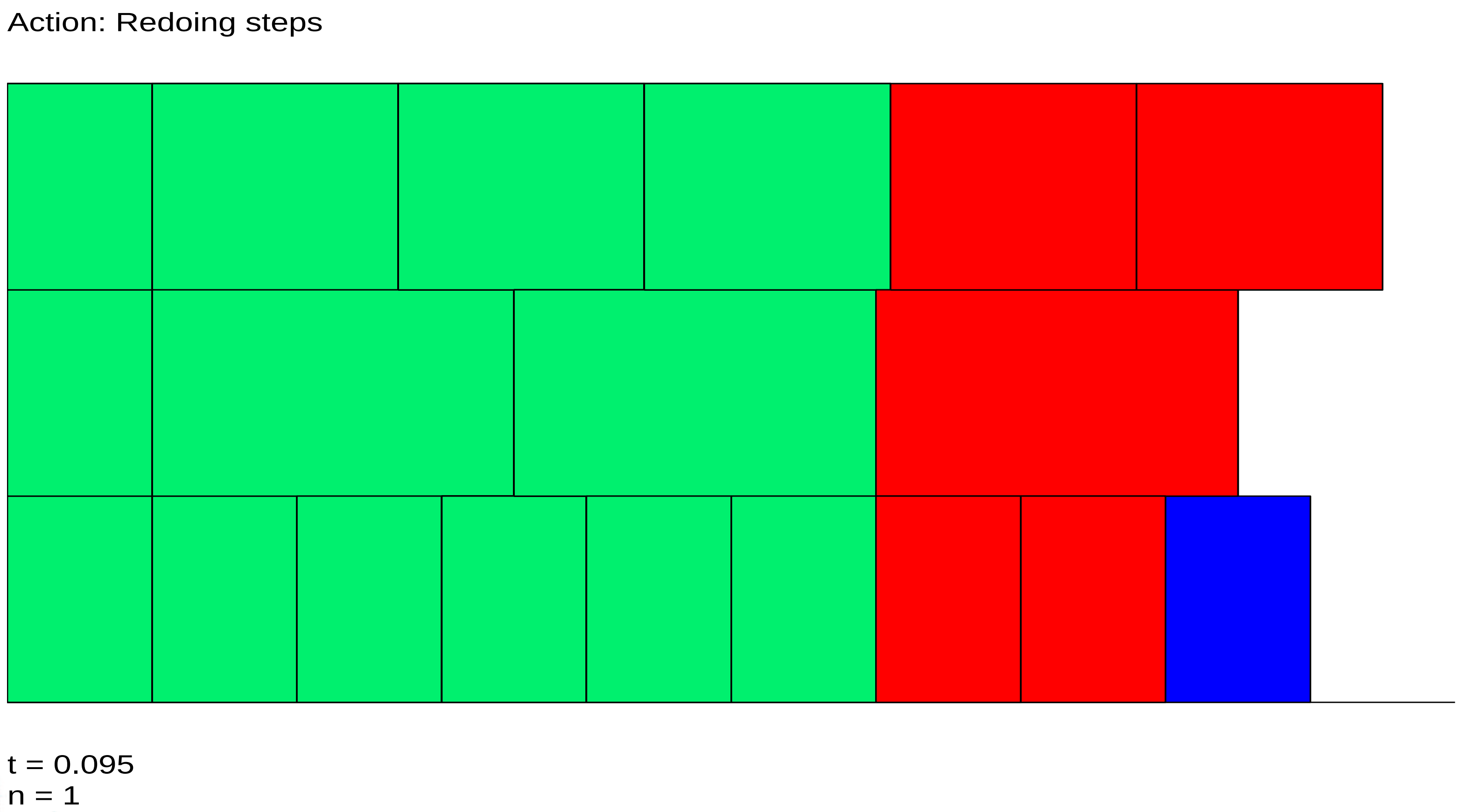
t = 0.08
n = 1

Multi-Adaptive Time-Stepping

Action: Redoing steps

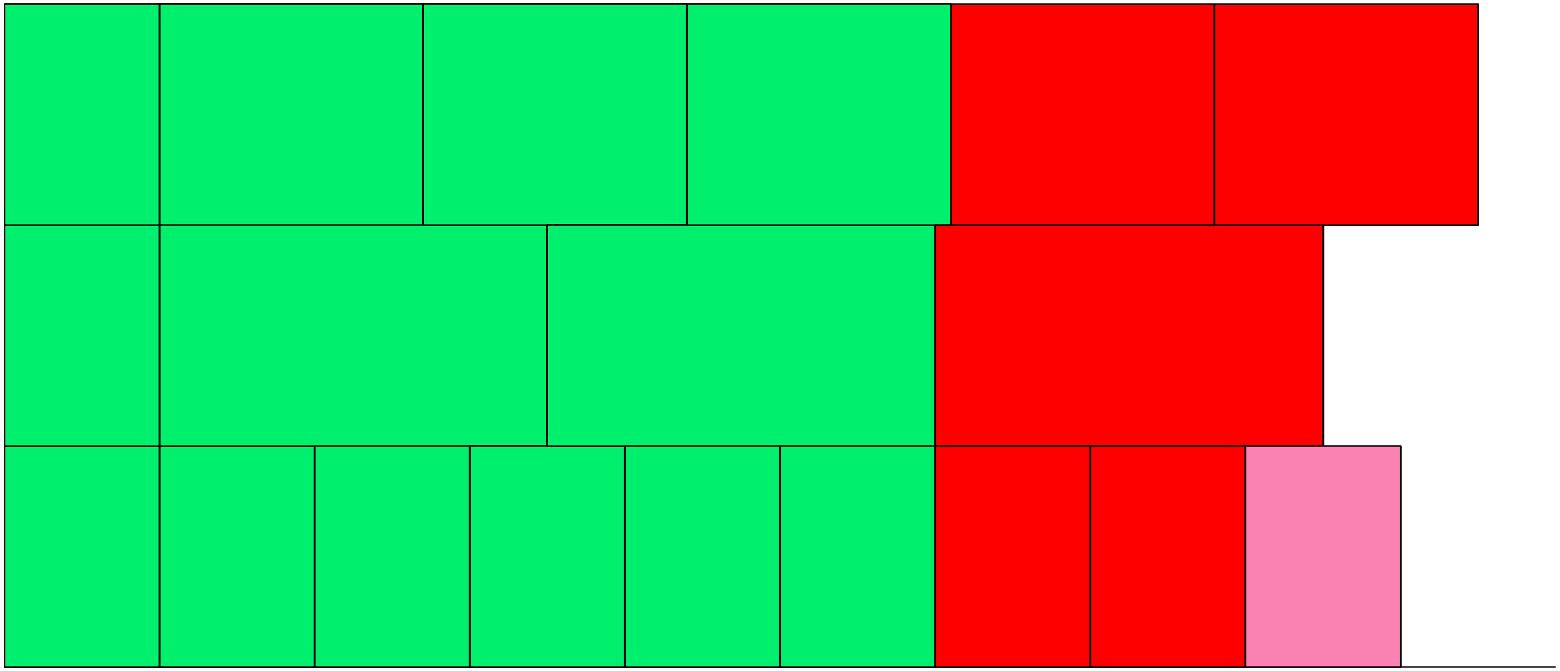
t = 0.08
n = 1

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.095
n = 1

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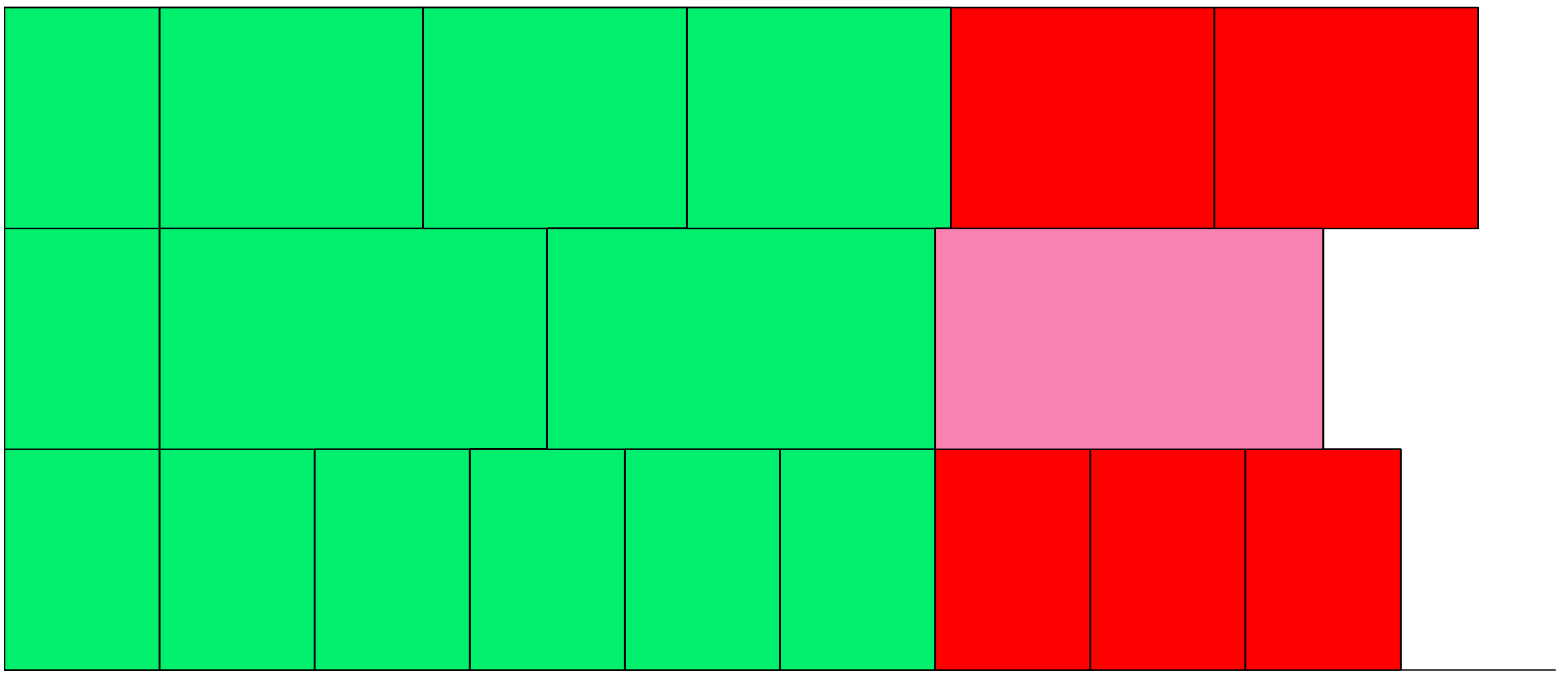
Multi-Adaptive Time-Stepping

Action: Redoing steps

t = 0.09
n = 1

Multi-Adaptive Time-Stepping

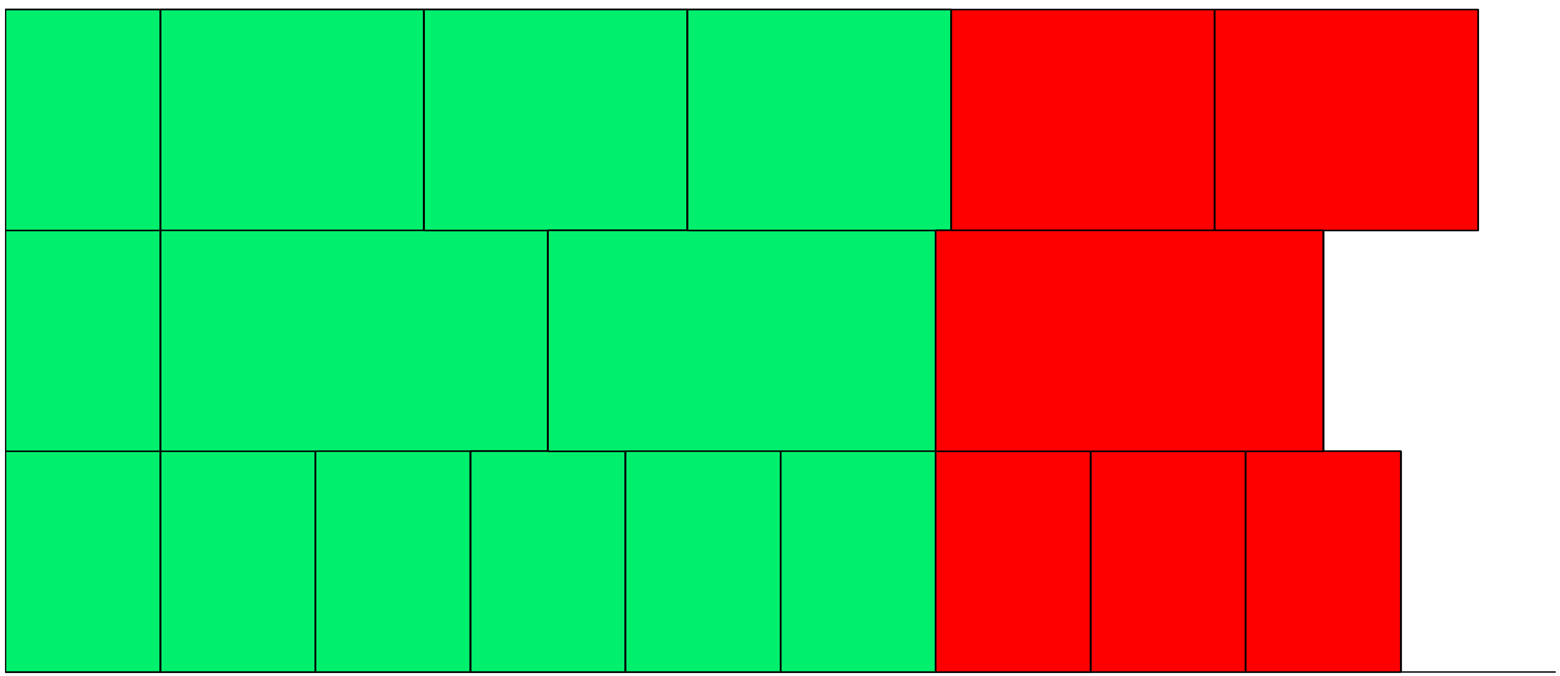
Action: Redoing steps



t = 0.09
n = 1

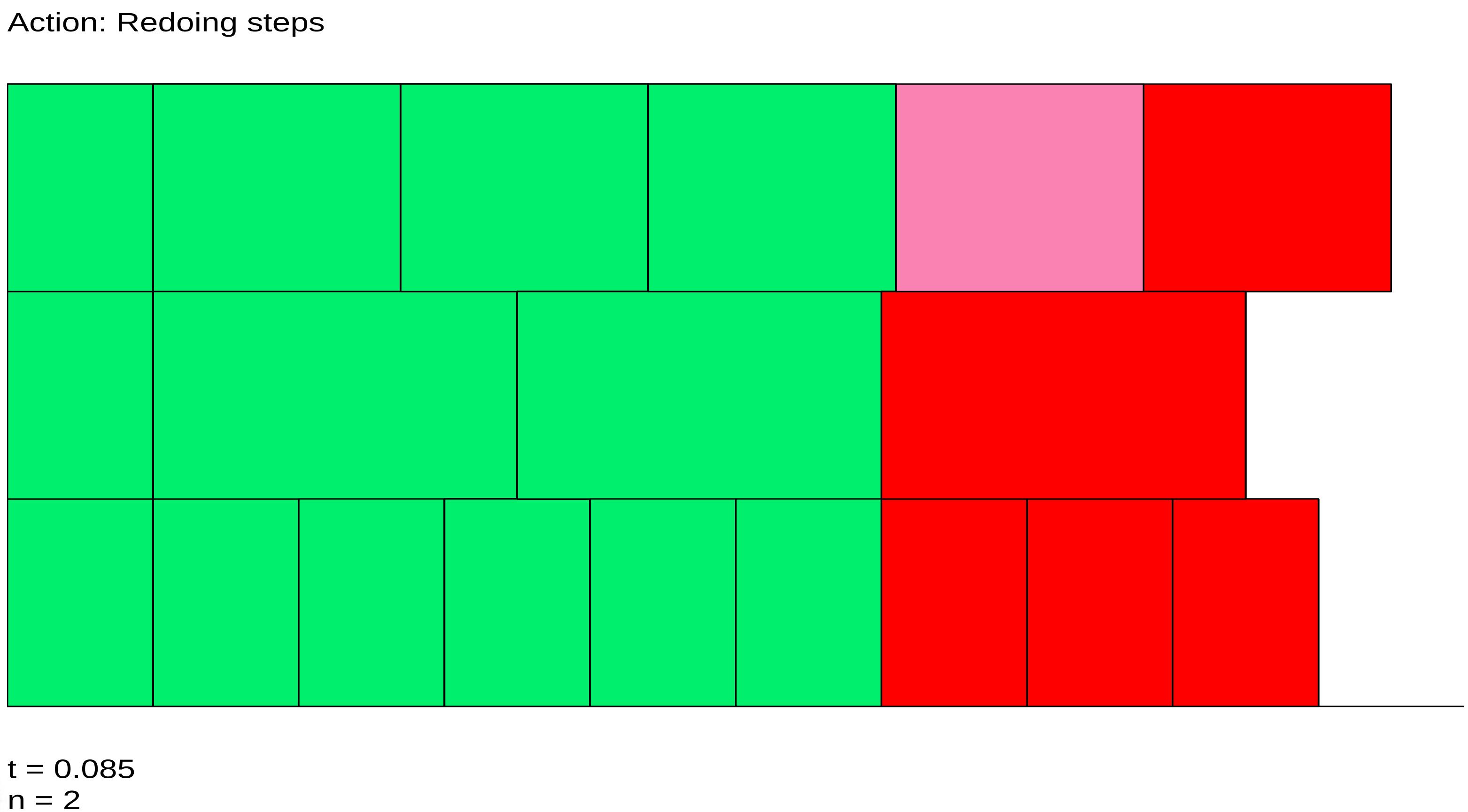
Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.085
n = 2

Multi-Adaptive Time-Stepping



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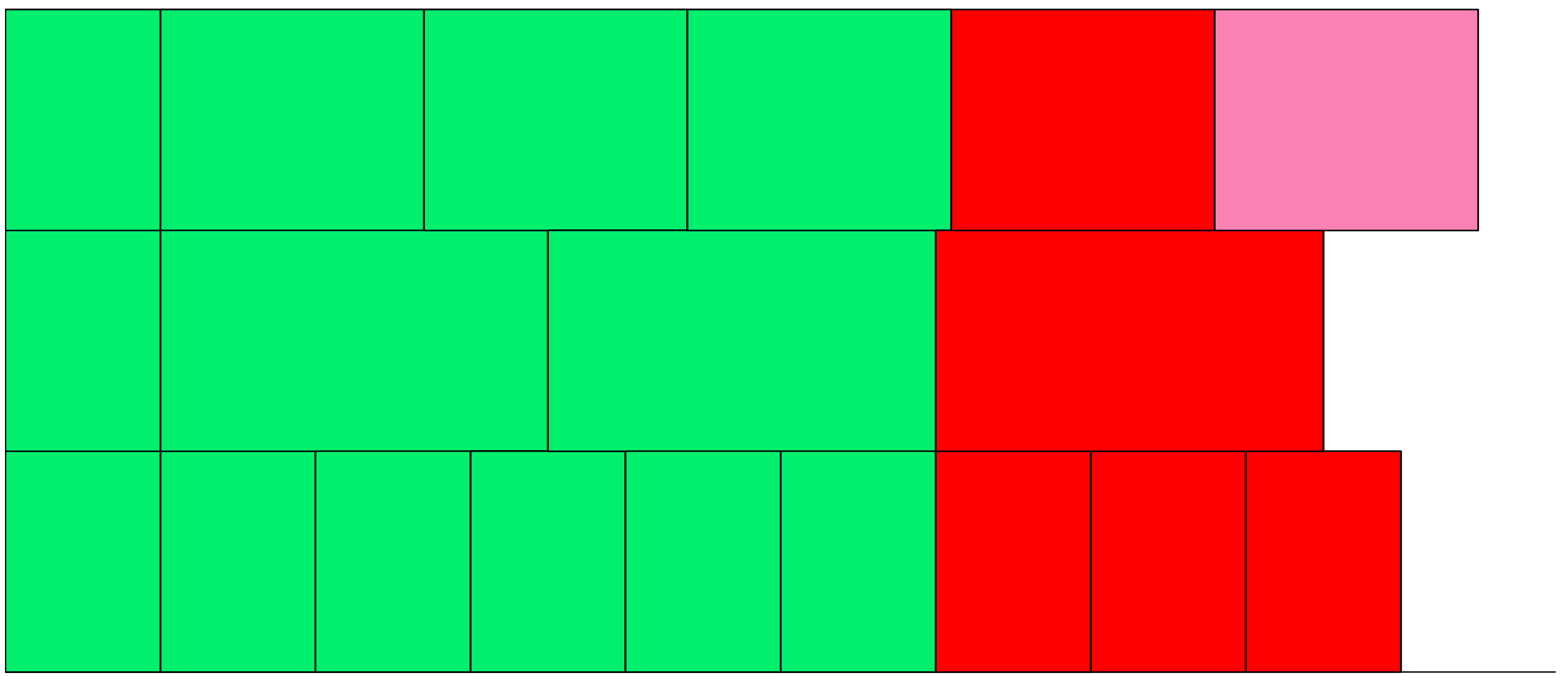
Multi-Adaptive Time-Stepping

Action: Redoing steps

t = 0.078
n = 2

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.078
n = 2

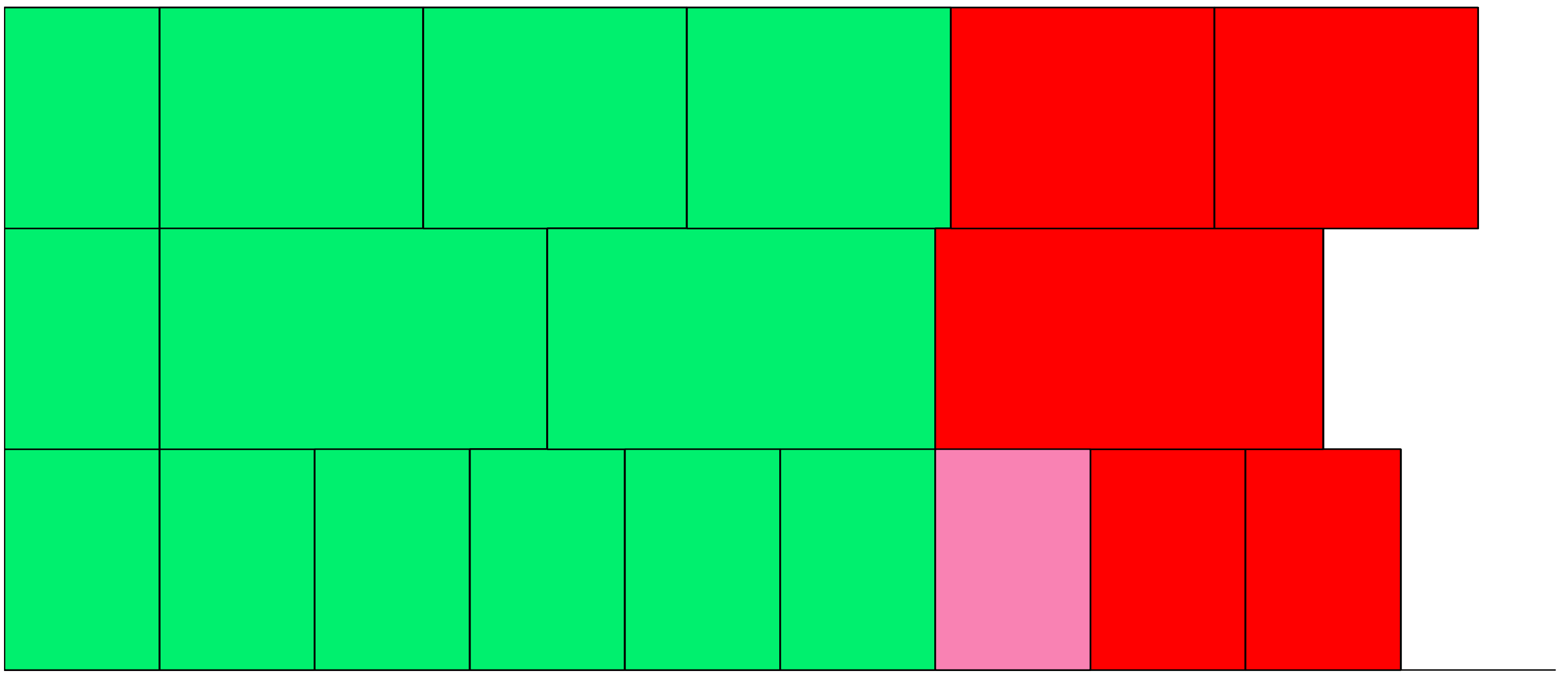
Multi-Adaptive Time-Stepping

Action: Redoing steps

t = 0.095
n = 2

Multi-Adaptive Time-Stepping

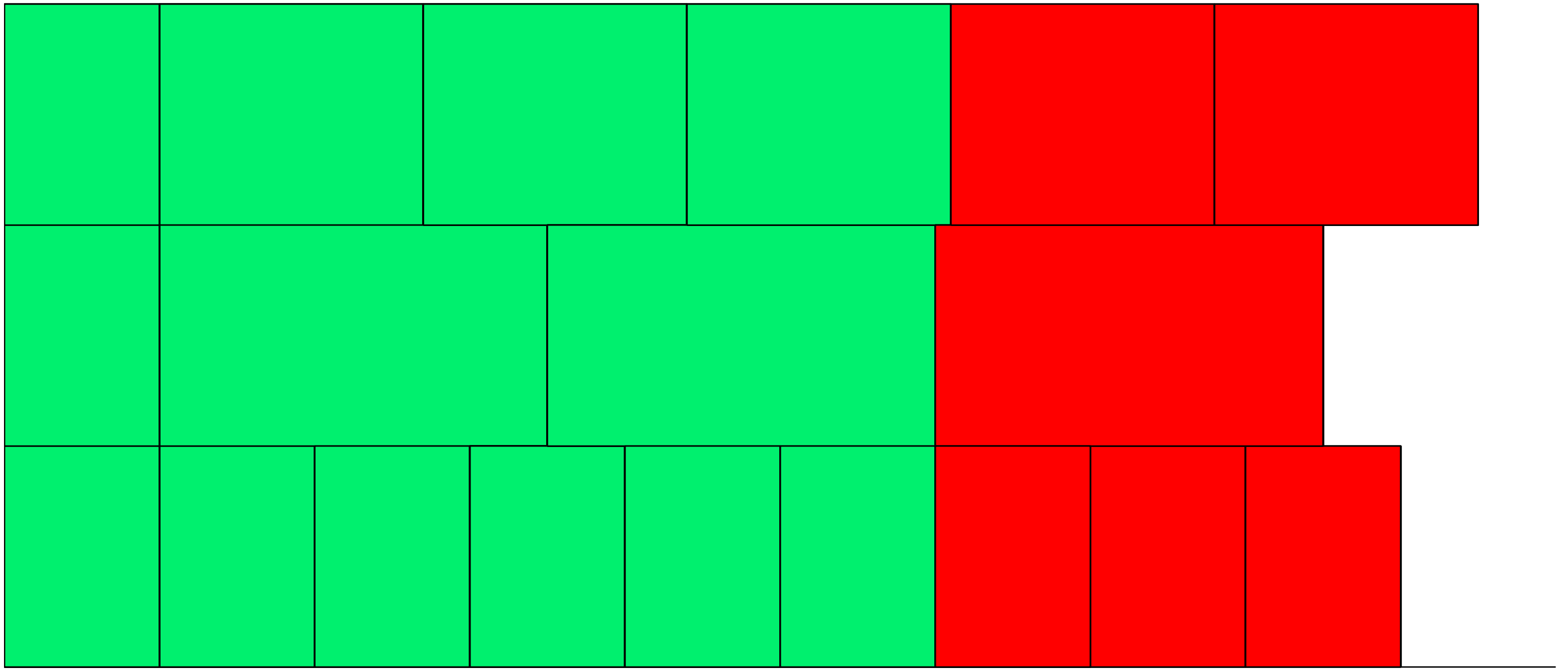
Action: Redoing steps



t = 0.095
n = 2

Multi-Adaptive Time-Stepping

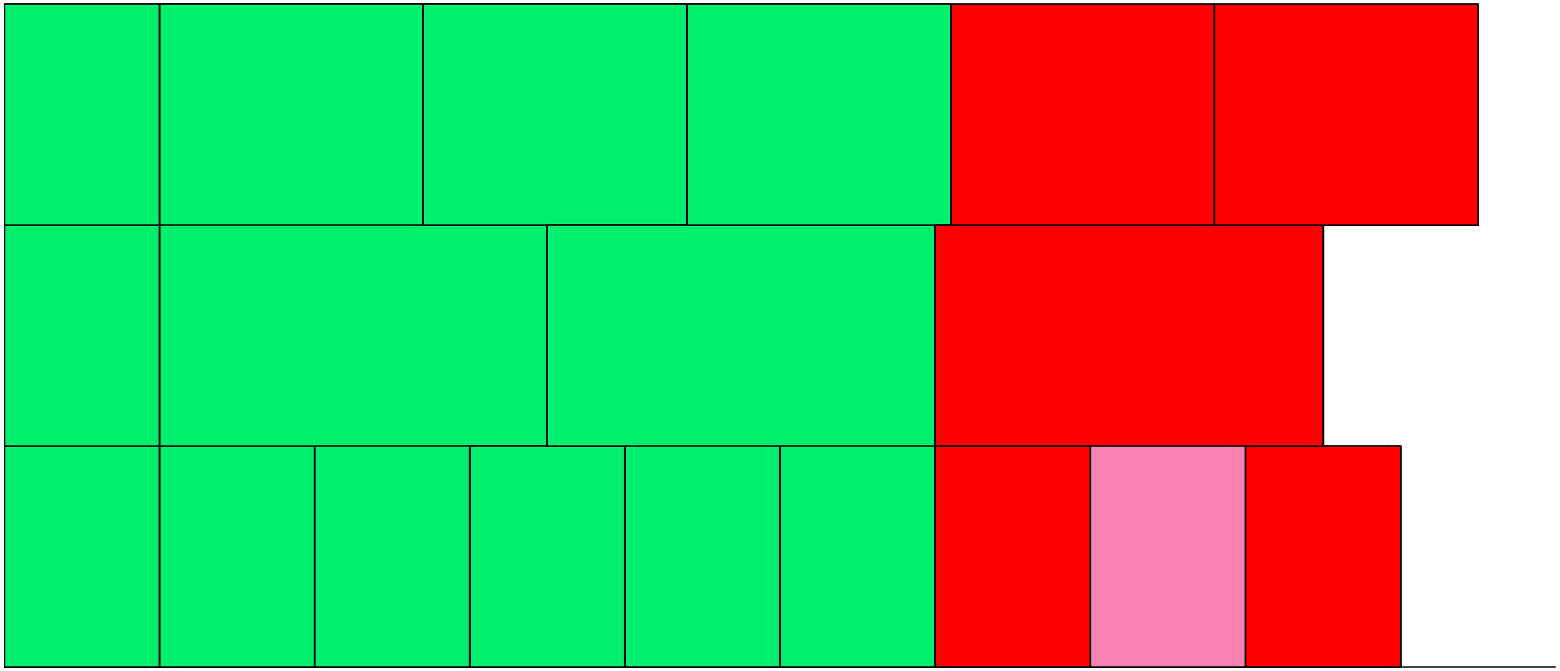
Action: Redoing steps



t = 0.07
n = 2.5

Multi-Adaptive Time-Stepping

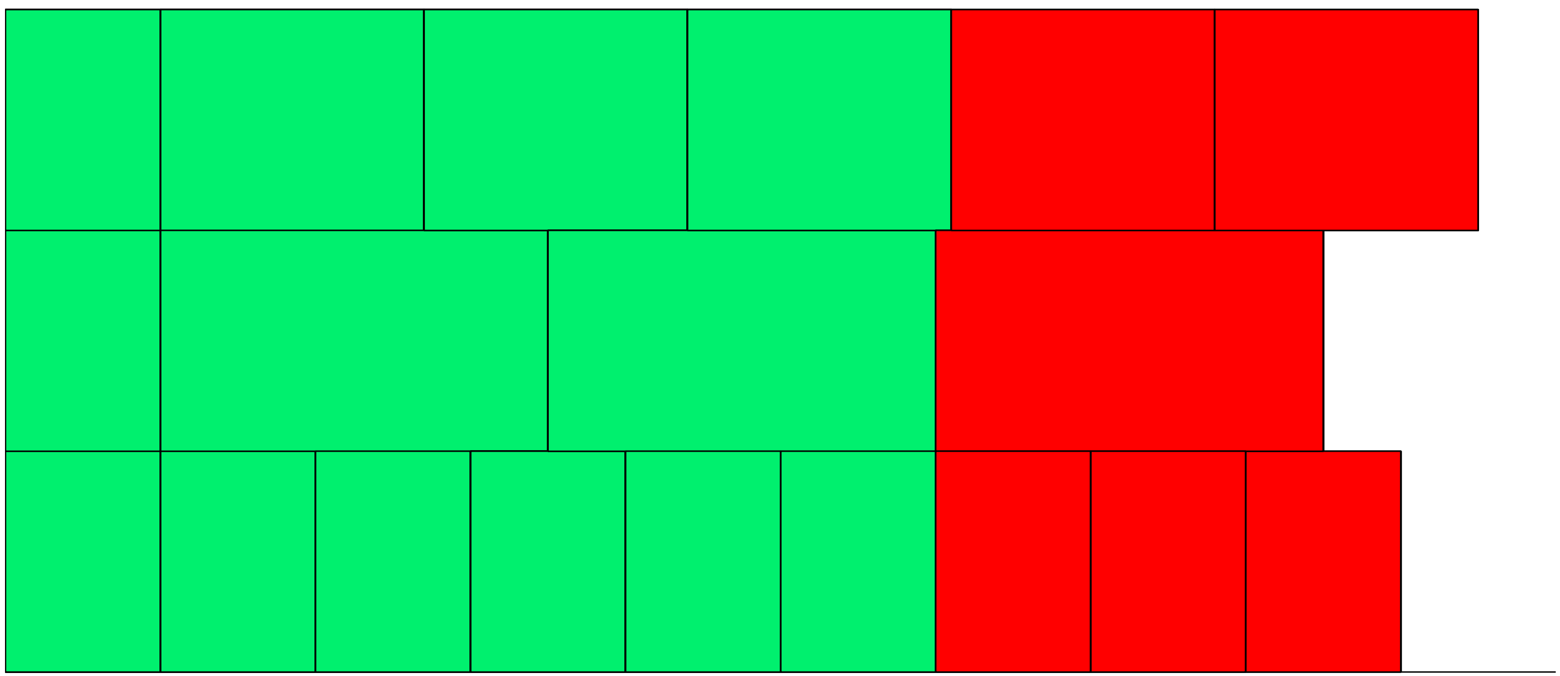
Action: Redoing steps



t = 0.07
n = 2.5

Multi-Adaptive Time-Stepping

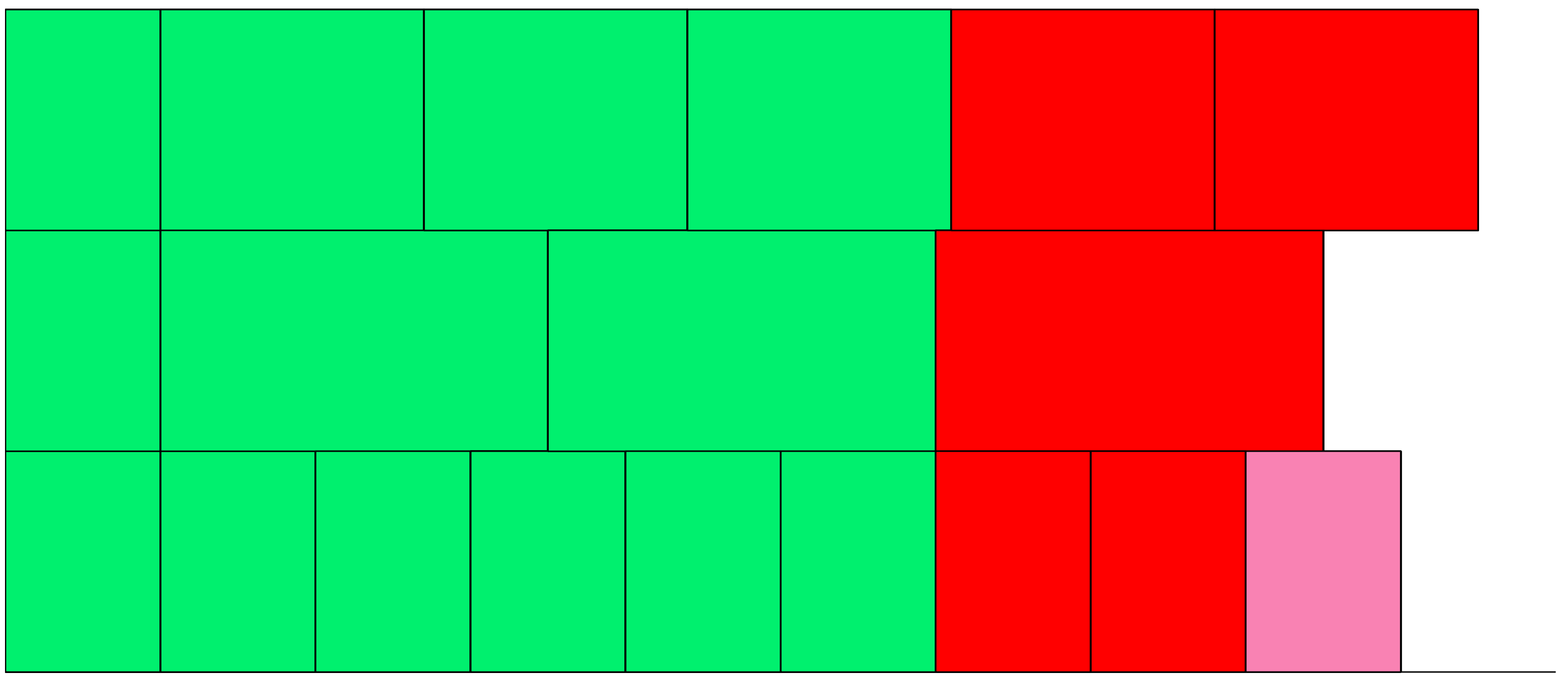
Action: Redoing steps



t = 0.08
n = 2.5

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.08
n = 2.5

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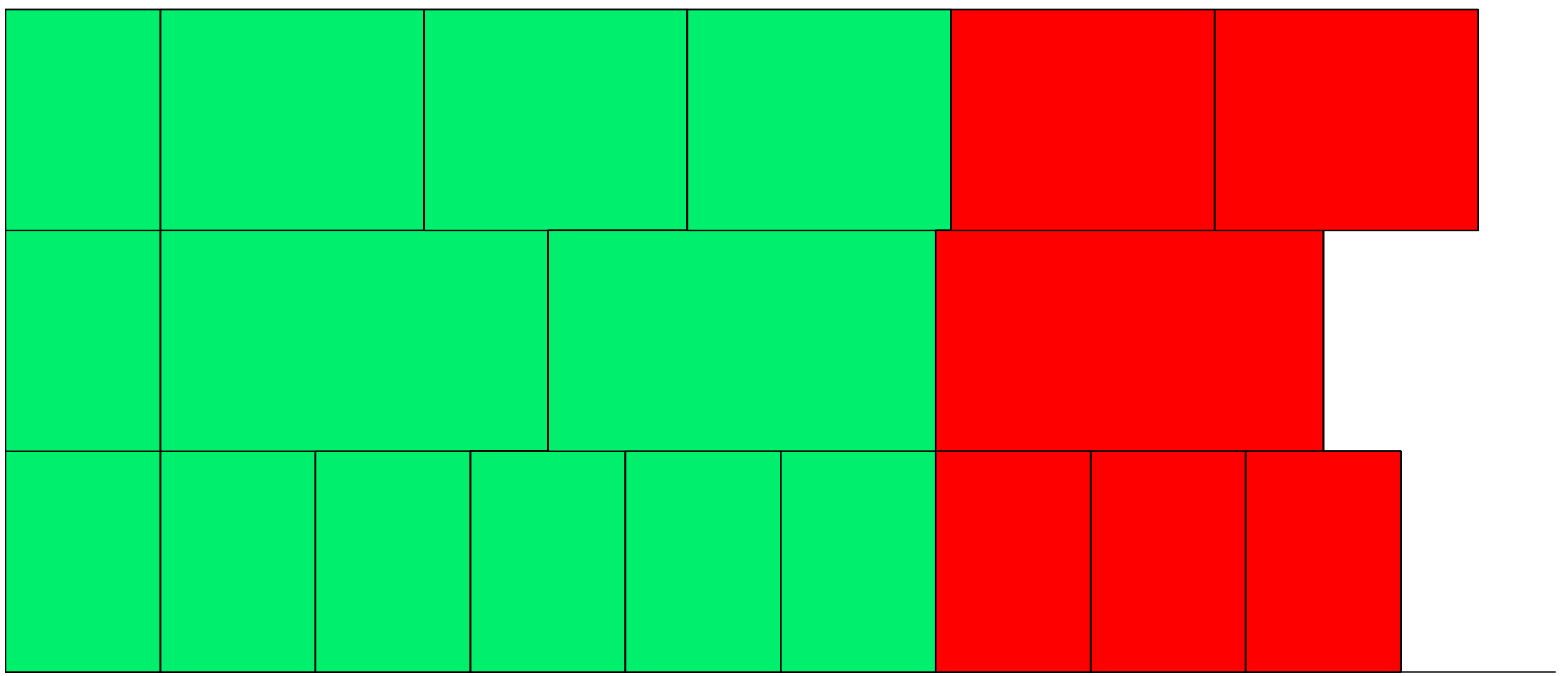
Multi-Adaptive Time-Stepping

