# Adaptive Finite Element methods for incompressible flow

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# 3 lectures

1. Adaptive FE methods for incompr. fluid flow
2. Hydrodynamic stability
3. Subgrid modeling & multi adaptivity

# Lecture 1

- Adaptivity & a posteriori error analysis
- Navier-Stokes equations (NSE)
- Discretization of NSE
- Discrete solvers
- A posteriori error analysis for NSE

# General setting

Mathematical model: $A(u) = f$ (MM)

where $A$ is a differential operator, $f$ is given data, and $u$ is the solution

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where $A$ is a differential operator, $f$ is given data, and $u$ is the solution

The model is pertubed from data $\hat{f}$, modeling $\hat{A}$, and discretization $U$, where $U$ is the numerical approximate solution to the pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$, with exact solution $\hat{u}$

# General setting

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

$\hat{A}$ may represent a turbulence model in applications to fluid flow

# General setting

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

Notation:

$u - \hat{u}$ = data/modeling error

$\hat{u} - U$ = discretization error

Total computational error = $u - U = u - \hat{u} + \hat{u} - U$

# General setting

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

Central questions:

- What quantities are computable?
  (point values, mean values,...)
- To what tolerance?
- And to what cost?

# Adaptive methods

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

Adaptive methods includes a feed-back process, where the computed solutions $U$ of (PP) are investigated with the objective of reducing the errors $u - \hat{u}$ and $\hat{u} - U$

# Adaptive methods

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

$u - \hat{u}$ is reduced by improving the model $\hat{A}$

$\hat{u} - U$ is typically reduced by modifying the local mesh size or increasing the approximation order

# Adaptive methods

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

Adaptive methods are based on a posteriori error estimates, in terms of computable residuals:

$$f - A(U) \quad \hat{f} - \hat{A}(U)$$

or estimated residuals involving $u$ or $\hat{u}$

# Adaptive methods

Stopping criterion:

- Sharp a posteriori error estimates of
  - error in solution (diff. norms & averages)
  - error in functionals (drag, lift,...)

Modification strategy:

- Chosen based on crude a posteriori error est:
  - refine/derefine local mesh size, inrease/decrease approximation order
  - modify turbulence model: e.g. modify local turbulent viscosity

# A posteriori error analysis

Galerkin method for $\hat{A}(\hat{u}) = \hat{f}$:

Find $U \in V$ s.t. $(\hat{A}(U), v) = (\hat{f}, v) \quad \forall v \in V_h$

where $V_h$ is a finite dimensional subspace of the Hilbert space $V$ with scalar product $(\cdot, \cdot)$ and

$$\hat{A} : V \to V$$

is Frechet differentiable with derivative

$$A' : V \to V$$

# A posteriori error analysis

To estimate $(e, \psi)$, $e = \hat{u} - U$ and $\psi \in V$, write

$$\hat{A}(\hat{u}) - \hat{A}(U) = \int_0^1 \frac{d}{ds} \hat{A}(s\hat{u} + (1-s)U)\, ds$$

$$\int_0^1 A'(s\hat{u} + (1-s)U)\, ds\; e \;\equiv\; A'(\hat{u}, U)e$$

$(e, \psi)$ may represent $L_2$-error ($\psi = e/\|e\|$), pointwise error in the point $x$ ($\psi = \delta_x$), error in linear functional $J(\cdot)$ ($\psi : (e, \psi) = J(\hat{u}) - J(U)$),...

# A posteriori error analysis

Let then $\phi \in V$ solve (the dual problem)

$$(A'(\hat{u}, U)w, \phi) = (w, \psi), \quad \forall w \in V$$

Setting $w = e$ gives the error representation

$$\begin{aligned}(e, \psi) &= (A'(\hat{u}, U)e, \phi) = (\hat{A}(\hat{u}) - \hat{A}(U), \phi) \\ &= (\hat{f} - \hat{A}(U), \phi) = (\hat{R}(U), \phi) \\ &= (\hat{R}(U), \phi - \Phi), \quad \Phi \in V_h\end{aligned}$$

since $(\hat{R}(U), \Phi) = 0$ by Galerkin orthogonality

# A posteriori error analysis

Basic example 1: Stationary linear problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with corresponding dual problem

$$\begin{aligned} -\Delta \phi &= \psi \quad \text{in } \Omega \\ \phi &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

# A posteriori error analysis

$$\begin{aligned}
|(e,\psi)| &= |\int_\Omega e\psi\,dx| = |\int_\Omega e\Delta\phi\,dx| = |\int_\Omega \Delta e\phi\,dx| \\
&= |\int_\Omega R(U)\phi\,dx| = |\int_\Omega R(U)(\phi-\Phi)\,dx| \\
&\leq \int_\Omega R(U)Ch^m|D^m\phi|\,dx
\end{aligned}$$

$R(U) = |f + \Delta U|$ and $\Phi \in V_h$ is an interpolant of $\phi$ satisfying $|\phi - \Phi| \leq Ch^m|D^m\phi|$

# A posteriori error analysis

$$
\begin{aligned}
|(e,\psi)| &= |\int_\Omega e\psi\; dx| = |\int_\Omega e\Delta\phi\; dx| = |\int_\Omega \Delta e\phi\; dx| \\
&= |\int_\Omega R(U)\phi\; dx| = |\int_\Omega R(U)(\phi - \Phi)\; dx| \\
&\leq \int_\Omega R(U)Ch^m|D^m\phi|\; dx
\end{aligned}
$$

We denote $|D^m\phi|$ a stability weight

# A posteriori error analysis

$$
\begin{aligned}
|(e,\psi)| &= |\int_\Omega e\psi\, dx| = |\int_\Omega e\Delta\phi\, dx| = |\int_\Omega \Delta e\phi\, dx| \\
&= |\int_\Omega R(U)\phi\, dx| = |\int_\Omega R(U)(\phi - \Phi)\, dx| \\
&\leq \int_\Omega R(U) C h^m |D^m\phi|\, dx \leq CS\|h^m R(U)\|
\end{aligned}
$$

Using Cauchy’s inequality we get a multiplicative stability factor $S = \|D^m\phi\|$

# Examples of dual data $\psi$

Error in $L_2$-norm:

$$\psi = e/\|e\|$$

Error in average over volume $\omega \subset \Omega$:

$$\psi = \chi_\omega/|\omega|$$

Error at the point $x \in \Omega$:

$$\psi = \delta_x$$

# A posteriori error analysis

Basic example 2: Transient linear problem

$$\begin{aligned} \dot{u} - \Delta u &= f \quad \text{in } \Omega \times [0,T] \\ u &= 0 \quad \text{on } \partial\Omega \times [0,T] \\ u(\cdot,0) &= u^0 \quad \text{in } \Omega \end{aligned}$$

with corresponding dual problem

$$\begin{aligned} -\dot{\phi} - \Delta \phi &= 0 \quad \text{in } \Omega \times [0,T] \\ \phi &= 0 \quad \text{on } \partial\Omega \times [0,T] \\ \phi(\cdot,T) &= \psi \quad \text{in } \Omega \end{aligned}$$

# A posteriori error analysis

$$
\begin{aligned}
|(e(T),\psi)| &= |\int_\Omega e(T)\psi\ dx| \\
&= |\int_\Omega e(T)\psi\ dx| + |\int_0^T \int_\Omega e(-\dot{\phi} - \Delta\phi)\ dx\ dt| \\
&= |\int_0^T \int_\Omega (\dot{e} - \Delta e)\phi\ dx\ dt| \quad (\text{ass. } e(0) = 0) \\
&= |\int_0^T \int_\Omega R(U)(\phi - \Phi)\ dx\ dt| \quad (\text{where } \Phi \in V_h)
\end{aligned}
$$

# A posteriori error analysis

$$
\begin{aligned}
|(e(T),\psi)| \;\leq\; & \int_0^T \int_\Omega R(U) C_1 k |\dot{\phi}| \, dx \, dt \\
+ \; & \int_0^T \int_\Omega R(U) C_1 h^m |D^m \phi| \, dx \, dt
\end{aligned}
$$

with $R(U) = |f - \dot{U} + \Delta U|$ and $\Phi \in V_h$ an interpolant of $\phi$ satisfying $|\phi - \Phi| \leq C_1 k |\dot{\phi}| + C_2 h^m |D^m \phi|$, $|\dot{\phi}|$ and $|D^m \phi|$ are stability weights

# A posteriori error analysis

$$
\begin{aligned}
|(e(T),\psi)| &\leq \int_0^T \int_\Omega R(U) C_1 k |\dot\phi| \, dx \, dt \\
&+ \int_0^T \int_\Omega R(U) C_2 h^m |D^m \phi| \, dx \, dt \\
&\leq S_1 C_1 \|kR(U)\|_I + S_2 C_2 \|h^m R(U)\|_I
\end{aligned}
$$

Using Cauchy's inequality we get multiplicative stability factors $S_1 = \|\dot\phi\|_I$ and $S_2 = \|D^m\phi\|_I$, with $\|\cdot\|_I = \|\cdot\|_{L_2(I;L_2(\Omega))}$

# Examples of dual data $\psi$

Error in $L_2$-norm at time $T$:

$$\psi = e/\|e\|$$

Error in average over volume $\omega \subset \Omega$ at time $T$:

$$\psi = \chi_\omega/|\omega|$$

Error in the point $x \in \Omega$ at time $T$:

$$\psi = \delta_x$$

# A posteriori error analysis

Basic example 3: Transient non linear problem

$$\begin{aligned} \dot{u} + f(u) &= 0 \quad \text{in } \Omega \times [0,T] \\ u &= 0 \quad \text{on } \partial\Omega \times [0,T] \\ u(\cdot,0) &= u^0 \quad \text{in } \Omega \end{aligned}$$

Dual problem with $f'(u,U)e = f(u) - f(U)$

$$\begin{aligned} -\dot{\phi} + f'(u,U)^*\phi &= 0 \quad \text{in } \Omega \times [0,T] \\ \phi &= 0 \quad \text{on } \partial\Omega \times [0,T] \\ \phi(\cdot,T) &= \psi \quad \text{in } \Omega \end{aligned}$$

# A posteriori error analysis

$$|(e(T),\psi)| = |\int_\Omega e(T)\psi\ dx|$$

$$= |\int_\Omega e(T)\psi\ dx| + |\int_0^T \int_\Omega e(-\dot{\phi} + f'(u,U)^*\phi)\ dx\ dt|$$

$$= |\int_0^T \int_\Omega (\dot{e} + f'(u,U)e)\phi\ dx\ dt| \quad (\text{ass. } e(0) = 0)$$

$$= |\int_0^T \int_\Omega R(U)(\phi - \Phi)\ dx\ dt| \quad (\text{where } \Phi \in V_h)$$

# A posteriori error analysis

$$
\begin{aligned}
|(e(T),\psi)| \;&\leq\; \int_0^T\int_\Omega R(U)C_1k|\dot{\phi}|\,dx\,dt \\
&+\; \int_0^T\int_\Omega R(U)C_2h^m|D^m\phi|\,dx\,dt
\end{aligned}
$$

with $R(U)=|\dot{U}+f(U)|$ and $\Phi\in V_h$ an interpolant of $\phi$ satisfying $|\phi-\Phi|\leq C_1k|\dot{\phi}|+C_2h^m|D^m\phi|$, $|\dot{\phi}|$ and $|D^m\phi|$ are stability weights