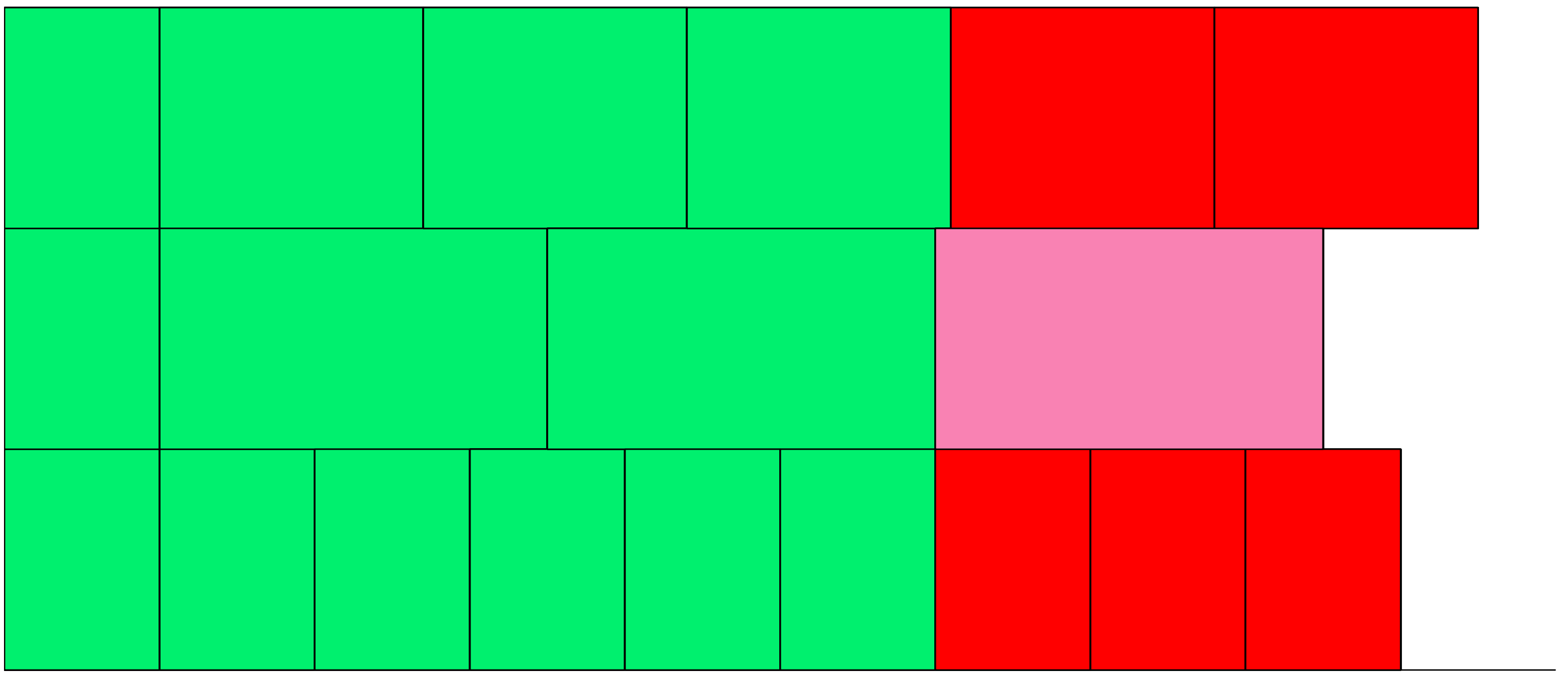
Action: Redoing steps



t = 0.09
n = 2.5

Multi-Adaptive Time-Stepping

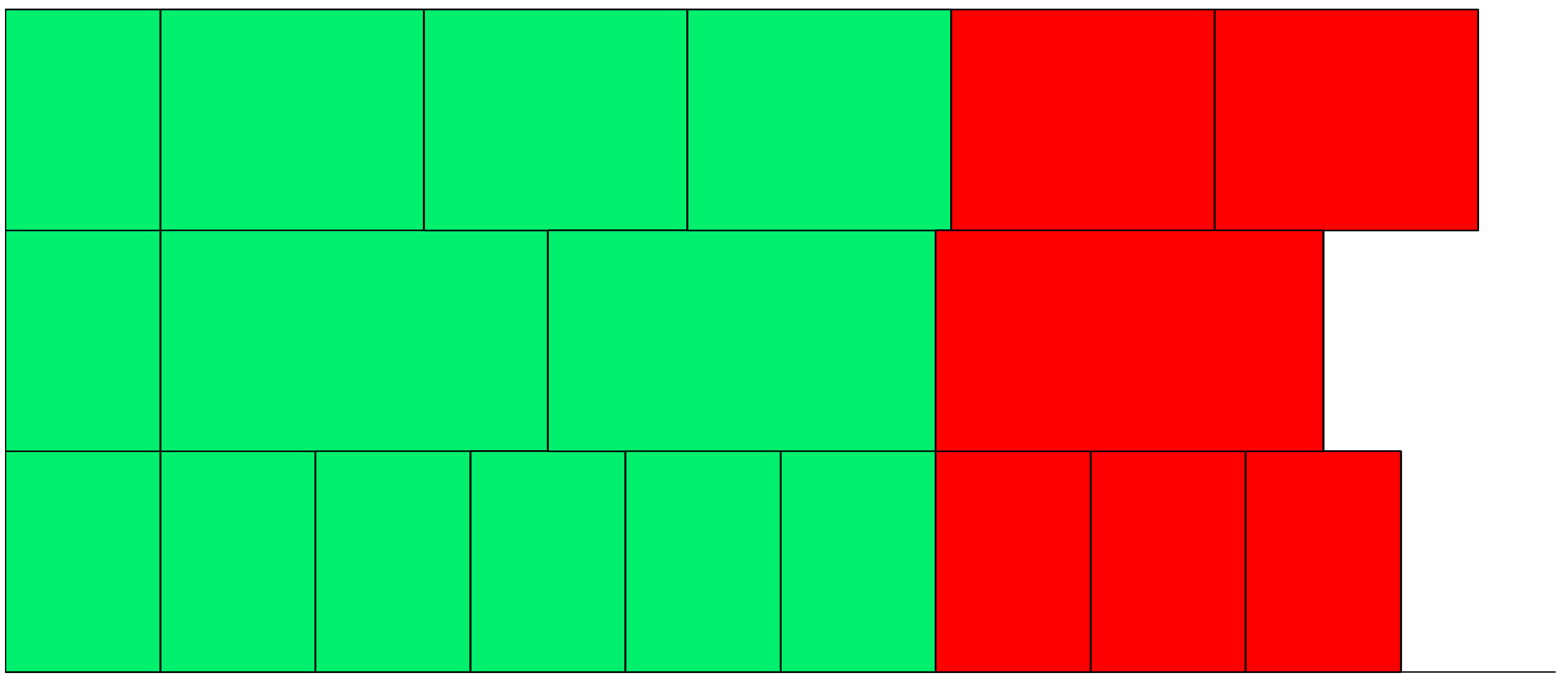
Action: Redoing steps



t = 0.09
n = 2.5

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.085
n = 3

Multi-Adaptive Time-Stepping

Action: Redoing steps

t = 0.085
n = 3

Multi-Adaptive Time-Stepping

Action: Redoing steps

t = 0.095
n = 3

Multi-Adaptive Time-Stepping

Action: Redoing steps

t = 0.095
n = 3

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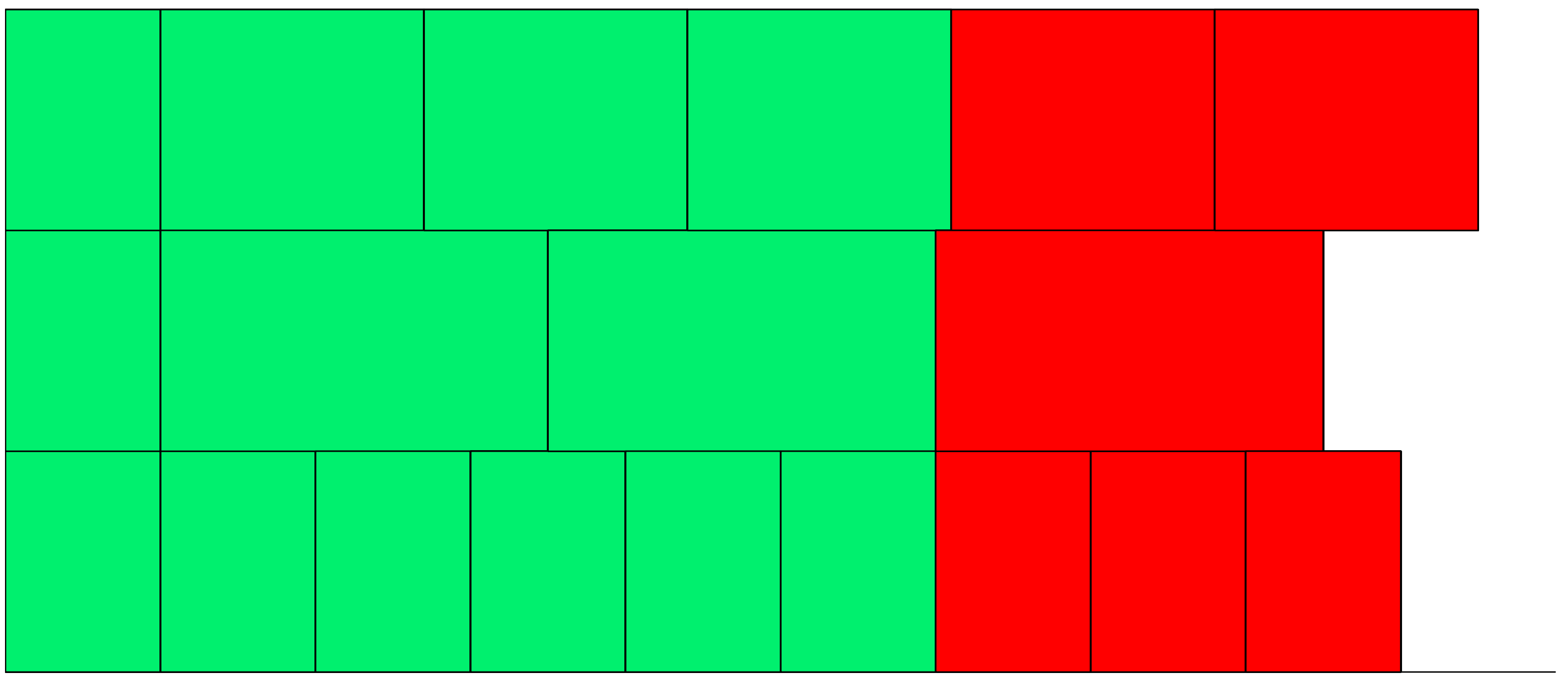
Multi-Adaptive Time-Stepping

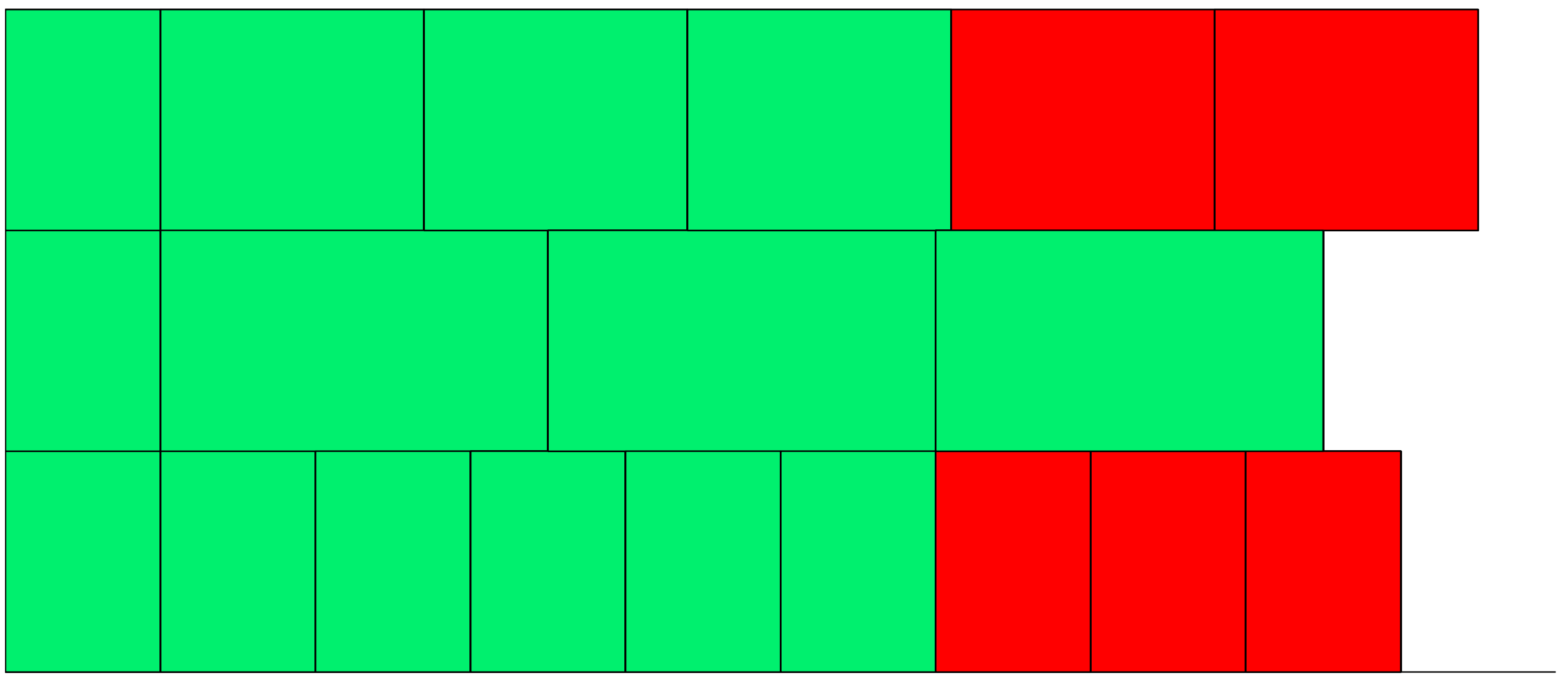
Action: Redoing steps



t = 0.09
n = 3.3333

Multi-Adaptive Time-Stepping

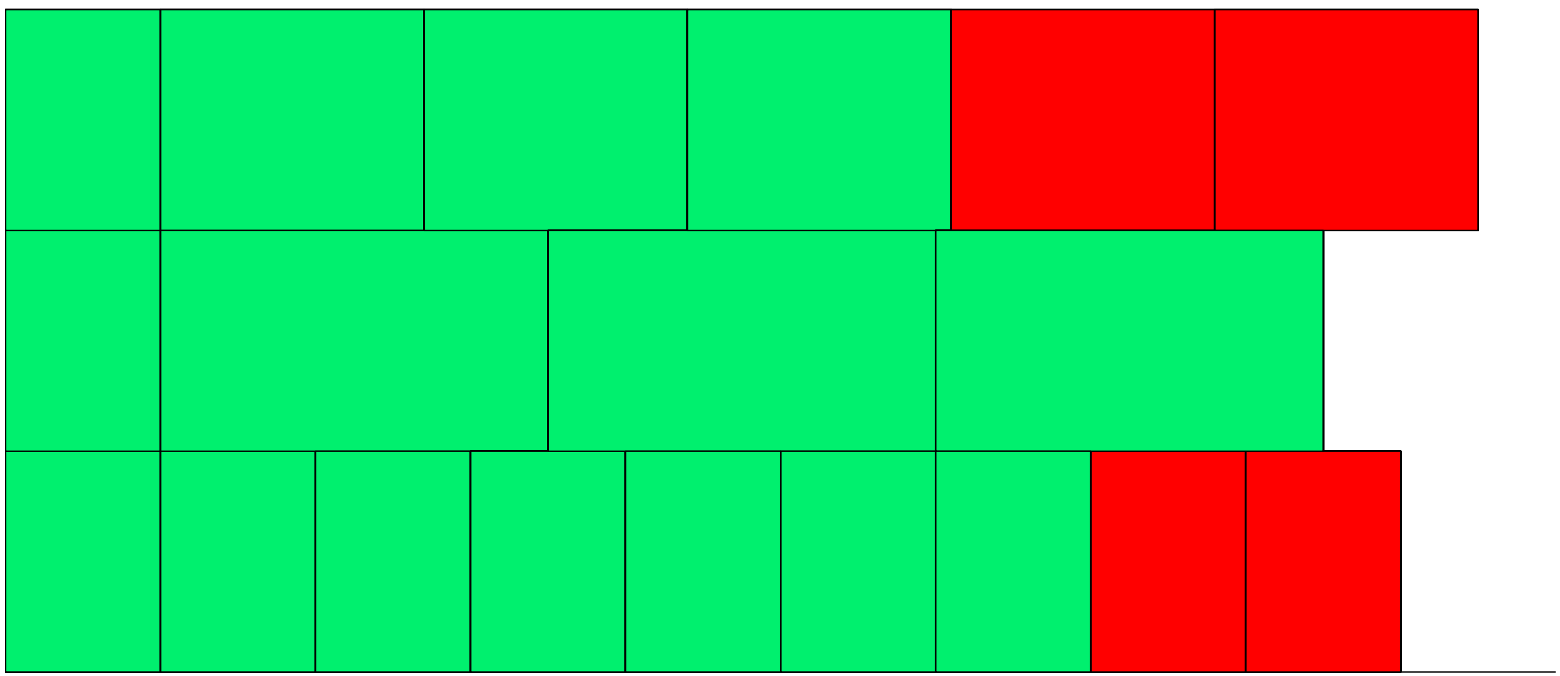
Action: Clearing steps



t = 0.085
n = 3.5

Multi-Adaptive Time-Stepping

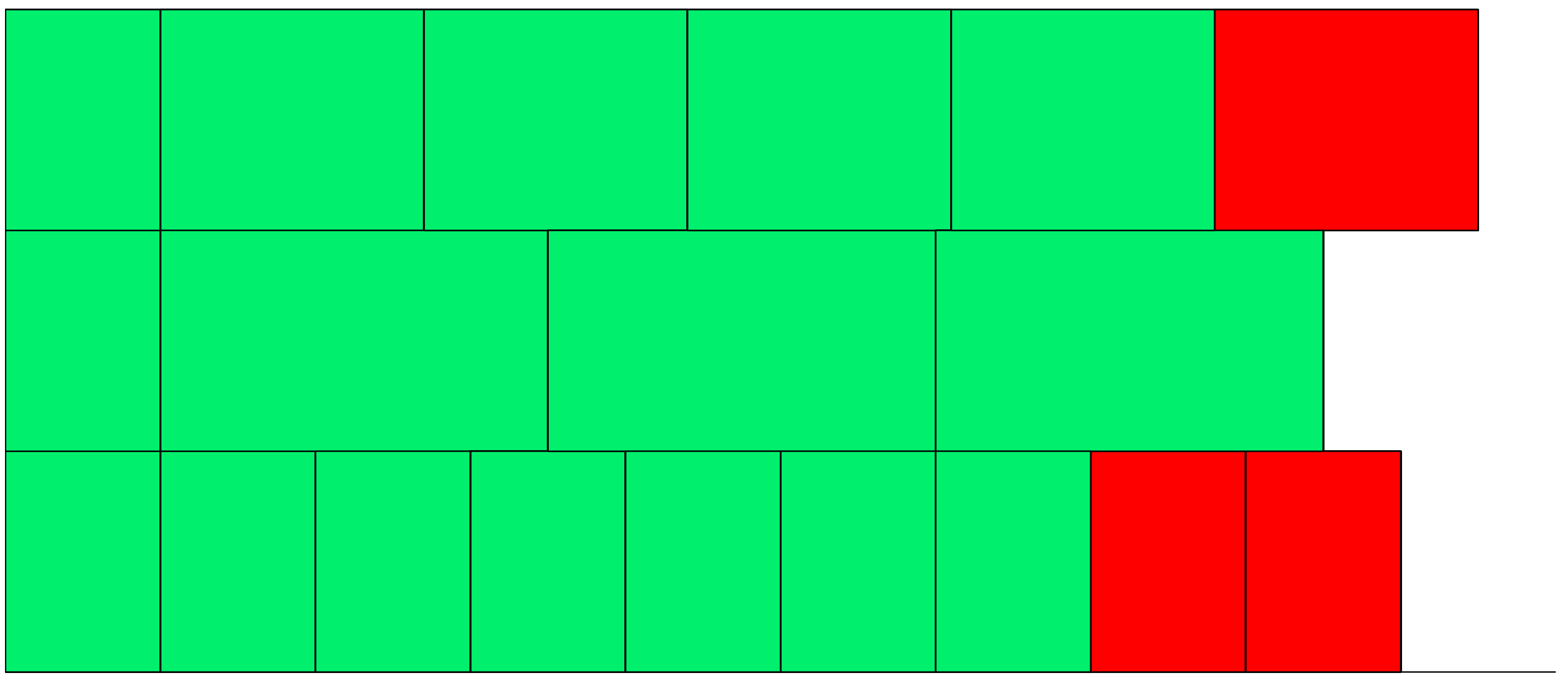
Action: Clearing steps



t = 0.07
n = 3.5

Multi-Adaptive Time-Stepping

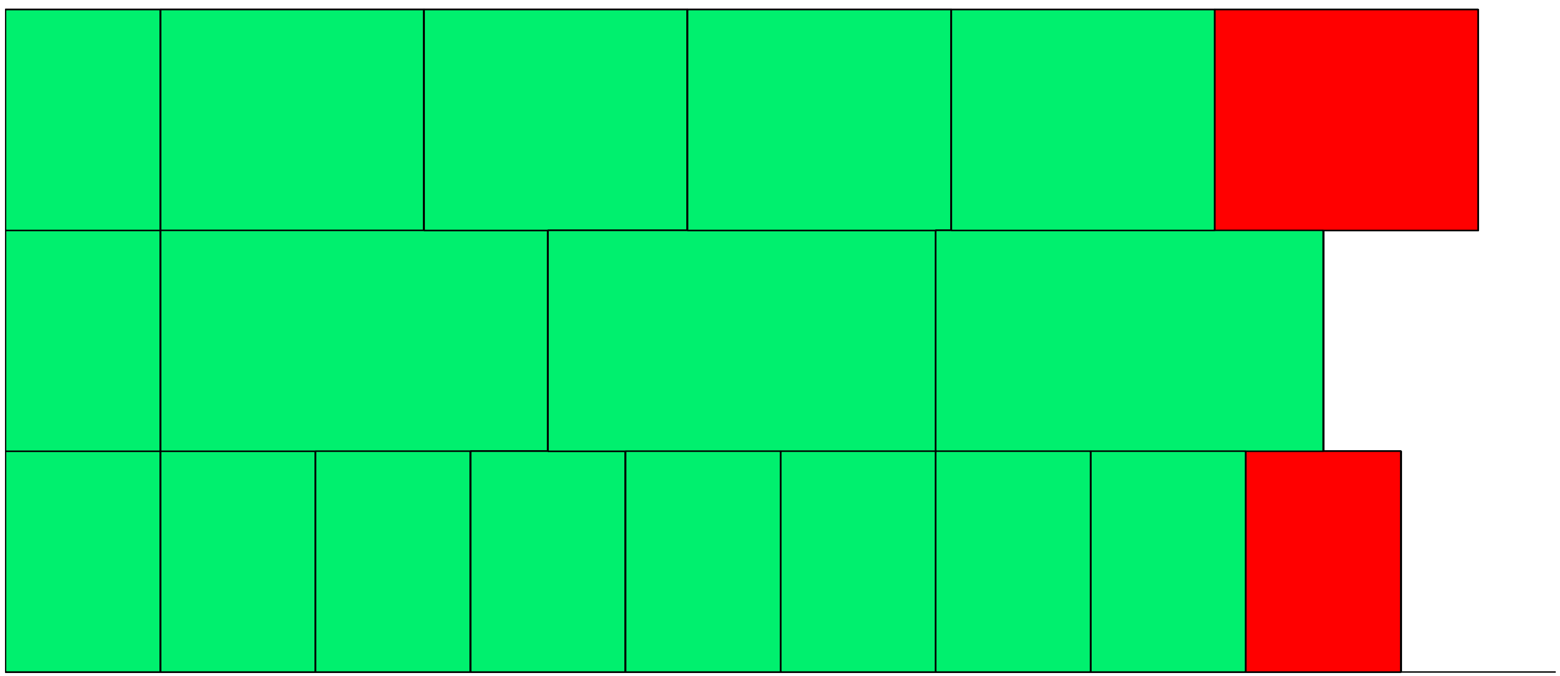
Action: Clearing steps



t = 0.078
n = 3.5

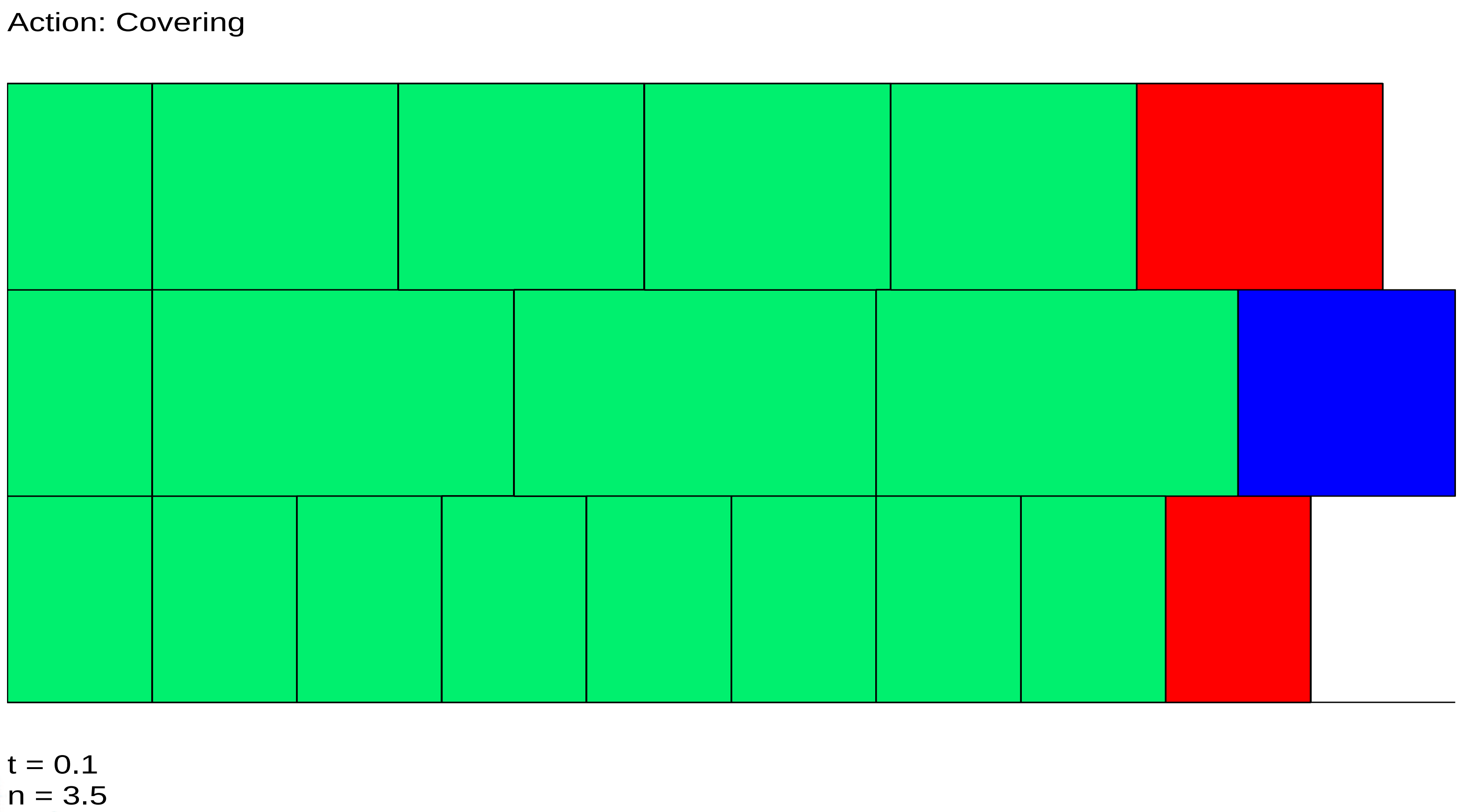
Multi-Adaptive Time-Stepping

Action: Clearing steps



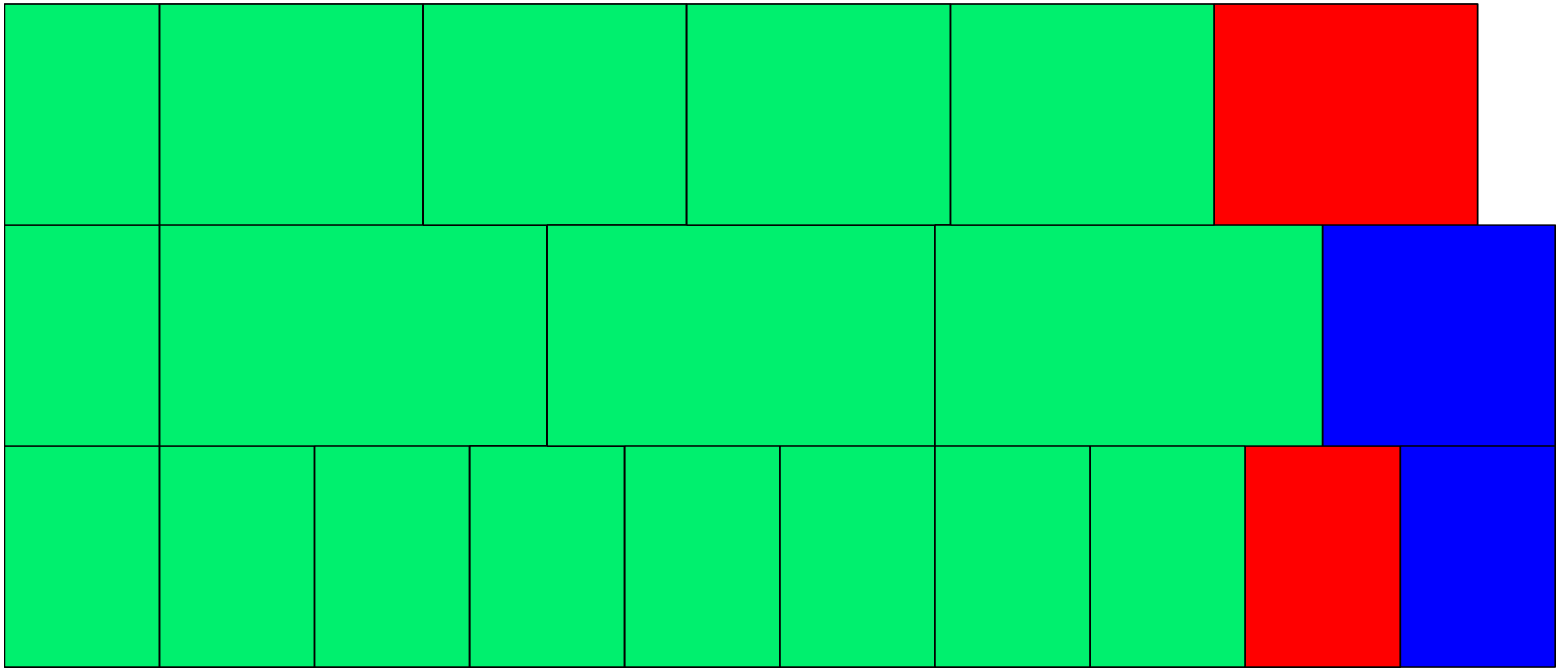
t = 0.08
n = 3.5

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

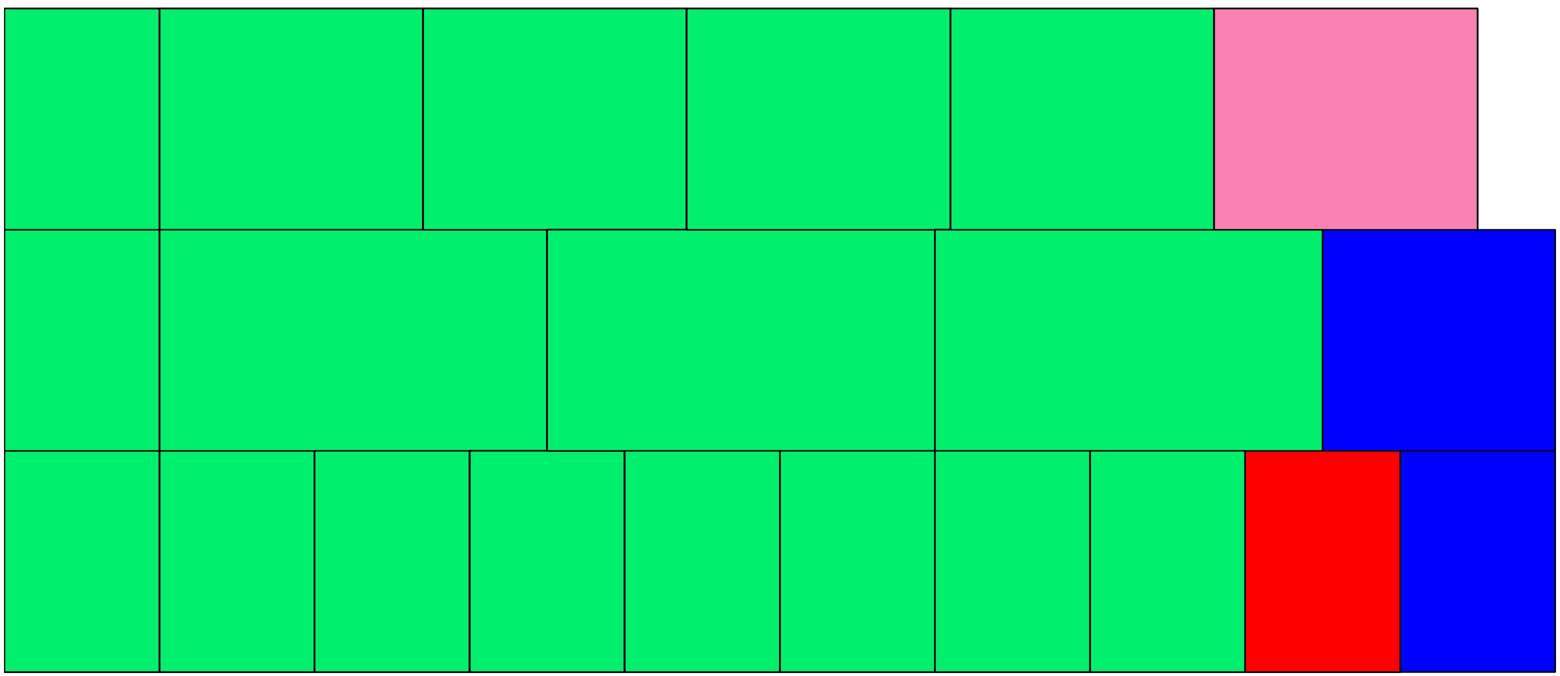
Action: Covering



t = 0.1
n = 3.5

Multi-Adaptive Time-Stepping

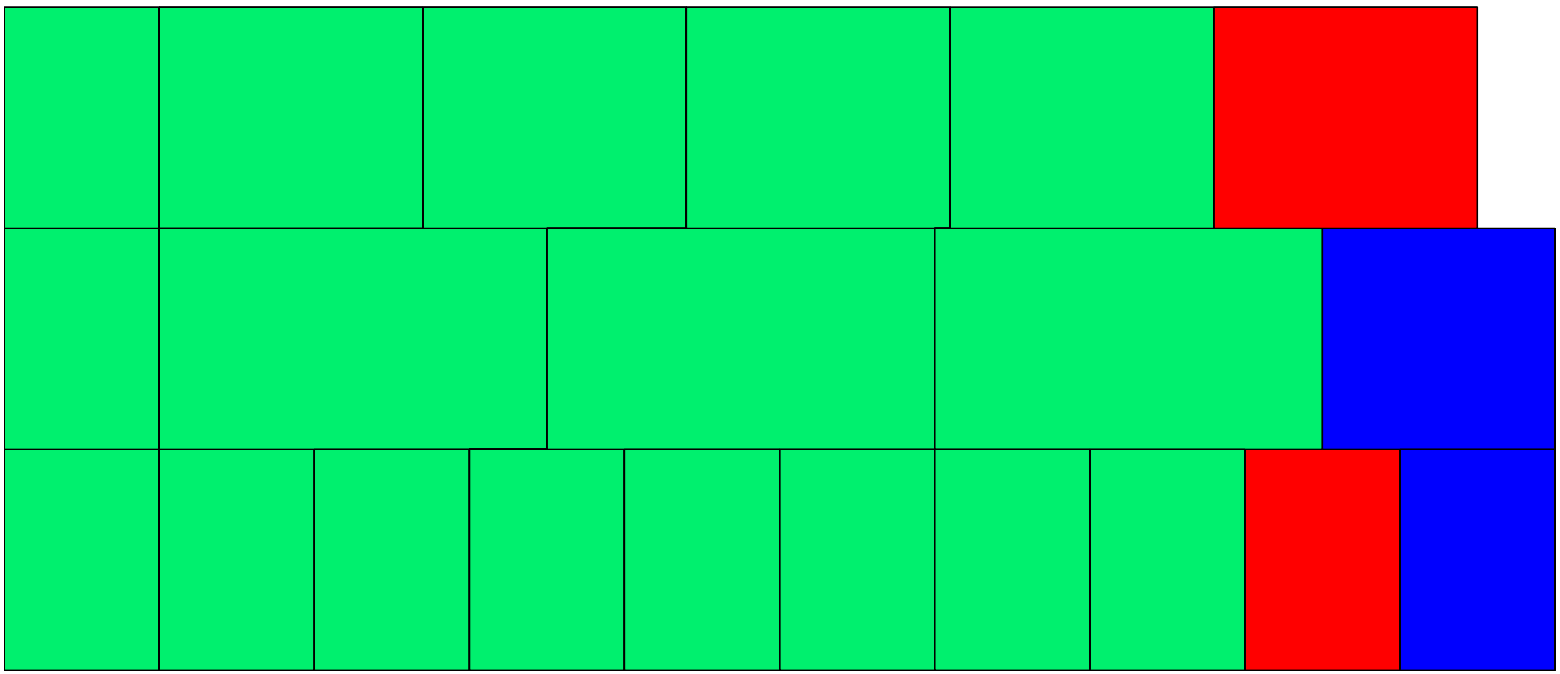
Action: Redoing steps



t = 0.1
n = 3.5

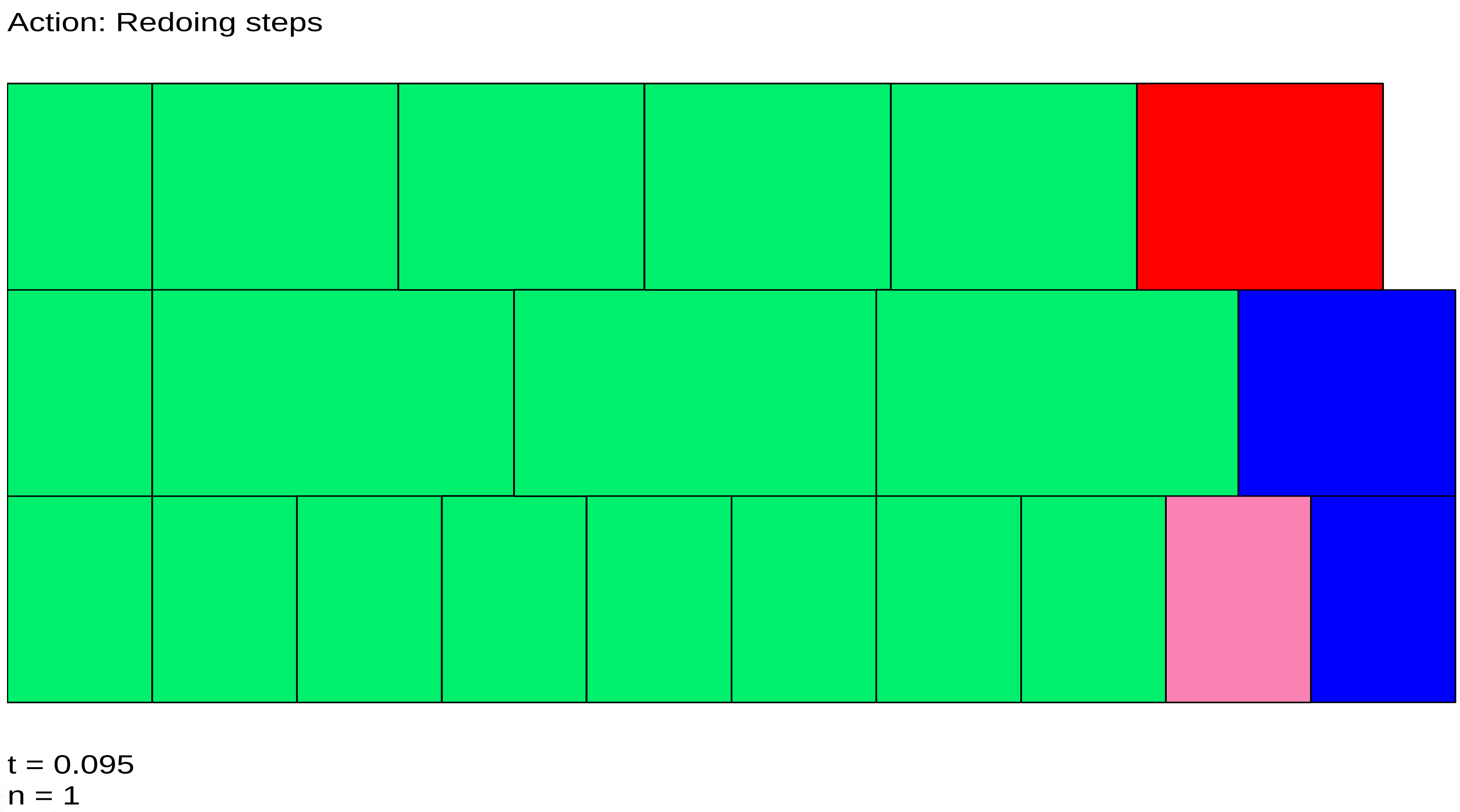
Multi-Adaptive Time-Stepping

Action: Redoing steps



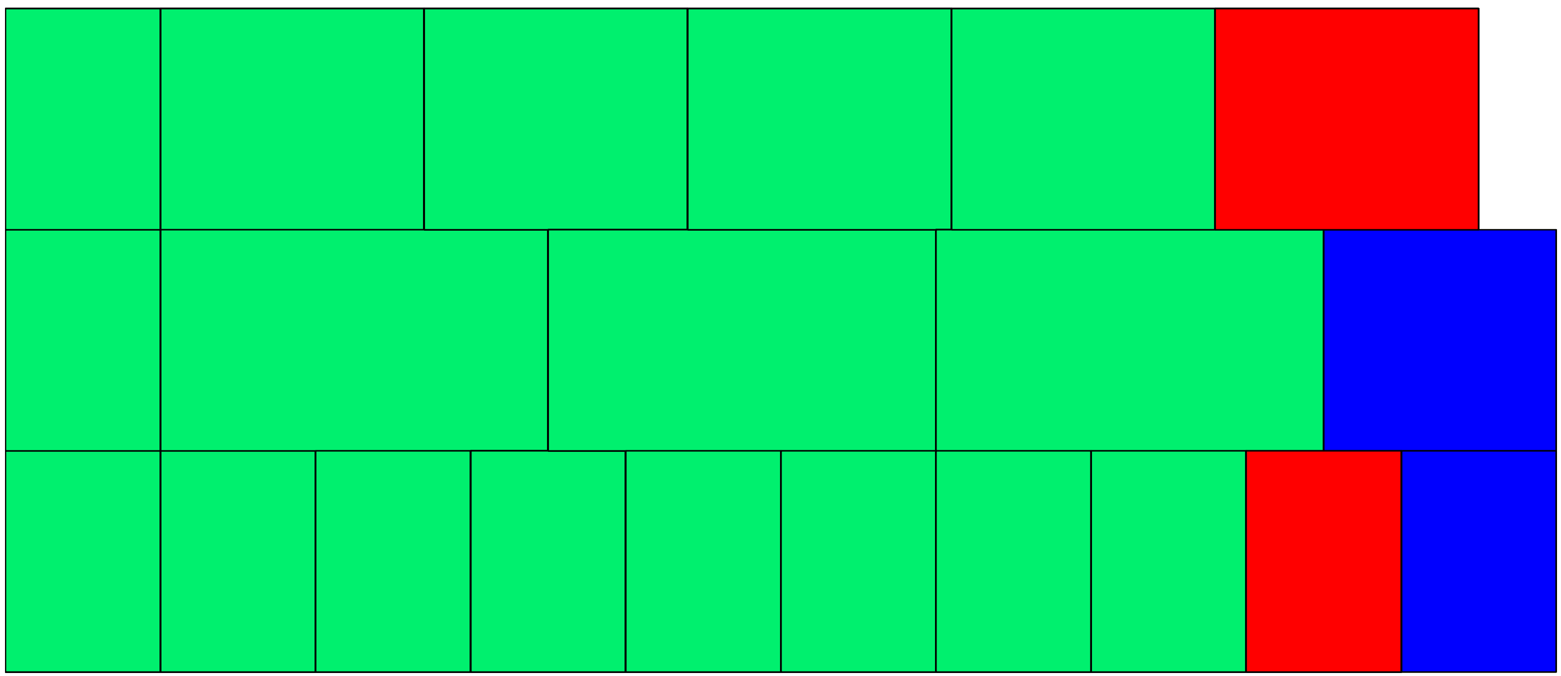
t = 0.095
n = 1

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

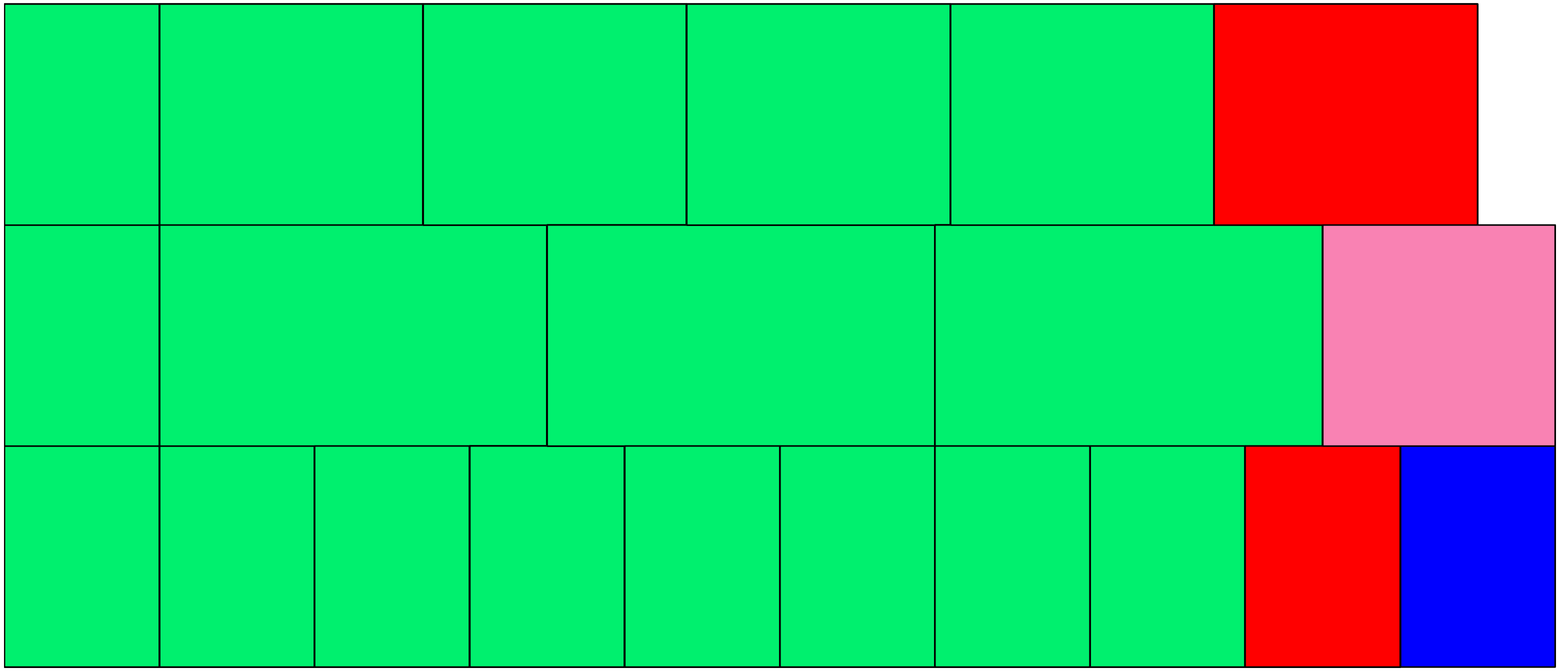
Action: Redoing steps



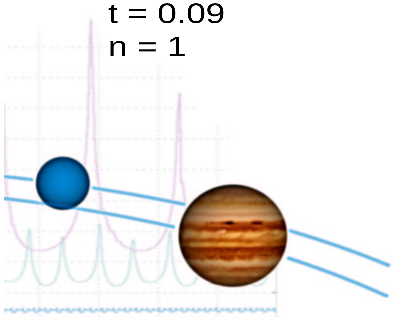
t = 0.09
n = 1

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.09
n = 1



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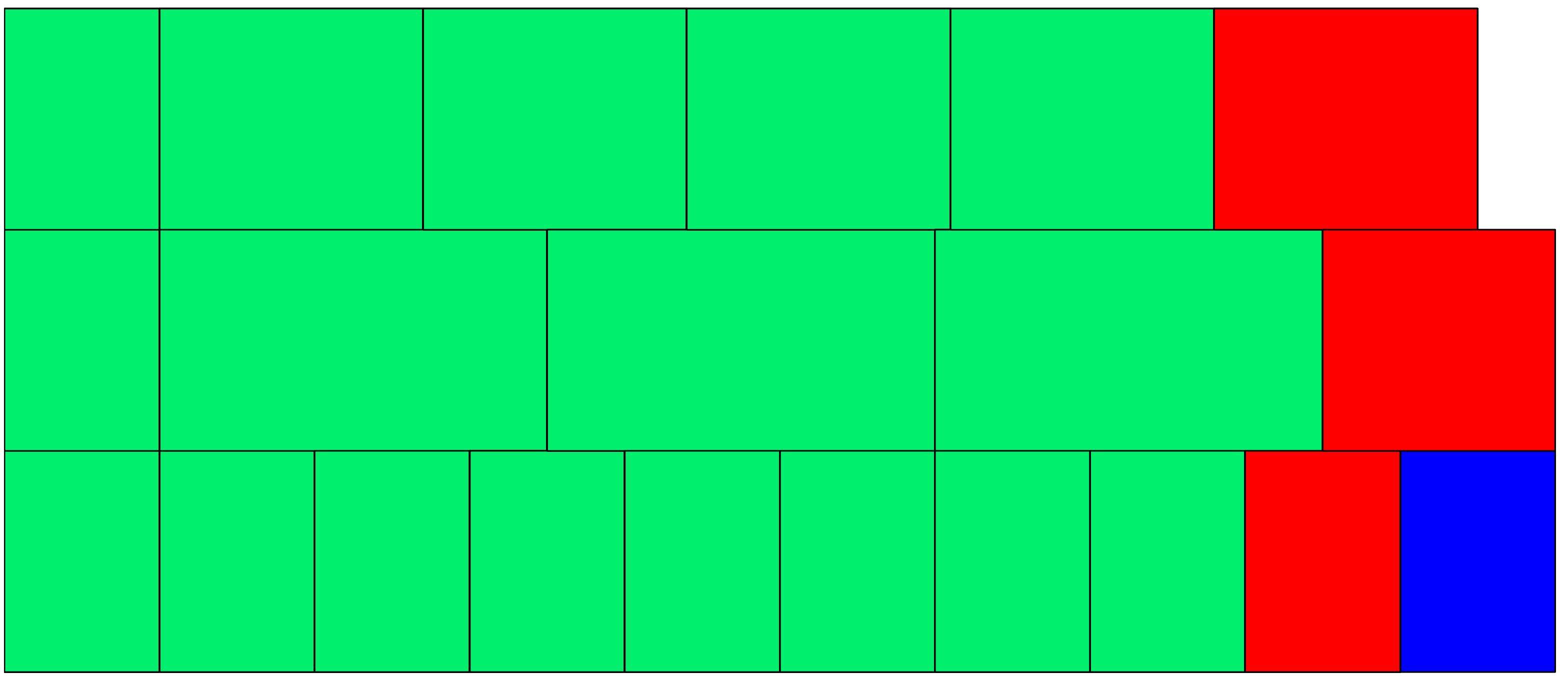
Multi-Adaptive Time-Stepping

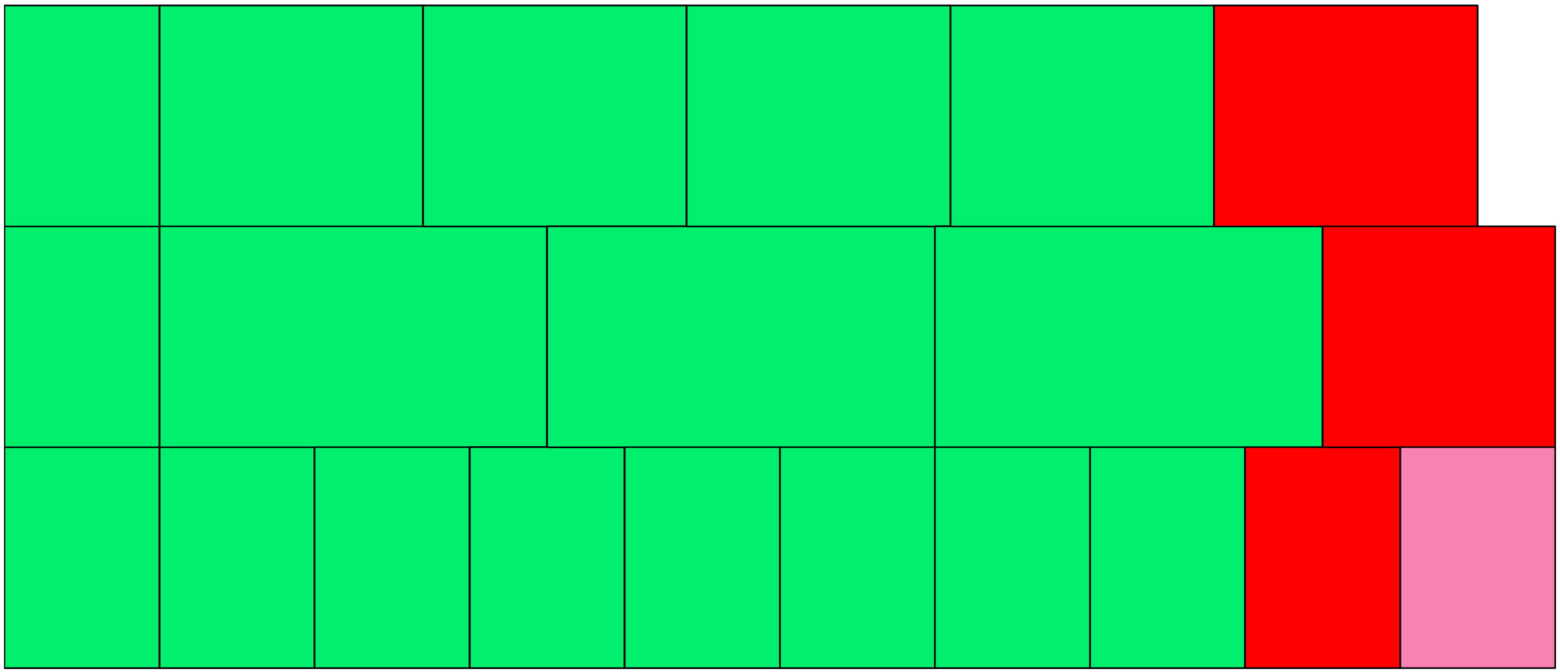
Action: Redoing steps



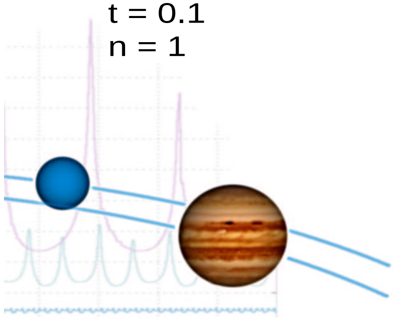
t = 0.1
n = 1

Multi-Adaptive Time-Stepping

Action: Redoing steps

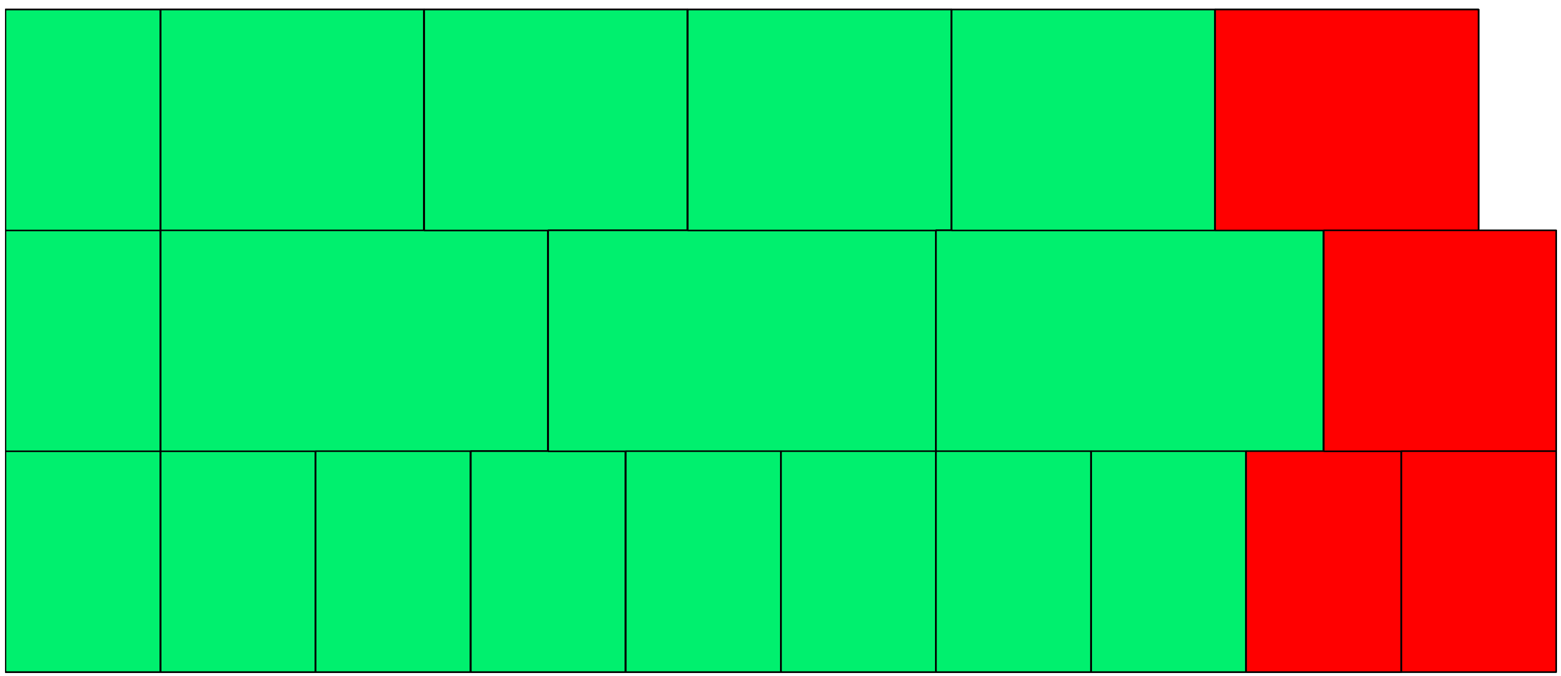


t = 0.1
n = 1



Multi-Adaptive Time-Stepping

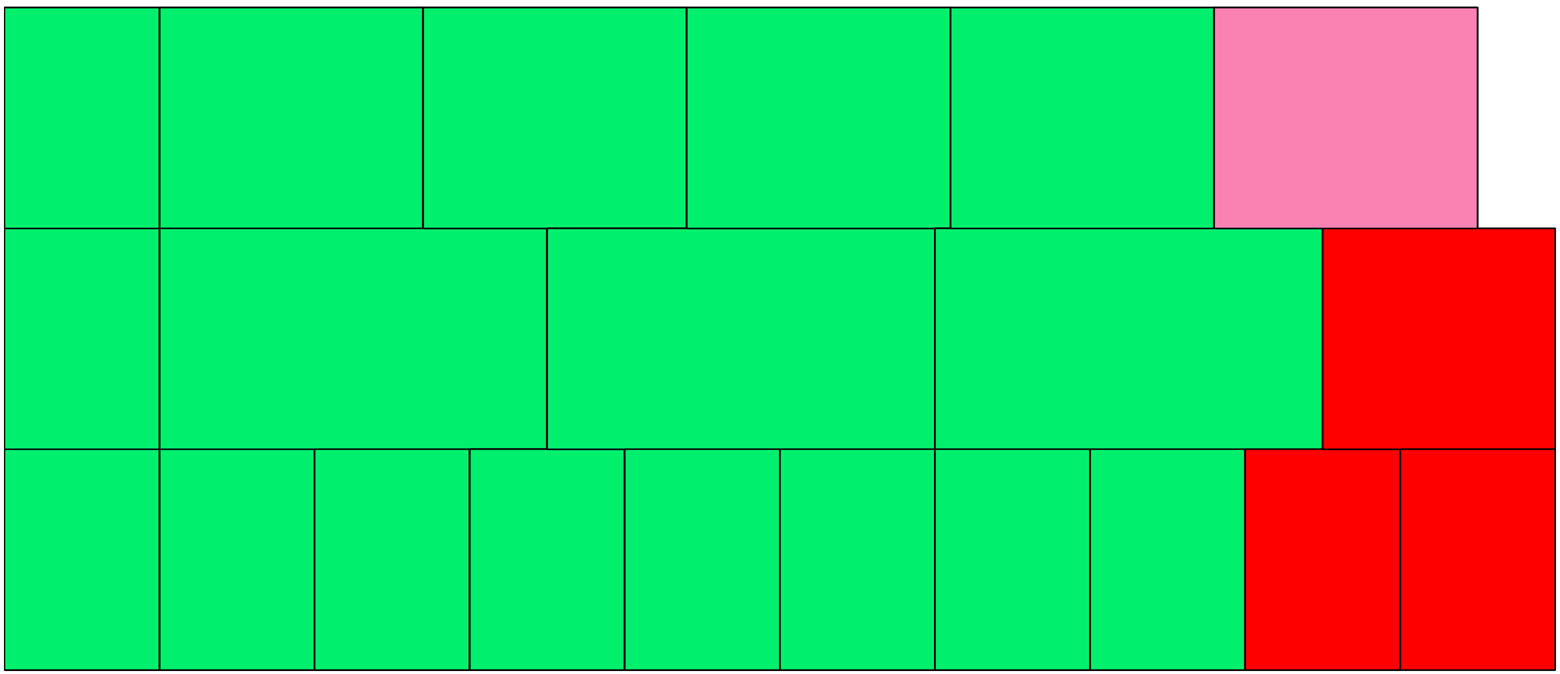
Action: Redoing steps



t = 0.1
n = 1

Multi-Adaptive Time-Stepping

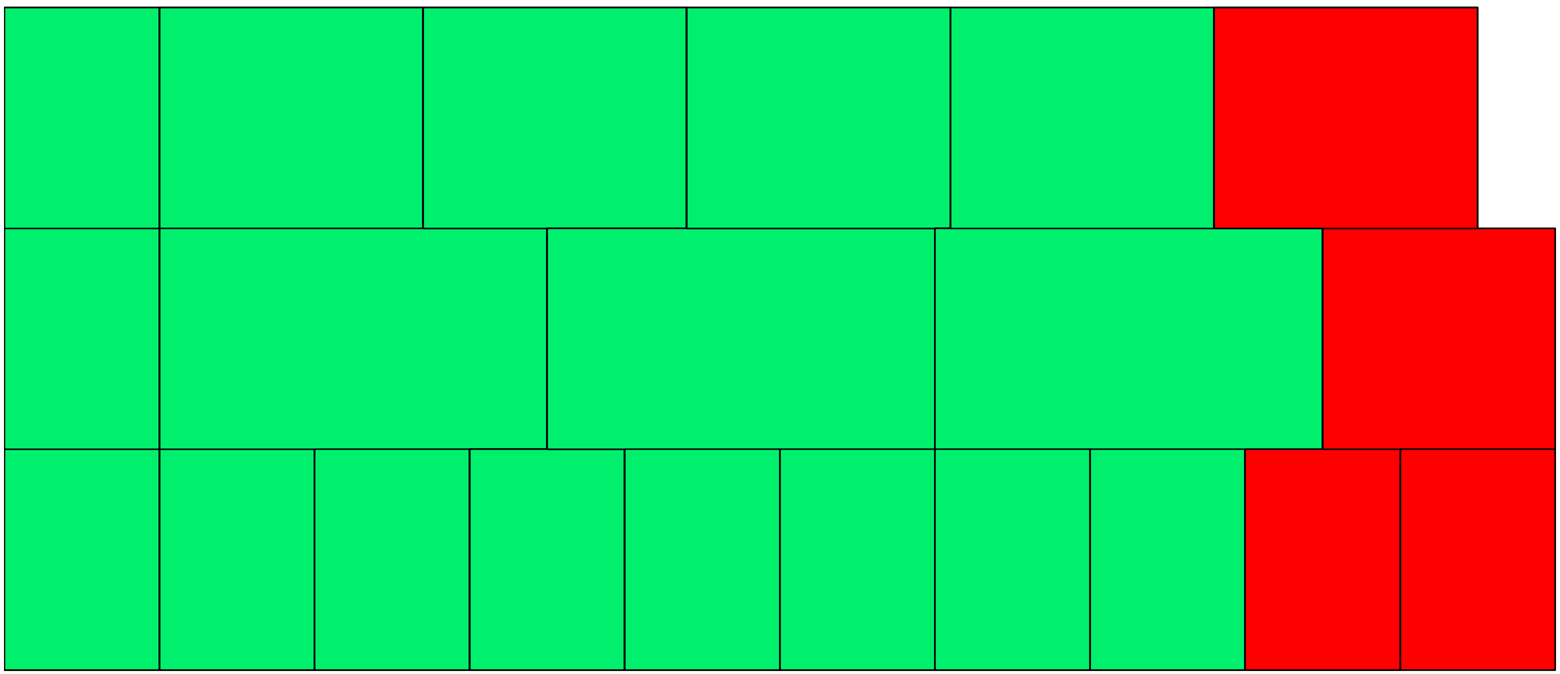
Action: Redoing steps



t = 0.1
n = 1

Multi-Adaptive Time-Stepping

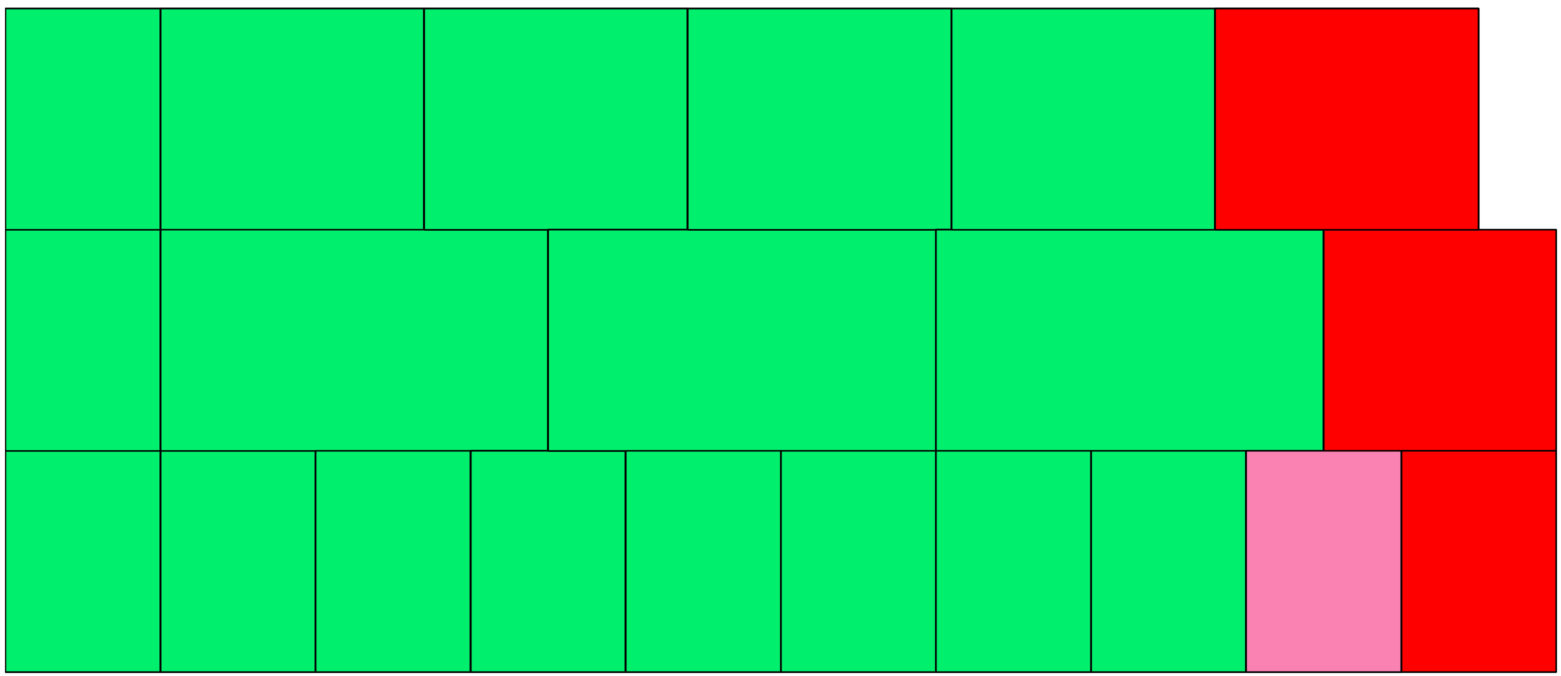
Action: Redoing steps



t = 0.095
n = 2

Multi-Adaptive Time-Stepping

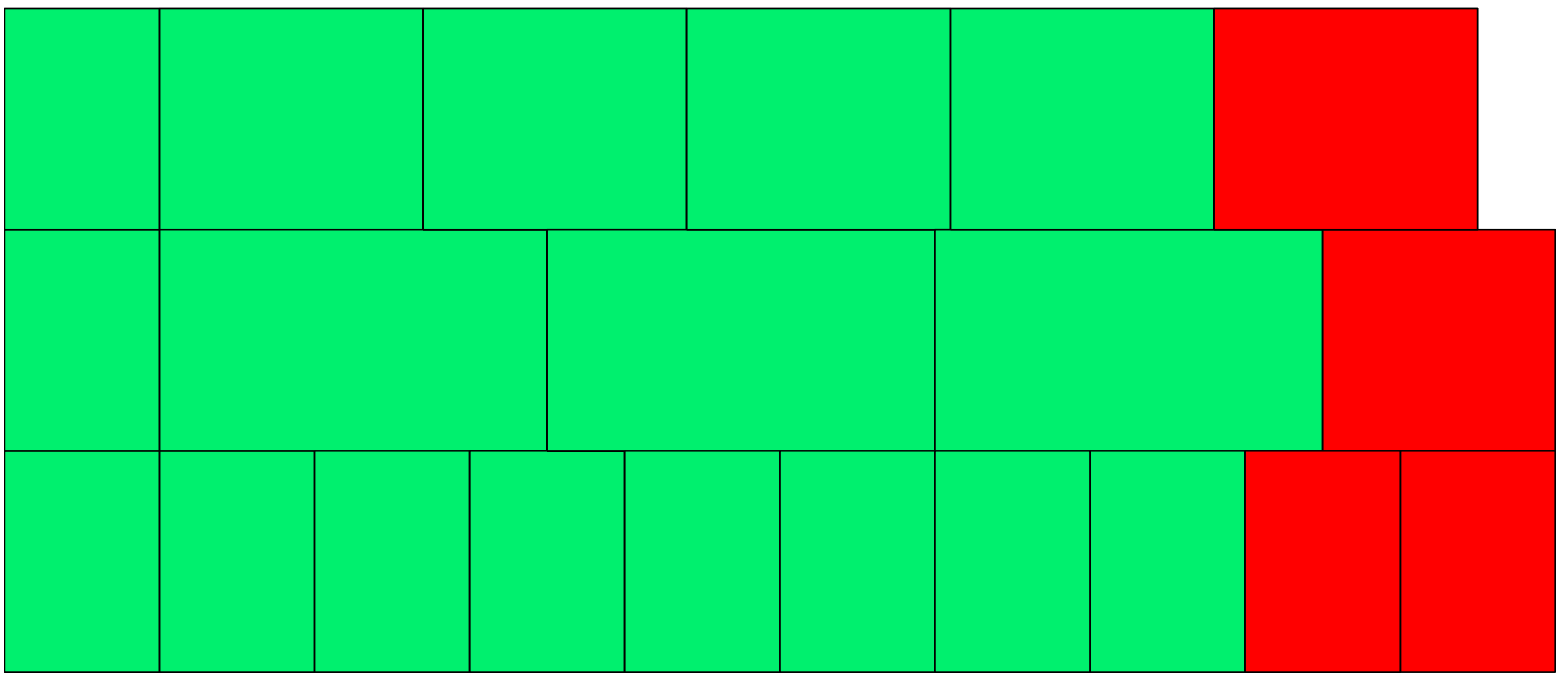
Action: Redoing steps



t = 0.095
n = 2

Multi-Adaptive Time-Stepping

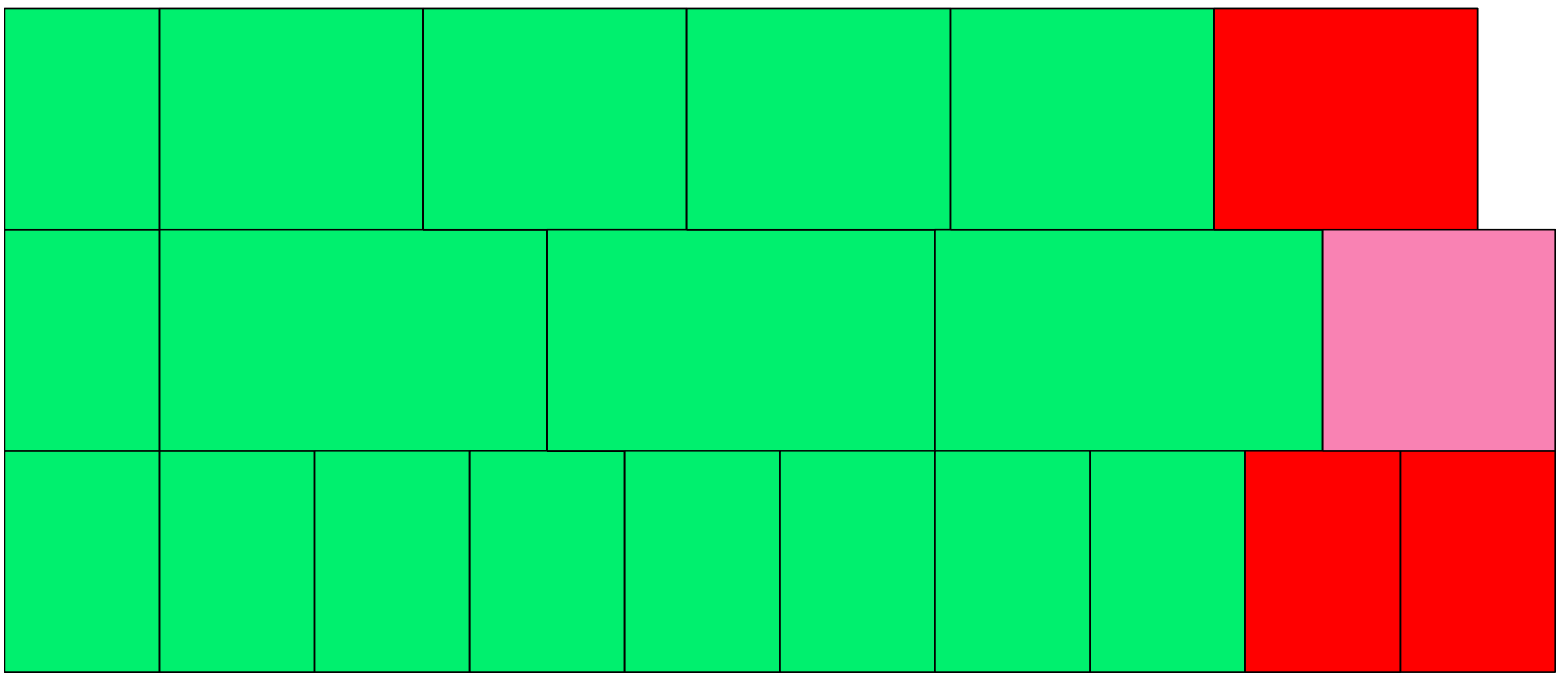
Action: Redoing steps



t = 0.09
n = 2

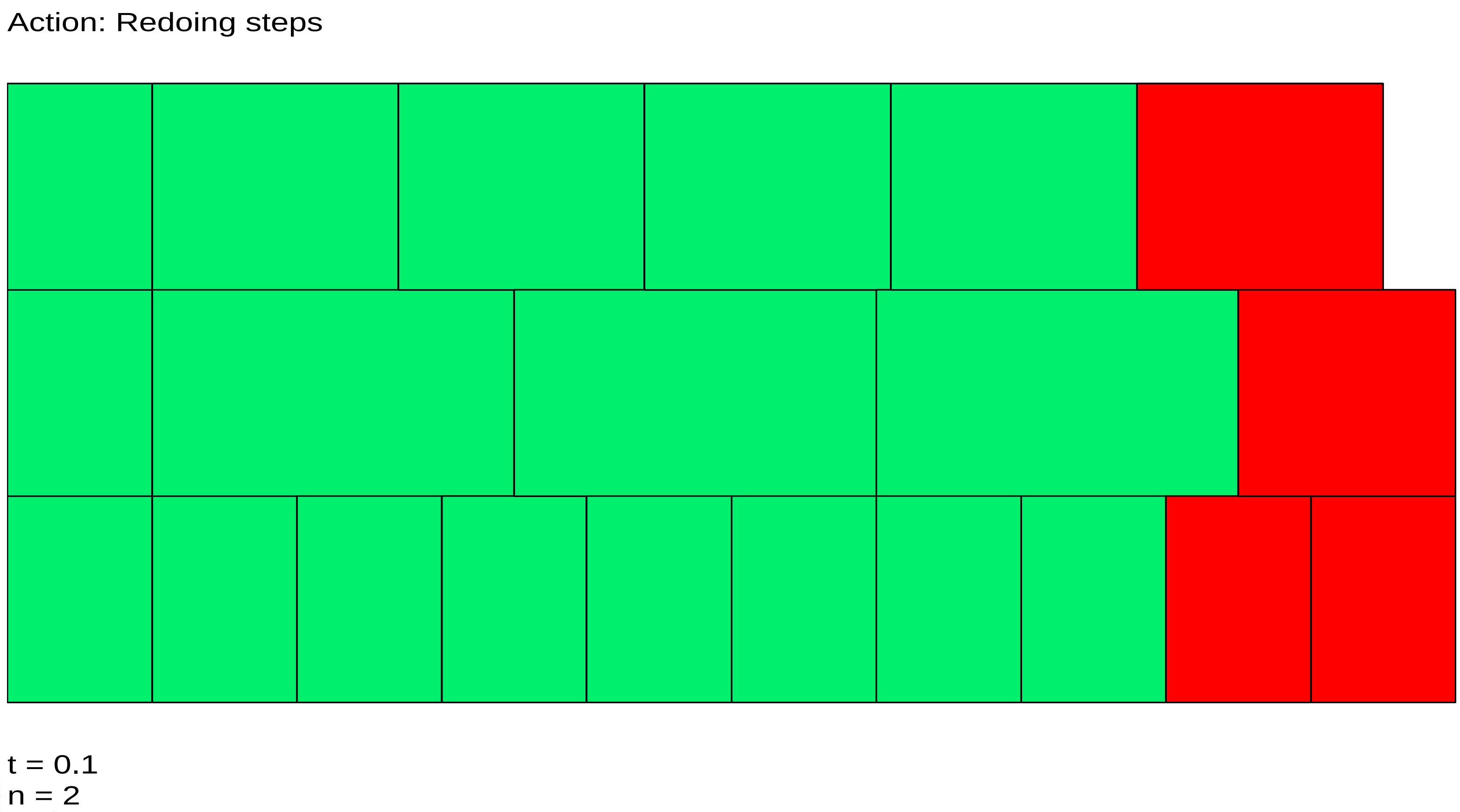
Multi-Adaptive Time-Stepping

Action: Redoing steps

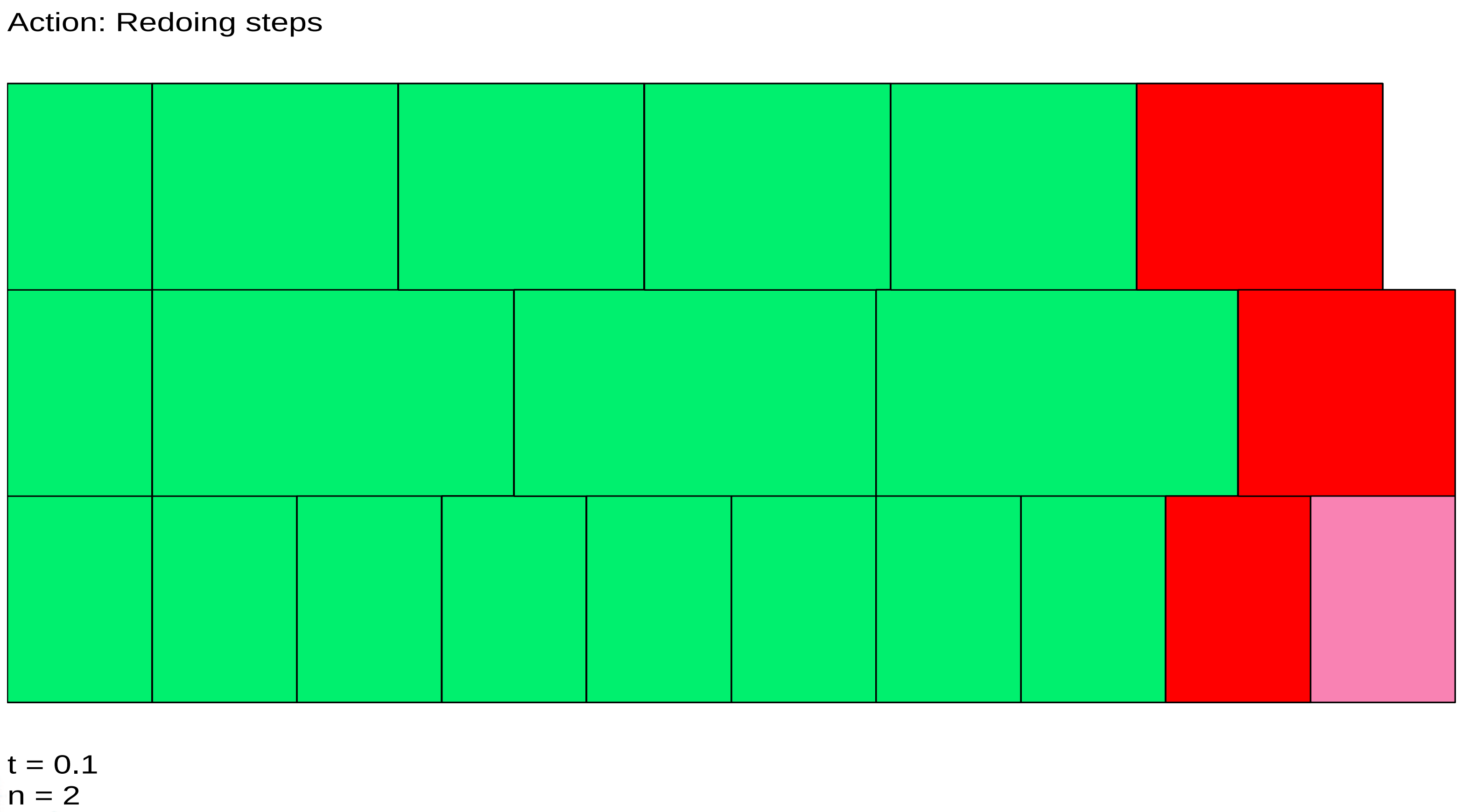


t = 0.09
n = 2

Multi-Adaptive Time-Stepping

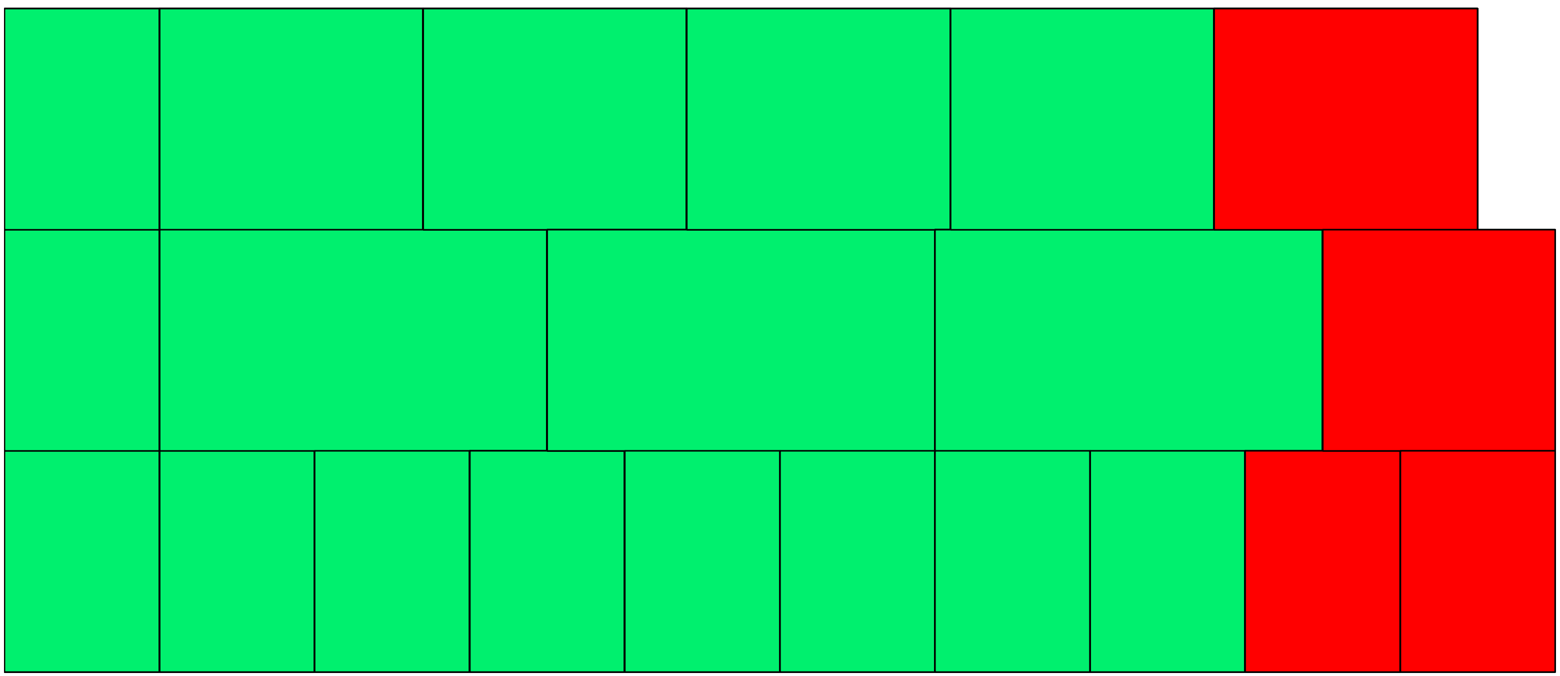


Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

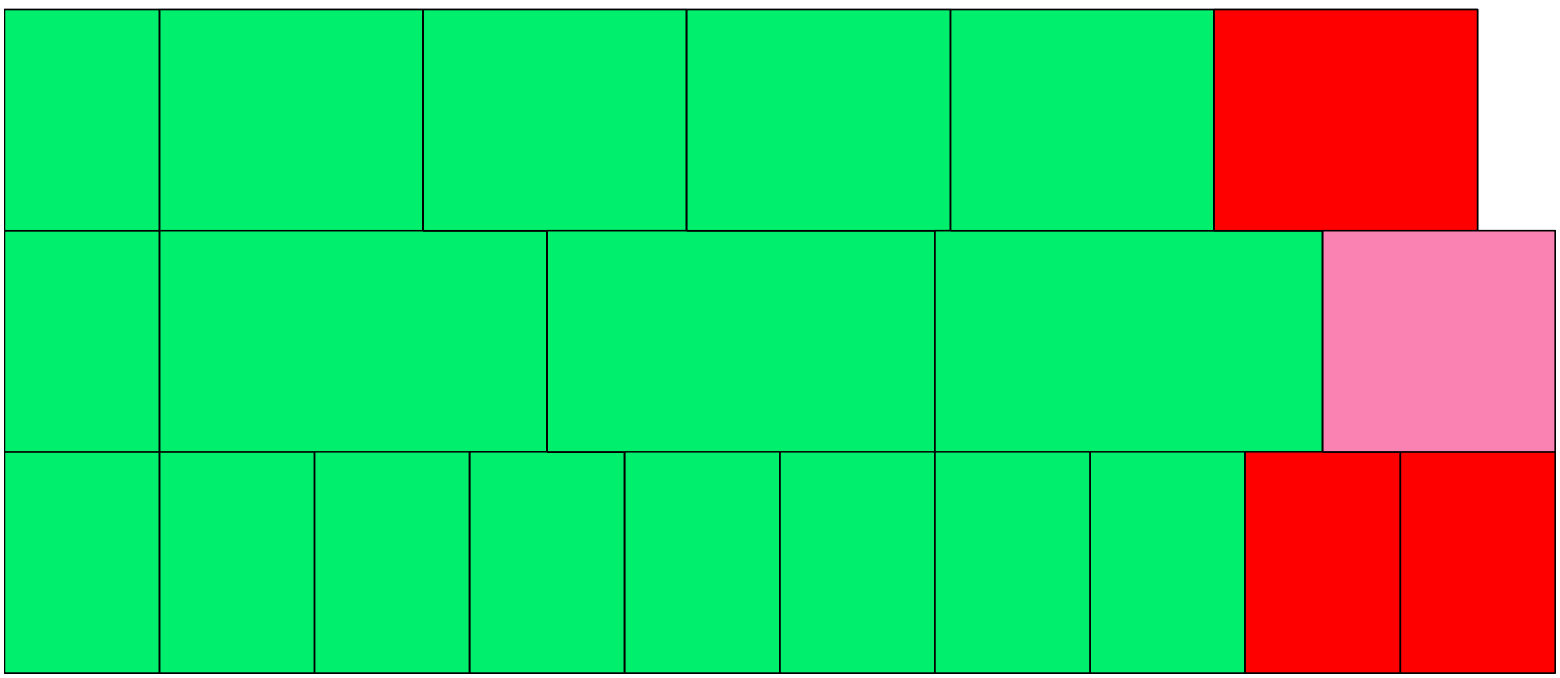
Action: Redoing steps



t = 0.1
n = 2.75

Multi-Adaptive Time-Stepping

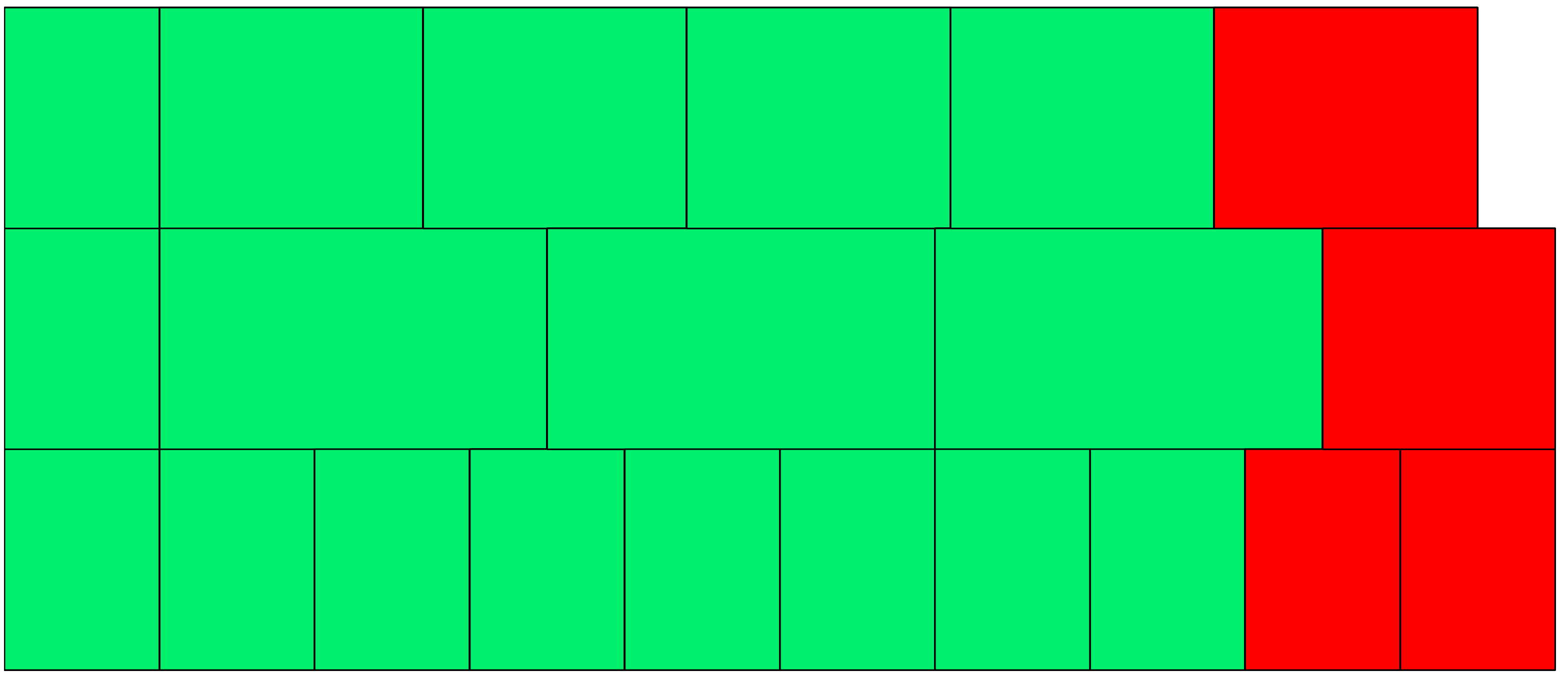
Action: Redoing steps



t = 0.1
n = 2.75

Multi-Adaptive Time-Stepping

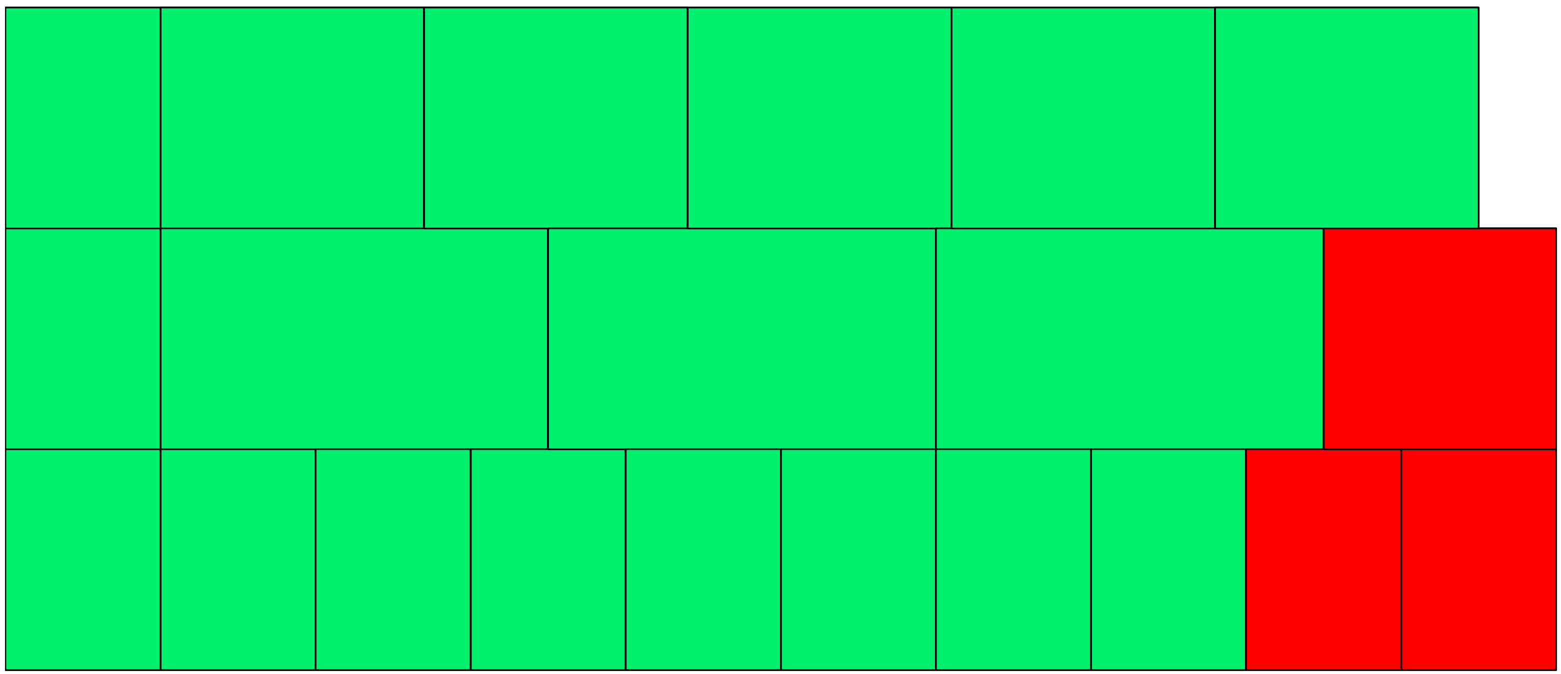
Action: Redoing steps



t = 0.1
n = 3

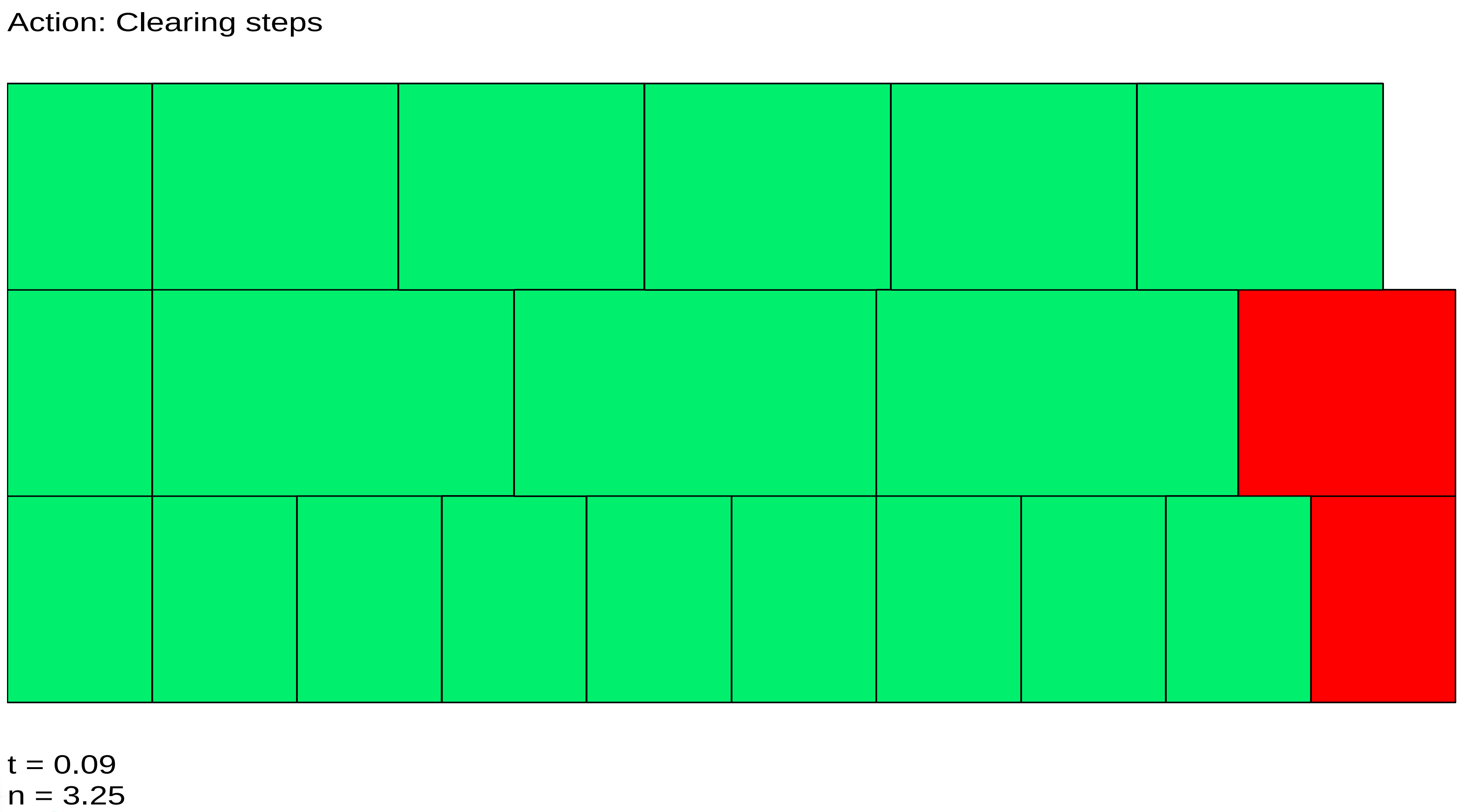
Multi-Adaptive Time-Stepping

Action: Clearing steps



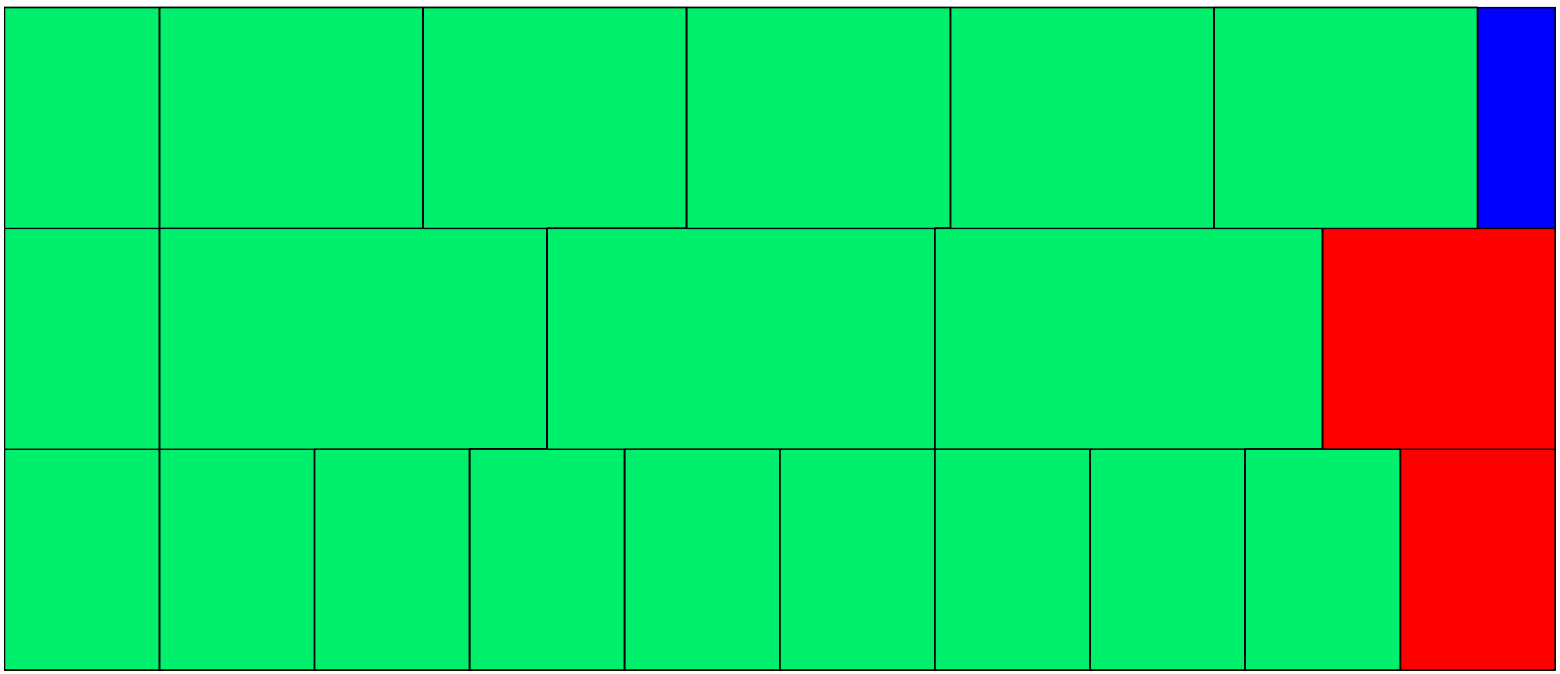
t = 0.095