# A posteriori error analysis

$$
\begin{aligned}
|(e(T),\psi)| &\leq \int_0^T \int_\Omega R(U) C_1 k |\dot{\phi}| \, dx \, dt \\
&+ \int_0^T \int_\Omega R(U) C_2 h^m |D^m \phi| \, dx \, dt \\
&\leq S_1 C_1 \|k R(U)\|_I + S_2 C_2 \|h^m R(U)\|_I
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Using Cauchy's inequality we get multiplicative stability factors $S_1 = \|\dot{\phi}\|_I$ and $S_2 = \|D^m \phi\|_I$, with $\|\cdot\|_I = \|\cdot\|_{L_2(I;L_2(\Omega))}$

# Examples of dual data $\psi$

Error in $L_2$-norm at time $T$:

$$\psi = e/||e||$$

Error in average over volume $\omega \subset \Omega$ at time $T$:

$$\psi = \chi_\omega/|\omega|$$

Error in the point $x \in \Omega$ at time $T$:

$$\psi = \delta_x$$

# A posteriori error analysis

Basic example 4: Estimating time averages

$$\begin{aligned} \dot{u} + f(u) &= 0 \quad \text{in } \Omega \times [0, T] \\ u &= 0 \quad \text{on } \partial\Omega \times [0, T] \\ u(\cdot, 0) &= u^0 \quad \text{in } \Omega \end{aligned}$$

Dual problem with $f'(u, U)e = f(u) - f(U)$

$$\begin{aligned} -\dot{\phi} + f'(u, U)^* \phi &= \psi \quad \text{in } \Omega \times [0, T] \\ \phi &= 0 \quad \text{on } \partial\Omega \times [0, T] \\ \phi(\cdot, T) &= 0 \quad \text{in } \Omega \end{aligned}$$

# A posteriori error analysis

$$
\begin{aligned}
|(e,\psi)| &= |\int_0^T \int_\Omega e\psi \; dx \; dt| \\
&= |\int_0^T \int_\Omega e(-\dot{\phi} + f'(u,U)^*\phi) \; dx \; dt| \\
&= |\int_0^T \int_\Omega (\dot{e} + f'(u,U)e)\phi \; dx \; dt| \\
&= |\int_0^T \int_\Omega R(U)(\phi - \Phi) \; dx \; dt|
\end{aligned}
$$

# A posteriori error analysis

$$
\begin{aligned}
|(e,\psi)| &\le \int_0^T \int_\Omega R(U) C_1 k |\dot{\phi}| \, dx \, dt \\
&+ \int_0^T \int_\Omega R(U) C_2 h^m |D^m \phi| \, dx \, dt
\end{aligned}
$$

with $R(U) = |\dot{U} + f(U)|$ and $\Phi \in V_h$ an interpolant of $\phi$ satisfying $|\phi - \Phi| \le C_1 k |\dot{\phi}| + C_2 h^m |D^m \phi|$, $|\dot{\phi}|$ and $|D^m \phi|$ are stability weights

# A posteriori error analysis

$$
\begin{aligned}
|(e,\psi)| &\leq \int_0^T \int_\Omega R(U) C_1 k |\dot{\phi}| \, dx \, dt \\
&+ \int_0^T \int_\Omega R(U) C_2 h^m |D^m \phi| \, dx \, dt \\
&\leq S_1 C_1 \|kR(U)\|_I + S_2 C_2 \|h^m R(U)\|_I
\end{aligned}
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Using Cauchy's inequality we get multiplicative stability factors $S_1 = \|\dot{\phi}\|_I$ and $S_2 = \|D^m \phi\|_I$, with $\|\cdot\|_I = \|\cdot\|_{L_2(I;L_2(\Omega))}$

# Examples of dual data $\psi$

Error in $L_2(0,T;L_2(\Omega))$-norm:

$$\psi = e/\|e\|_{L_2(0,T;L_2(\Omega))}$$

Error in average over $\omega \times I \subset \Omega \times [0,T]$:

$$\psi = \chi_{\omega\times I}/|\omega \times I|$$

# A posteriori error analysis

- In computations of the linearized dual problem we have to use $U$ instead of $\hat{u}$
$\Rightarrow$ we get a linearization error
- More on this in Lecture 3!

# Incompr. Navier-Stokes eqns (NSE)

$$
\begin{aligned}
\dot{u} + u \cdot \nabla u - \nu \Delta u + \nabla q &= f && \text{in } \Omega \times I \\
\nabla \cdot u &= 0 && \text{in } \Omega \times I \\
u|_{\Gamma_D} &= w && \text{in } \partial\Omega \times I \\
u(\cdot, 0) &= u^0 && \text{on } \Omega
\end{aligned}
$$

# Incompr. Navier-Stokes eqns (NSE)

$$\begin{aligned} \dot{u} + u \cdot \nabla u - \nu \Delta u + \nabla q &= f \quad \text{in } \Omega \times I \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \times I \\ u|_{\Gamma_D} &= w \quad \text{in } \partial\Omega \times I \\ u(\cdot, 0) &= u^0 \quad \text{on } \Omega \end{aligned}$$

$\nu \Delta u - \nabla p$ represents the total fluid force, which we may write $\nu \Delta u - \nabla p = \text{div}\, \sigma(u, p)$, where $\sigma_{ij} = 2\nu\epsilon_{ij}(u) - p\delta_{ij}$ is the stress tensor, and $\epsilon_{ij}(u) = (u_{i,j} + u_{j,i})/2$ is the strain tensor

# Incompr. Navier-Stokes eqns (NSE)

$$\begin{aligned} \dot{u} + u \cdot \nabla u - \nu \Delta u + \nabla q &= f \quad \text{in } \Omega \times I \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \times I \\ u|_{\Gamma_D} &= w \quad \text{in } \partial\Omega \times I \\ u(\cdot, 0) &= u^0 \quad \text{on } \Omega \end{aligned}$$

If we normalize so that the reference velocity and typical length both are equal to one we get that $Re = \nu^{-1}$

# Existence and uniqueness of solutions

- Existence and uniqueness of the 3d incompressible NSE is on the Clay Institute's list of $\$10^6$ prize problems
- Little progress since Leray 1934, proving existence, but not uniquness, of a certain type of weak solution
- Existence and uniquness of certain regularized NSE possible by standard teqniques

# Existence and uniqueness of solutions

A common regularization:

$$\hat{\nu} = \nu + h^2|\epsilon(u)|^2$$

where $h$ is the smallest spatial scale gives ($f = w = 0$)

$$\int_\Omega |u(x,T)|^2\,dx + 4\int_0^T\int_\Omega \nu|\epsilon(u)|^2 + h^2|\epsilon(u)|^4\,dx\,dt$$
$$\leq \int_\Omega |u^0|^2\,dx$$

# Existence and uniqueness of solutions

Grönwall's inequality and the Sobolev inequality

$$
\begin{aligned}
(w \cdot \nabla u, w) &\leq C\|\nabla w\|^{3/2}\|w\|^{1/2}\|\epsilon(u)\| \\
&\leq \frac{\nu}{2}\|\nabla w\|^2 + \frac{C}{\nu^3}\|\epsilon(u)\|^4\|w\|^2
\end{aligned}
$$

is strong enough since we have a bound on $|\epsilon(u)|^4$ from the regularization

# Existence and uniqueness of solutions

The resulting estimate of $w = u - v$, where $u$ and $v$ are two solutions to NSE, is

$$\|w(t)\|^2 \leq C \exp(\frac{C}{h^2\nu^3})\|w(0)\|^2$$

Note that the growth factor is extremely large, which puts doubts in the physical meaningfulness of the uniqueness proof

# Existence and uniqueness of solutions

The simplest (Smagorinsky) turbulence model is of the form

$$\hat{\nu} = \nu + h^2|\epsilon(u)|^2$$

indicating that regularized NSE with non-linear viscosity are more relevant than the standard case with constant viscosity

# Existence and uniqueness of solutions

- The uniqueness is intimately connected to hydrodynamic stability, which concerns the growth of pertuations in NSE.
- Hydrodynamic stability has a long tradition, but few concrete results of significance
- In general, both pertubation growth and the complexity of the flow increases with the Reynolds number $Re$
- More on this in Lecture 2!

# Discretization: $G^2$-method

(General space-time Galerkin least squares stabilized finite element method)

General Galerkin $G^2$-method is a flexible methodology for discretization of NSE applicable to flow problems form creeping viscous flow to slightly viscous flows, including moving boundaries

# Discretization: $G^2$-method

The $G^2$ method includes:

- Streamline diffusion method on Eulerian space-time meshes
- Characteristic Galerkin method on Lagrangean space-time meshes (orientation along particle trajectories)
- Arbitrary Lagrangian-Eulerian ALE methods

# Discretization: $G^2$-method

The least-squares stabilization takes care of

- instabilities from discretization of convection terms in Eulerian methods
- pressure instabilities in equal order interpolation of velocity and pressure

# Discretization: $G^2$-method

$$0 = t_0 < t_1 < ... < t_N = T, \quad I_n = (t_{n-1}, t_n]$$
$$k_n = t_n - t_{n-1}, \quad S_n = \Omega \times I_n$$

$H^1(\Omega) \supset W_n = \{$FEM space consisting of p.w. polynomials of order $p$ on a mesh $\tau_n = \{\kappa\}$ of mesh size $h_n(x)\}$

$W_{0n}$ is the space of functions in $W_n$ that are zero on $\Gamma$

# Discretization: G$^2$-method

For a given velocity field $\beta$ on $S_n$ vanishing on $\Gamma \times I_n$, define particle paths $x(\bar{x}, \bar{t})$ by

$$\frac{dx}{dt} = \beta(x, \bar{t}), \quad \bar{t} \in I_n$$
$$x(\bar{x}, t_n) = \bar{x}, \quad \bar{x} \in \Omega$$

Define corresponding mapping $F_n^\beta : S_n \to S_n$ by $(x, t) = F_n^\beta(\bar{x}, \bar{t}) = (x(\bar{x}, \bar{t}), t)$

# Discretization: $G^2$-method

For $q \geq 0$, define the spaces

$$\bar{V}_n^\beta = \{\bar{v} \in H^1(S_n)^3 : \bar{v}(\bar{x},\bar{t}) = \sum_{j=0}^{q} (\bar{t} - t_n)^j U_j(\bar{x}),$$

$$U_j \in [W_{0n}]^3\},$$

$$\bar{Q}_n^\beta = \{\bar{q} \in H^1(S_n) : \bar{q}(\bar{x},\bar{t}) = \sum_{j=0}^{q} (\bar{t} - t_n)^j q_j(\bar{x}),$$

$$q_j \in W_{0n}\}$$

# Discretization: G$^2$-method

For $q \geq 0$, define the spaces

$$V_n^\beta = \{v : \bar{v} \in \bar{V}_n^\beta\},$$
$$Q_n^\beta = \{q : \bar{q} \in \bar{Q}_n^\beta\},$$

where $v(x,t) = \bar{v}(\bar{x},\bar{t})$ and $q(x,t) = \bar{q}(\bar{x},\bar{t})$, and

$$V^\beta \times Q^\beta = \prod_n V_n^\beta \times Q_n^\beta$$

# Discretization: $G^2$-method

Find $(U,P) \in V^\beta \times Q^\beta$ s.t. for $n = 1, 2, ..., N$

$$\begin{aligned} & (\dot{U} + (U\cdot\nabla)U, v)_n - (P, \mathrm{div}\, v)_n + (q, \mathrm{div}\, U)_n \\ + \quad & (2\nu\epsilon(U), \epsilon(v))_n + (\delta_1 a(U; U, P), a(U; v, q))_n \\ + \quad & (\delta_2 \mathrm{div}\, U, \mathrm{div}\, v)_n + ([U^{n-1}], v_+^{n-1}) \\ = \quad & (f, v + \delta_1 a(U; v, q))_n \quad \forall (v,q) \in V_n^\beta \times Q_n^\beta \end{aligned}$$

$$a(w; v, q) = \dot{v} + (w\cdot\nabla)v + \nabla q - \nu\Delta v$$

$$(v,w)_n = \int_{I_n} (v,w)dt, \quad (v,w) = \sum_{K\in T_n} \int_K v\cdot w\, dx$$