n = 3.25

Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

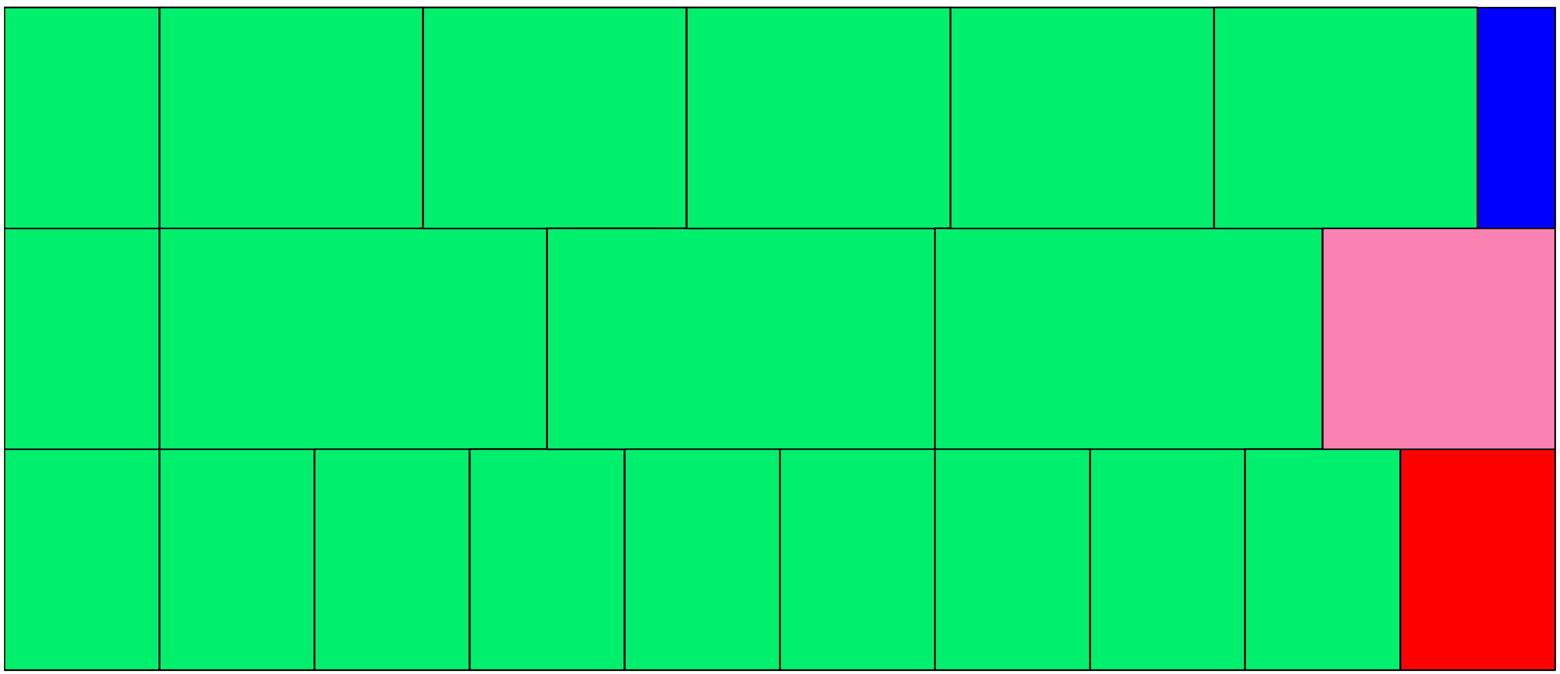
Action: Covering



t = 0.1
n = 3.25

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.1
n = 3.25

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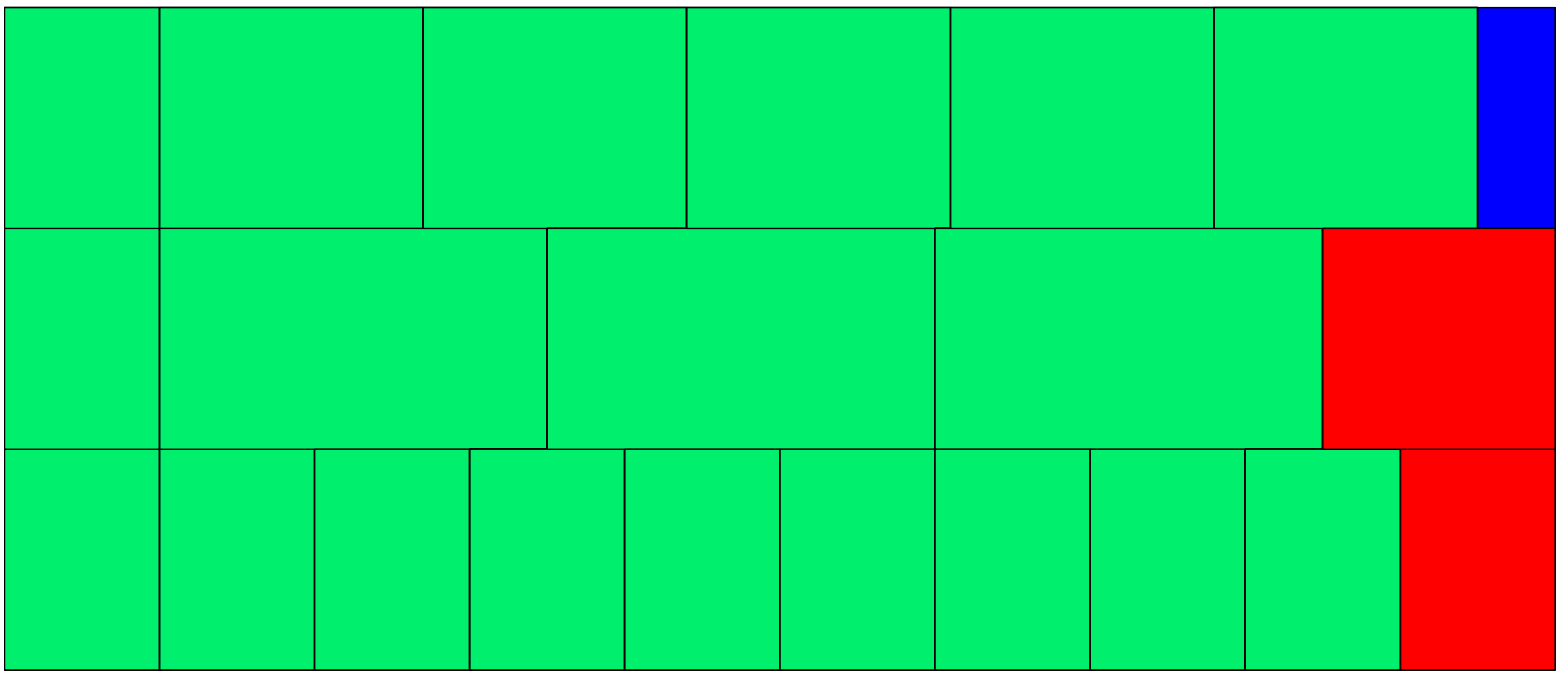
Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.1
n = 1

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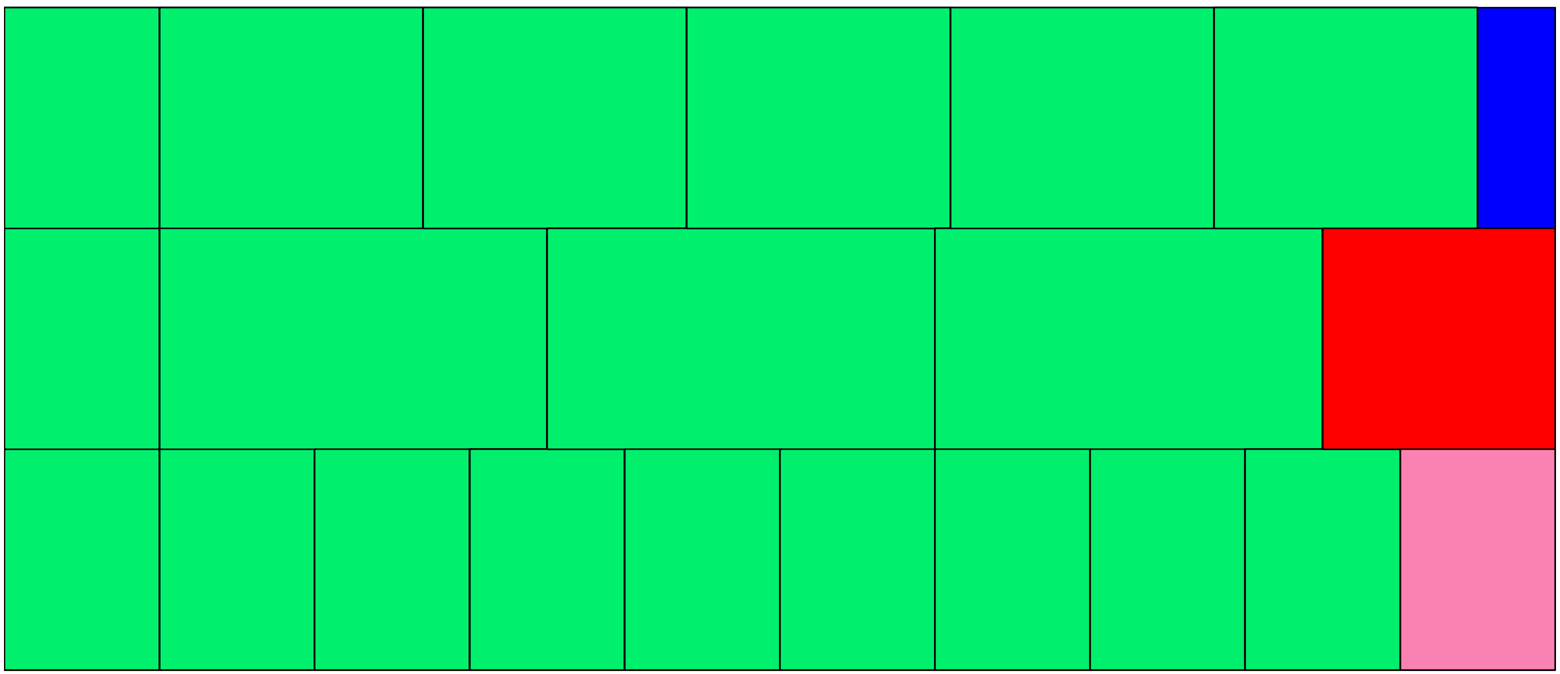
Multi-Adaptive Time-Stepping

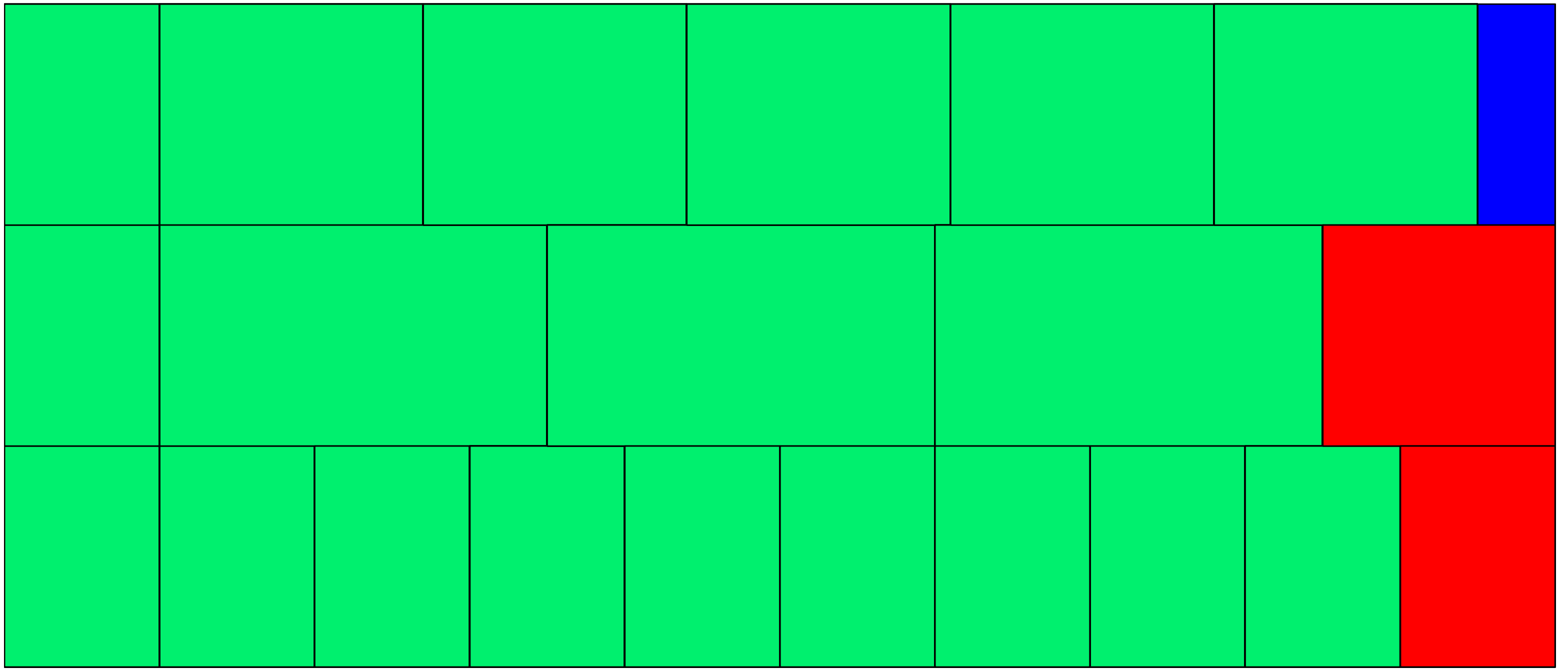
Action: Redoing steps



t = 0.1
n = 1

Multi-Adaptive Time-Stepping

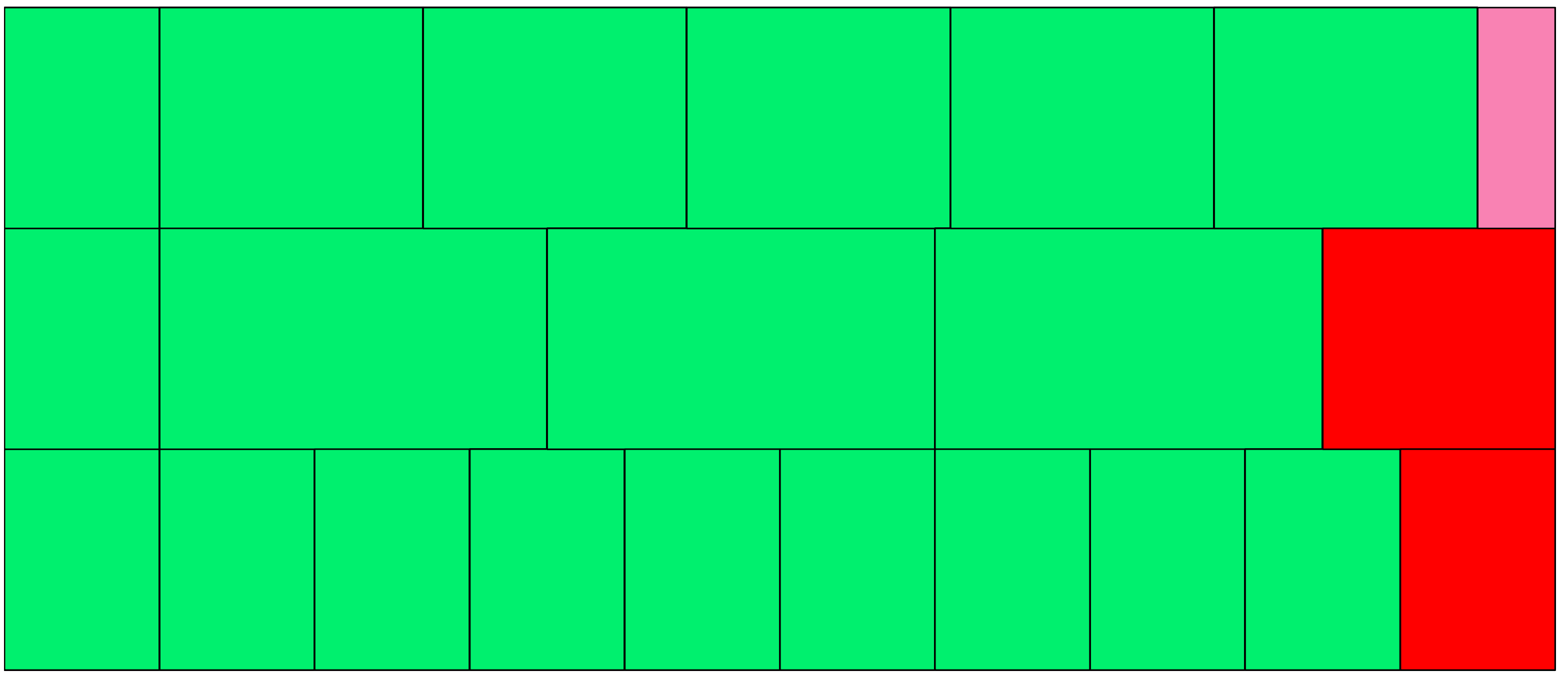
Action: Redoing steps



t = 0.1
n = 1

Multi-Adaptive Time-Stepping

Action: Redoing steps



t = 0.1
n = 1

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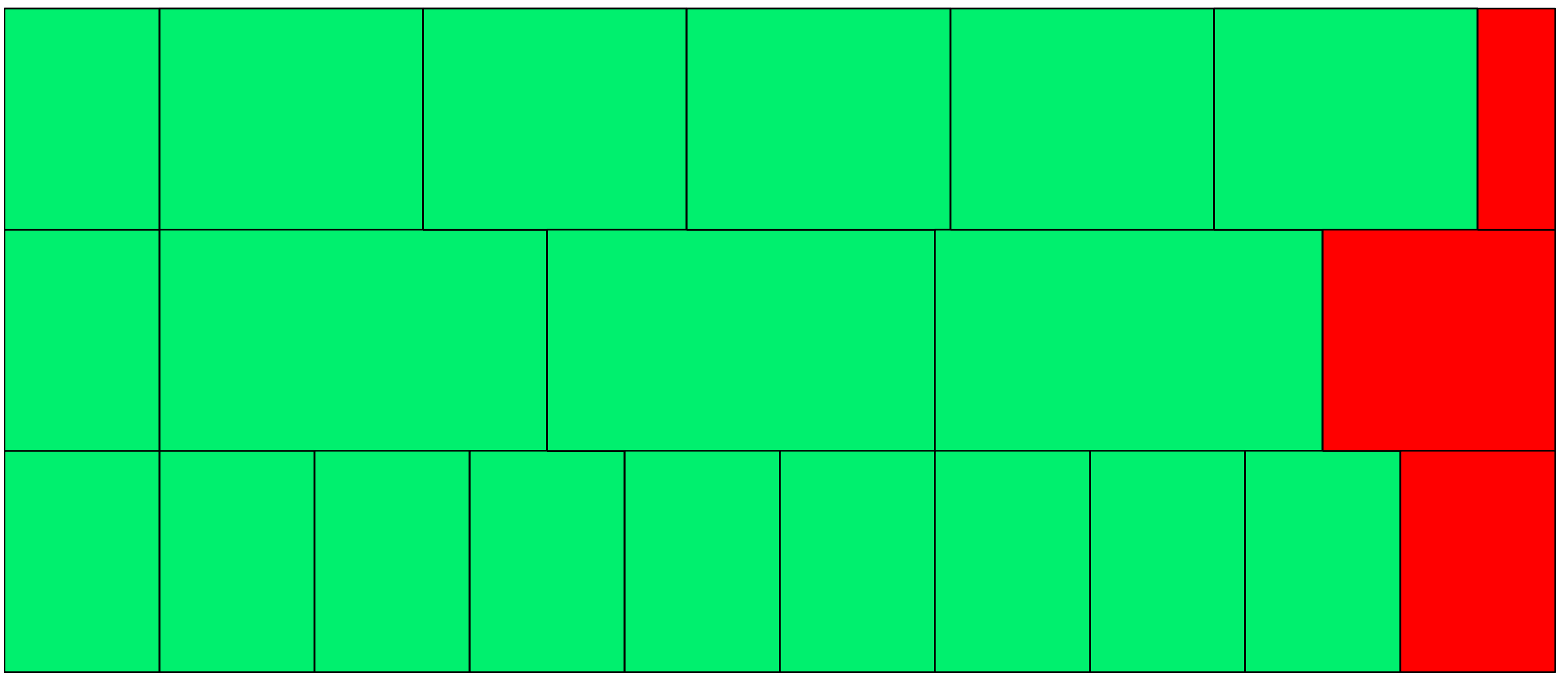
Multi-Adaptive Time-Stepping

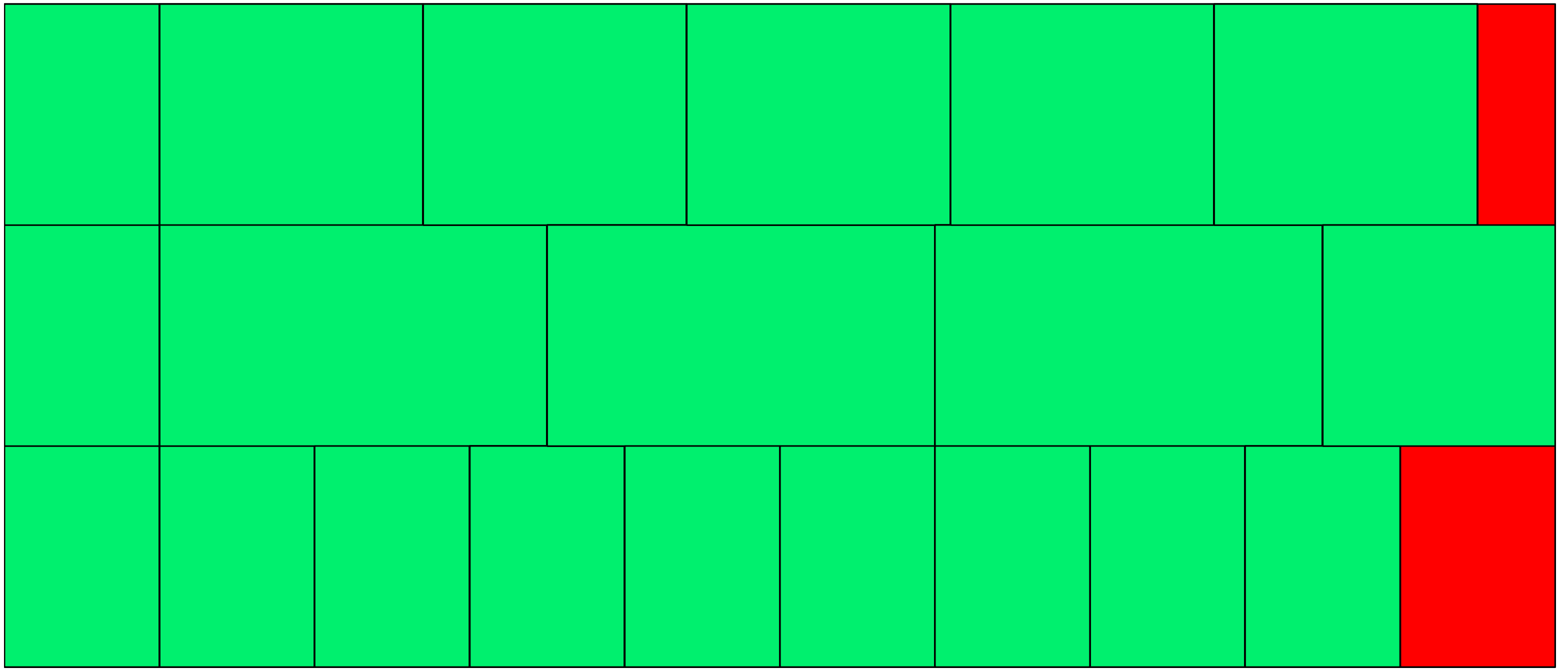
Action: Redoing steps



t = 0.1
n = 1

Multi-Adaptive Time-Stepping

Action: Clearing steps



t = 0.1
n = 2

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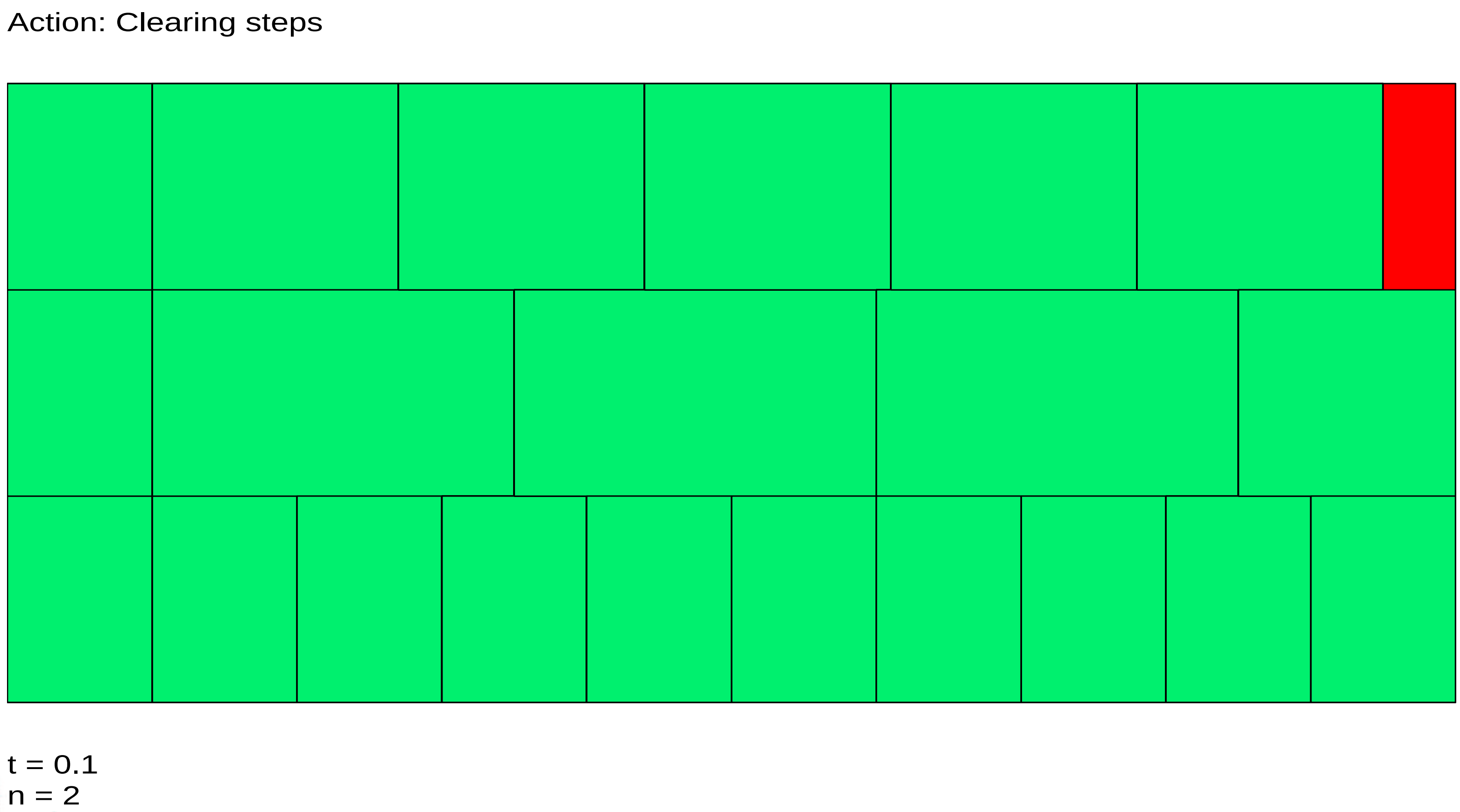
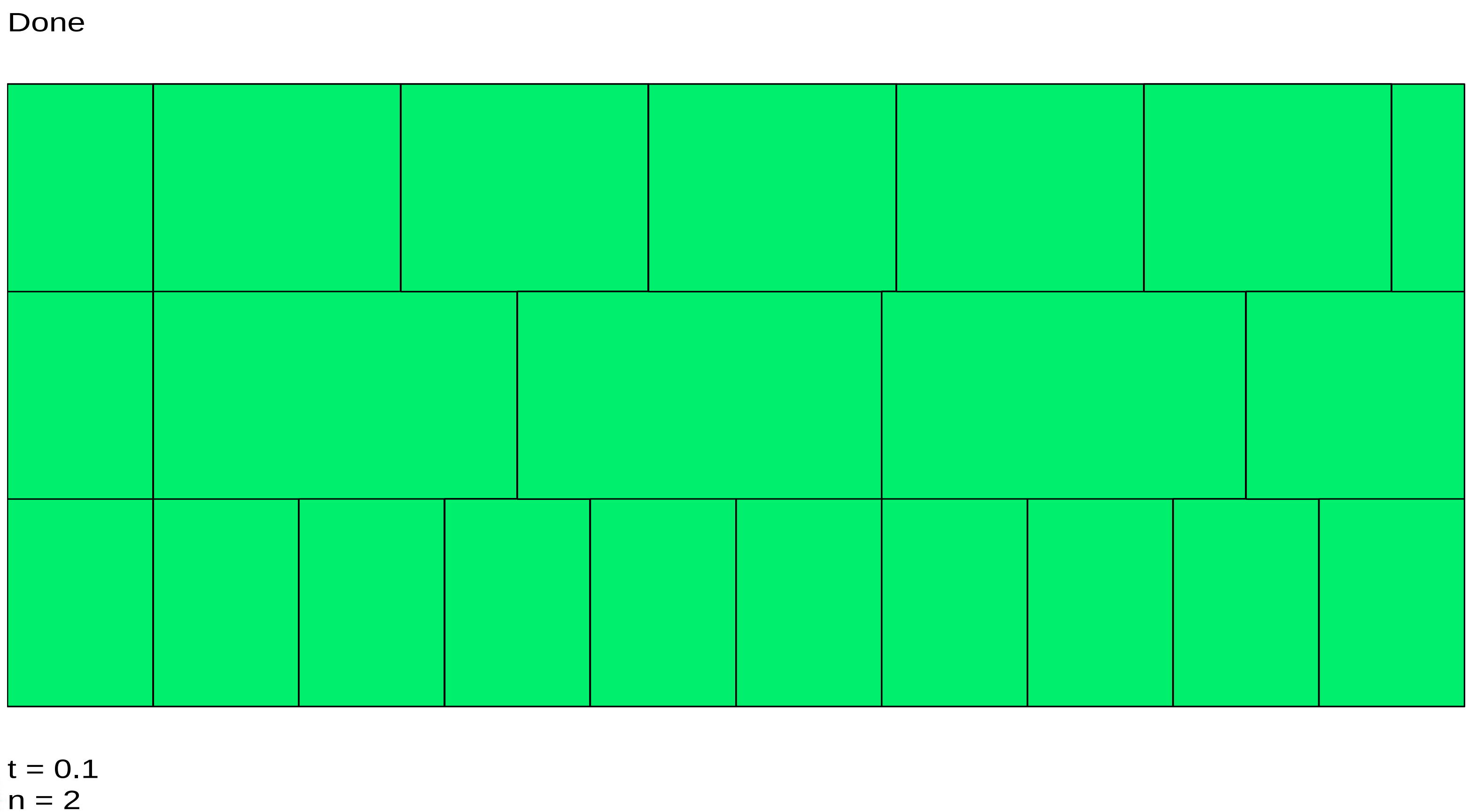
Multi-Adaptive Time-Stepping



Multi-Adaptive Time-Stepping

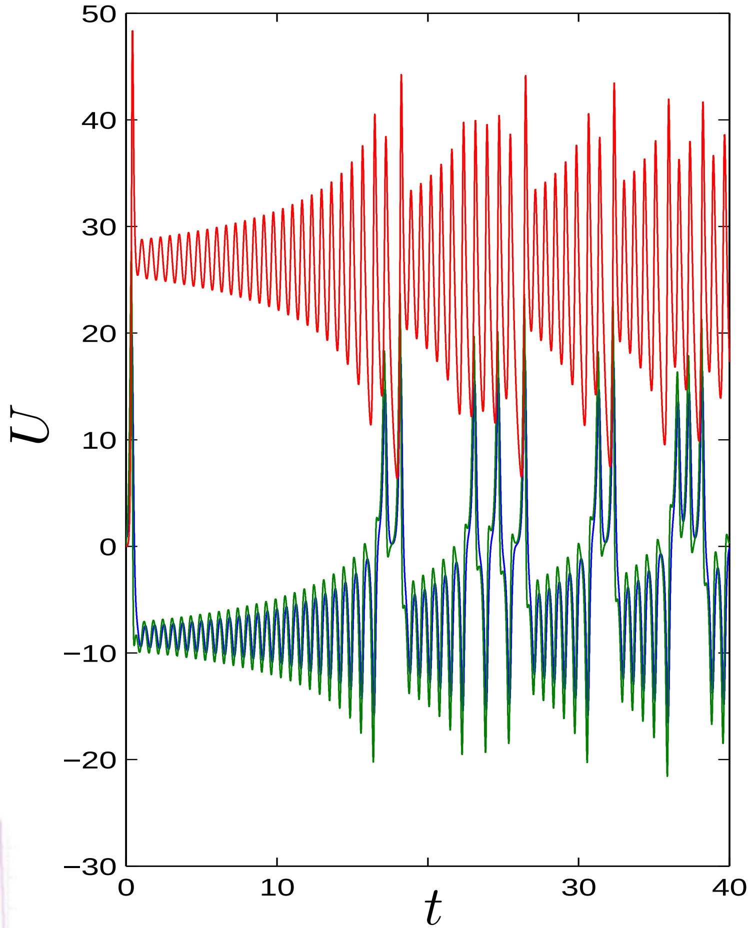


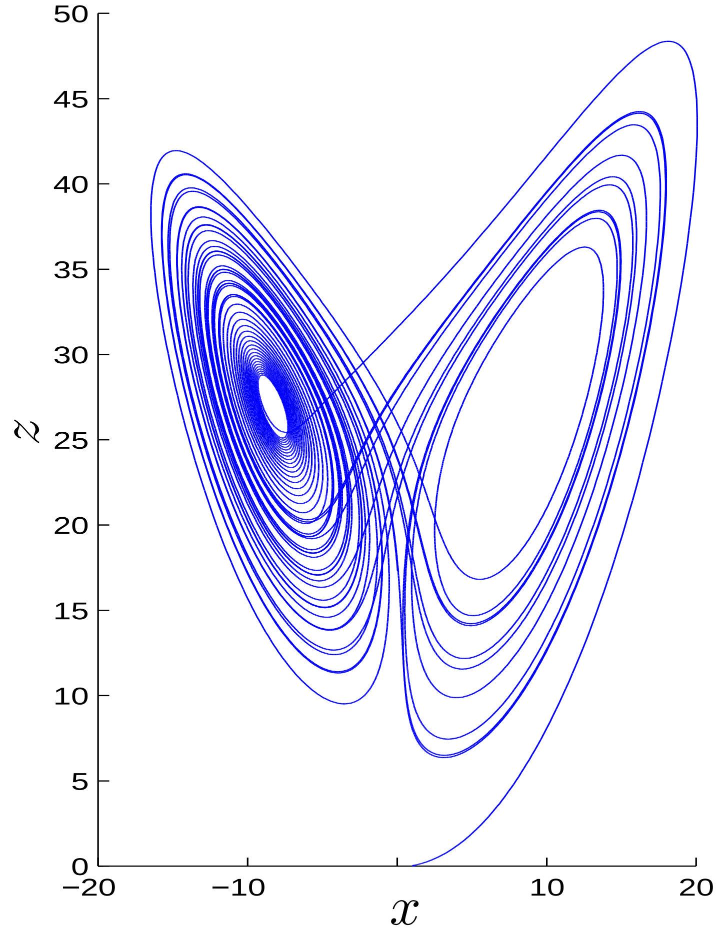
The Lorenz System

$$
\begin{cases}
\dot{x} = \sigma(y - x), \\
\dot{y} = rx - y - xz, \\
\dot{z} = xy - bz,
\end{cases}
$$

where we, as usual, take $(x_0, y_0, z_0) = (1, 0, 0)$, $\sigma = 10$, $b = 8/3$ and $r = 28$.

The Solution?





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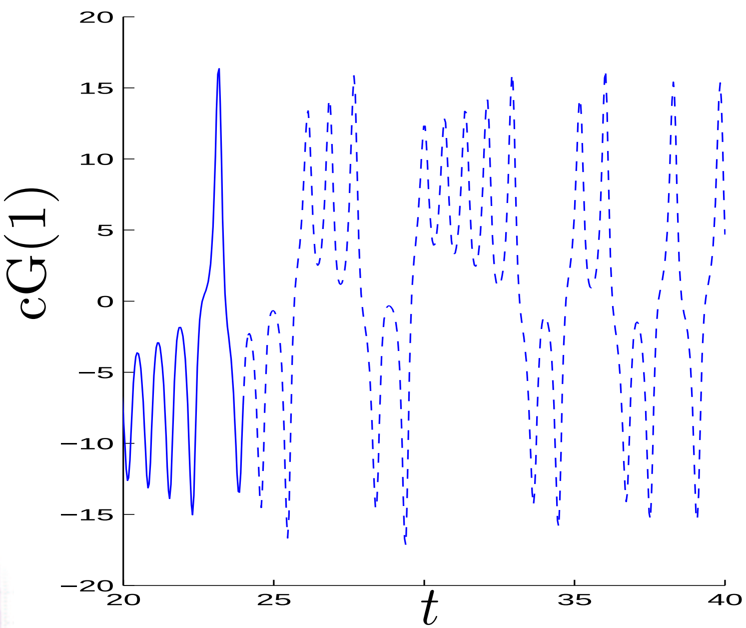
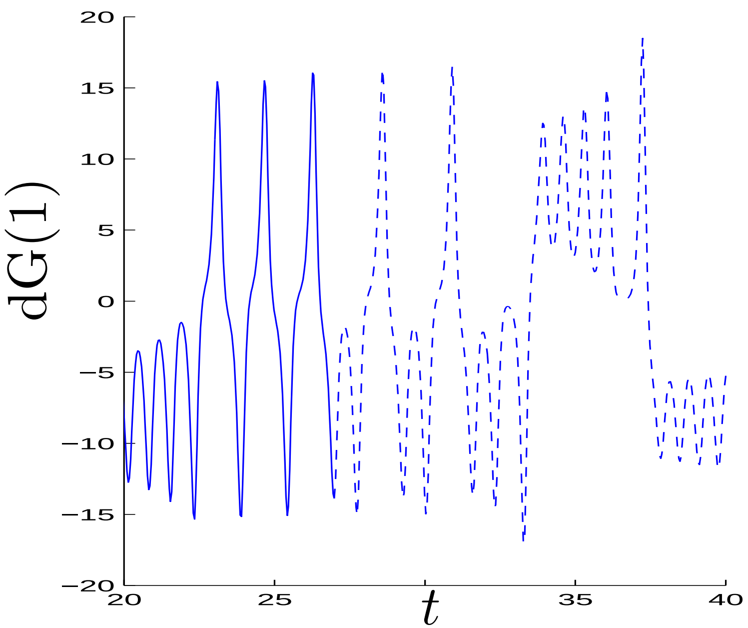
A Simple Experiment

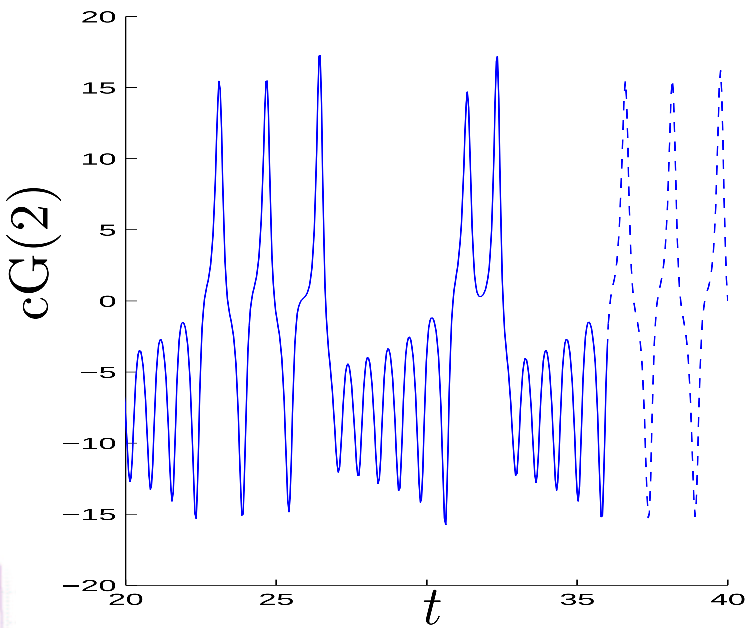
- $T = 40$
- $k = 0.001$

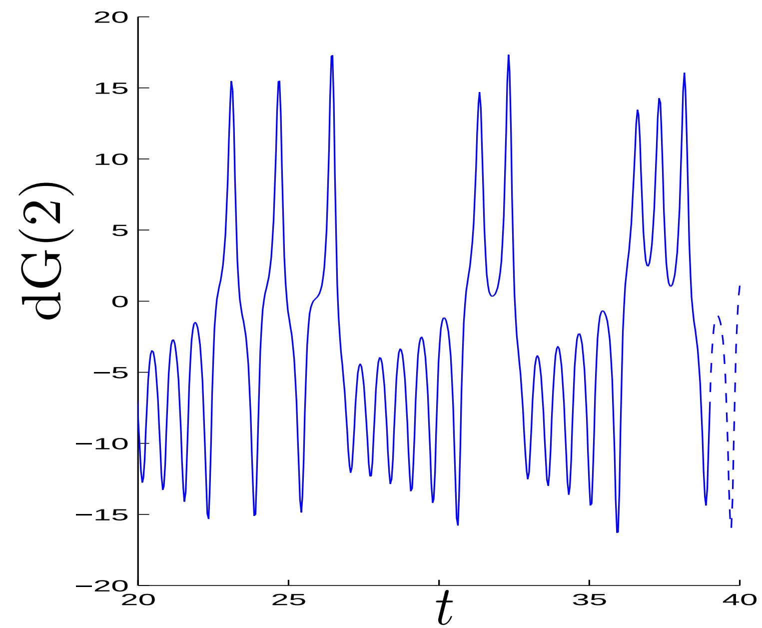




A Simple Experiment

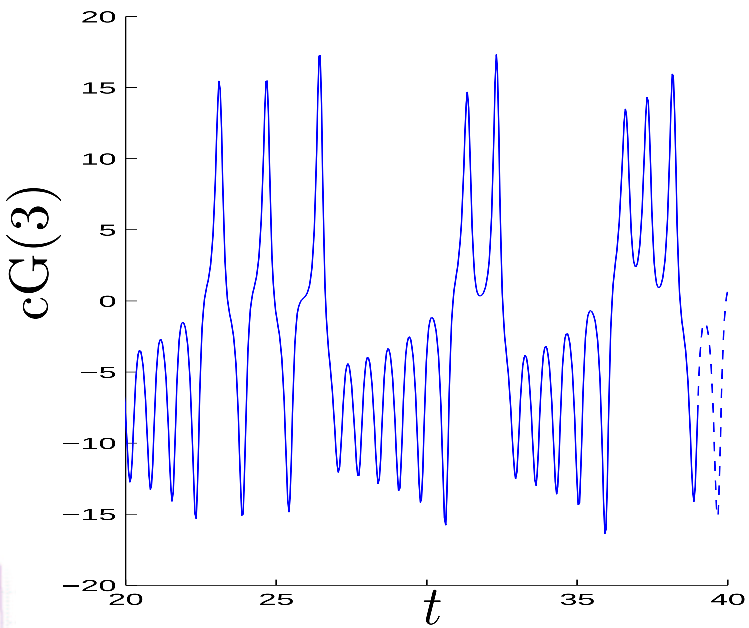
- $T = 40$
- $k = 0.001$

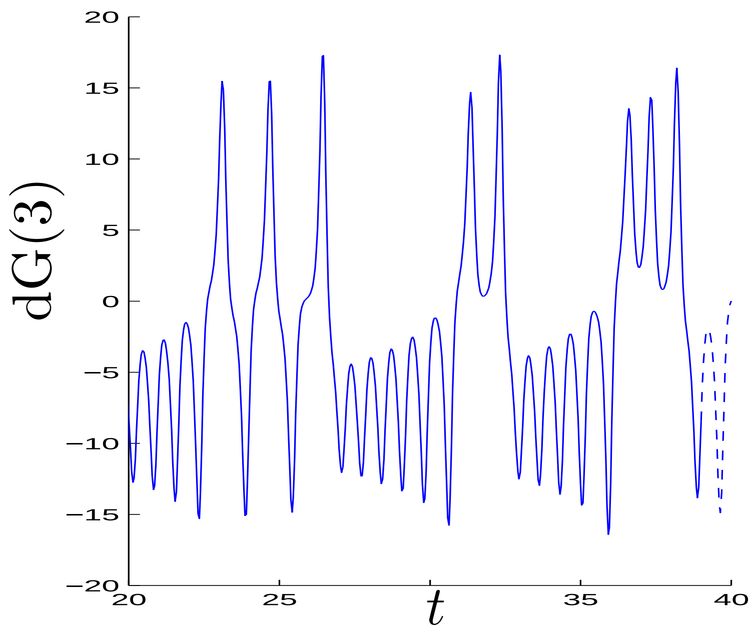


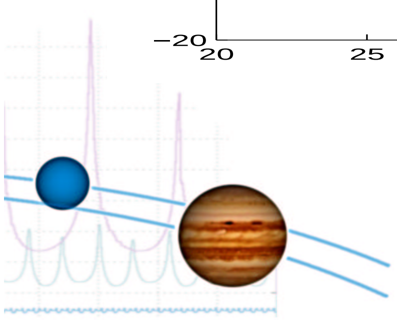


A Simple Experiment

- $T = 40$
- $k = 0.001$

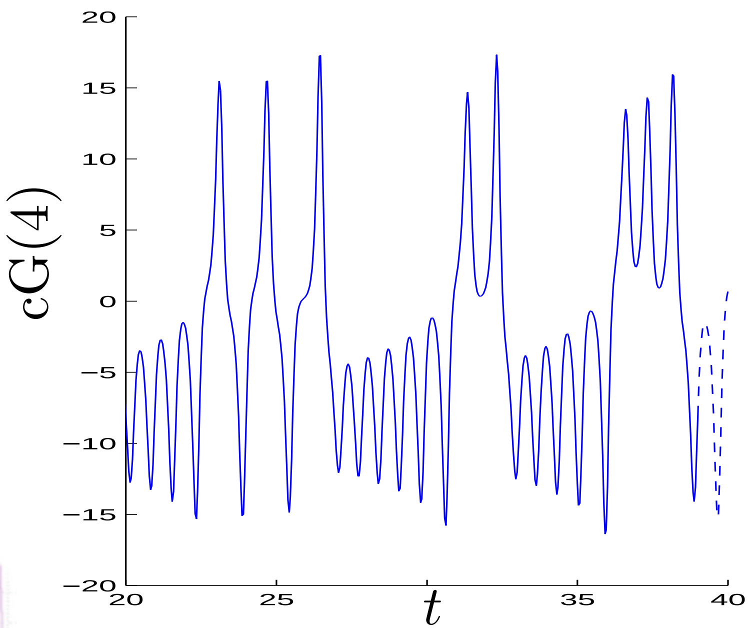


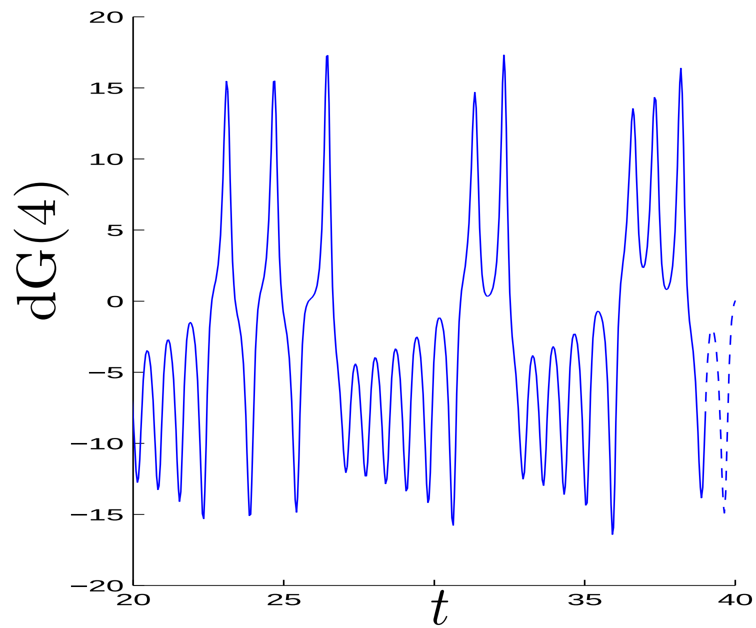




A Simple Experiment

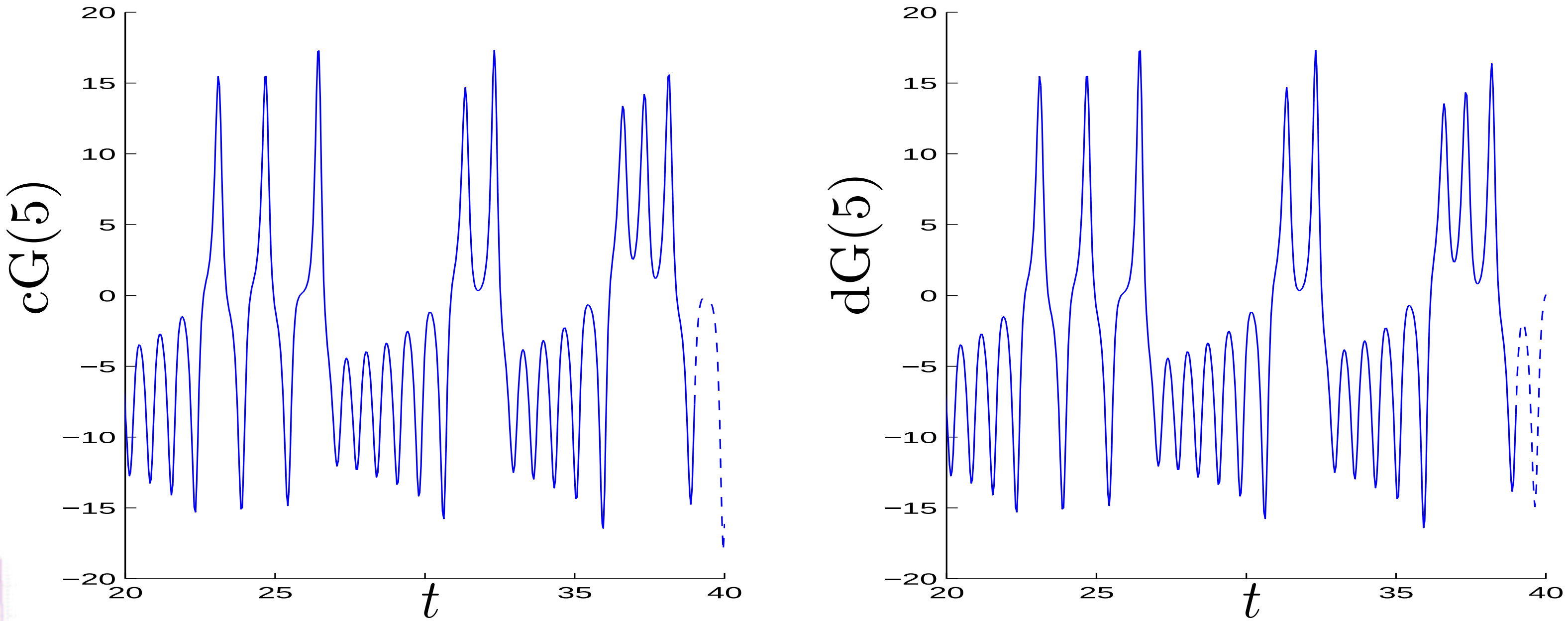
- $T = 40$
- $k = 0.001$





A Simple Experiment

- $T = 40$
- $k = 0.001$



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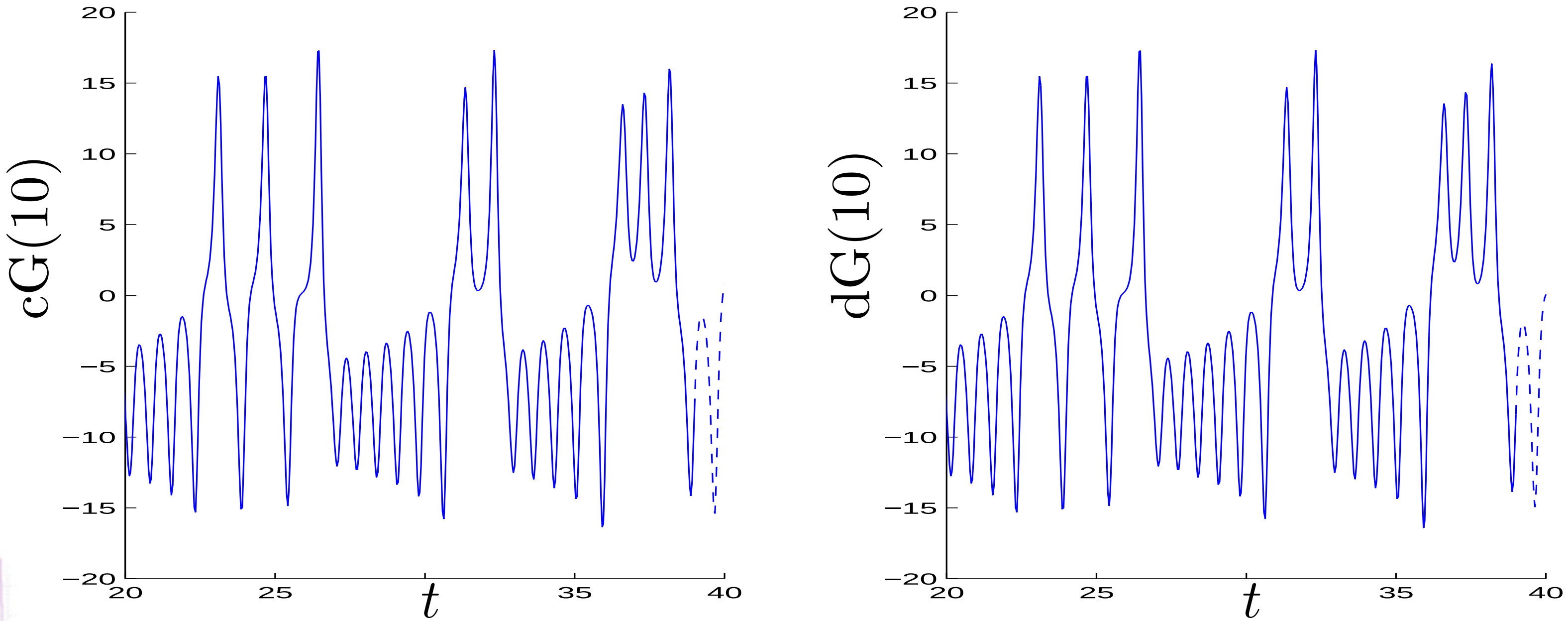
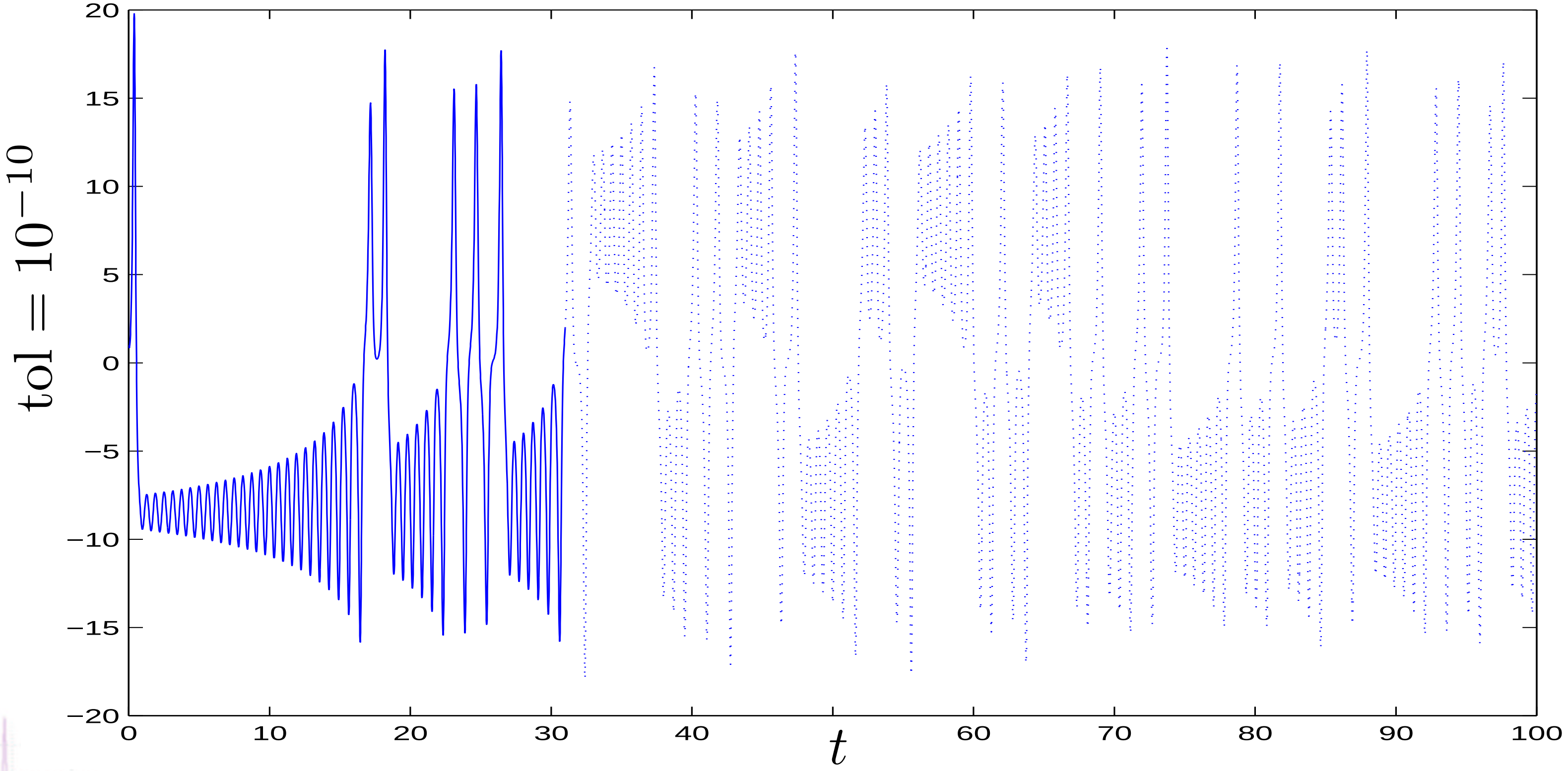
A Simple Experiment

- $T = 40$
- $k = 0.001$



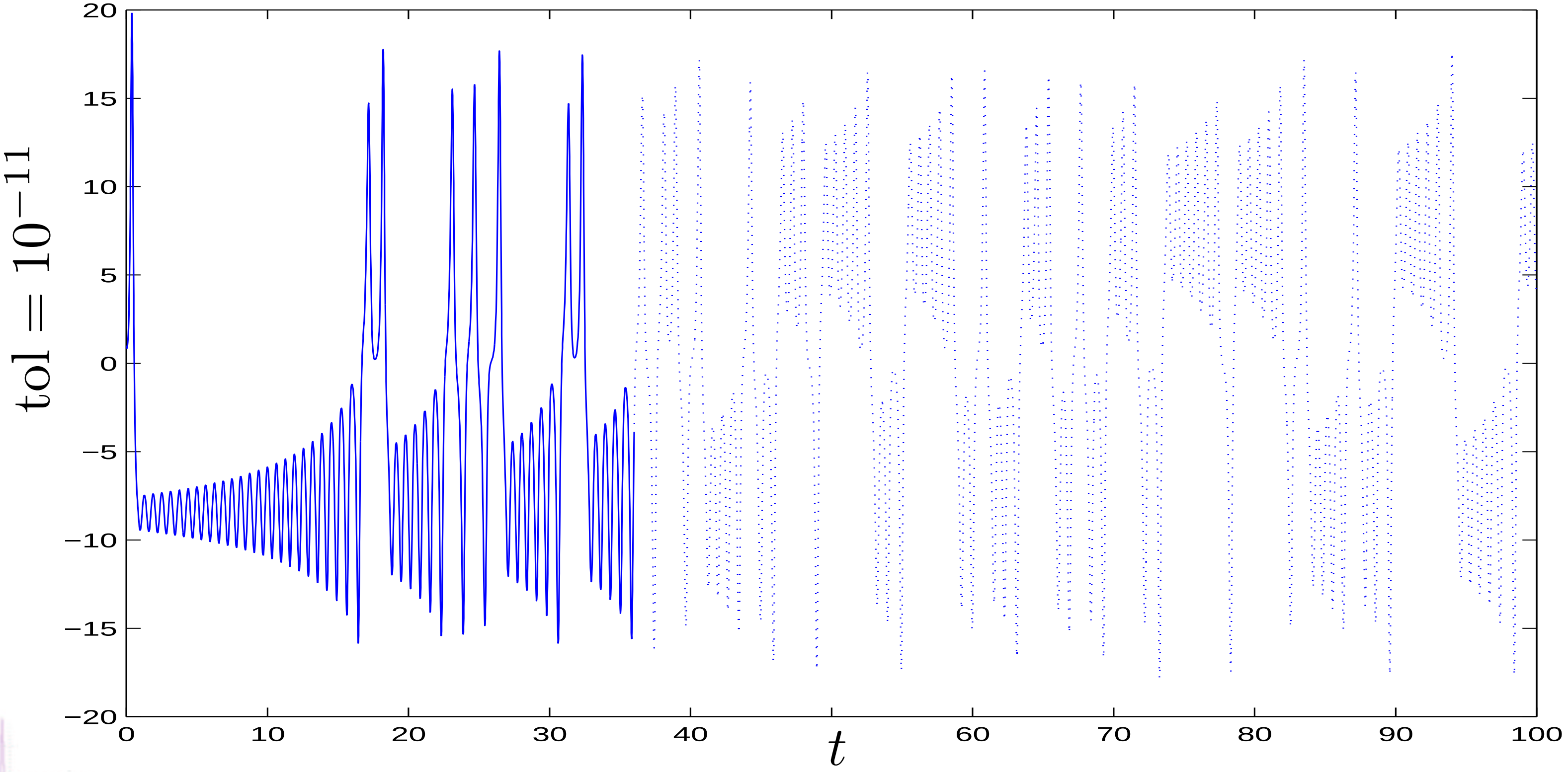
ode45

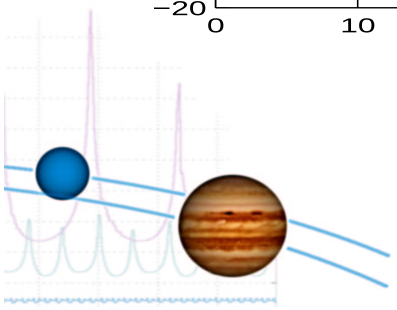
Trying the same thing with Matlabs ode45:



ode45

Trying the same thing with Matlabs `ode45`:





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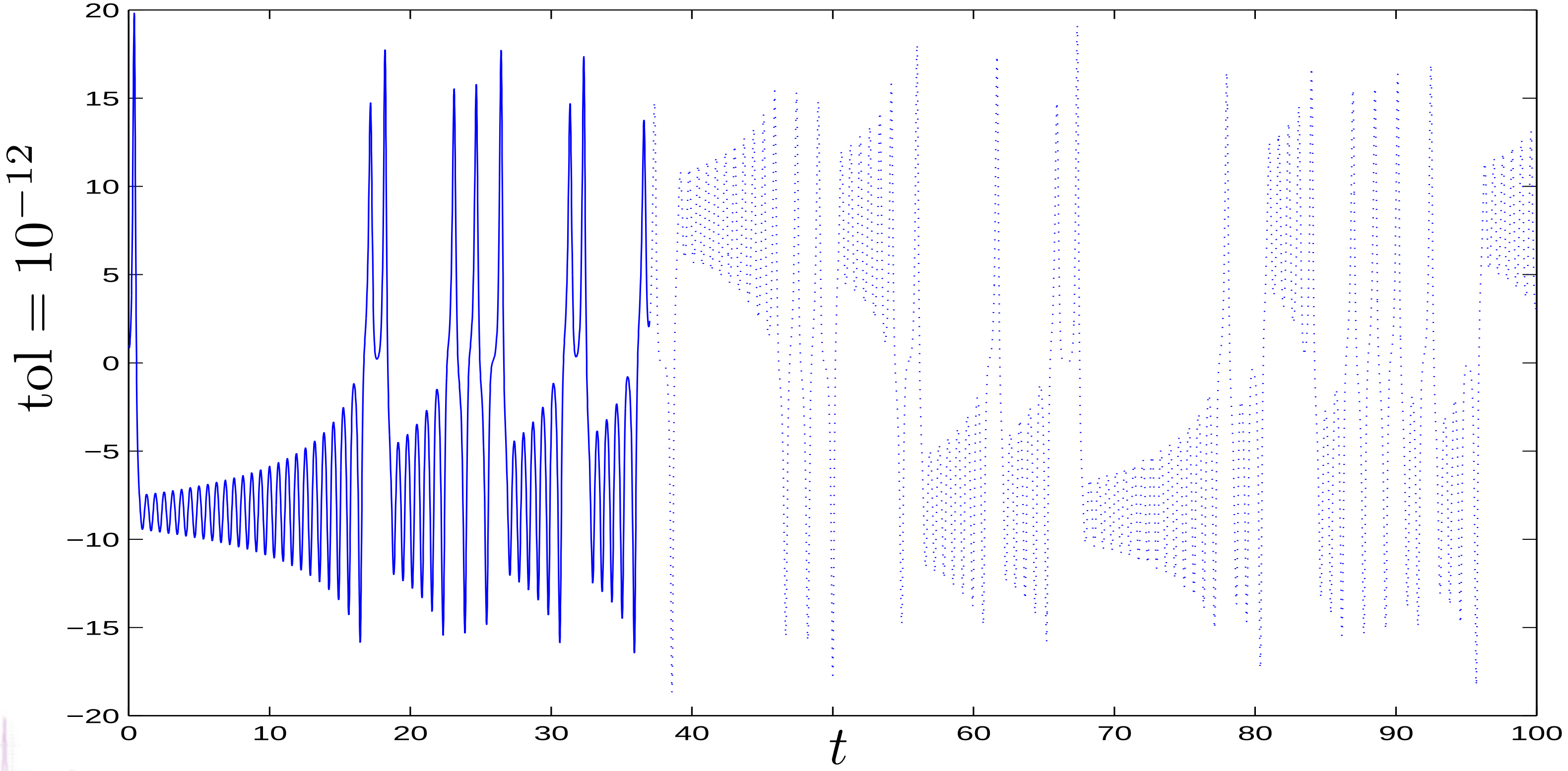
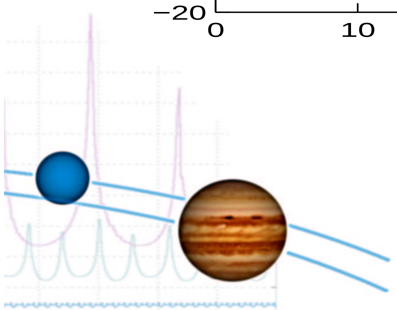
ode45

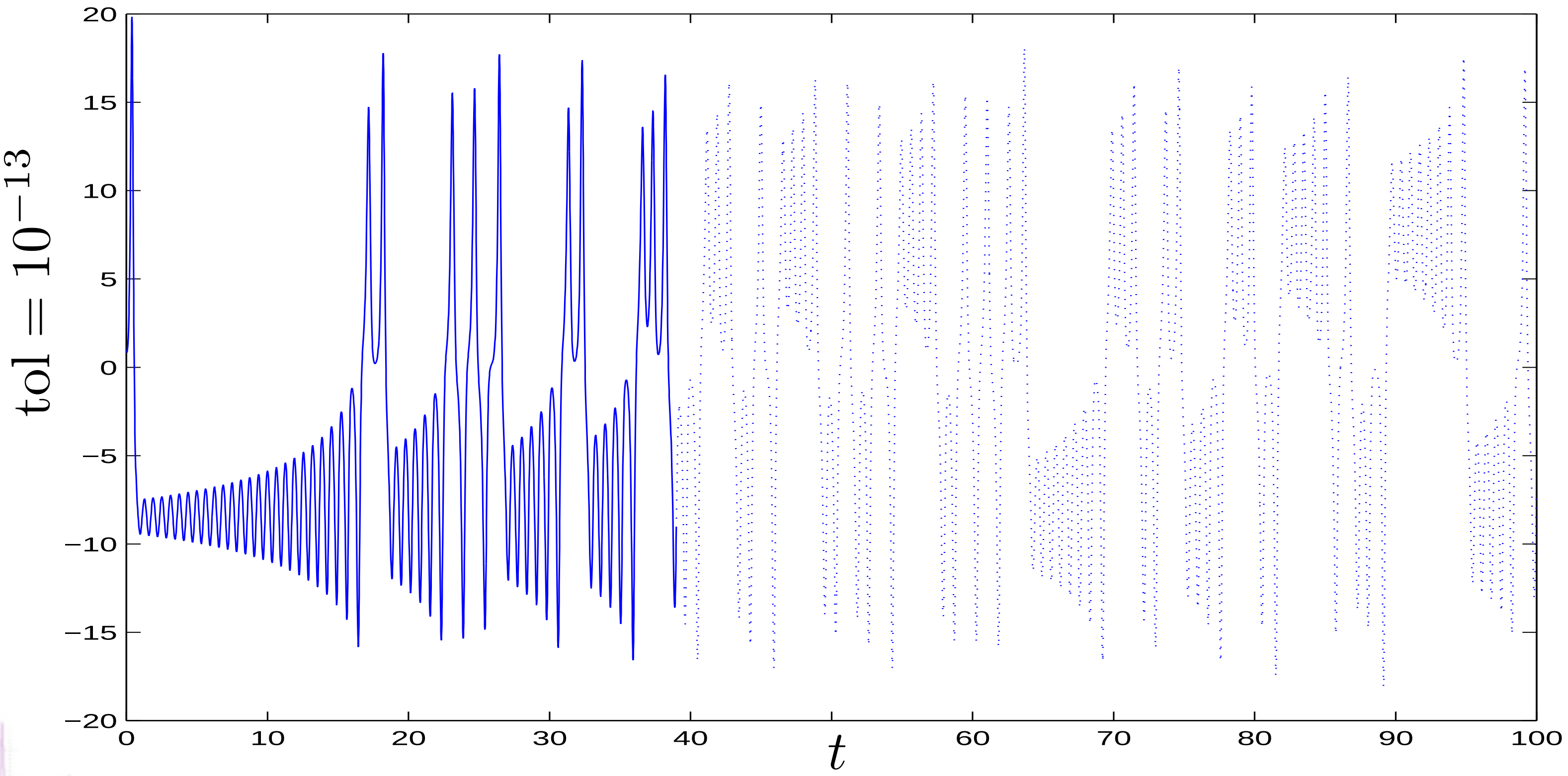
Trying the same thing with Matlabs `ode45`:

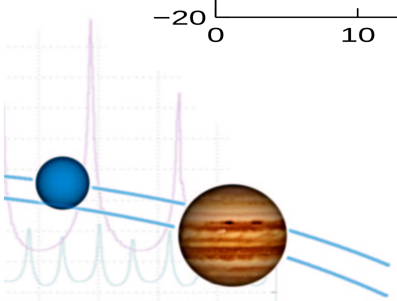




ode45

Trying the same thing with Matlabs `ode45`:





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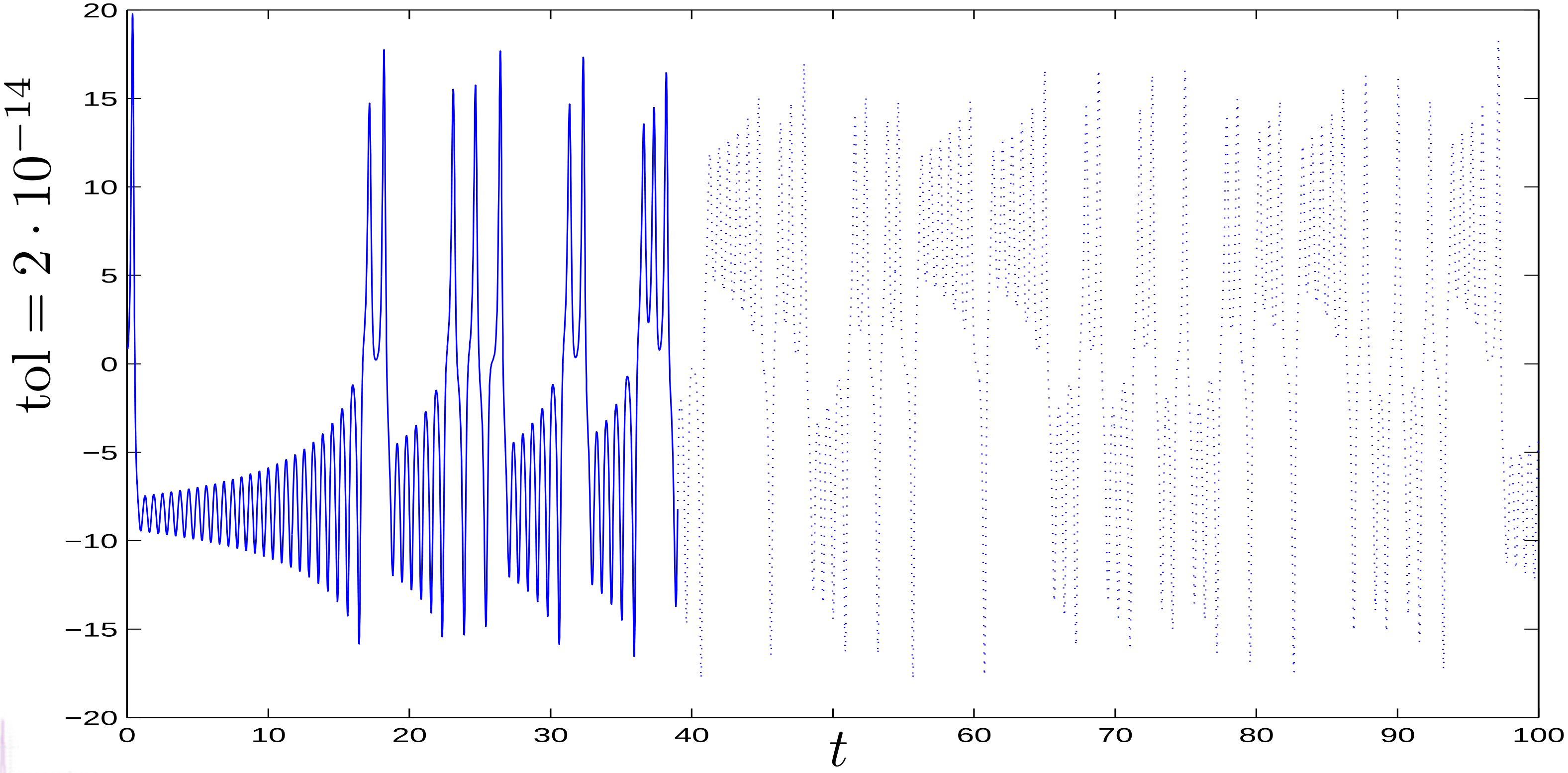
CHALMERS FINITE ELEMENT CENTER

ode45

Trying the same thing with Matlabs `ode45`:



Getting Further

- Small Galerkin error
- Large computational error

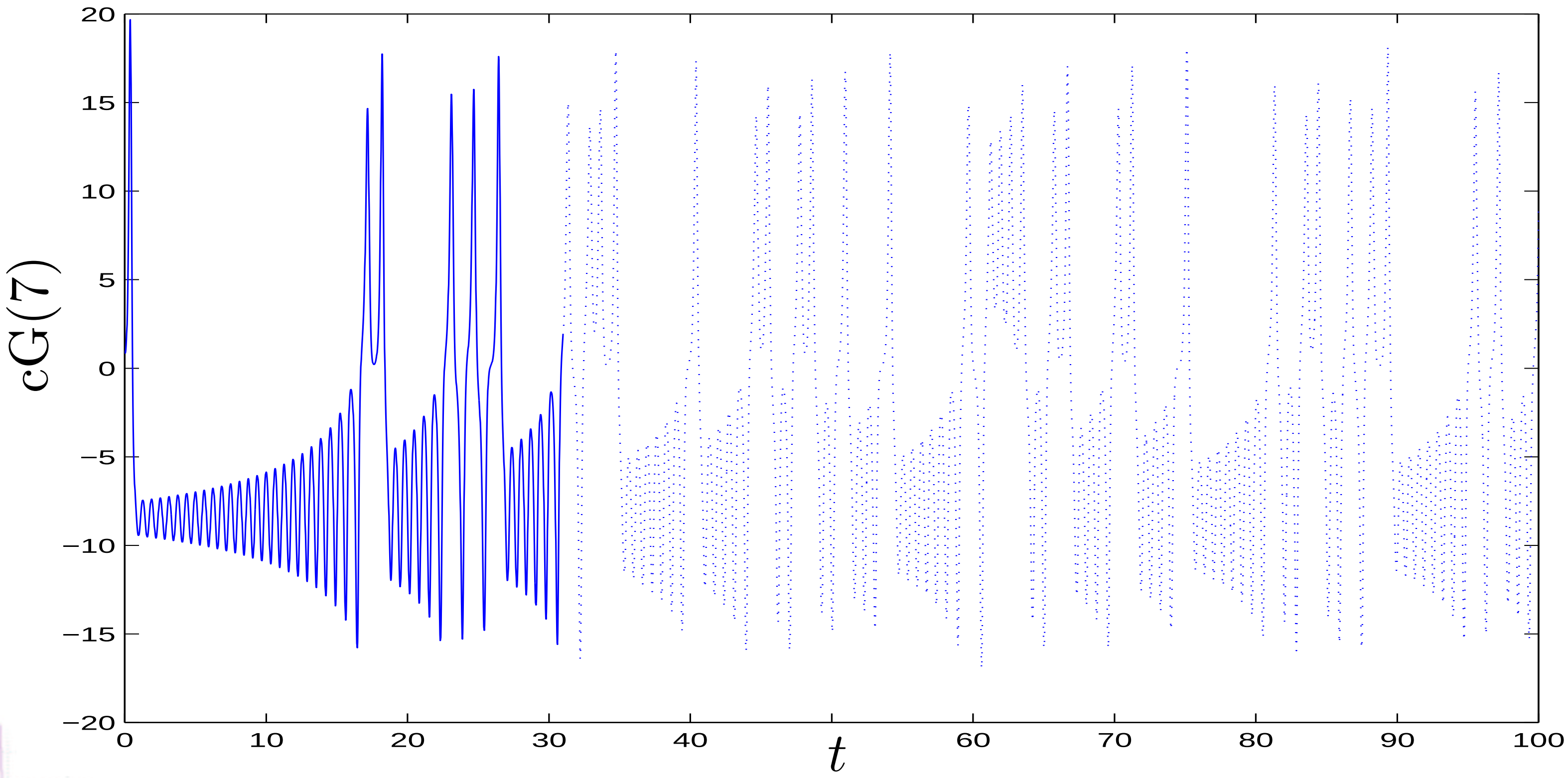
We must thus decrease the computational error, which means decreasing the discrete residual,

$$R^C \approx \frac{10^{-16}}{k}.$$

This in turn means that we have to *increase* the time-step!