$[v^n] = v_+^n - v_-^n$ is the jump across the time level $t_n$

# Discretization: $G^2$-method

$$\delta_1 = \tfrac{1}{2}(k_n^{-2} + |U|^2 h_n^{-2})^{-1/2} \text{ if } \epsilon < CUh_n, \quad \kappa_1 h^2 \text{ else}$$

$$\delta_2 = \kappa_2 h_n \text{ if } \epsilon < CUh_n, \quad \kappa_2 h^2 \text{ else}$$

# Example: cG(1)dG(0)-method

For $n = 1, ..., N$, find $(U^n, P^n) \in V_n^0 \times Q_n^0$ s.t.

$$
\begin{aligned}
& (\frac{U^n - U^{n-1}}{k_n}, v) + (\nu \nabla U^n, \nabla v) + (\nabla \cdot U^n, q) \\
+ \; & (U^n \cdot \nabla U^n + \nabla P^n, v + \delta_1 (U^n \cdot \nabla v + \nabla q)) \\
= \; & (f^n, v + \delta_1 (U^n \cdot \nabla v + \nabla q)) \quad \forall (v, q) \in V_n^0 \times Q_n^0,
\end{aligned}
$$

# Example: cG(1)dG(0)-method

Letting $v$ vary while choosing $q = 0$, we get the discrete momentum equation:

$$(\frac{U^n - U^{n-1}}{k_n}, v) + (U^n \cdot \nabla U^n + \nabla P^n, v + \delta_1 U^n \cdot \nabla v)$$

$$+ (\nu \nabla U^n, \nabla v) = (f^n, v + \delta_1 U^n \cdot \nabla v) \quad \forall v \in V_n^0,$$

# Example: cG(1)dG(0)-method

Letting $q$ vary while setting $v = 0$, we get the discrete “pressure equation”

$$
\begin{aligned}
& (\delta_1 \nabla P^n, \nabla q)) = -(\delta_1 U^n \cdot \nabla U^n, \nabla q) \\
- \; & (\nabla \cdot U^n, q) + (\delta_1 f^n, \nabla q) \quad \forall q \in Q_n^0.
\end{aligned}
$$

# Example: cG(1)dG(0)-method

The cG(1)dG(0) has a backward Euler first order accurate time stepping, and thus in general is too dissipative.

# Example: cG(1)cG(1)-method

For $n = 1, ..., N$, find $(U^n, P^n) \in V_n^0 \times Q_n^0$ s.t.

$$\begin{aligned} & (\frac{U^n - U^{n-1}}{k_n}, v) + (\nu \nabla \hat{U}^n, \nabla v) + (\nabla \cdot \hat{U}^n, q) \\ + & \ (\hat{U}^n \cdot \nabla \hat{U}^n + \nabla P^n, v + \delta_1 (\hat{U}^n \cdot \nabla v + \nabla q)) \\ = & \ (f^n, v + \delta_1 (\hat{U}^n \cdot \nabla v + \nabla q)) \quad \forall (v, q) \in V_n^0 \times Q_n^0, \end{aligned}$$

where $\hat{U}^n = \frac{1}{2}(U^n + U^{n-1})$

# Example: cG(1)cG(1)-method

This method corresponds to a second order accurate Crank-Nicolson time-stepping

The stabilization suffers from an inconsistency up to the term $\delta_1 \dot{u}$ resulting from the piecewise constant test functions in time

This inconsistency seems to be acceptible unless $\dot{u}$ is large

# Neumann boundary conditions

Assume Neumann b.c. $\sigma \cdot n = g$ on $\Gamma_1 \subset \partial\Omega$

$W_{0n}$ is chosen to be functions in $W_n$ vanishing on the remaining Dirichlet boundary $\Gamma_0$, and the right hand side is supplemented with an integral of $g \cdot v$ over $\Gamma_1$

# Neumann boundary conditions

This implements the boundary condition in weak form through the term $(-P, \text{div } v) + (2\nu\epsilon(U), \epsilon(v)) = (\sigma, \epsilon(v))$ on the left hand side, which when integrated by parts generates the integral of $(\sigma \cdot n) \cdot v$ over $\Gamma_1$

If the viscous term appears in the form $(\nu\nabla U, \nabla v)_n$ the corresponding Neumann boundary condition has the form $\nu\dfrac{\partial u}{\partial n} - pn = 0$

# Outflow boundary conditions

We may use a Neumann b.c. with $g = 0$ corresponding to a zero force at outflow, simulating outflow into a large empty reservoir.

Alternatively we may use the slightly different condition $\nu \frac{\partial u}{\partial n} - pn = 0$ as an approximation of a transparent outflow b.c.

# Example: 3d Step down

- Channel flow with no slip walls, $1 \times 1$ rectangular cross section and length 4, with a step down of height and length 0.5
- $x_1 = 0$ inflow boundary with inflow condition $u_1 = 64(y - 0.5)(1 - y)z(1 - z)$ for $y \geq 0.5$
- Transparent outflow condition
- $u^0 = 0$
- $\nu = 1/1000$

# Example: 3d Step down

start animation

# Solution of the discrete system

The cG(1)cG(1)-method with $\delta_1$-stabilization:

$$\begin{aligned} AU^n + k_n BP^n &= k_n F^n \\ -B^T U^n + CP^n &= G^n \end{aligned}$$

in each step of an outer fixed point iteration with the convection velocity being given from the previous iteration

# Solution of the discrete system

The cG(1)cG(1)-method with $\delta_1$-stabilization:

$$\begin{aligned} AU^n + k_n BP^n &= k_n F^n \\ -B^T U^n + CP^n &= G^n \end{aligned}$$

where $A = M_n + k_n N_n - k_n \nu \Delta_n$, with $M_n$ a mass matrix, $N_n$ a discrete analog of convection term with frozen velocity from previous iteration, $\Delta_n$ a discrete Laplacian, $B$ a discrete gradient, $B^T$ a discrete divergence, and $C = -\delta_1 \Delta_n$

# Solution of the discrete system

First solve for $P^{n,j+1}$ in terms of $U^{n,j}$ from

$$CP^{n,j+1} = G^n + B^T U^{n,j}$$

using multigrid, then solve for $U^{n,j+1}$ from

$$AU^{n,j+1} = k_n F^n - k_n B P^{n,j+1}$$

using GMRES, converges if $k_n/\delta_1$ is small enough

We have used $k_n \sim \delta_1 \sim h_n$

# Solution of the discrete system

Alternatively apply GMRES directly to

$$AU^n + k_n BP^n \; = \; k_n F^n,$$

with $P^n$ solved in terms of $U^n$ from

$$-B^T U^n + CP^n \; = \; G^n,$$

using multigrid

# Solution of the discrete system

Convergence depends on the condition number for the matrix

$$M_n + k_n N_n - k_n \nu \Delta_n + \frac{k_n}{\delta_1} B \Delta_n^{-1} B^T$$

which is bounded by $k_n/h_n$, $k_n\nu/h_n^2$ , $k_n/\delta_1$

# A posteriori error estimates for NSE

We derive a posteriori error estimates for cG(1)cG(1), with $\delta_1 = 0$ for simplicity

We are interested in the error measure at $t = T$, that is $(e(T), \psi)$

# A posteriori error estimates for NSE

Introduce linearized dual problem:

Find $(\varphi, \theta) \in L_2(I; [H_0^1(\Omega)]^3 \times L_2(\Omega))$ s.t.

$$\begin{aligned} -\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U \cdot \varphi + \nabla\theta - \epsilon\Delta\varphi &= 0 && \text{in } Q \\ \text{div}\, \varphi &= 0 && \text{in } Q \\ \varphi &= 0 && \text{on } \Gamma \times I \\ \varphi(\cdot, T) &= \psi && \text{in } \Omega \end{aligned}$$

where $Q = \Omega \times I$, and $I = (0, T)$

# A posteriori error estimates for NSE

Multiplying first equation by $e$, integrating over $Q$ together with integration by parts, using that

$$(u \cdot \nabla)u - (U \cdot \nabla)U = (u \cdot \nabla)e + (e \cdot \nabla)U$$

# A posteriori error estimates for NSE

$$(e(T), \psi) = \sum_{n=0}^{N} \{(-\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U \cdot \varphi, e)_n$$

$$+ (\nabla \theta, e)_n + (\nu \nabla \varphi, \nabla e)_n\}$$

# A posteriori error estimates for NSE

$$(e(T),\psi) = \sum_{n=0}^{N}\{(-\dot{\varphi} - (u\cdot\nabla)\varphi + \nabla U\cdot\varphi, e)_n$$

$$+ (\nabla\theta, e)_n + (\nu\nabla\varphi, \nabla e)_n\}$$

$$= \sum_{n=1}^{N}\{(\varphi, e_t)_n + ((u\cdot\nabla)e,\varphi)_n + ((e\cdot\nabla)U,\varphi)_n$$

$$- (\theta, \text{div}\, e)_n + (\nu\nabla\varphi, \nabla e)_n - (p - P, \text{div}\,\varphi)_n\}$$

# A posteriori error estimates for NSE

$$= \sum_{n=1}^{N} \{(\dot{u} + u \cdot \nabla u + \nabla p, \varphi)_n + (\epsilon \nabla u, \nabla \varphi)_n$$

$$- (\dot{U} + U \cdot \nabla U + \nabla P, \varphi)_n - (\nu \nabla U, \nabla \varphi)_n$$

$$+ (\theta, \mathrm{div}\, U)_n\}$$

# A posteriori error estimates for NSE

$$= \sum_{n=1}^{N} \{(\dot{u} + u \cdot \nabla u + \nabla p, \varphi)_n + (\epsilon \nabla u, \nabla \varphi)_n$$

$$- (\dot{U} + U \cdot \nabla U + \nabla P, \varphi)_n - (\nu \nabla U, \nabla \varphi)_n$$

$$+ (\theta, \text{div}\, U)_n\}$$

$$= - \sum_{n=1}^{N} \{(\dot{U} + U \cdot \nabla U + \nabla P - f, \varphi - \Phi)_n$$

$$- (\nu \nabla U, \nabla (\varphi - \Phi))_n + (\text{div}\, U, \theta - \Theta)_n\}$$

# A posteriori error estimates for NSE

Estimate interpol. errors $\varphi - \Phi$ and $\theta - \Theta$ gives

$$|(e(T),\psi)| \leq \sum_{i=1}^{2} \int_Q R_i(U)(Ch^m|D^m\varphi| + Ck|\dot{\varphi}|)dx\,dt$$
$$+ \int_Q R_3(U)(Ch^m|D^m\theta| + Ck|\dot{\theta}|)dx\,dt,$$

for $m = 1, 2$, where $D^m$ measures derivatives w.r.t. $x$ of order $m$, $C$ repr. interpolation constants.