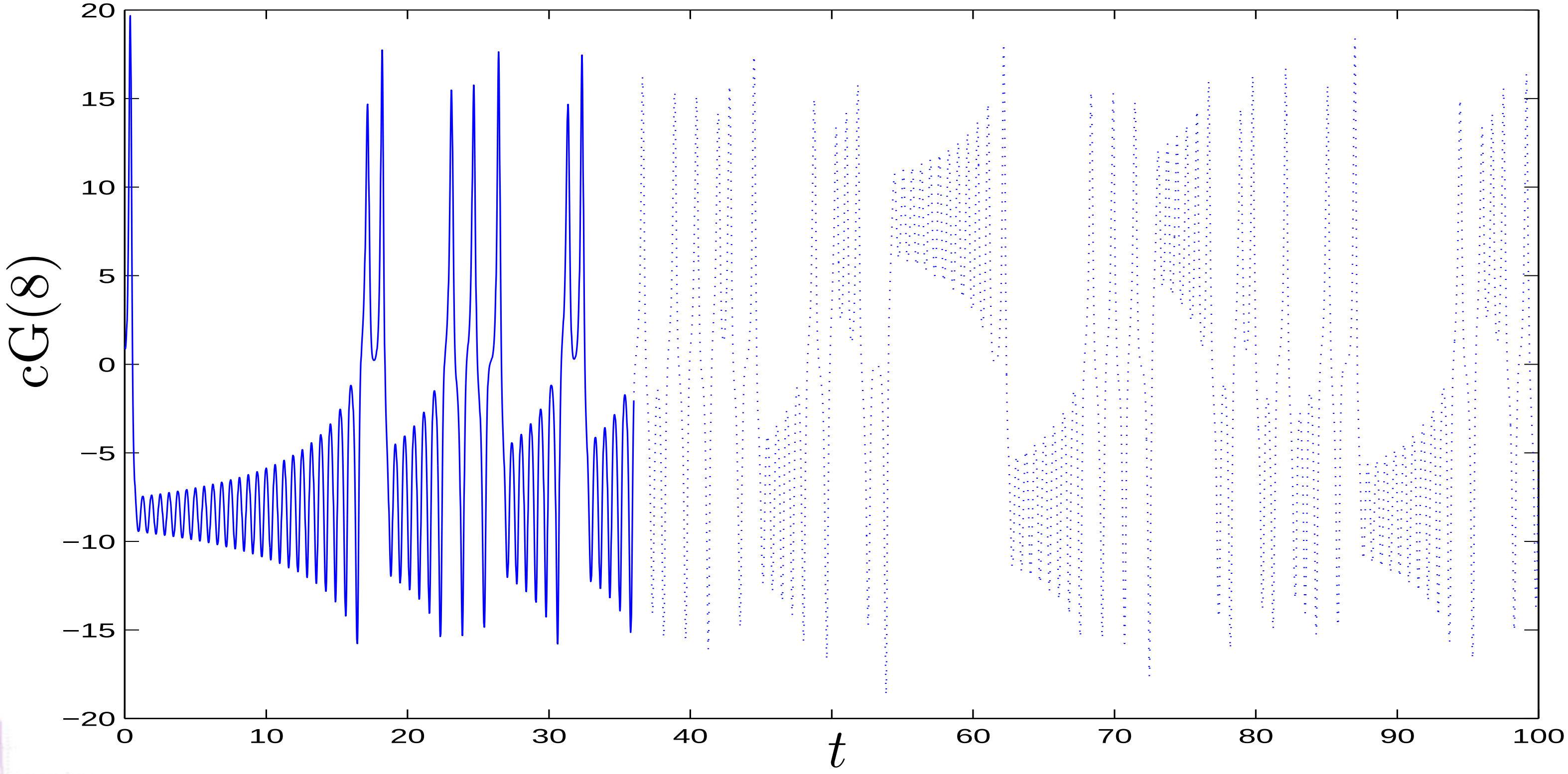
A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



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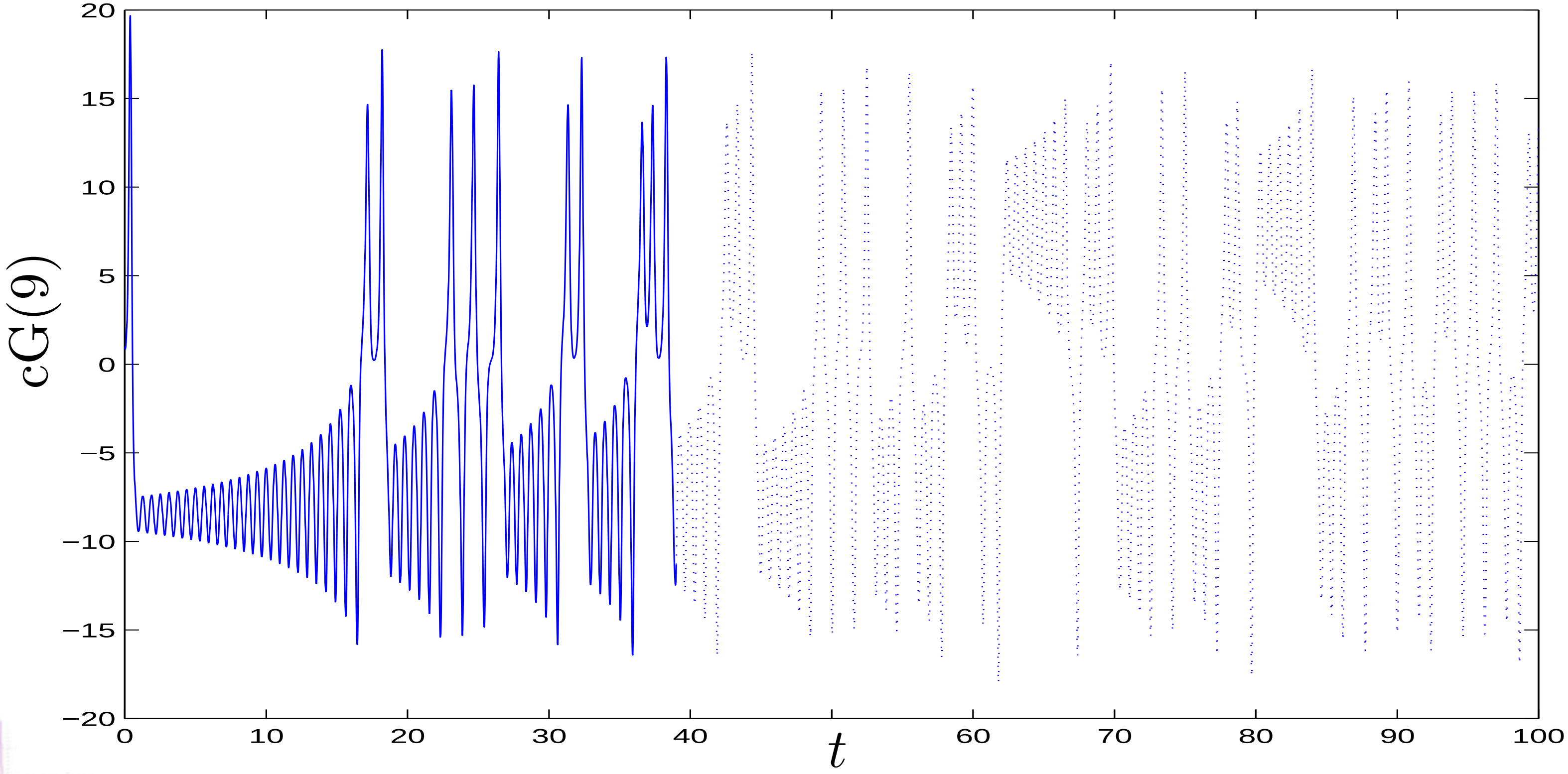
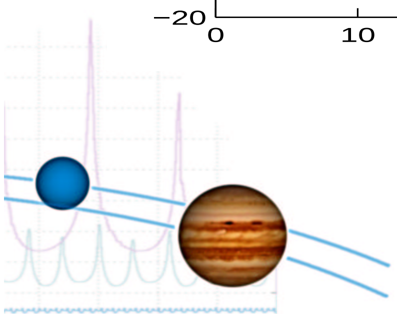
A Simple Experiment — *continued*

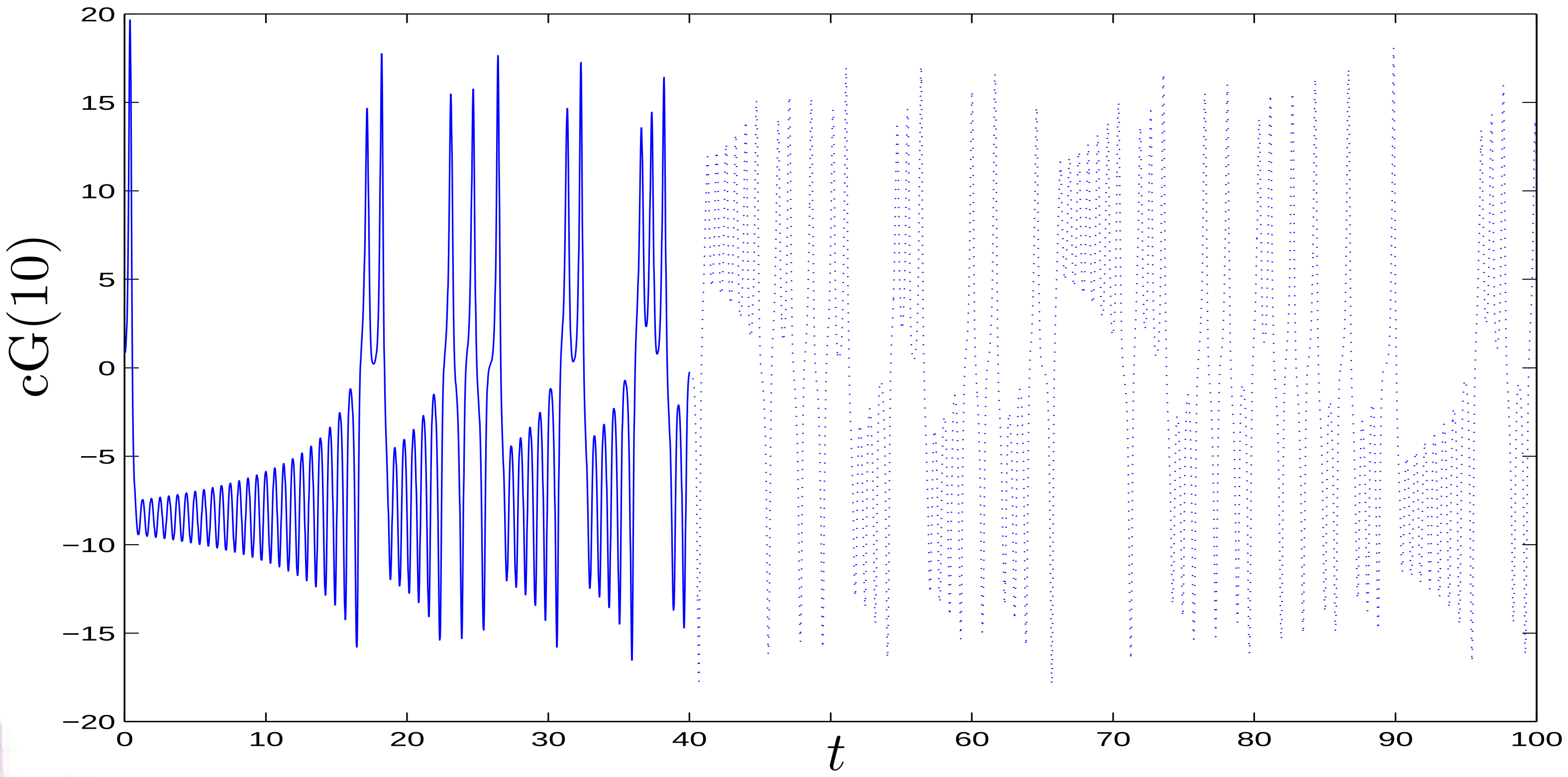
Increasing the time-step to $k = 0.1$:

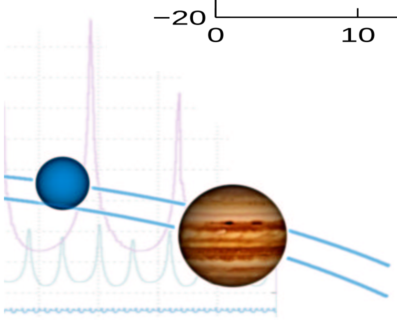




A Simple Experiment — *continued*

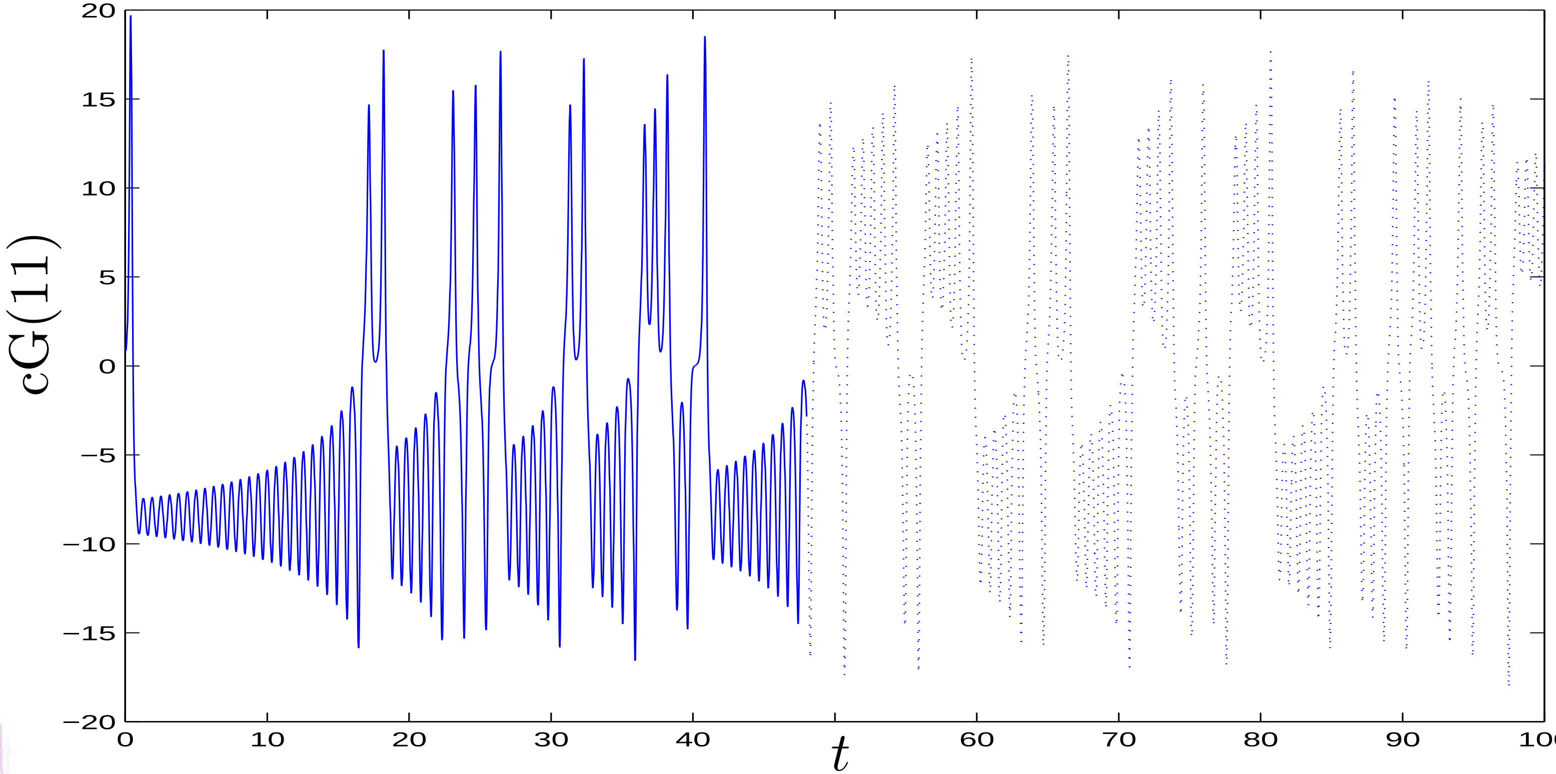
Increasing the time-step to $k = 0.1$:





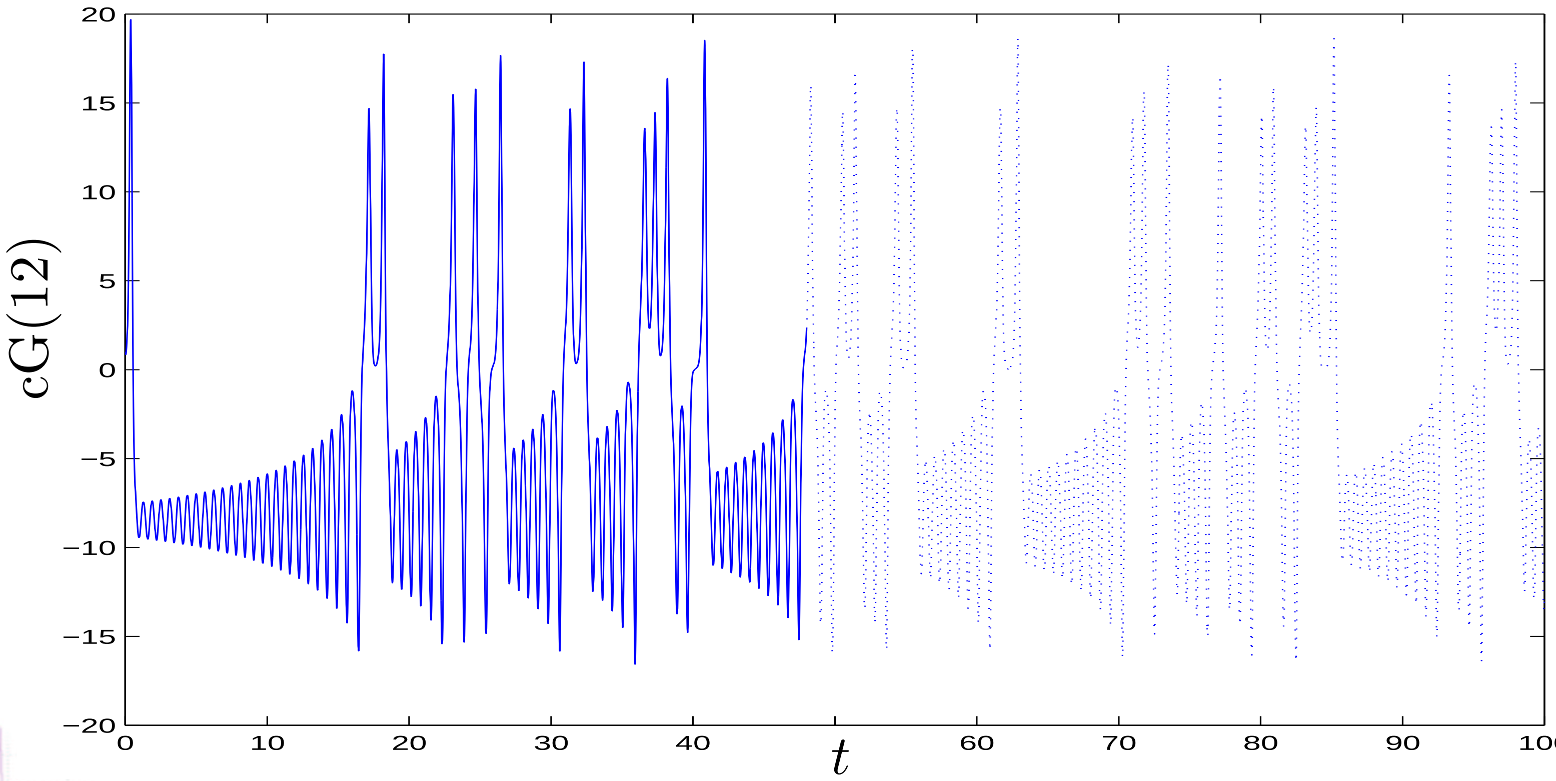
A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



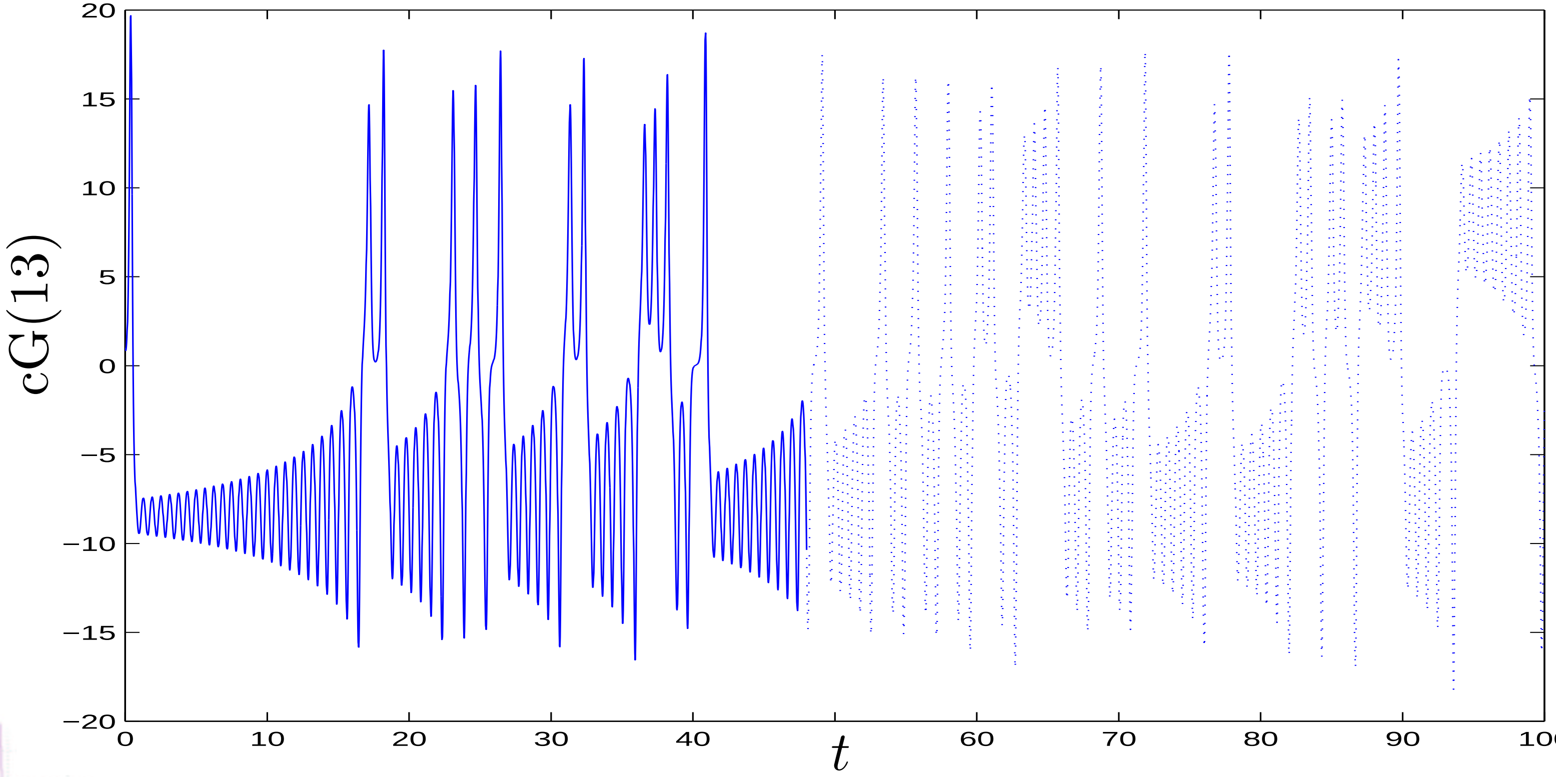
A Simple Experiment — *continued*

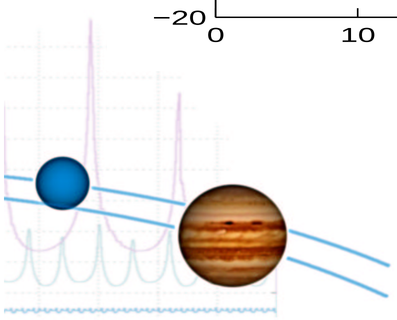
Increasing the time-step to $k = 0.1$:



A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:





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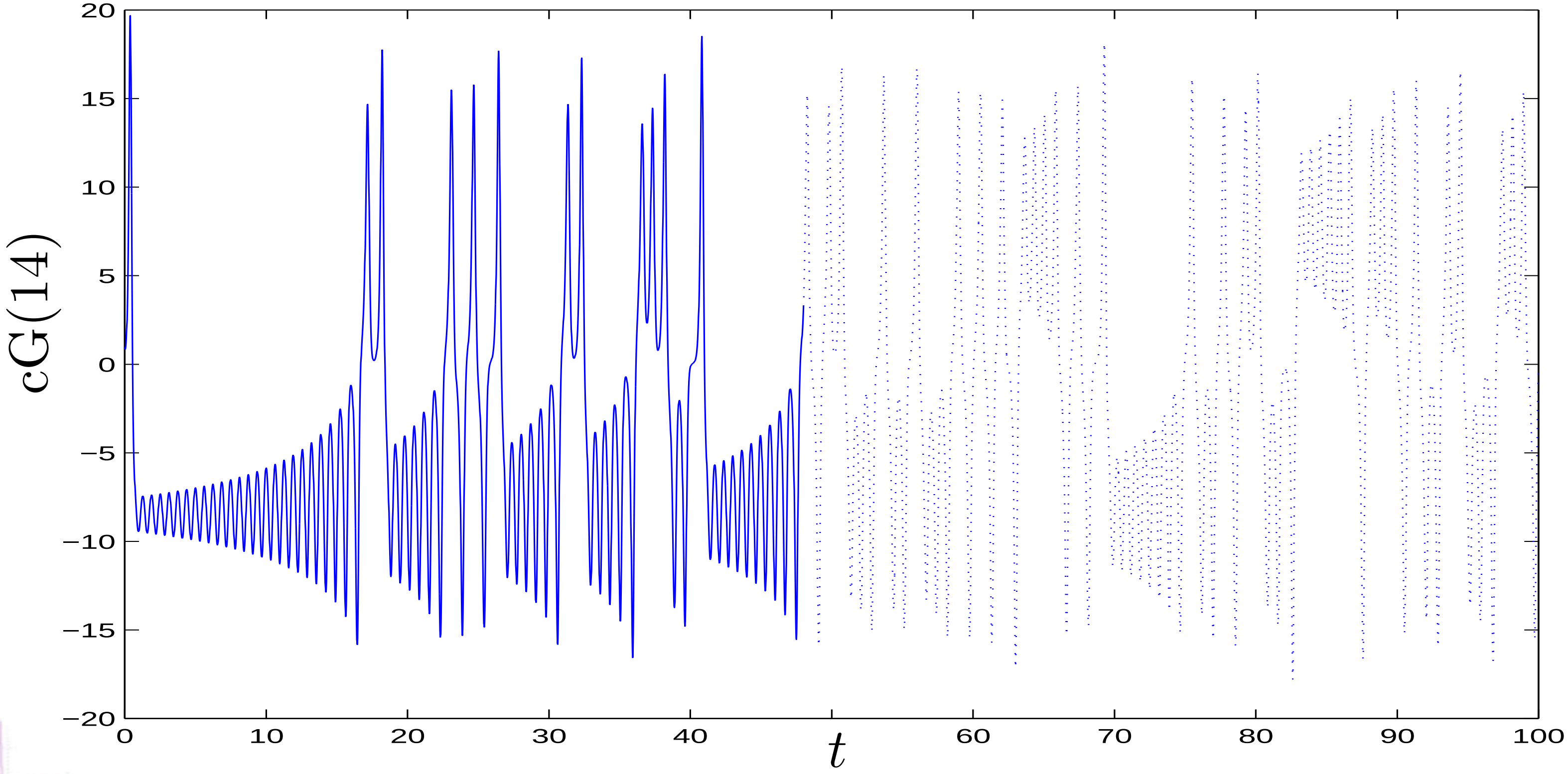
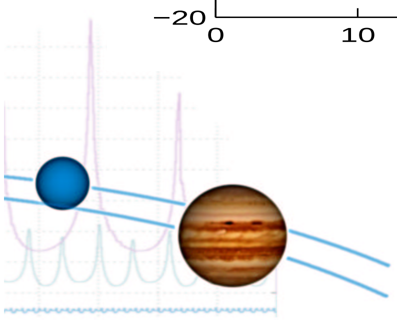
A Simple Experiment — *continued*

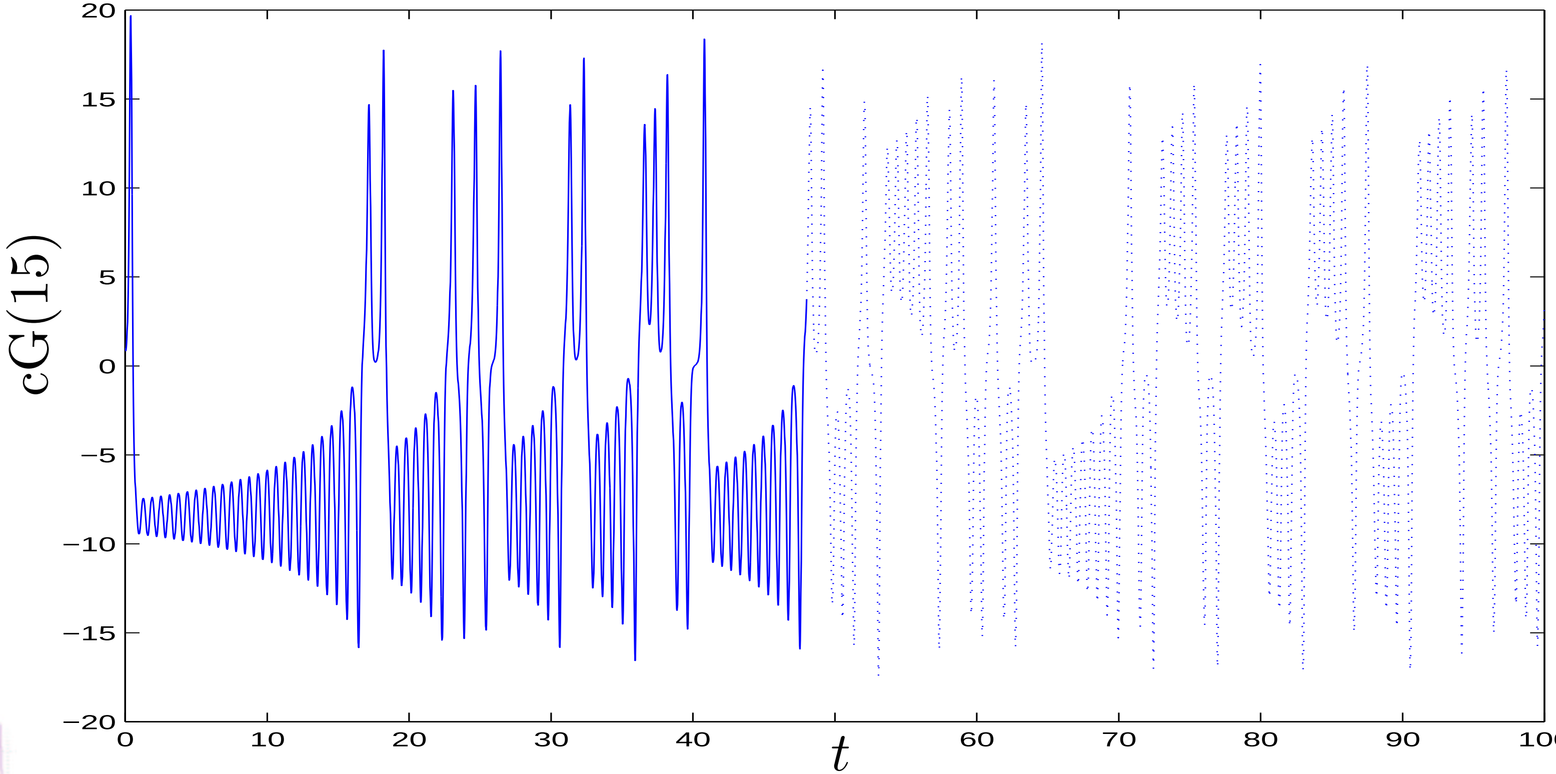
Increasing the time-step to $k = 0.1$:



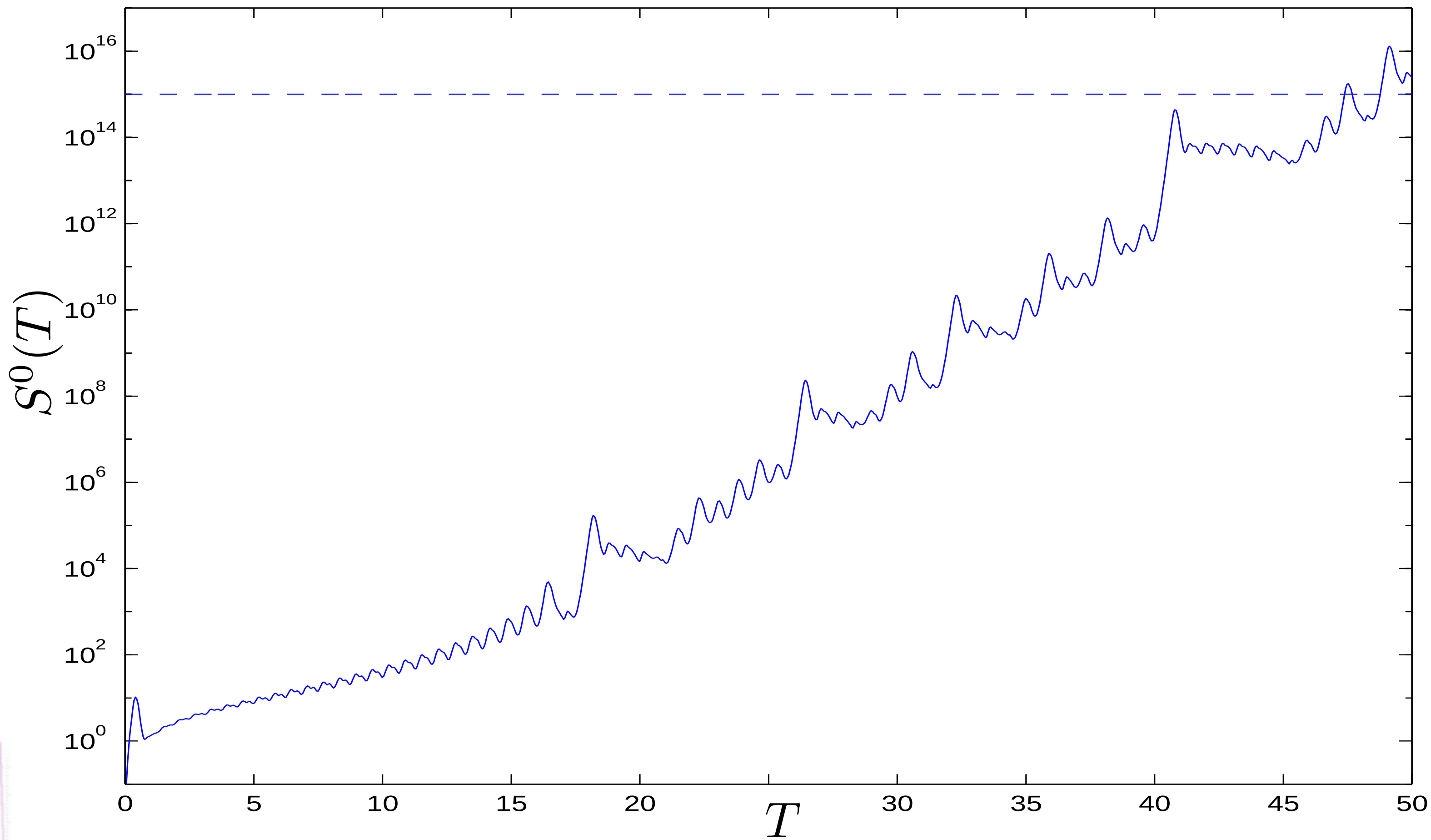


A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



The Stability Factor



A Simple Estimate

A simple estimate for the stability factors is

$$S^{[q]}(T) \approx 10^{q+T/3},$$

so that

$$E_C(T) \approx \frac{10^{-16}}{k} 10^{T/3} = 10^{T/3-16}/k.$$

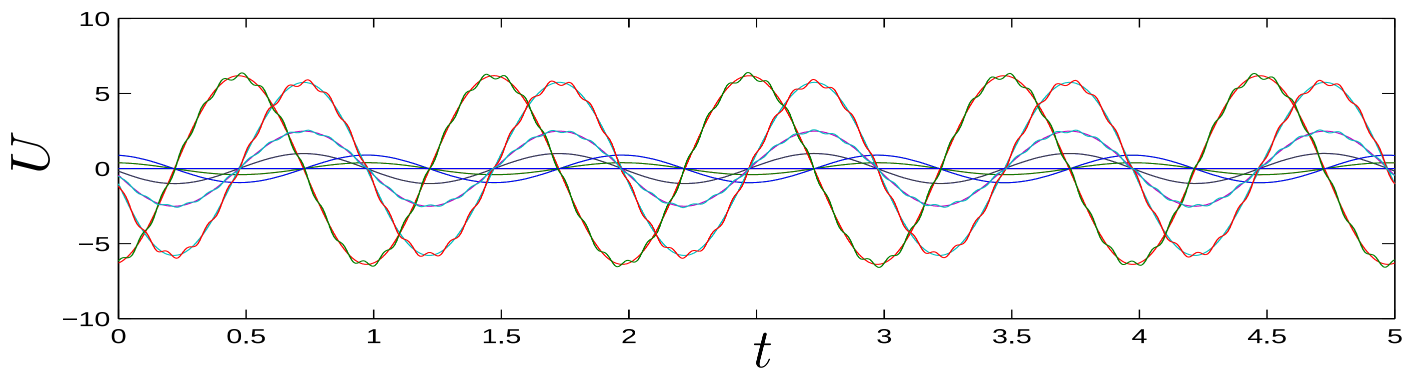
How far we get

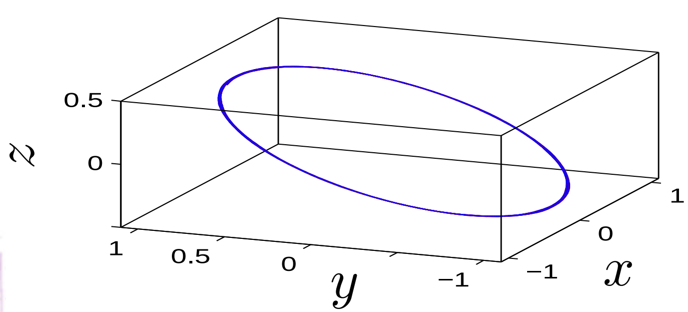
Keeping $E_C < 1$ we find the maximum value of T:

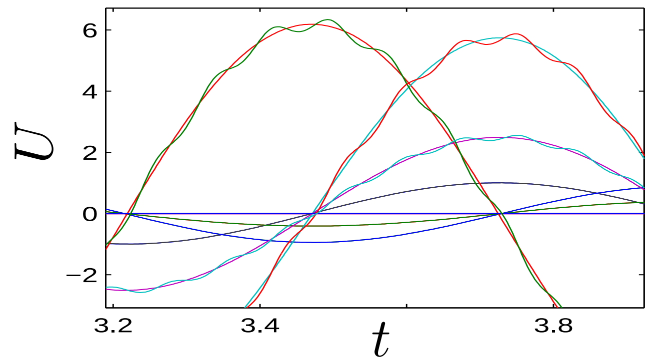
- $T = 3 \cdot 13 = 39$ with $k = 0.001$
- $T = 3 \cdot 15 = 45$ with $k = 0.1$
- not much further with larger time-step, since soon we will have $k > T$

The Solar System

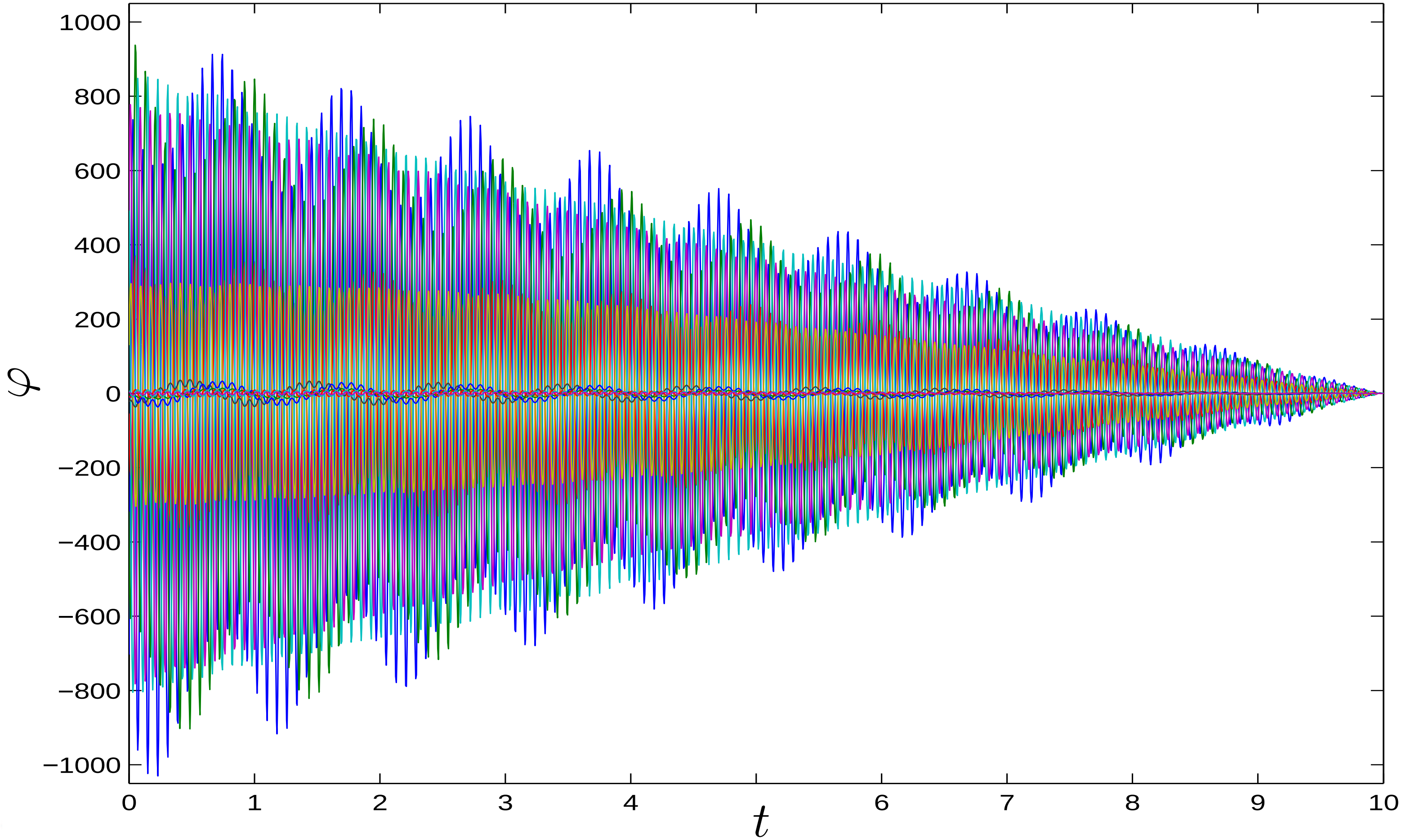
$$m_i \ddot{x}_i = \sum_{j \neq i} \frac{G m_i m_j}{|x_j - x_i|^3}(x_j - x_i)$$







The Dual



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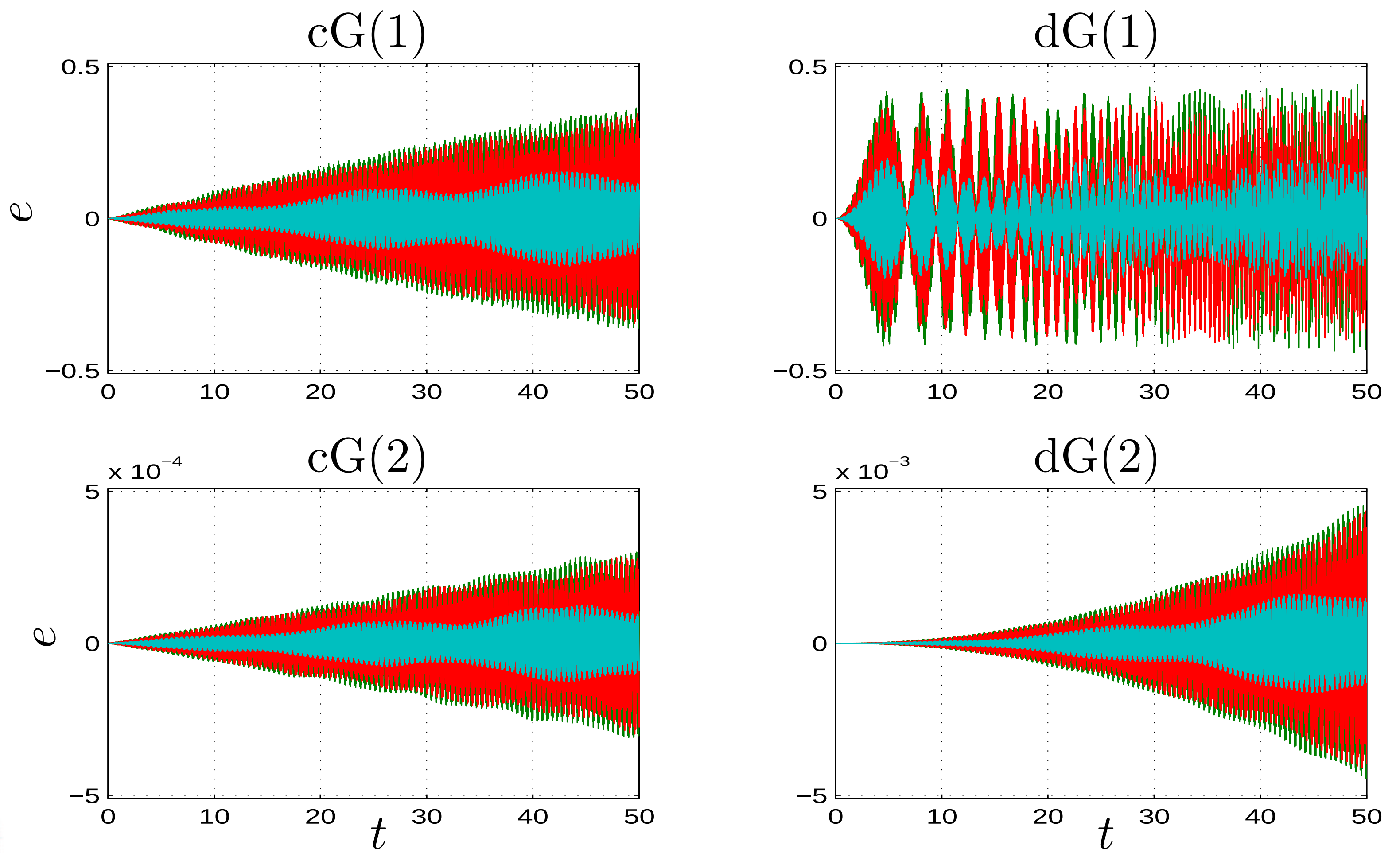
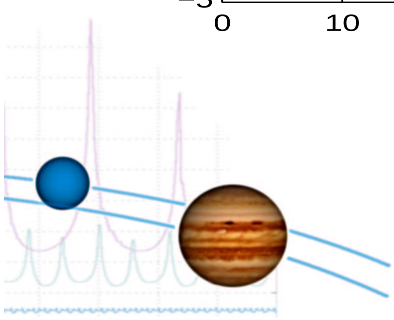
Error Growth





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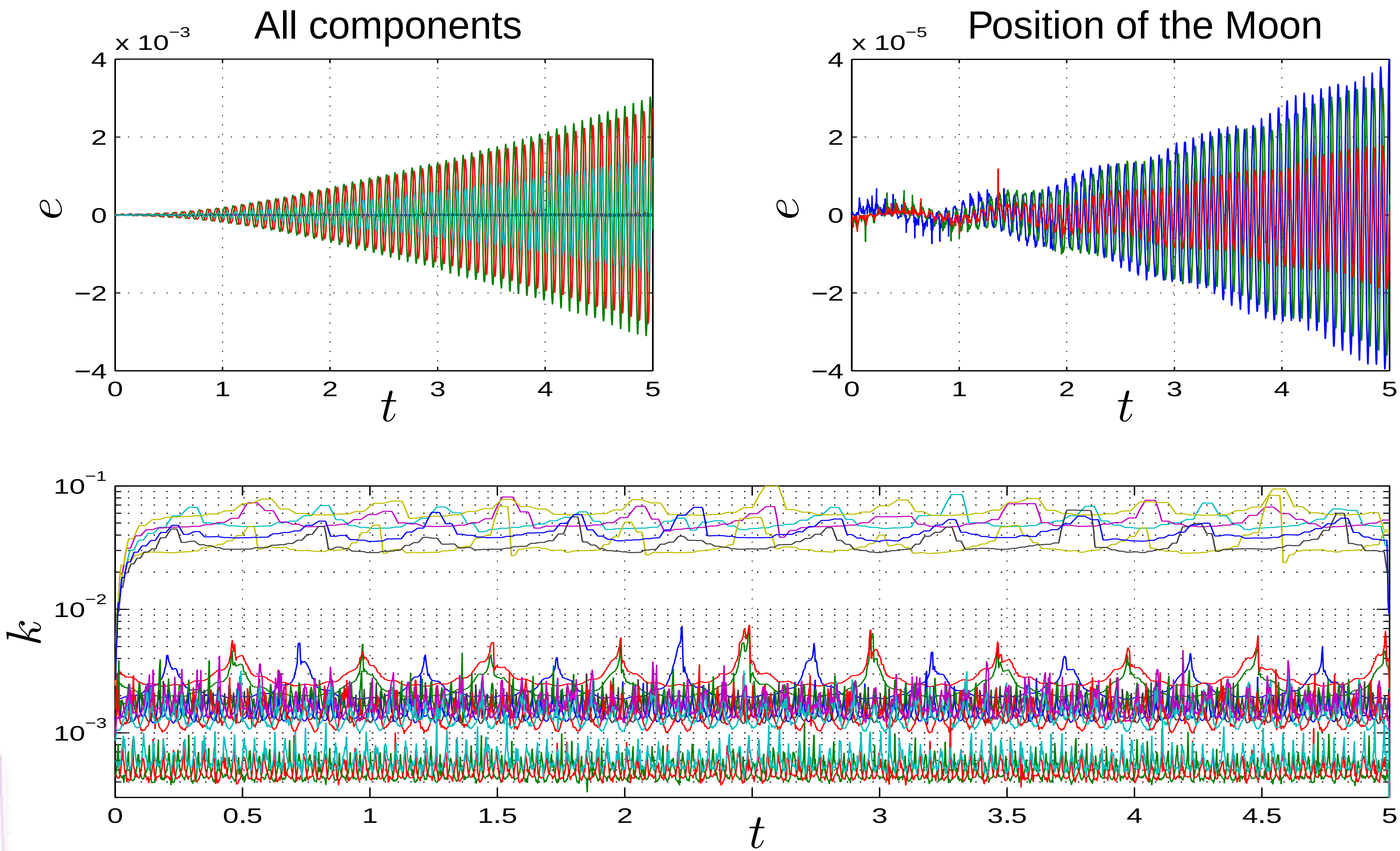
Error Growth



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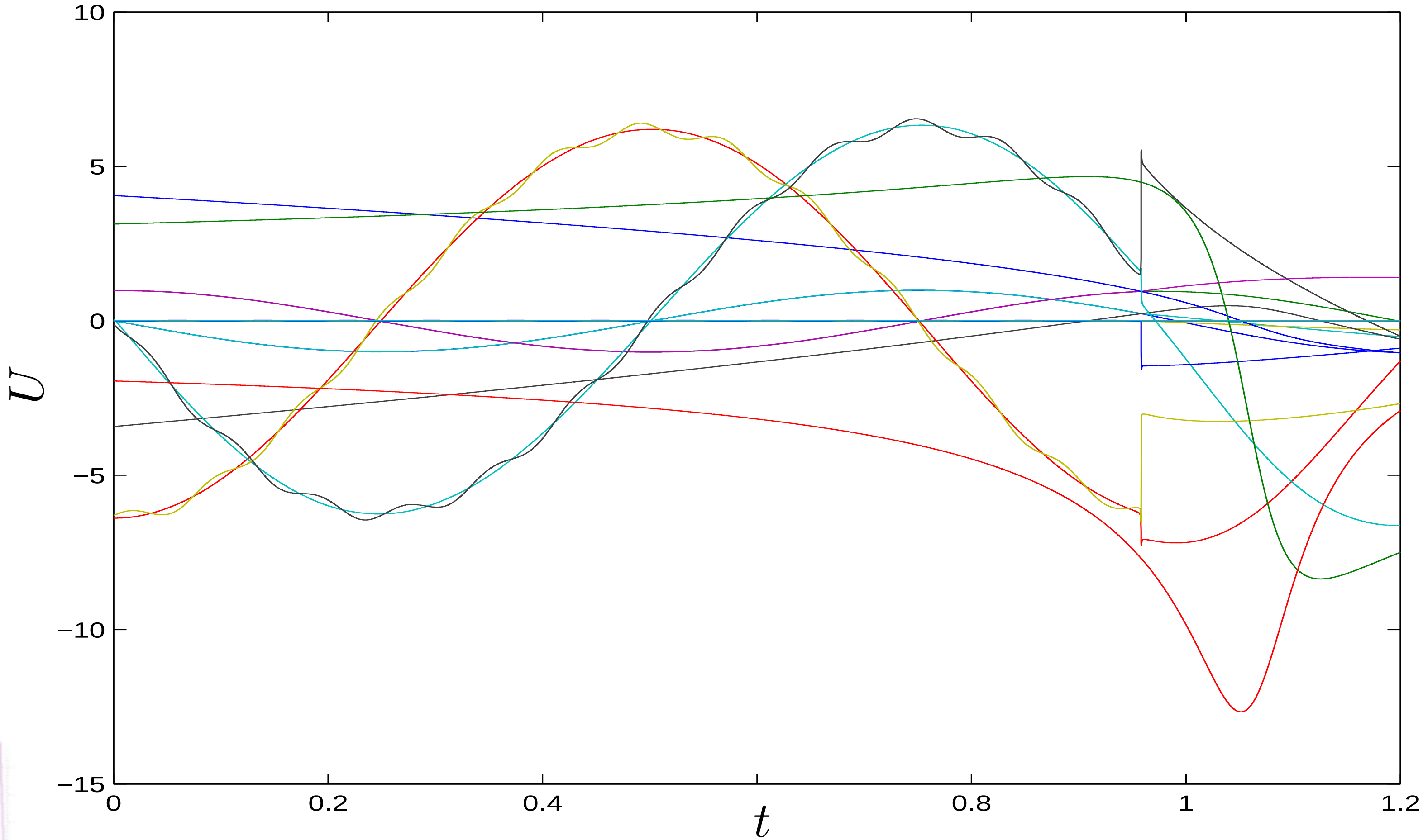
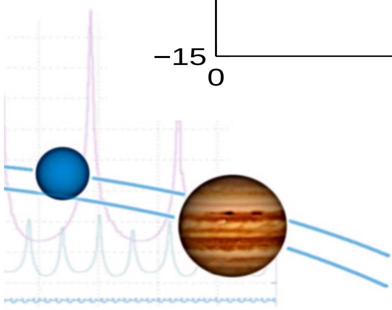
The Earth, the Moon and the Comet





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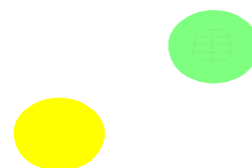

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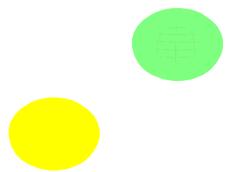

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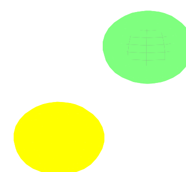

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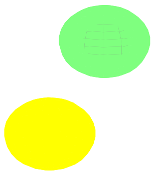

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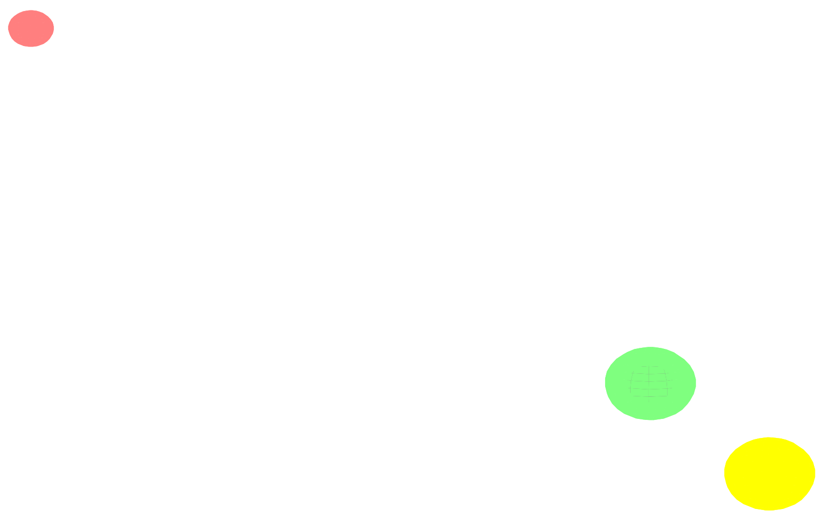

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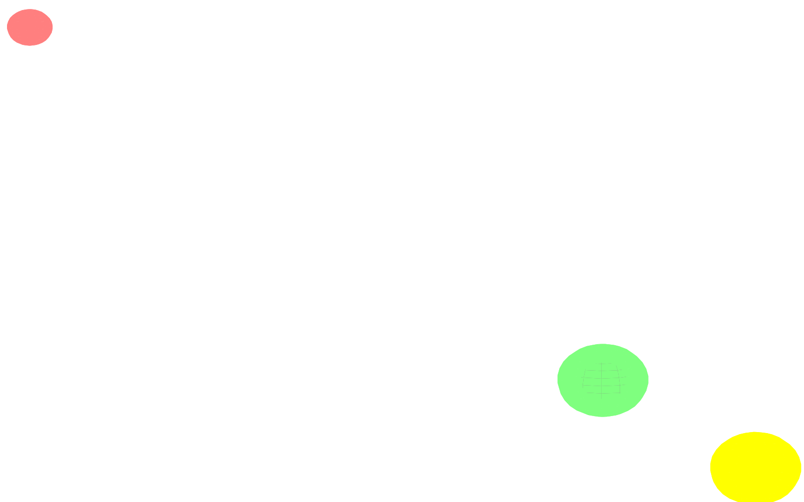

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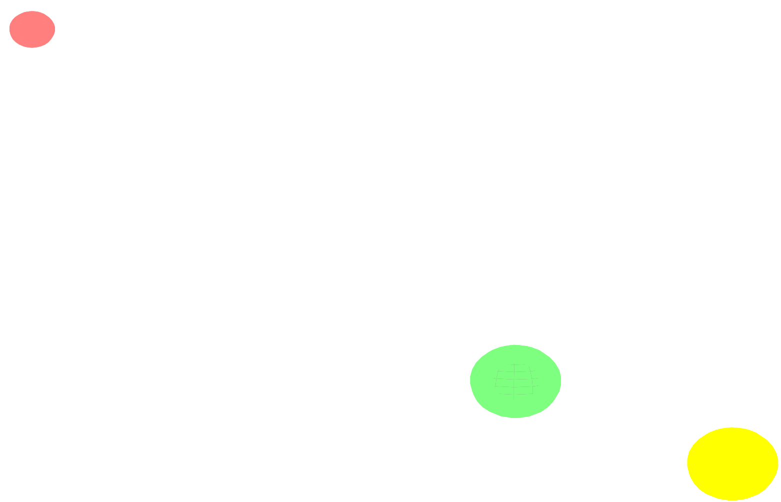

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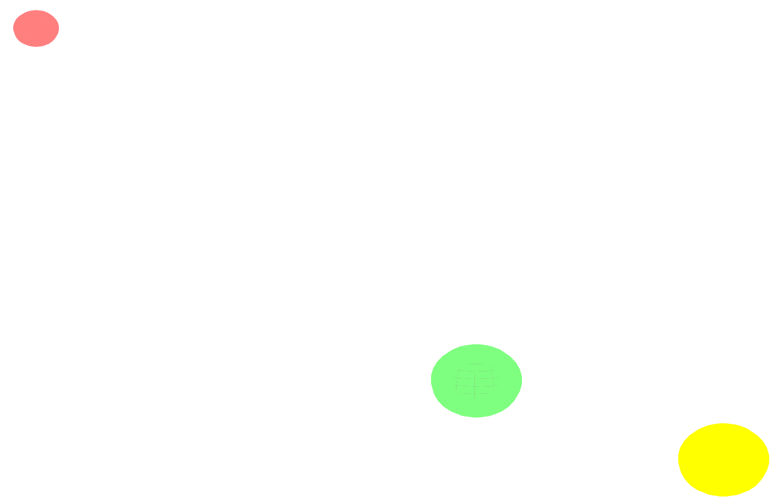

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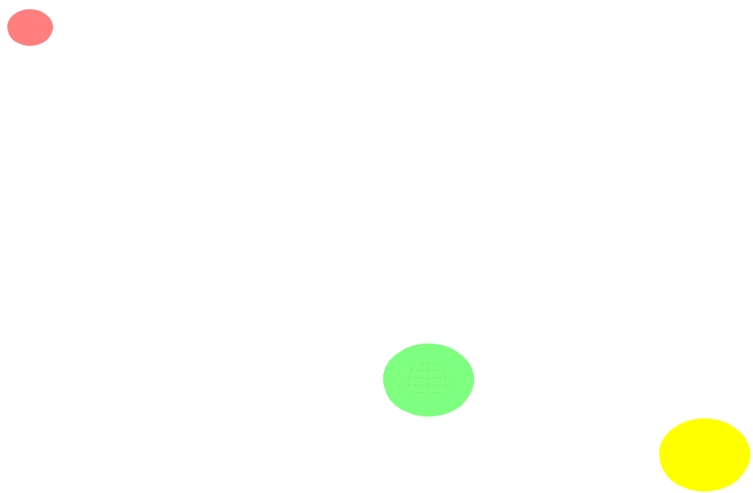

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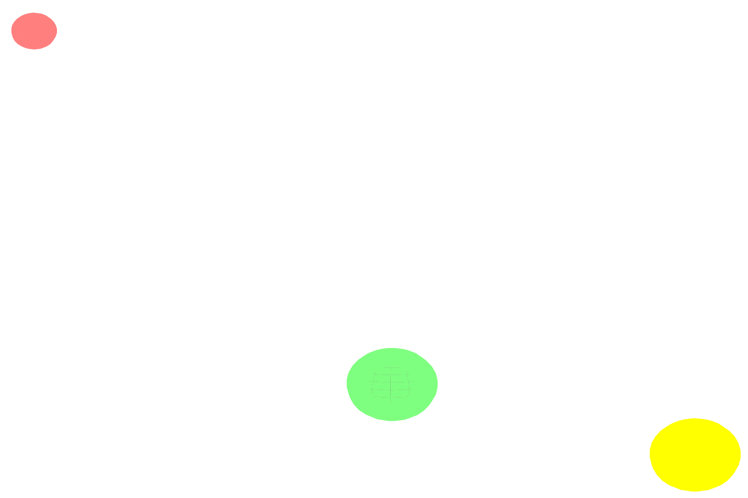

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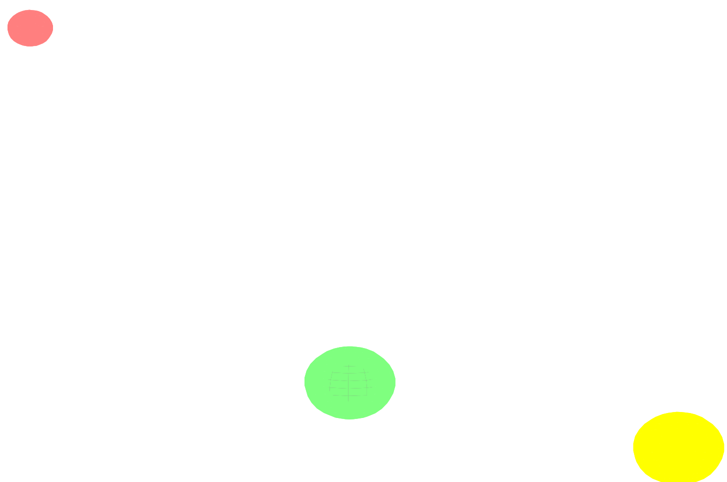

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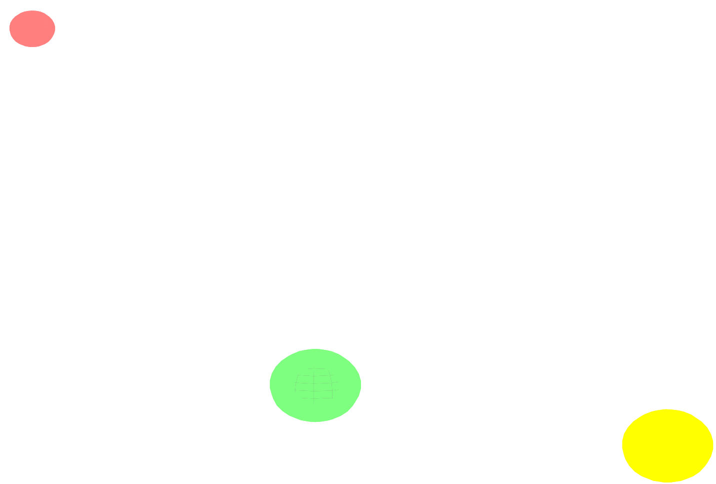

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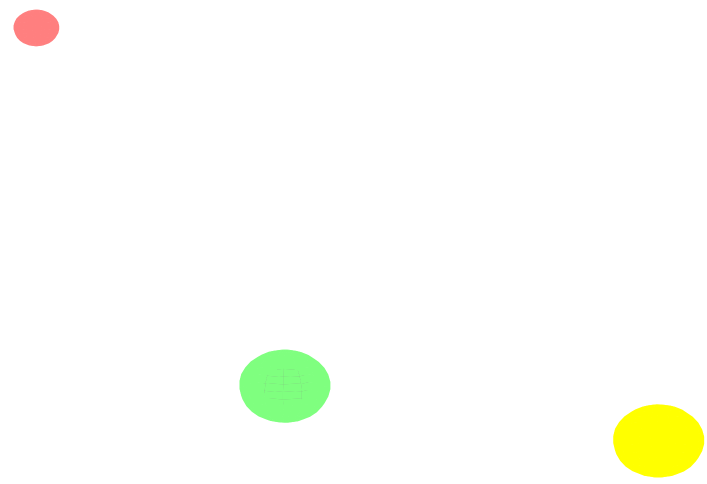

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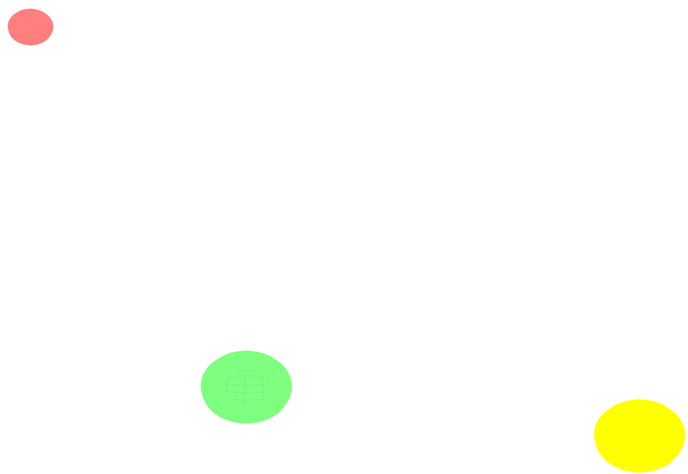

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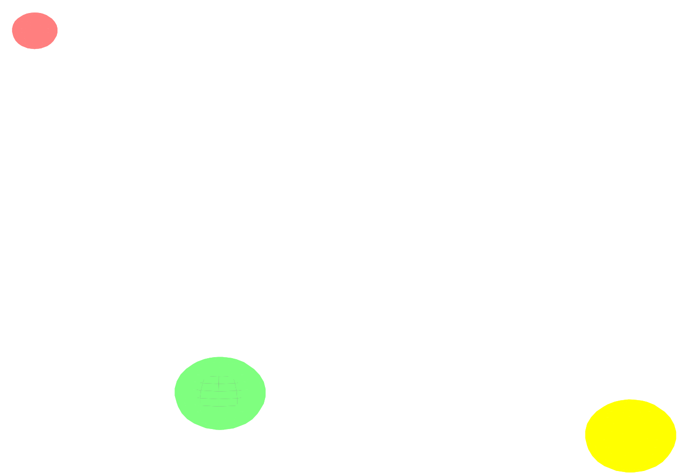

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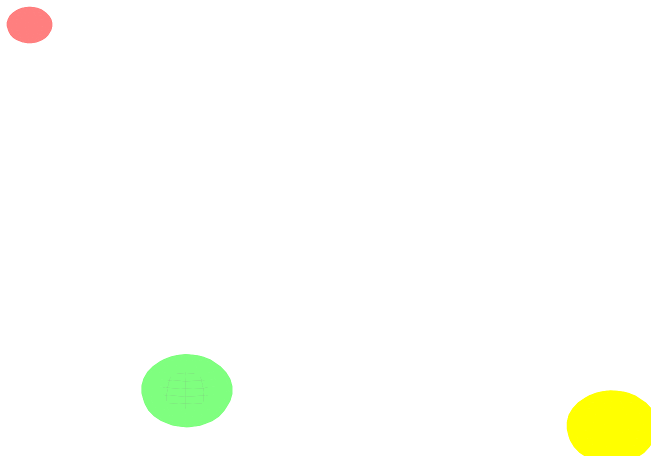

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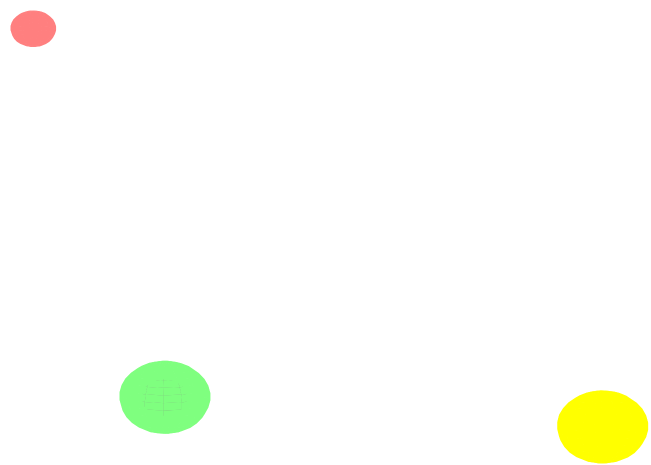

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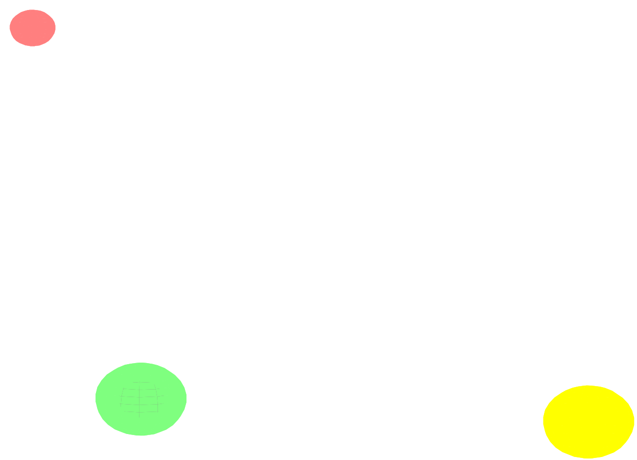

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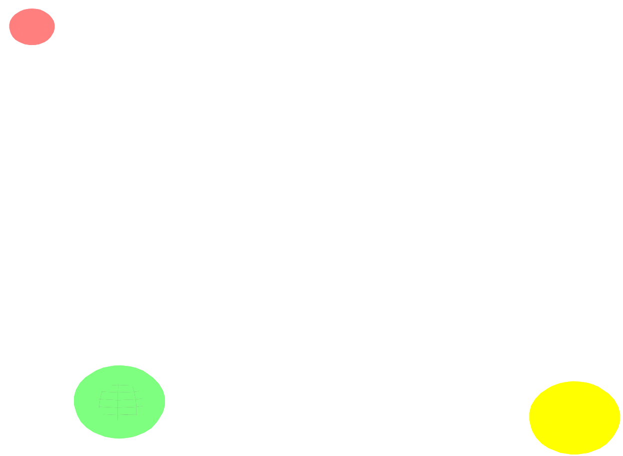

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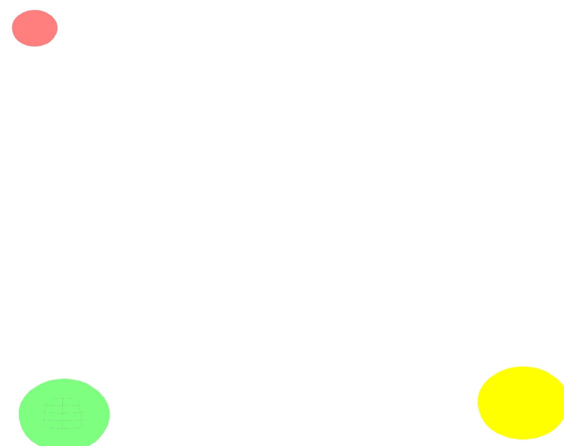

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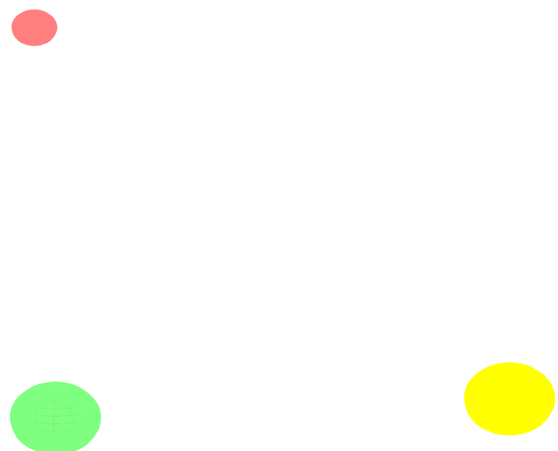

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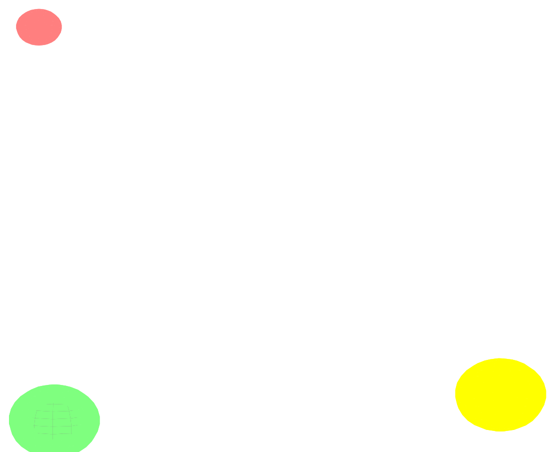

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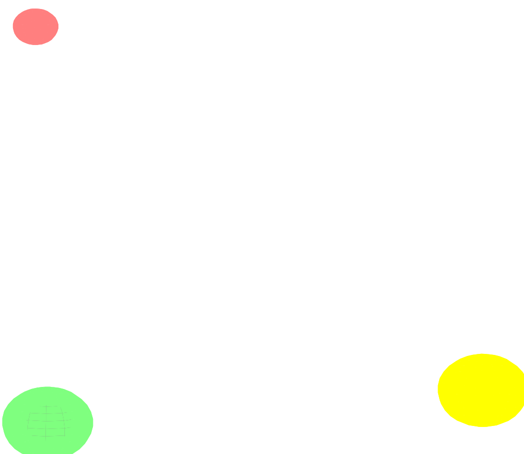

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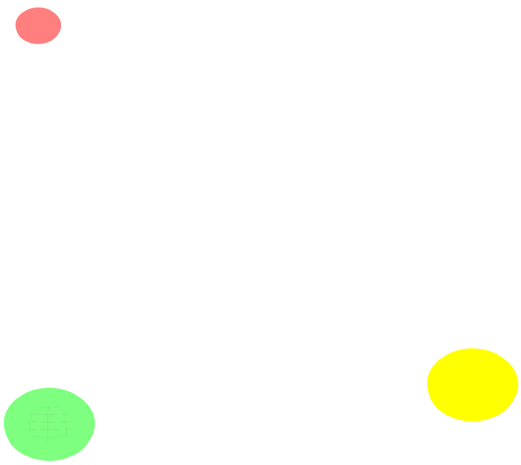

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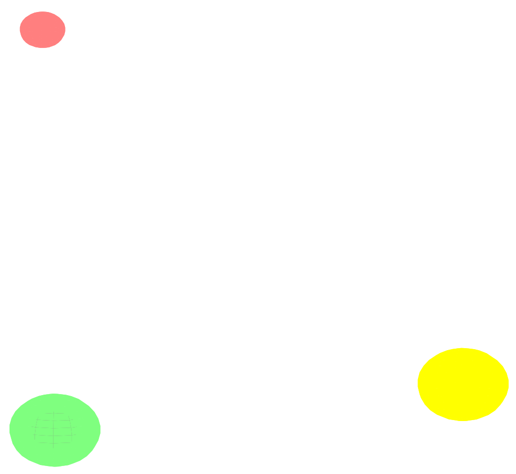

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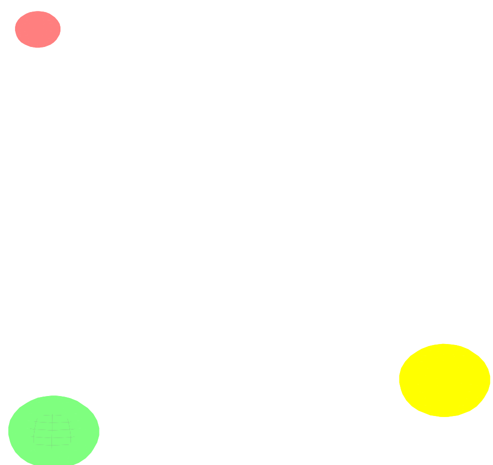

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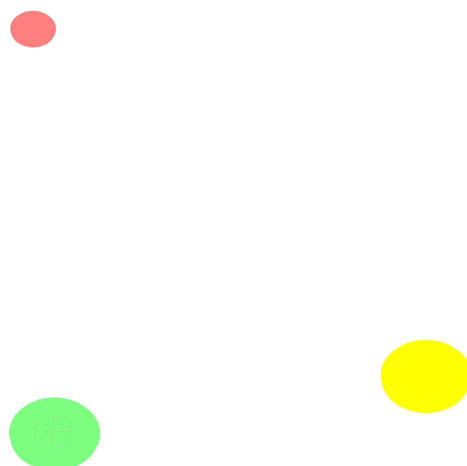

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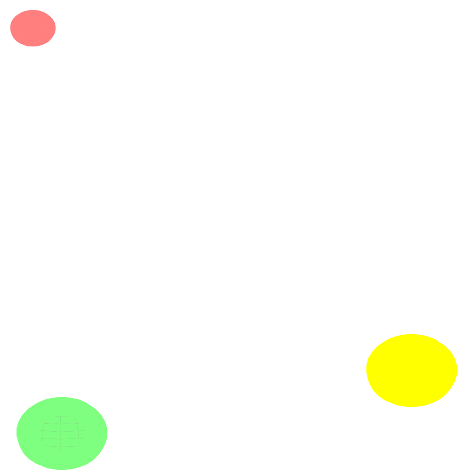

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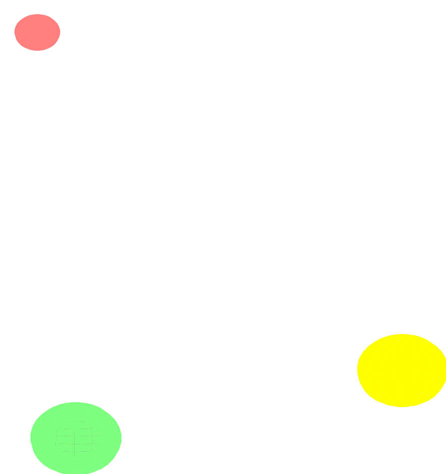

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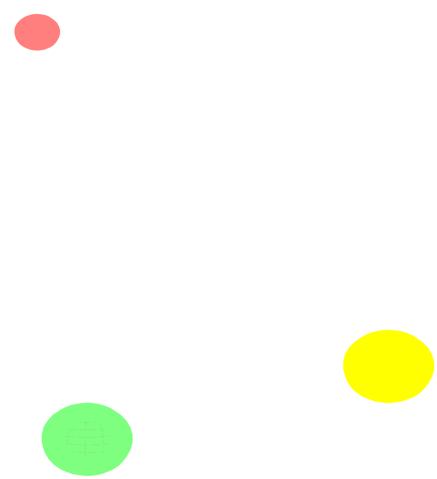

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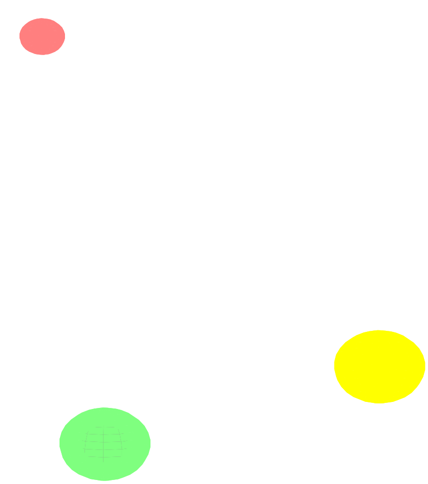

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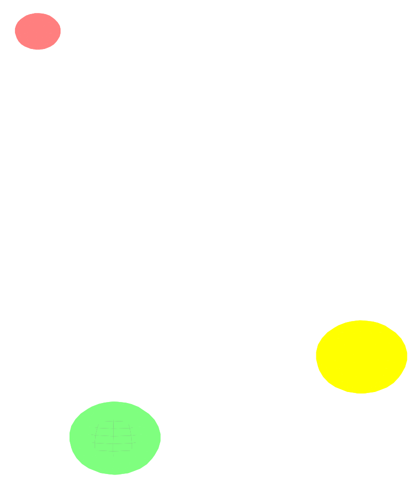

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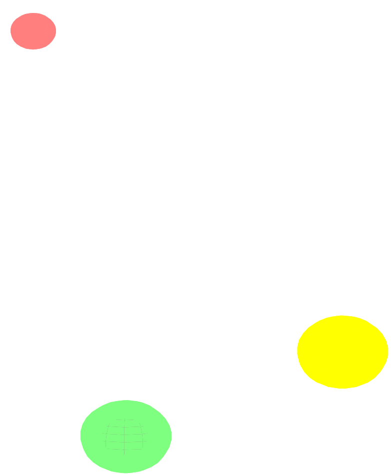

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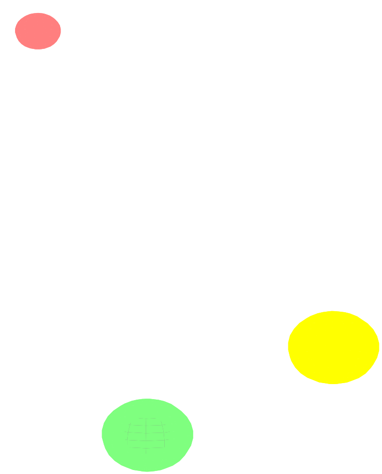

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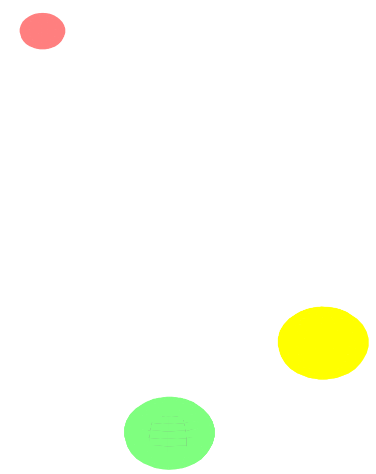

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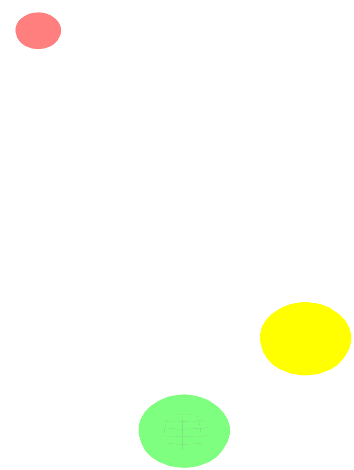

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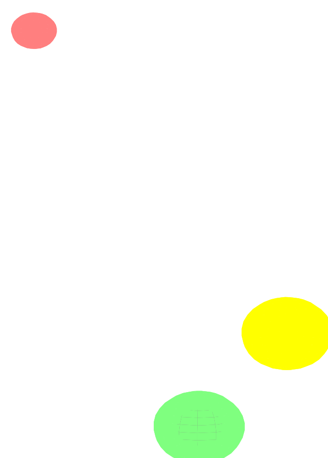

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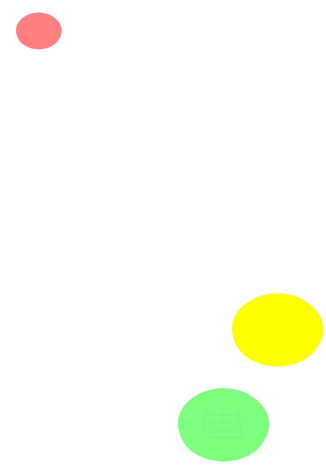

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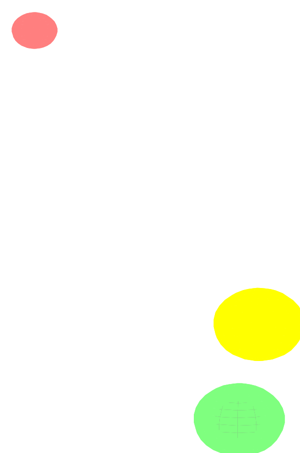

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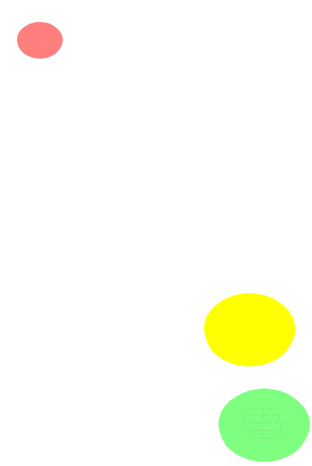

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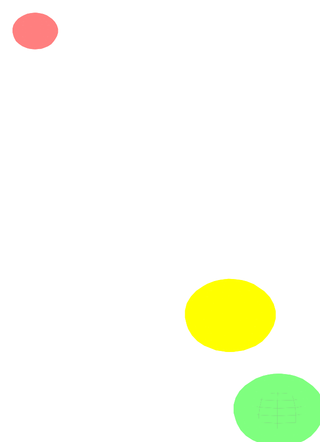

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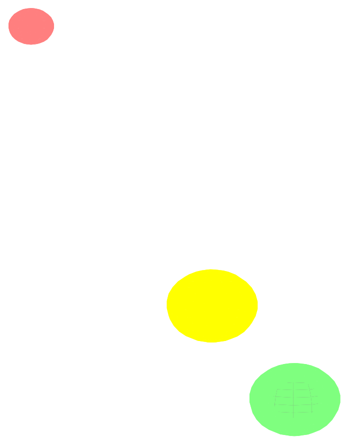

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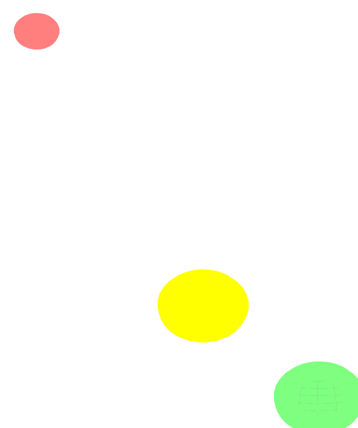

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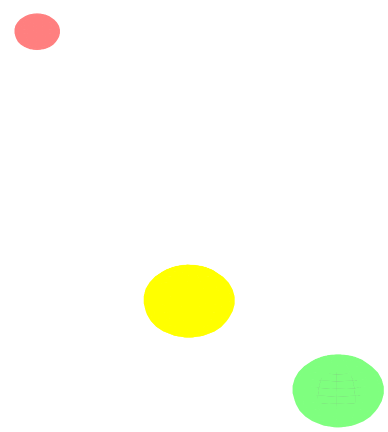

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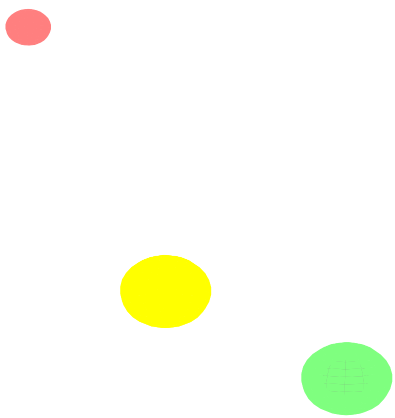

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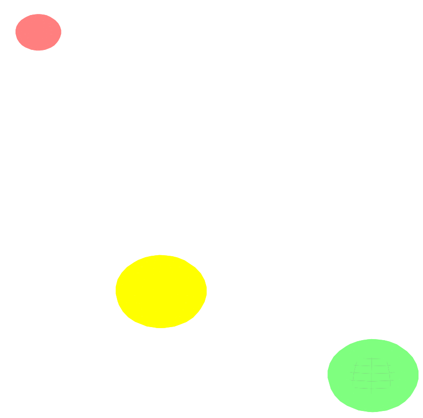

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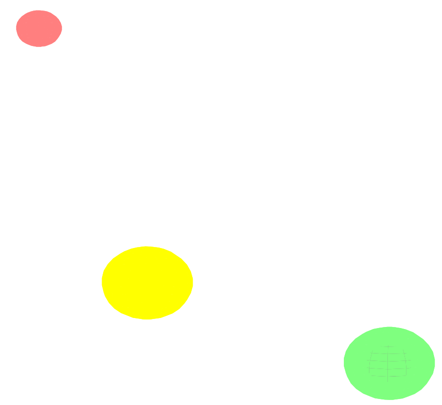

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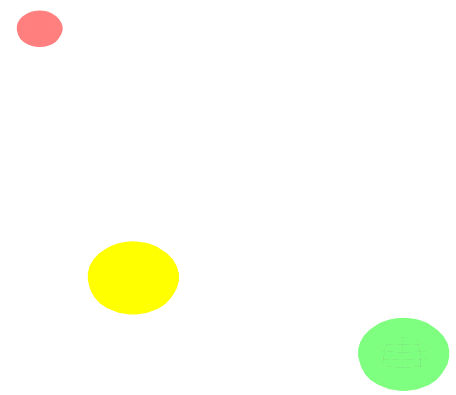

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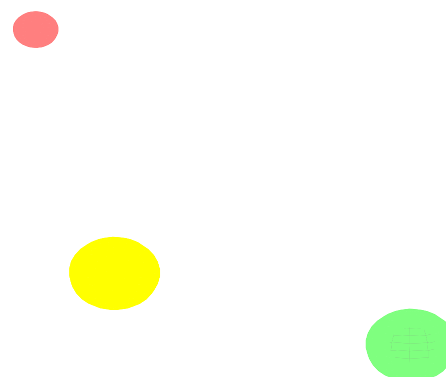

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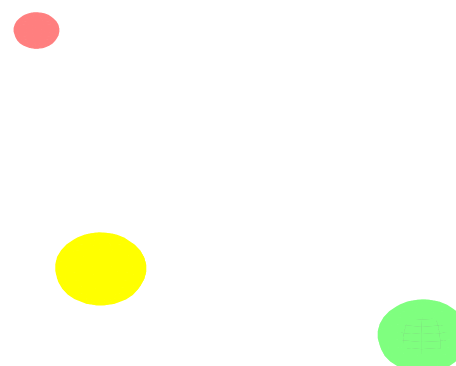