# A posteriori error estimates for NSE

$$
\begin{aligned}
R_1(U,P) &= |\dot{U} + U\cdot\nabla U + \nabla P - f - \nu\Delta U| \\
R_2(U,P) &= \nu D_2(U) \\
R_3(U,P) &= |\mathrm{div}\, U|
\end{aligned}
$$

$$
D_2(U)(x,t) = \max_{y\in\partial K}(h_n(x))^{-1}|[\frac{\partial U}{\partial n}(y,t)]|,\ x\in K
$$

$R_1 + R_2$ bounds the residual for the momentum equation

# A posteriori error estimates for NSE

For concrete a posteriori error estimates:

Solve dual problems numerically and compute approximations of the derivatives of the dual solutions

With adaptive choice of meshing:

Choose $h_n(x)$ and $k_n$ from a principle of equidistribution with the derivatives of the dual solution entering as weights.

# A posteriori error estimates for NSE

Estimating the space-time integrals e.g. using Cauchy's inequality gives

$$|(e(T),\psi)| \leq C\|\dot{\varphi}\|_I \sum_{i=1}^{2} \|kR_i(U,P)\|_I$$

$$+ C\|D^2\varphi\|_I \sum_{i=1}^{2} \|h^2 R_i(U,P)\|_I + C\|\dot{\theta}\|_I \|kR_3(U,P)\|_I$$

$$+ C\|D^1\theta\|_I \|hR_3(U,P)\|_I$$

# A posteriori error estimates for NSE

$\|\dot{\varphi}\|_I$, $\|D^2\varphi\|_I$, $\|\dot{\theta}\|_I$ and $\|D^1\theta\|_I$ are multiplicative stability factors.

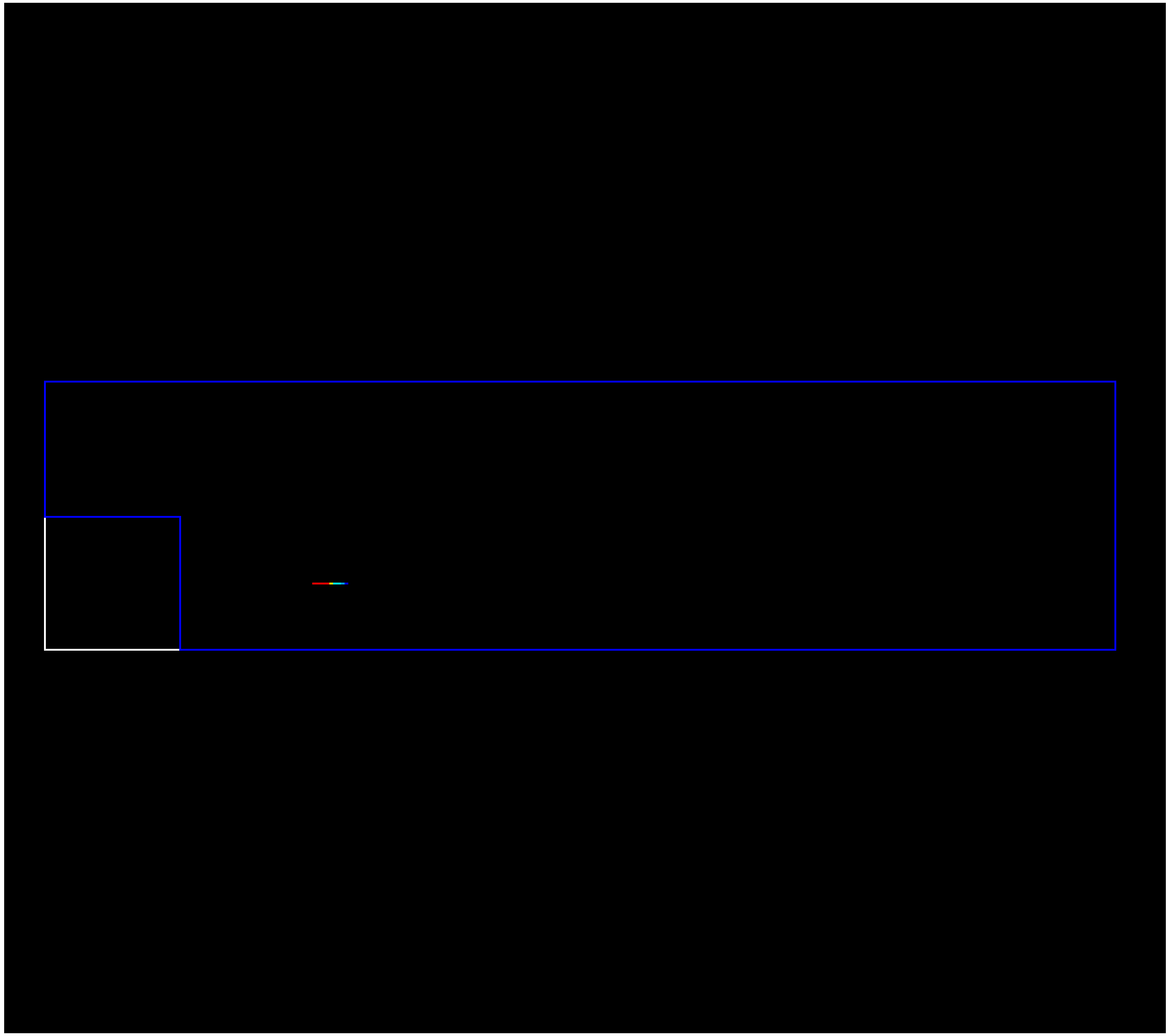
The equidistribution now works on e.g. the product $h^2R_1$ with a mesh size factor and a residual factor.

# The linearized dual problem

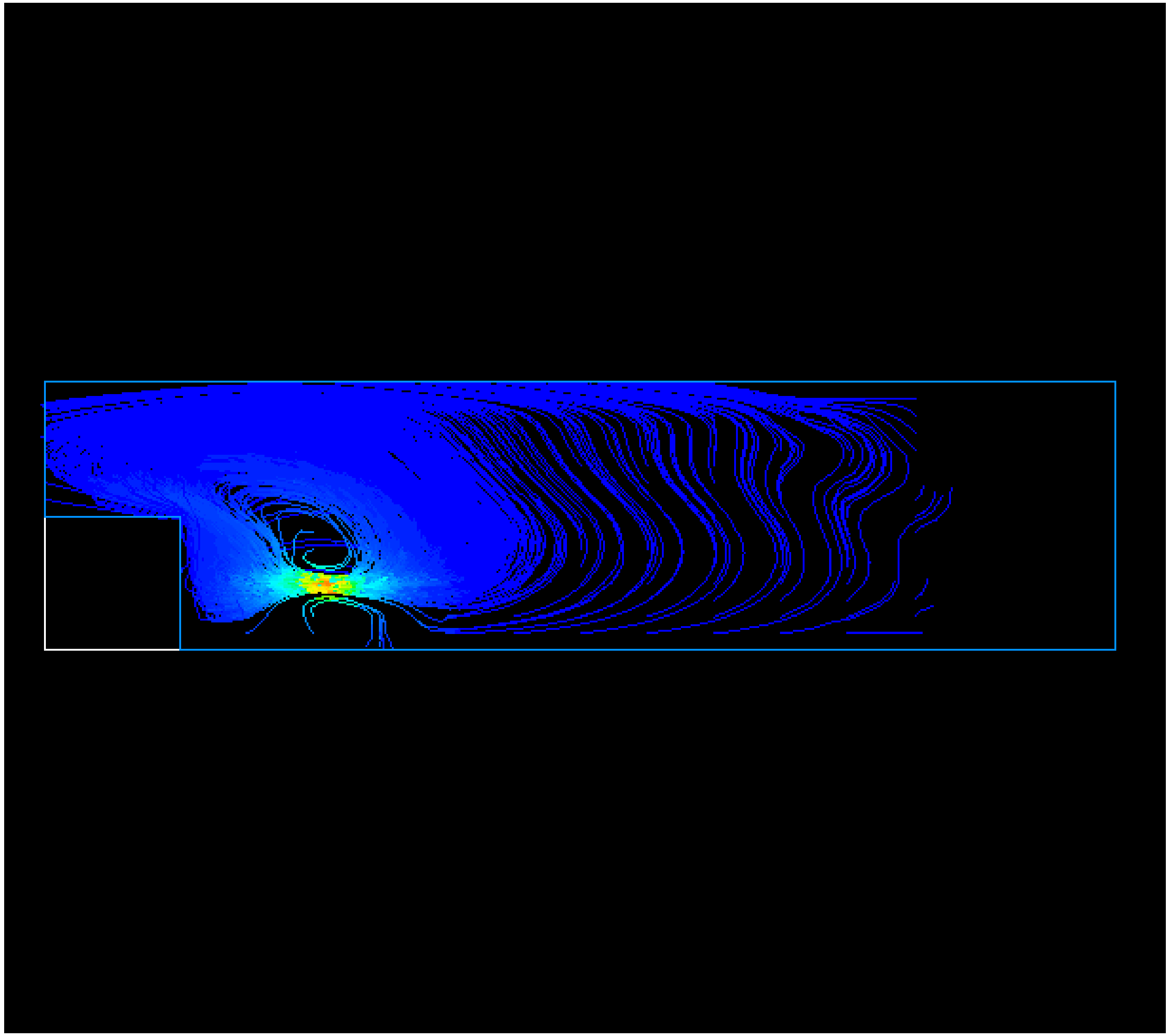
We aim at error control of a mean value of $u_1$ over $\omega \subset \Omega$ at time $T = 2$ (initial state from $t = 8$)

- Final data for dual problem is $\psi = \chi_\omega / |\omega|$, where $|\omega|$ the volume of $\omega$
- $\omega$ is a square with side length $d(\omega)$
- Solved using the cG(1)cG(1)-method with $h = 1/32$

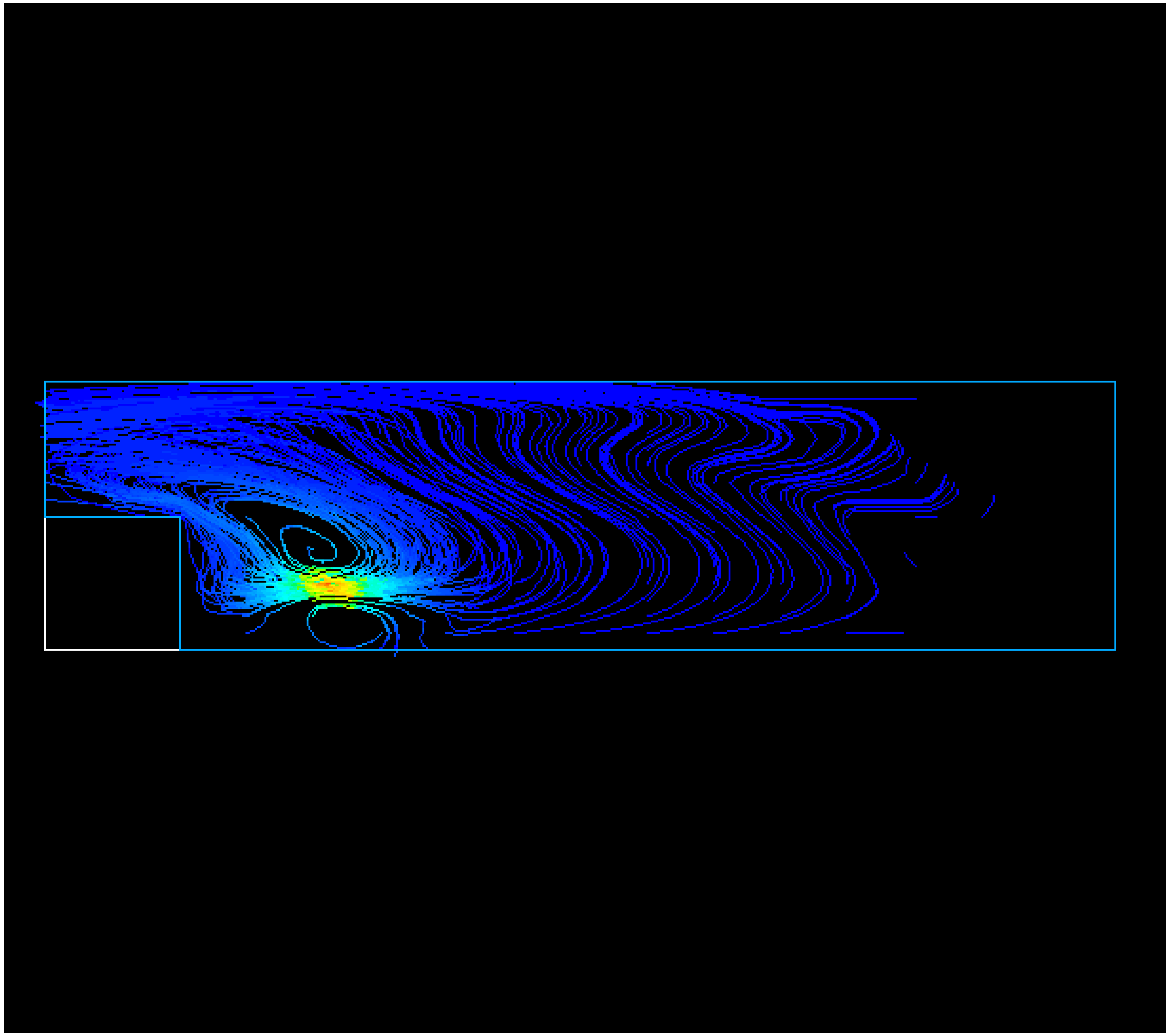
# Dual solution at $t = 2$, $d(\omega) = 0.0625$



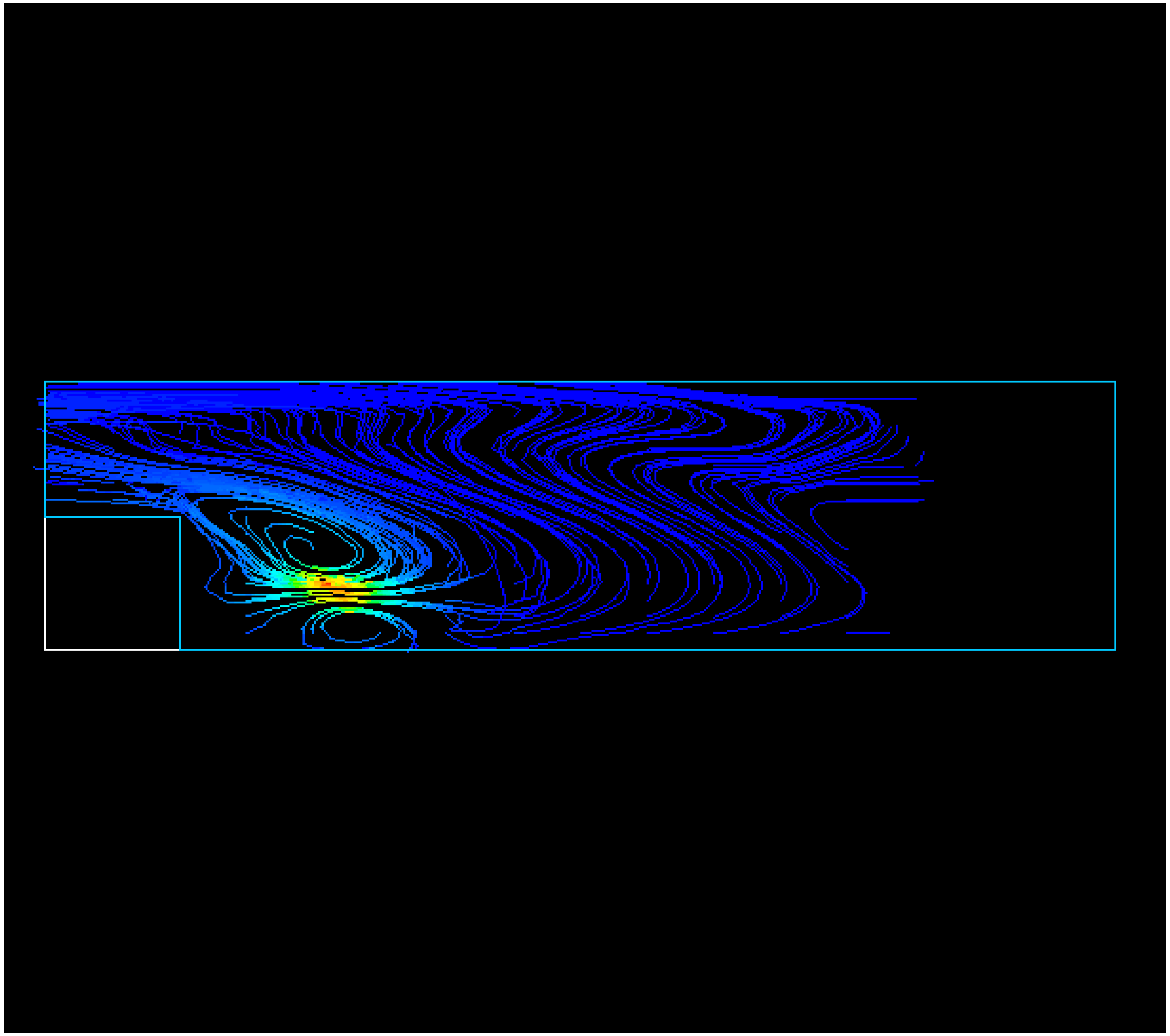
# Dual solution at $t = 1.5$, $d(\omega) = 0.0625$



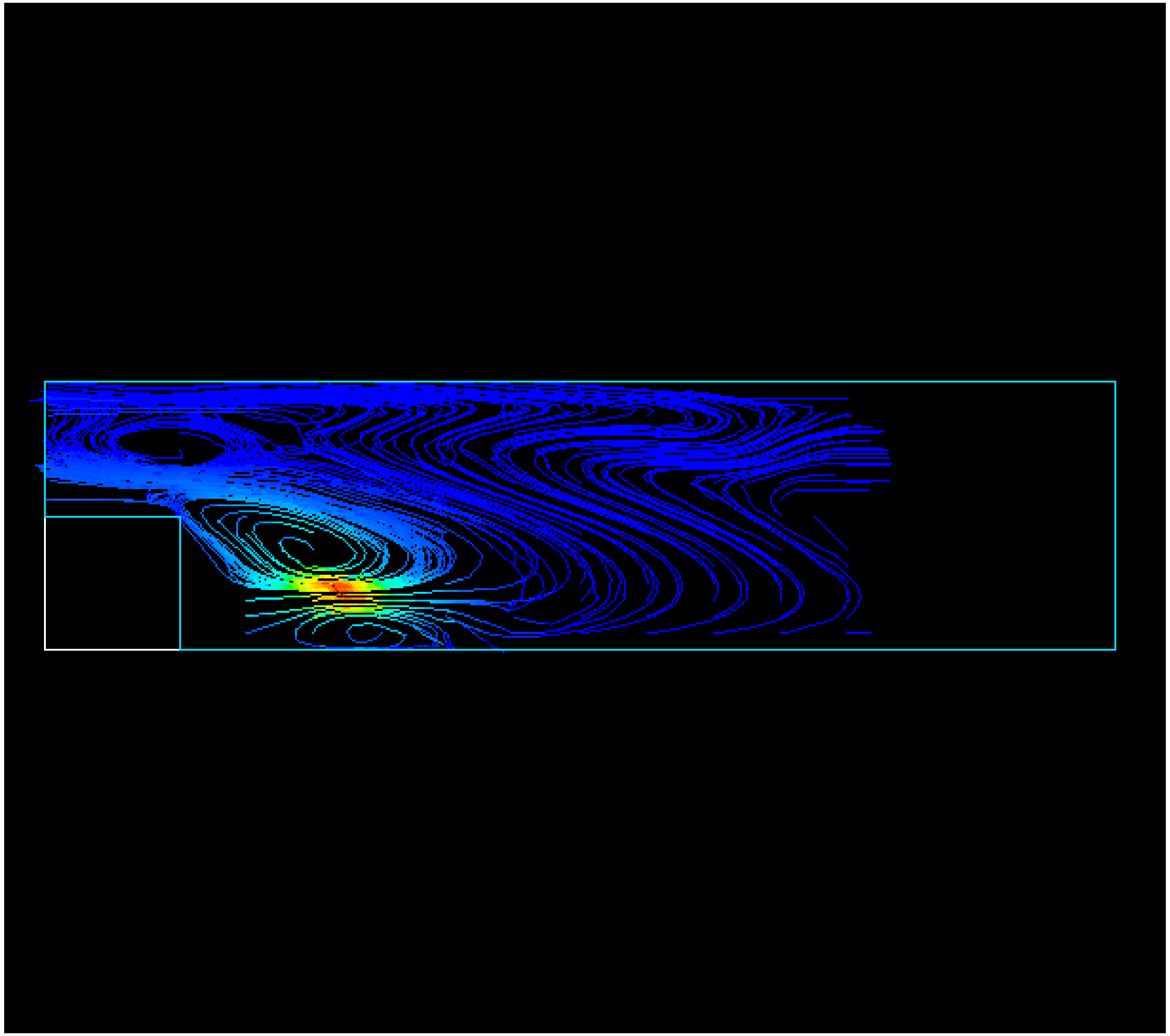
# Dual solution at $t = 1$, $d(\omega) = 0.0625$