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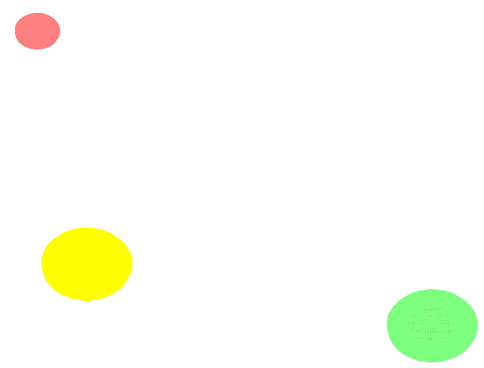

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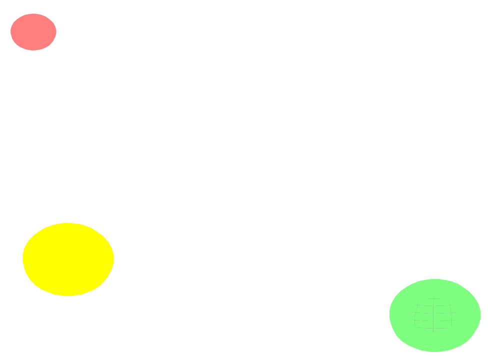

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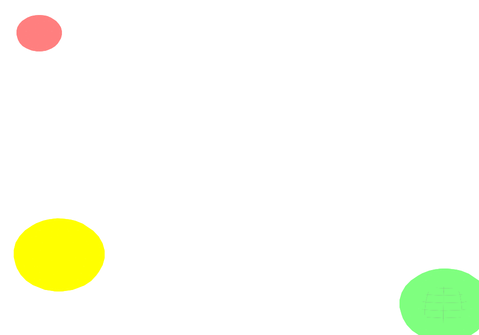

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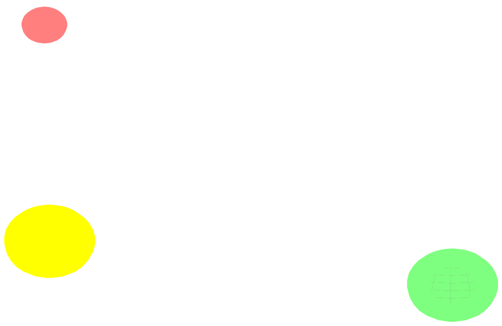

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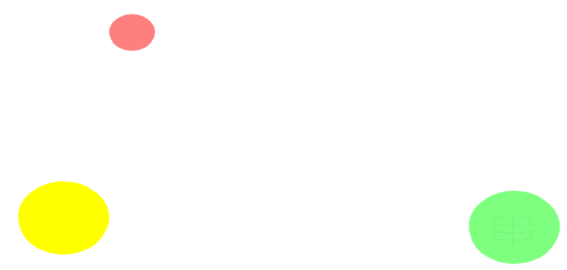

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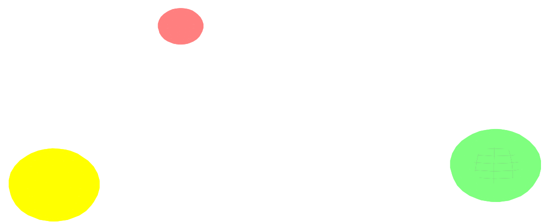

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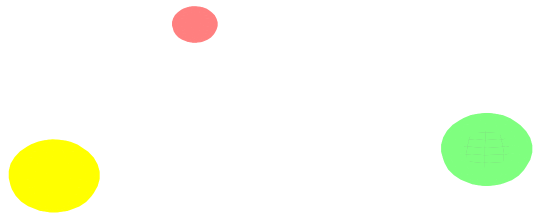

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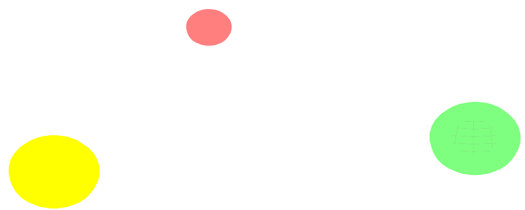

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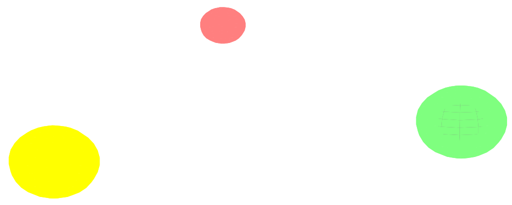

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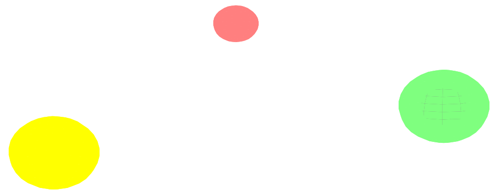

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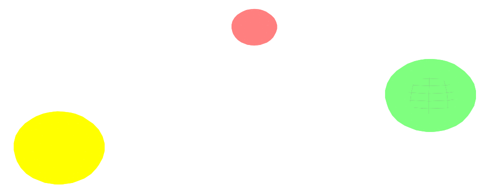

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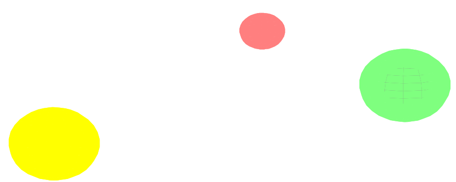

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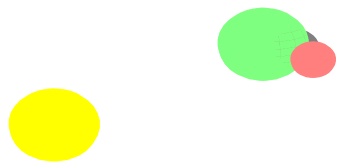

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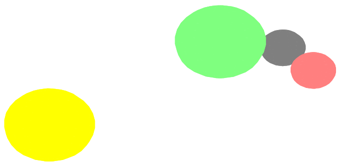

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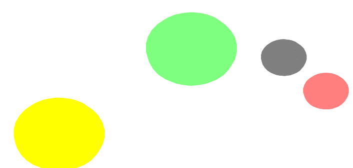

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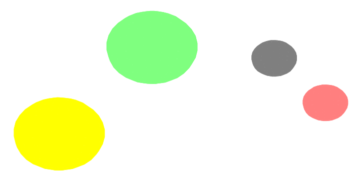

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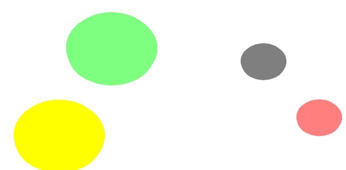

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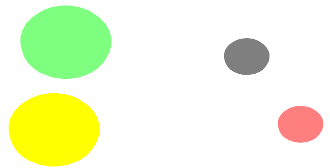

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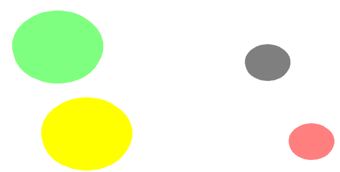

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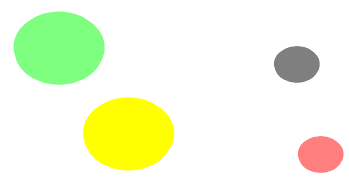

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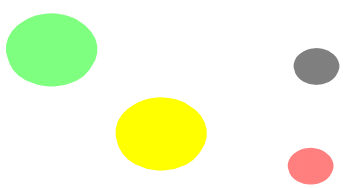

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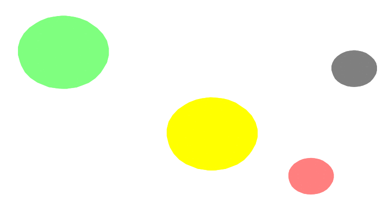

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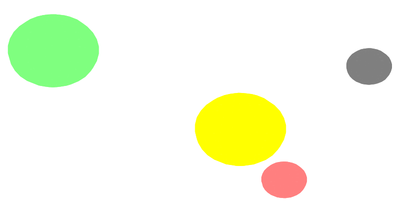

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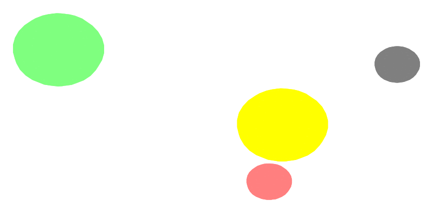

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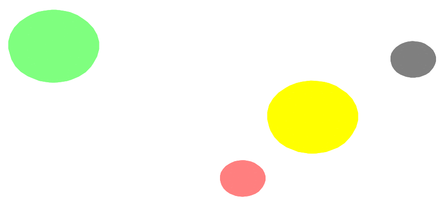

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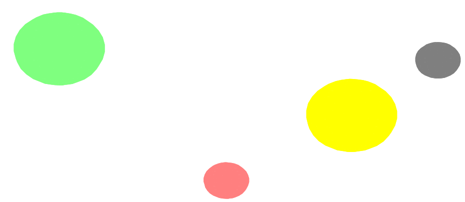

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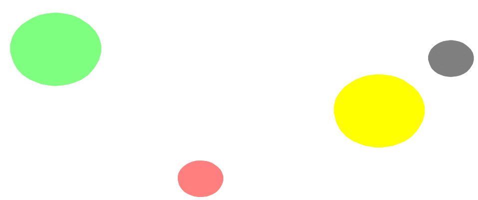

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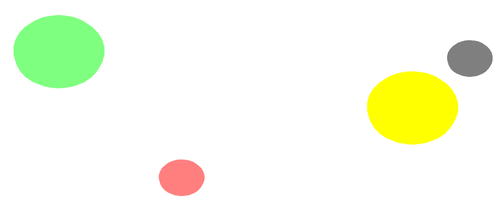

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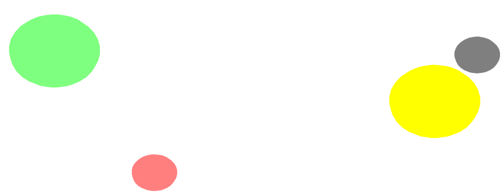

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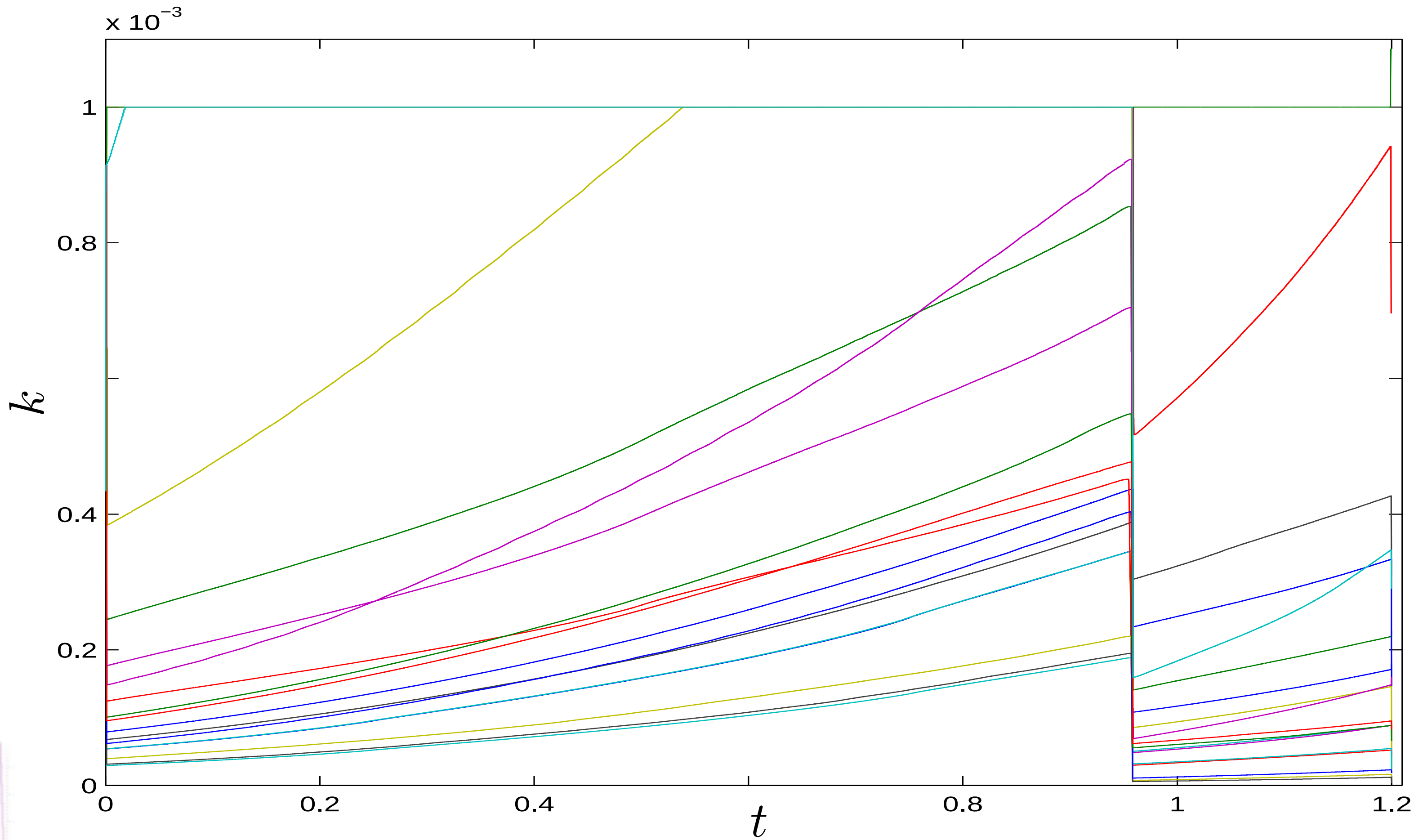
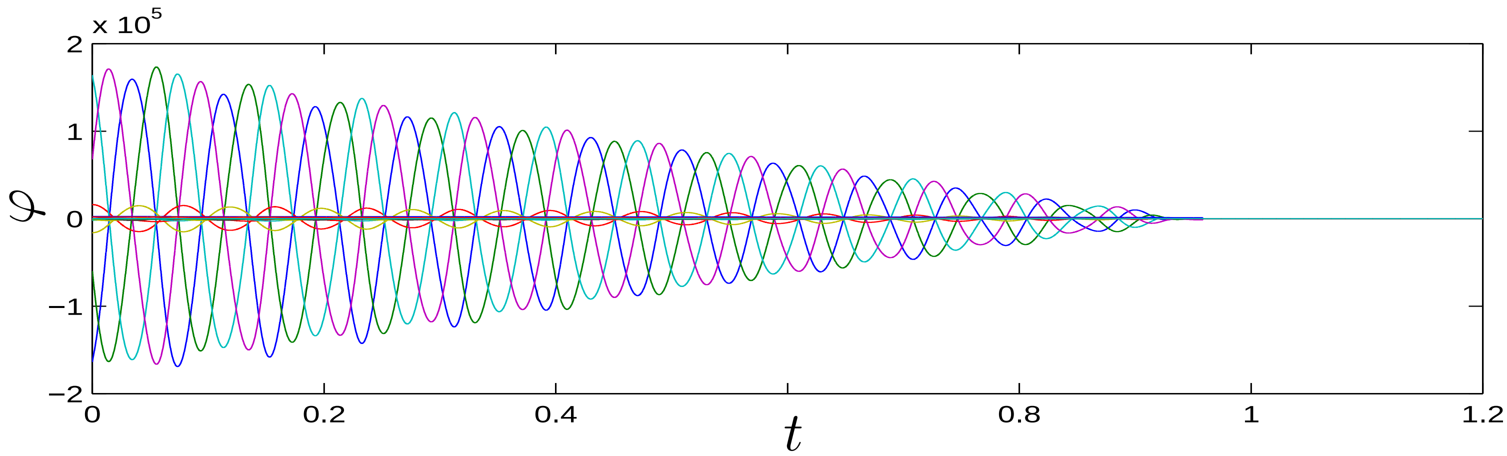
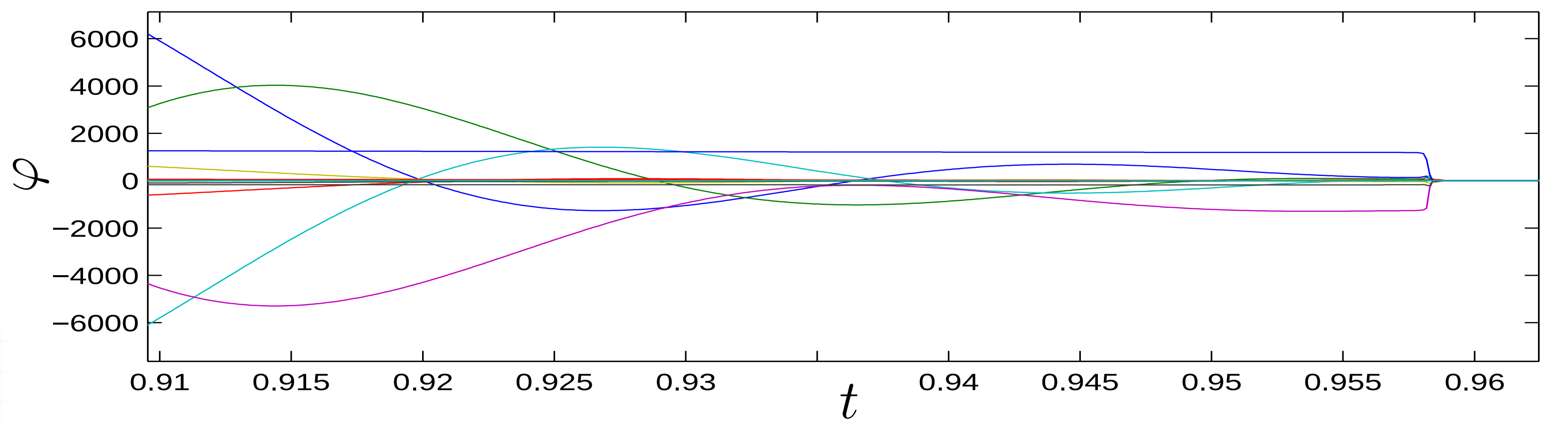
Time-Steps



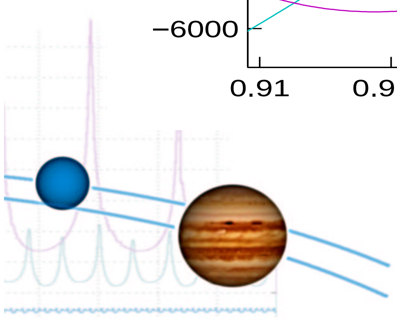
The Dual



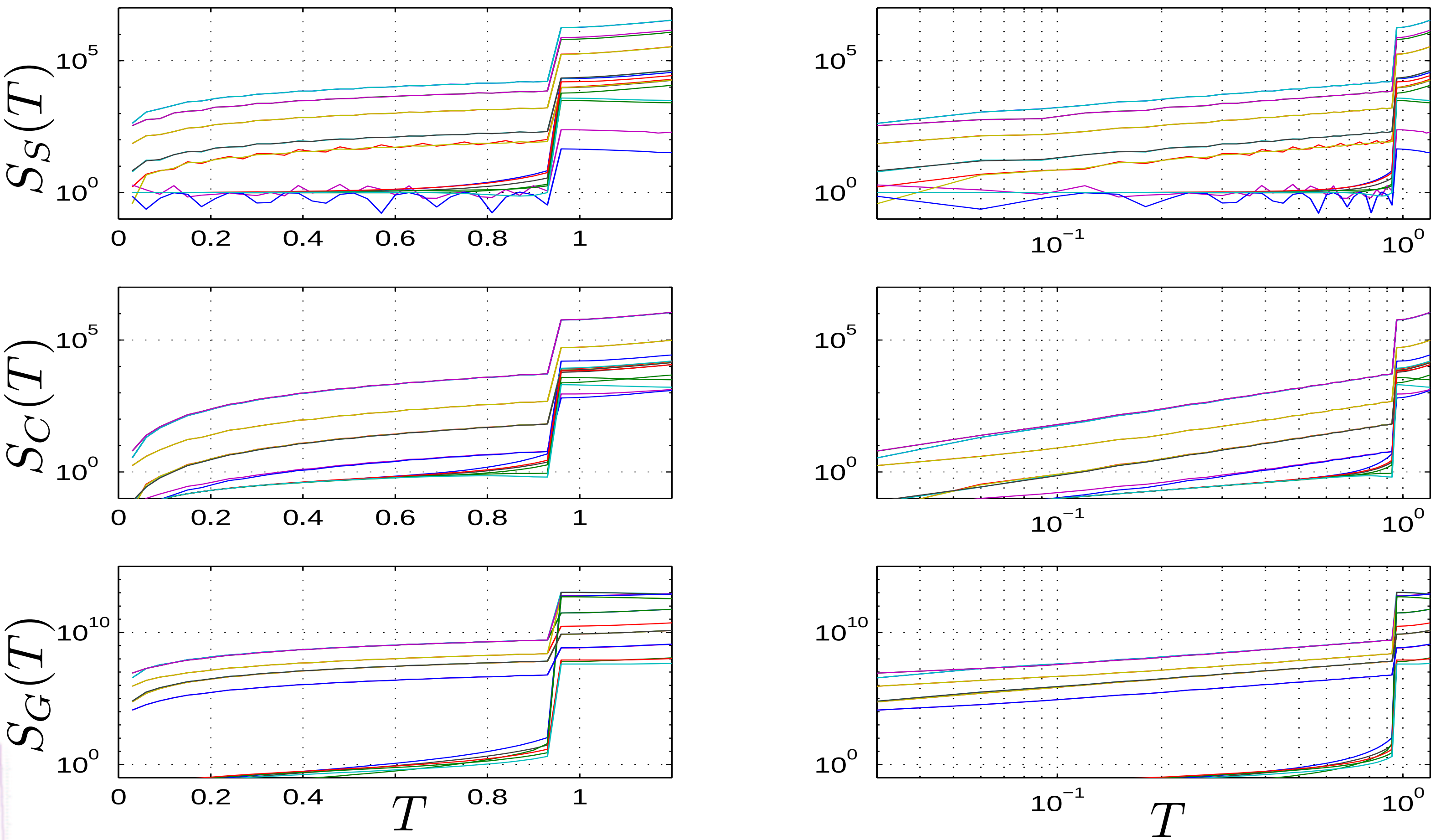
x 10^5
2
1
0
−1
−2
φ
0
0.2
0.4
0.8
1
1.2
t



6000
4000
2000
0
−2000
−4000
−6000
φ
0.91
0.915
0.92
0.925
0.93
0.94
0.945
0.95
0.955
0.96
t



Stability Factors



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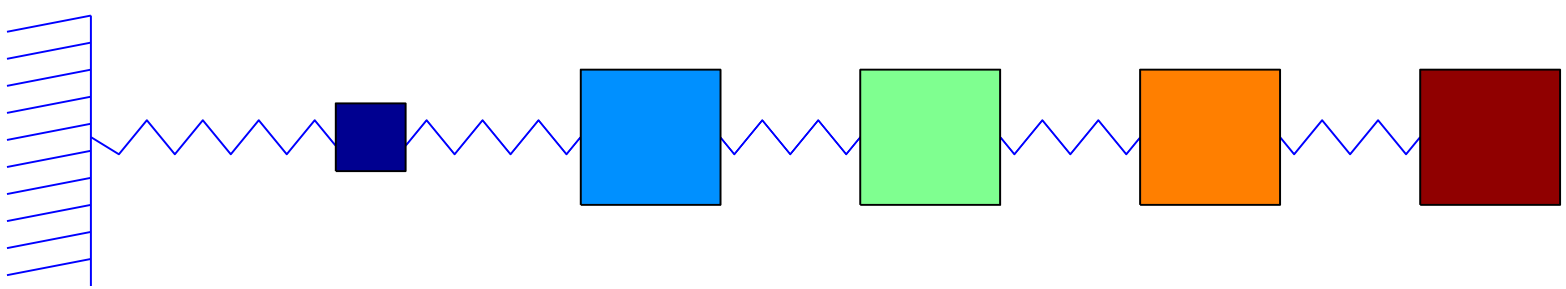
Planetjakten

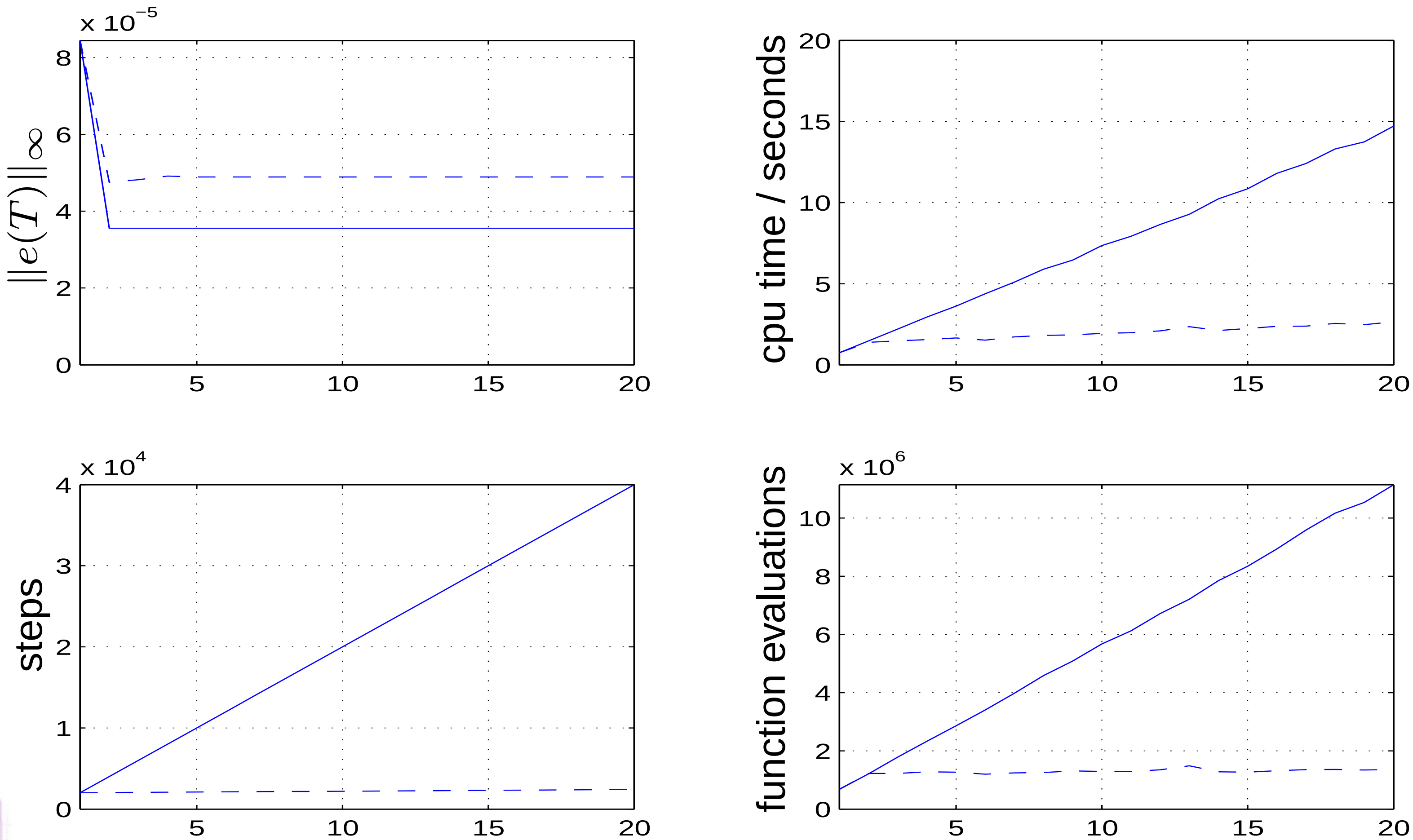
Starta!

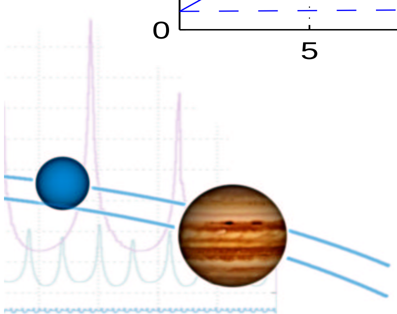
A Mechanical Multi-Scale System

$$\begin{cases} m_i\ddot{x}_i &= k(x_{i+1}-x_i) - kx_i, & i = 1, \\ m_i\ddot{x}_i &= k(x_{i+1}-x_i) - k(x_i - x_{i-1}), & 1 < i < N, \\ m_i\ddot{x}_i &= -k(x_i - x_{i-1}), & i = N \end{cases}$$



Work





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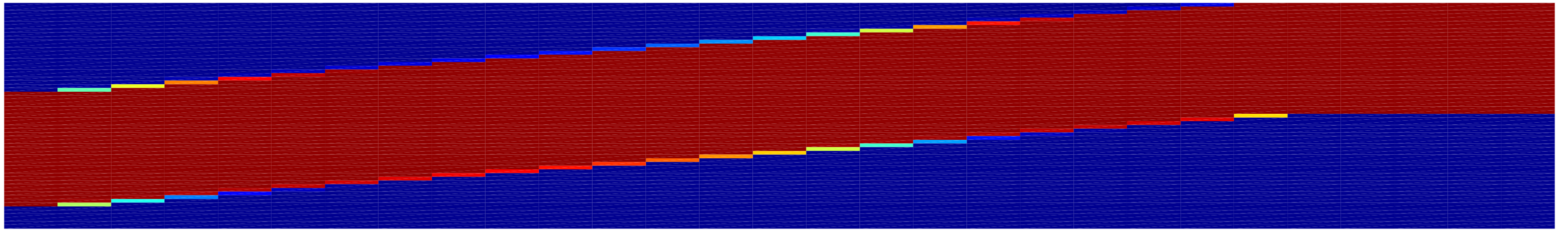
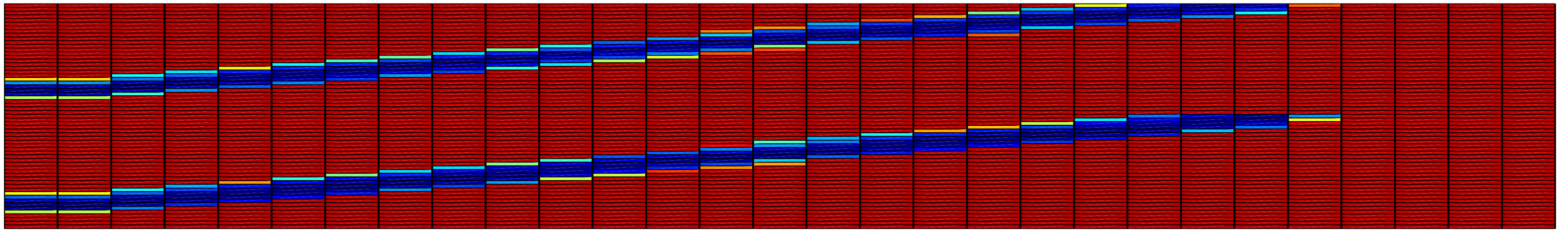
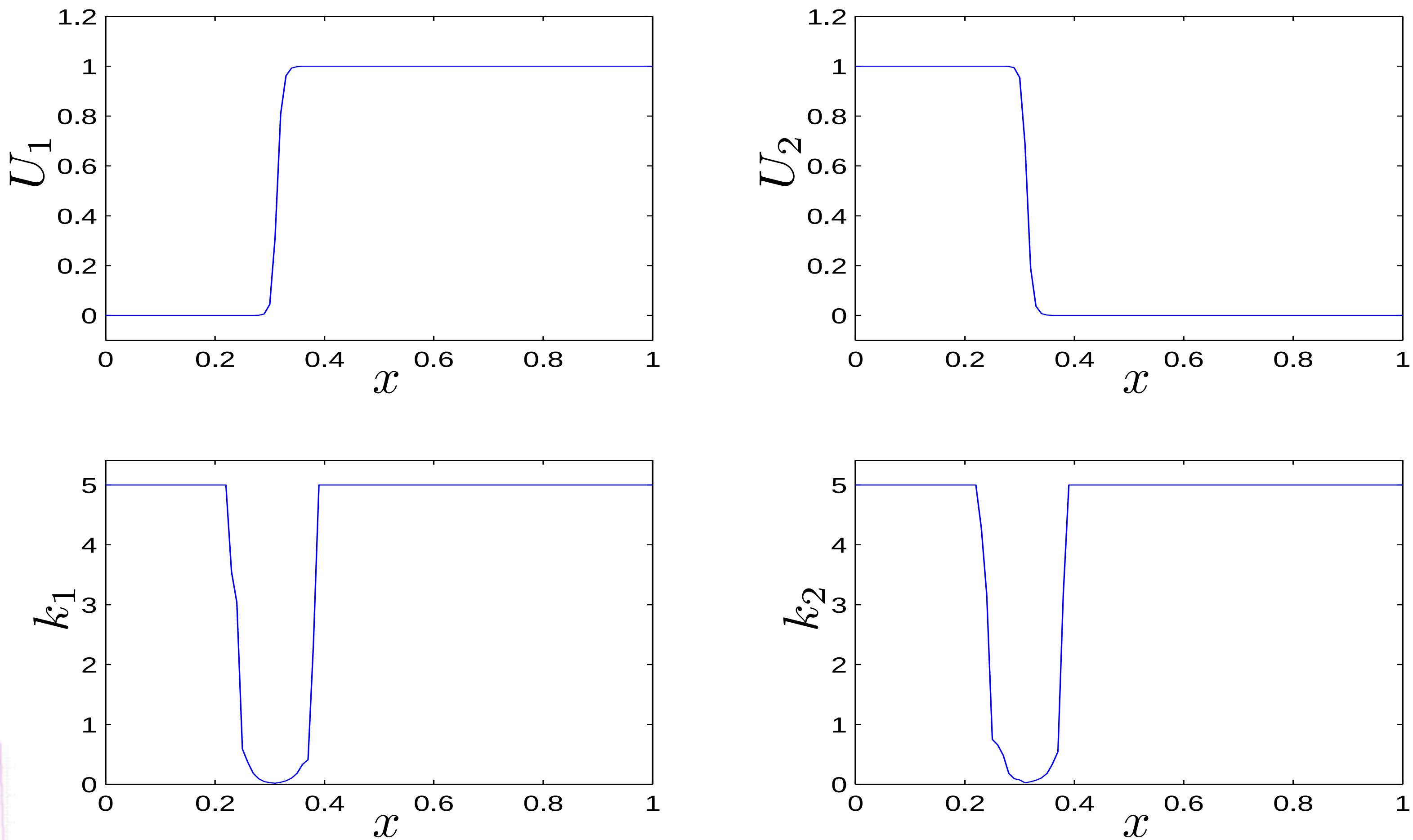
A Propagating Front Problem

$$\begin{cases} \dot{u}_1 - \epsilon u_1'' & = & -u_1 u_2^2 \\ \dot{u}_2 - \epsilon u_2'' & = & u_1 u_2^2 \end{cases}$$





A Propagating Front Problem



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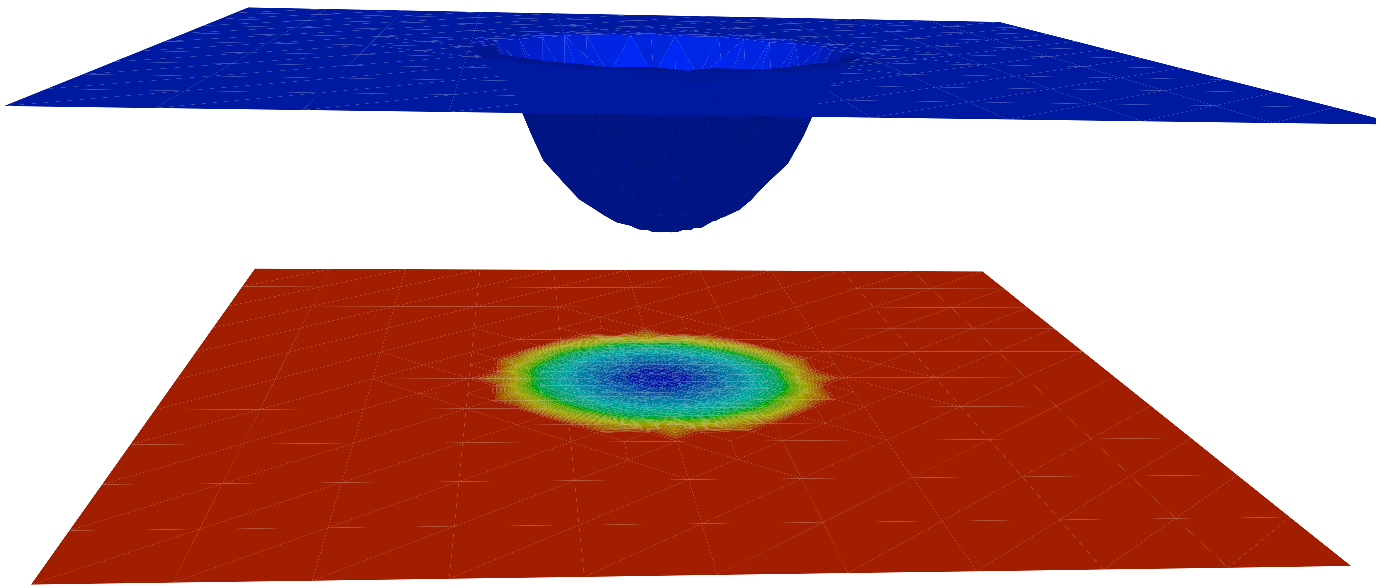
The Heat Equation

$$\dot{u}(x,t) - \Delta u(x,t) = f(x,t)$$

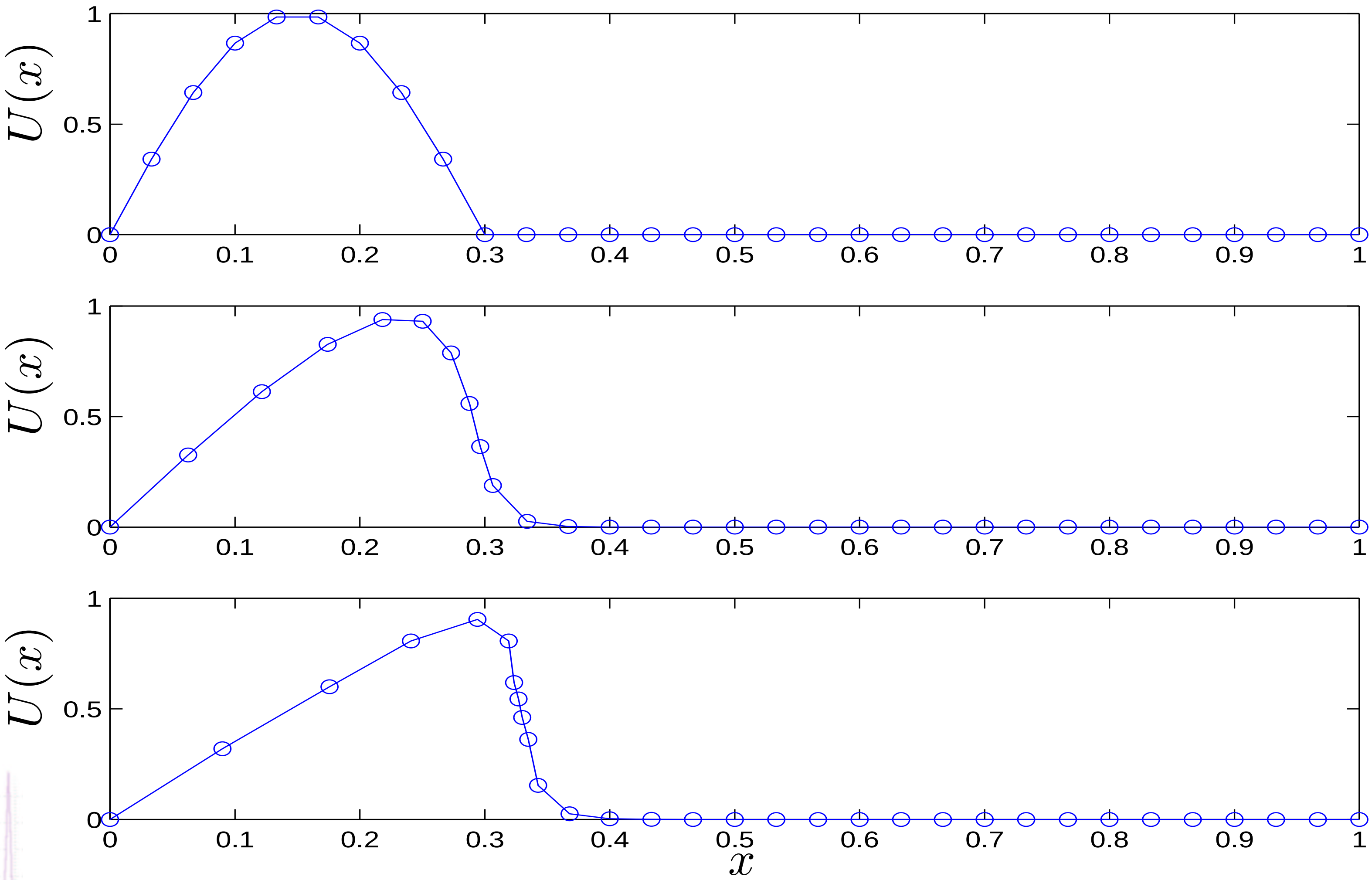


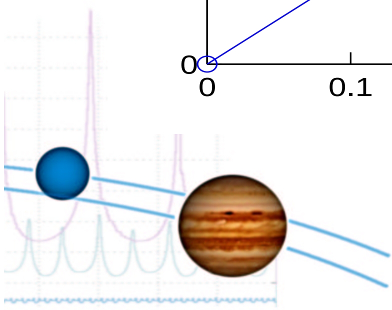
Burger's Equation with Moving Nodes

$$\dot{u} + \mu u u' - \epsilon u'' = 0$$

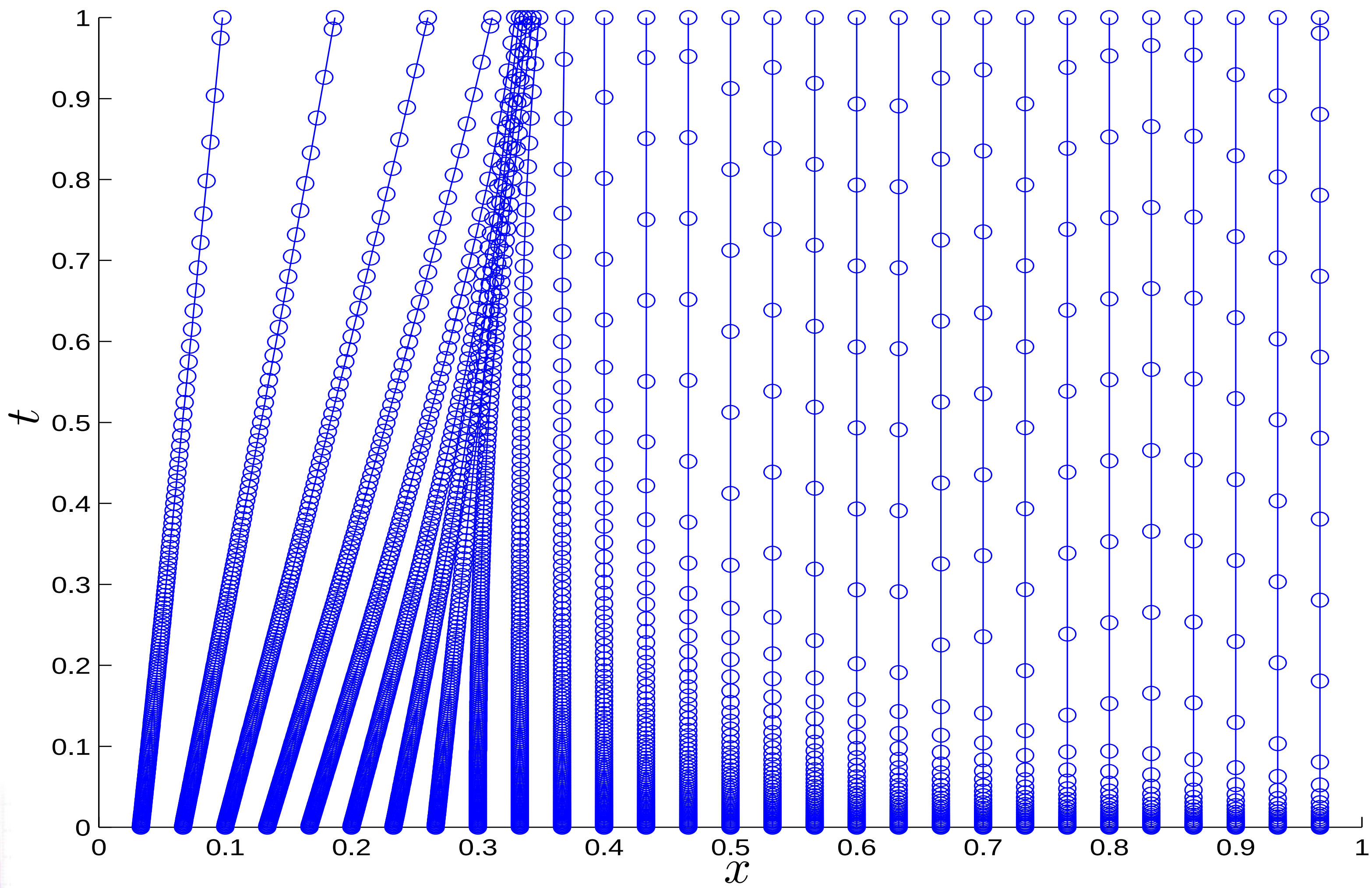
$$u_0(x) = \begin{cases} \sin(\pi x / x_0), & 0 \leq x \leq x_0, \\ 0, & \text{elsewhere} \end{cases}$$

Solution





Node Paths



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Future Work

- Further development of the iterative strategy
- Extension to PDEs (in more detail)
- Extension to DAEs
- More applications

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