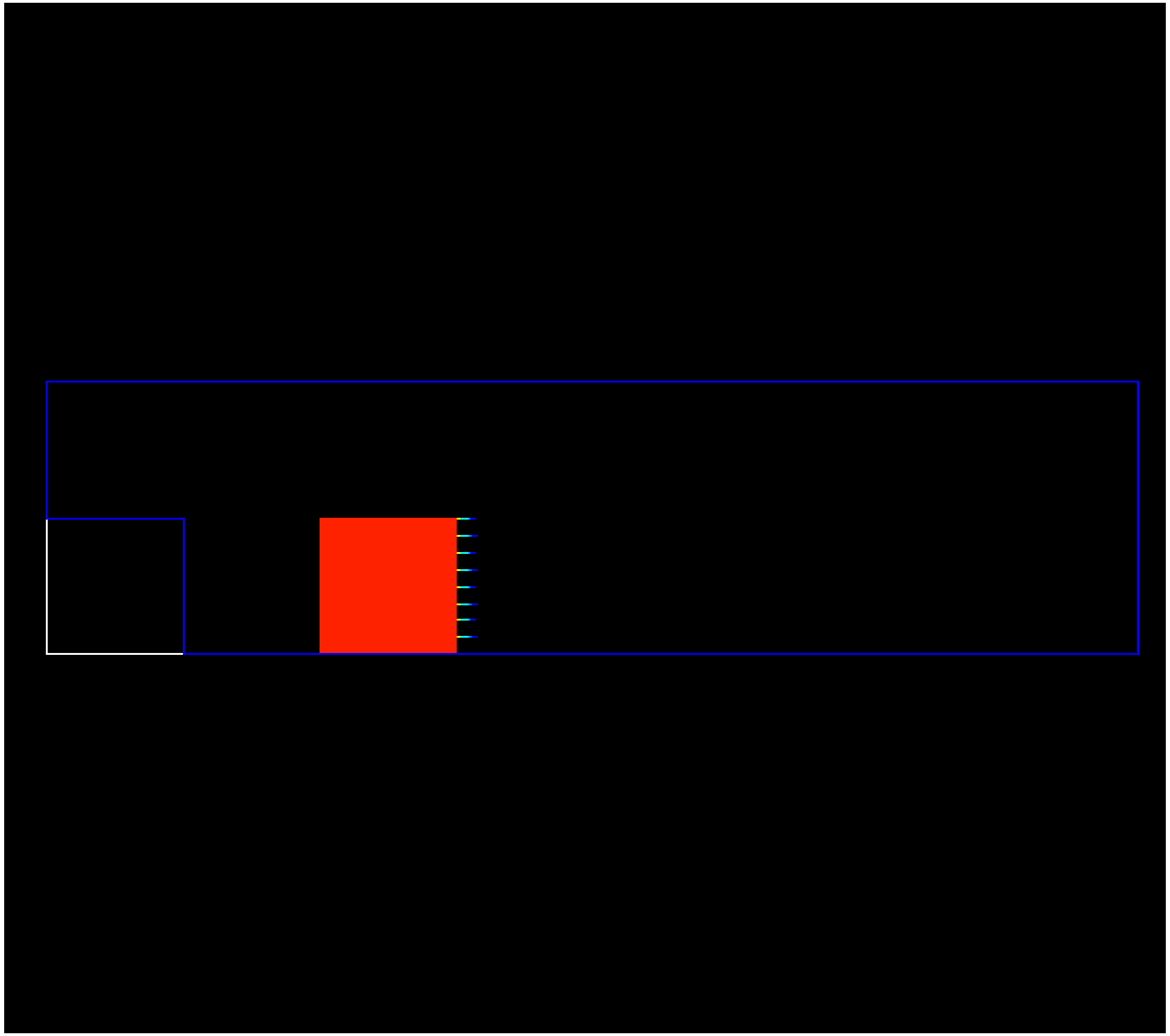
# Dual solution at $t = 0.5$, $d(\omega) = 0.0625$



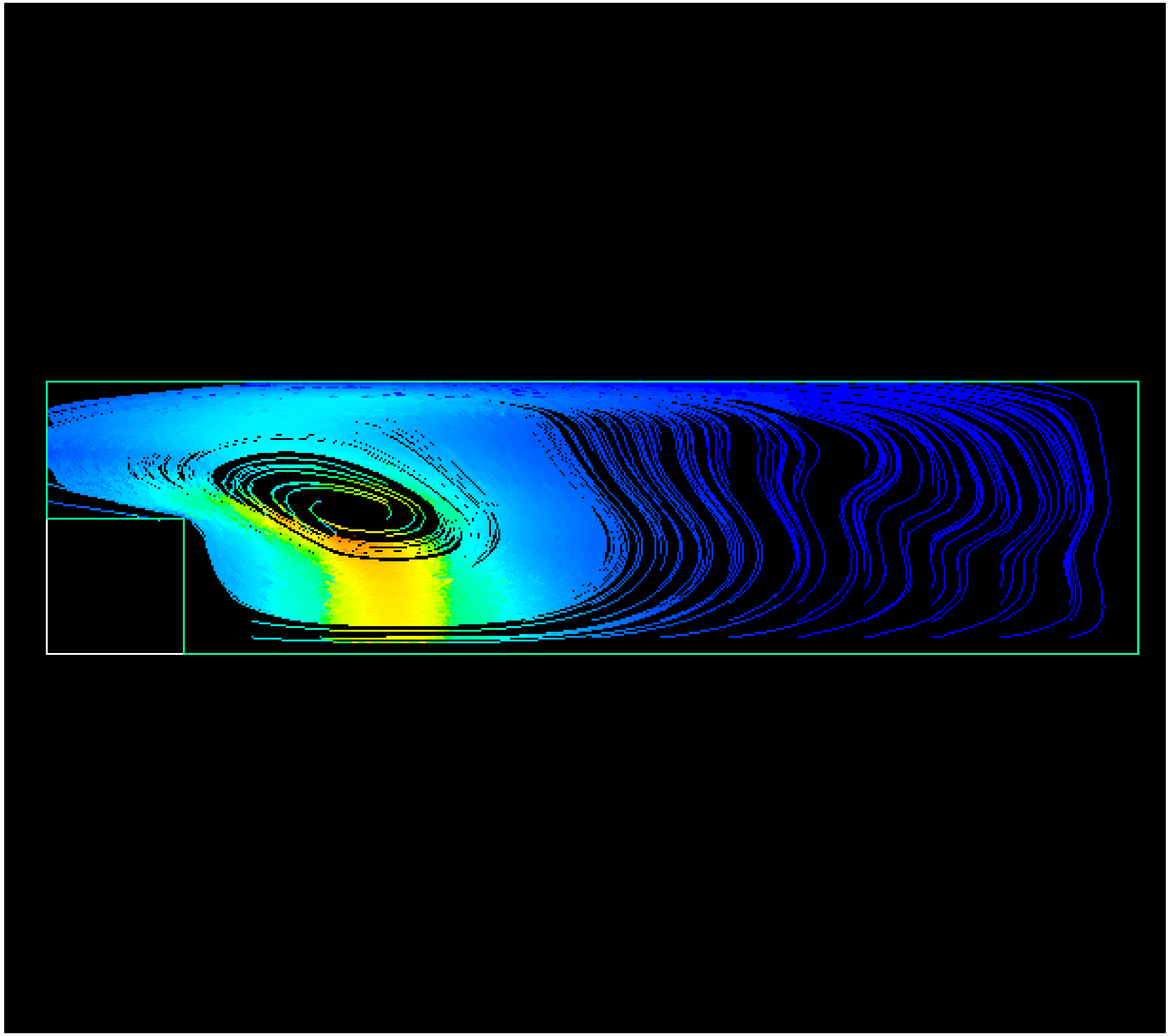
# Dual solution at $t = 0$, $d(\omega) = 0.0625$



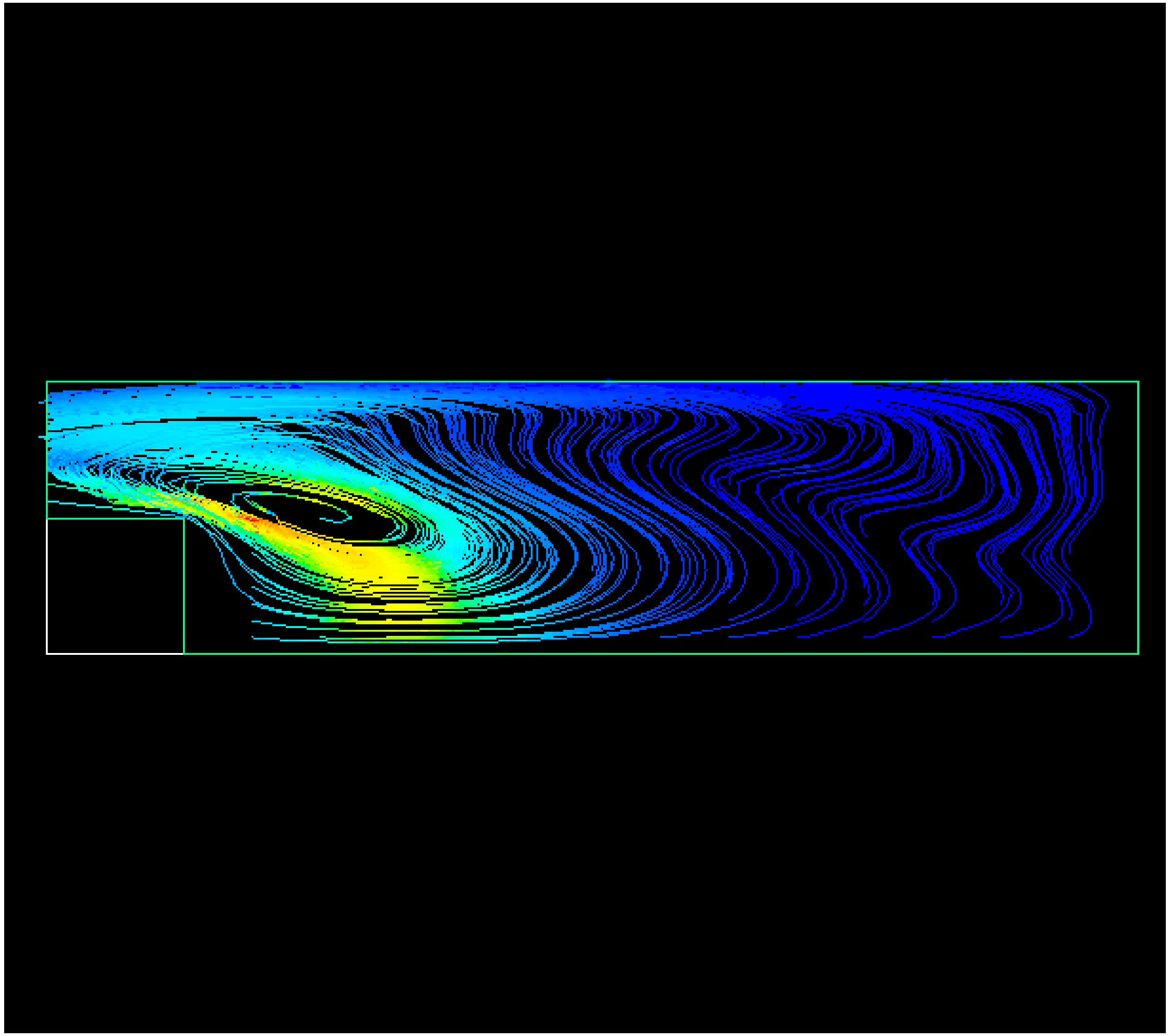
# Dual solution at $t = 2$, $d(\omega) = 0.5$



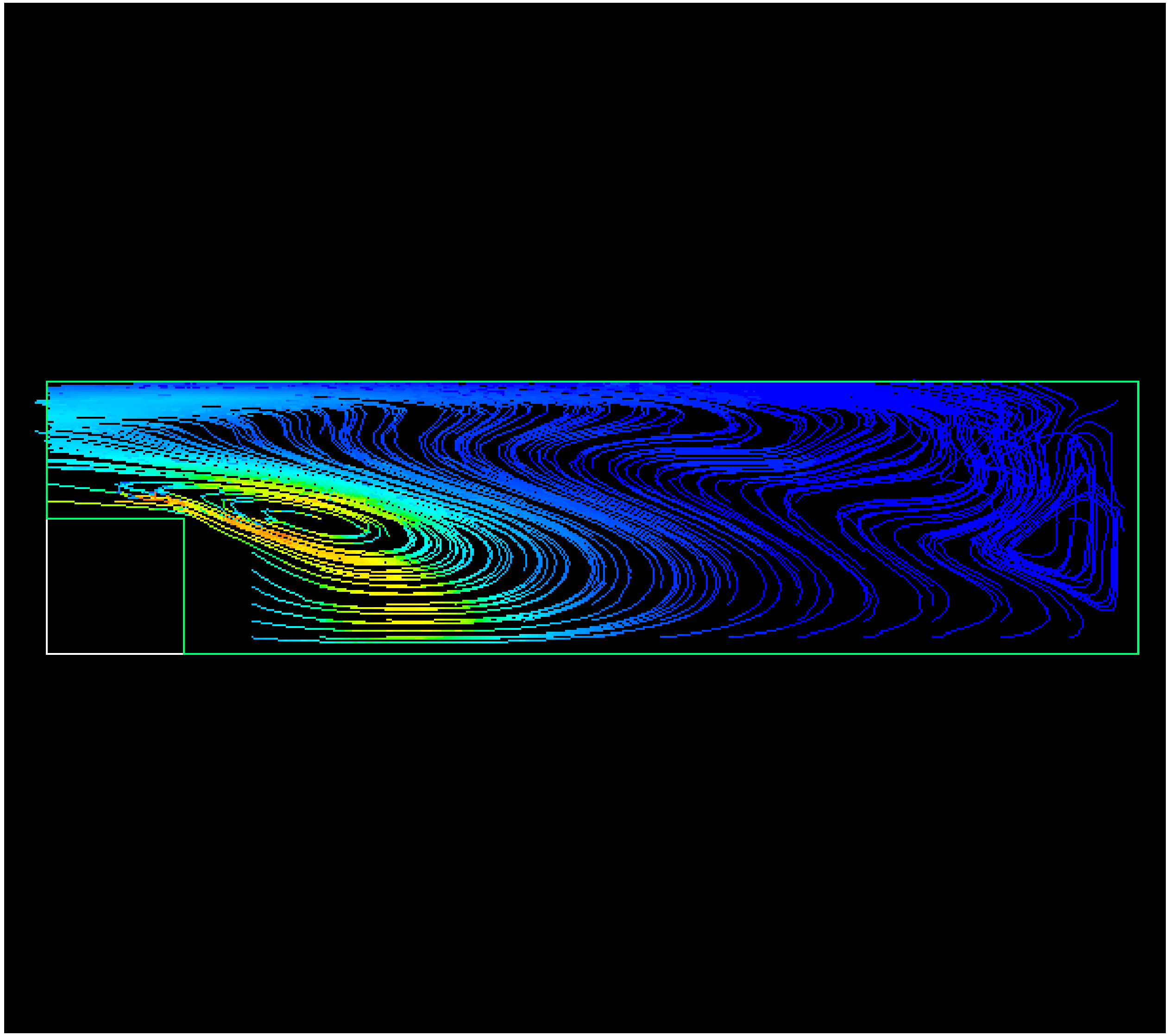
# Dual solution at $t = 1.5$, $d(\omega) = 0.5$



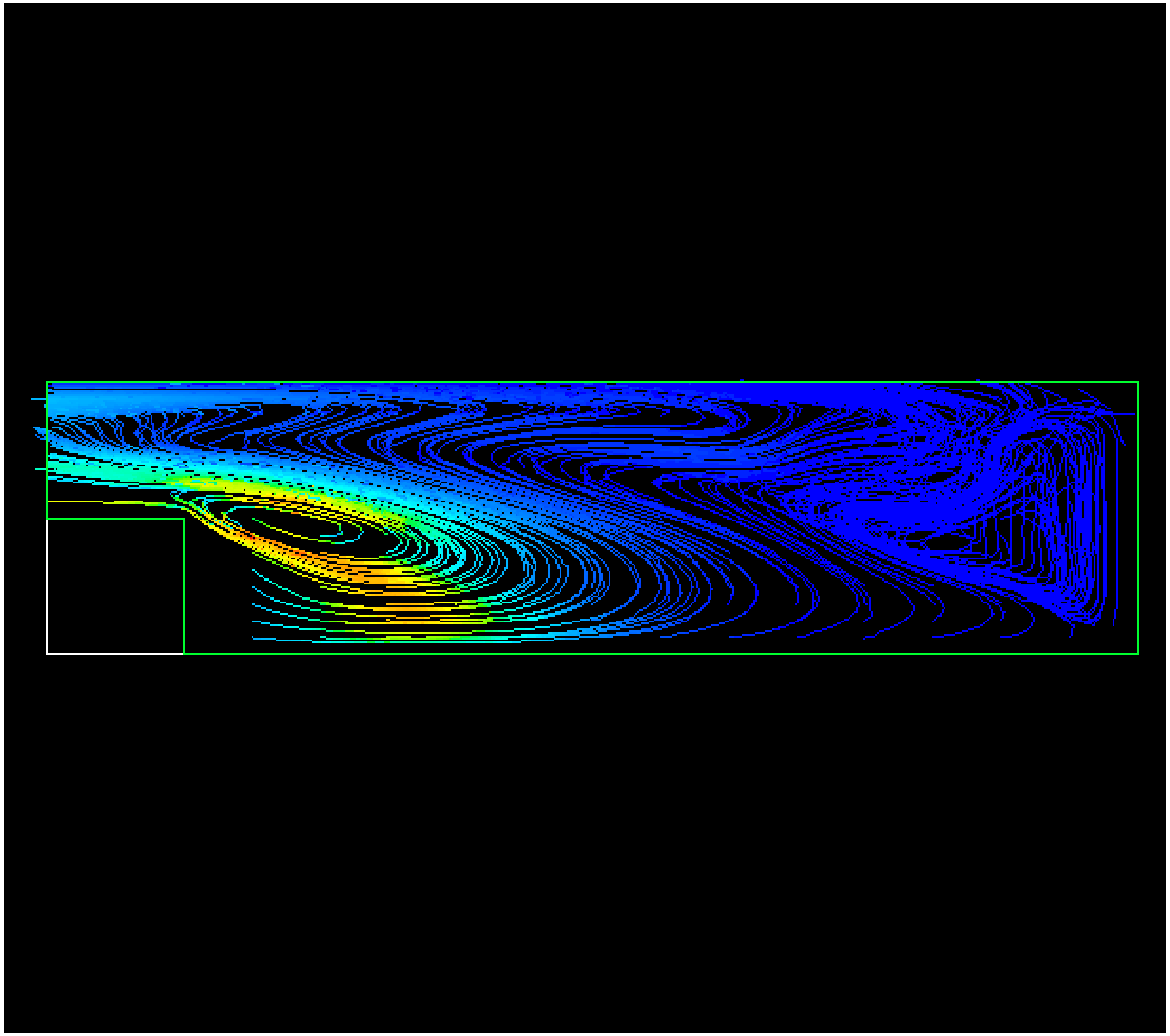
# Dual solution at $t = 1$, $d(\omega) = 0.5$



# Dual solution at $t = 0.5$, $d(\omega) = 0.5$



# Dual solution at $t = 0$, $d(\omega) = 0.5$



# Stability factors in $L_1(I; L_1(\Omega))$-norm

$$S_{0,1}(T) = ||\varphi||_I$$

$$S_{1,1}(T) = ||\nabla\varphi||_I$$

$$S_{1,2}(T) = ||\dot{\varphi}||_I$$

$$S_{1,3}(T) = ||\nabla\theta||_I$$

$$S_{2,1}(T) = ||\Delta\varphi||_I$$

$||\cdot||_I$ is the $L_1(I; L1(\Omega))$-norm

# Stability factors in $L_1(I; L_1(\Omega))$-norm

| $d(\omega)$ | $S_{0,1}$ | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{2,1}$ |
|---|---|---|---|---|---|
| 0.0625 | 20.1 | 411.7 | 37.0 | 13.6 | 8119.7 |
| 0.125 | 10.7 | 200.2 | 24.6 | 10.0 | 3539.2 |
| 0.25 | 6.4 | 98.9 | 11.4 | 5.0 | 1683.9 |
| 0.5 | 3.3 | 44.9 | 6.5 | 3.0 | 797.0 |

Stability factors proportional to $1/d(\omega)$, where $d(\omega)$ is the side length of $\omega$.

# Stability factors in $L_1(I; L_1(\Omega))$-norm

| $d(\omega)$ | $S_{0,1}$ | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{2,1}$ |
|---|---|---|---|---|---|
| 0.0625 | 20.1 | 411.7 | 37.0 | 13.6 | 8119.7 |
| 0.125 | 10.7 | 200.2 | 24.6 | 10.0 | 3539.2 |
| 0.25 | 6.4 | 98.9 | 11.4 | 5.0 | 1683.9 |
| 0.5 | 3.3 | 44.9 | 6.5 | 3.0 | 797.0 |

Residuals are of order 0.1 $\Rightarrow$ a small average appear uncomputable on the current mesh, while a large average is computable to a fine tolerance

# Computation of lift and drag

We want to approximate the quantity

$$N(\sigma(u,p)) = \frac{1}{T}\int_0^T \int_{\Gamma_1} \sum_{i,j=1}^{3} \sigma_{i,j}(u,p) n_j \psi_i \, ds,$$

where $\Gamma = \Gamma_0 \cup \Gamma_1$ is a decomposition of the boundary $\Gamma$, and $\psi = (\psi_i)$ a given function on $\Gamma_1$.

# Computation of lift and drag

$N(\sigma)$ may represent the mean value over $[0, T]$ of the drag or lift on a body with boundary $\Gamma_1$ immersed in a flow, dep. on the choice of $\psi$.

Example: $\psi_1 = 1, \psi_2 = \psi_3 = 0$ gives the mean of the drag force, if the $x_1$ is oriented in the stream-wise direction

# Computation of lift and drag

Instead of directly using the expression for $N(\sigma)$, we may use the following alternative expression with the idea of increasing the precision (Süli,Giles,Larsson,...)

$$N(\sigma(u,p)) = \frac{1}{T}\int_0^T (\dot{u} + u\cdot\nabla u, \psi) - (p, \text{div}\,\psi) + (2\nu\epsilon(u), \epsilon(\psi)))\,dt$$

with $\psi$ an extension of $\psi$ into $\Omega$ with $\psi = 0$ on $\Gamma_0$

# Computation of lift and drag

We approximate $N(\sigma(u,p))$ by the quantity

$$N_h(\sigma(U,P)) = \frac{1}{T}\int_0^T \{(\dot{U} + U\cdot\nabla U, \Psi) - (P, \text{div}\,\Psi) + (2\nu\epsilon(U), \epsilon(\Psi))\}\,dt$$

where $\Psi$ is a finite element function satisfying $\Psi = \psi$ on $\Gamma_1$, assuming $\psi$ is the restriction to $\Gamma_1$ of a finite element function and $(U,P)$ is a finite element solution of NSE.

# Computation of lift and drag

Let now $(\varphi, \theta)$ be the solution of the linearized dual problem with $\varphi(T) = 0$ and $\varphi(\cdot, t) = \psi$ on $\Gamma_1$ and $\varphi(\cdot, t) = 0$ on $\Gamma_0$ for $t \in [0, T]$.

We then obtain an a posteriori error estimate for $N(\sigma(u, p)) - N_h(\sigma(U, P))$ of the same form as above with corresponding associated stability factors.

# Example: Bluff body

- Channel flow with slip walls, $1 \times 1$ rectangular cross section and length 4.
- Cubic body of side length 0.25 centered at $(0.5, 0, 0)$
- $x_1 = 0$ inflow boundary with inflow condition $u = (1, 0, 0)$
- Transparent outflow condition
- cG(1)cG(1), $h = 1/32$
- $\nu = 1/1000$

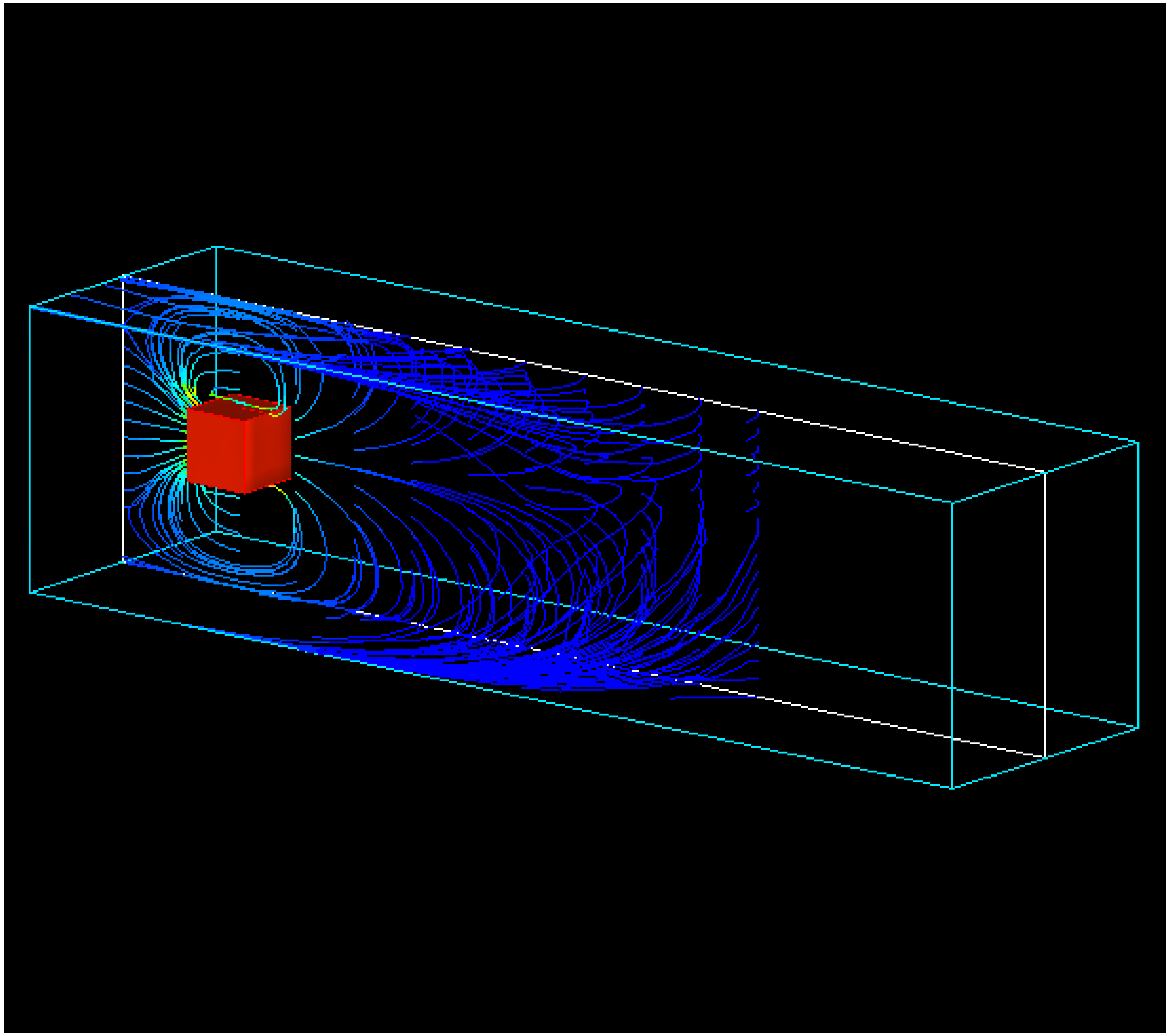
# Bluff body

start animation

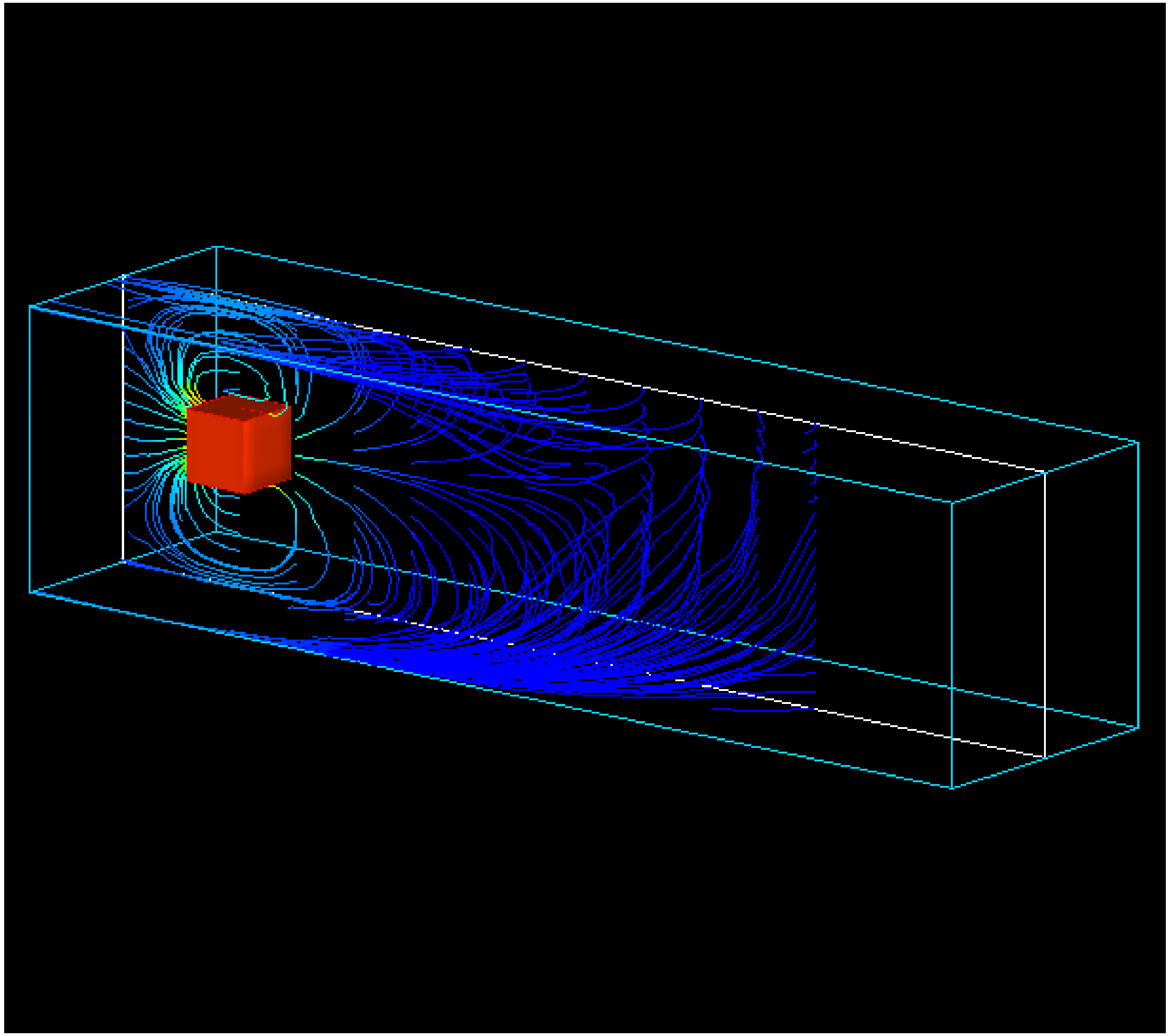
# Bluff body - Computation of the drag

- Take the state at $t = 18$ as initial data at $t = 0$
- Want to compute the mean drag force over $I = [0, 2]$, which corresponds to a boundary condition $u = (1/|I|, 0, 0)$ on the surface of the body

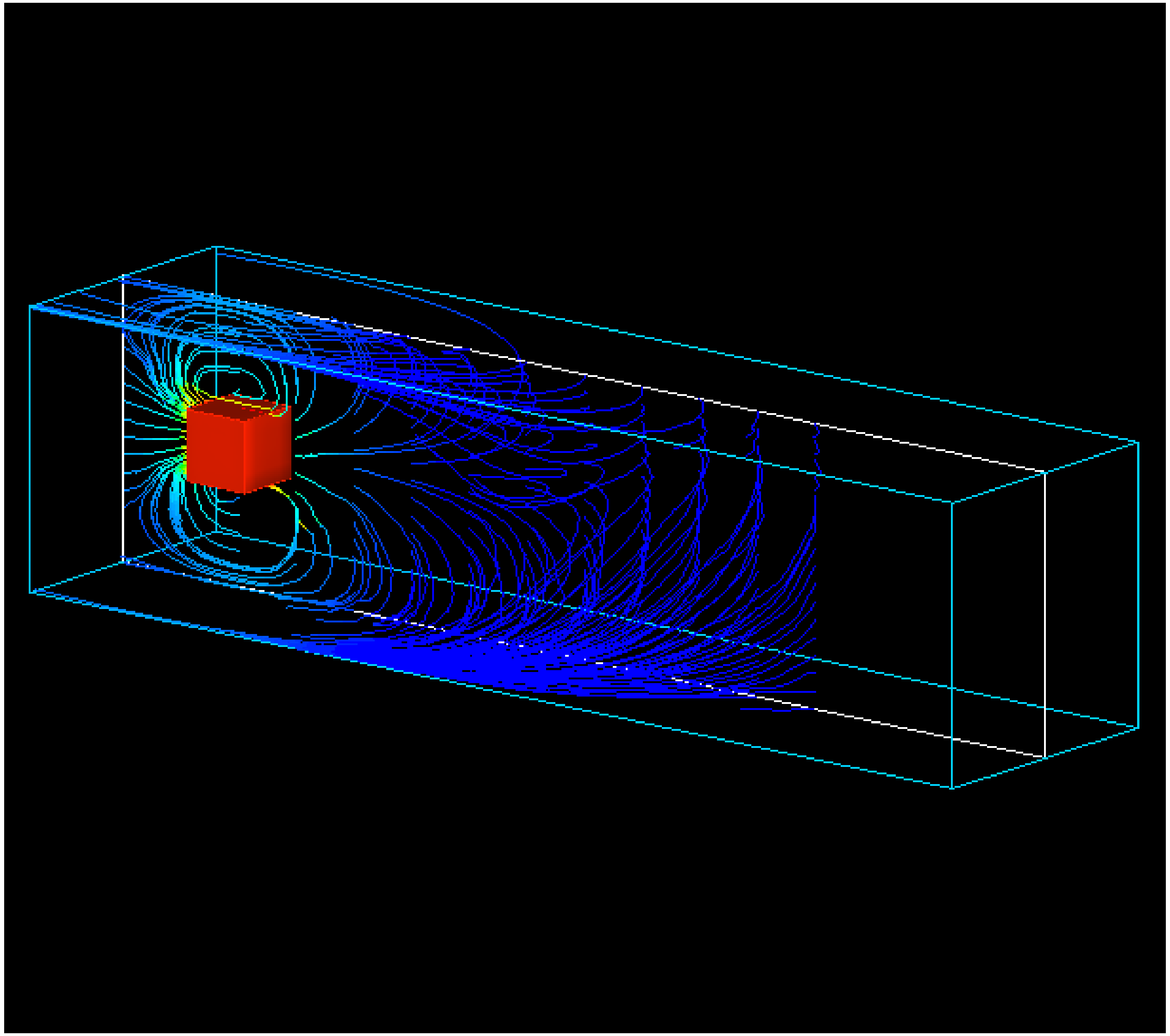
# Dual solution, $t = 1.75$



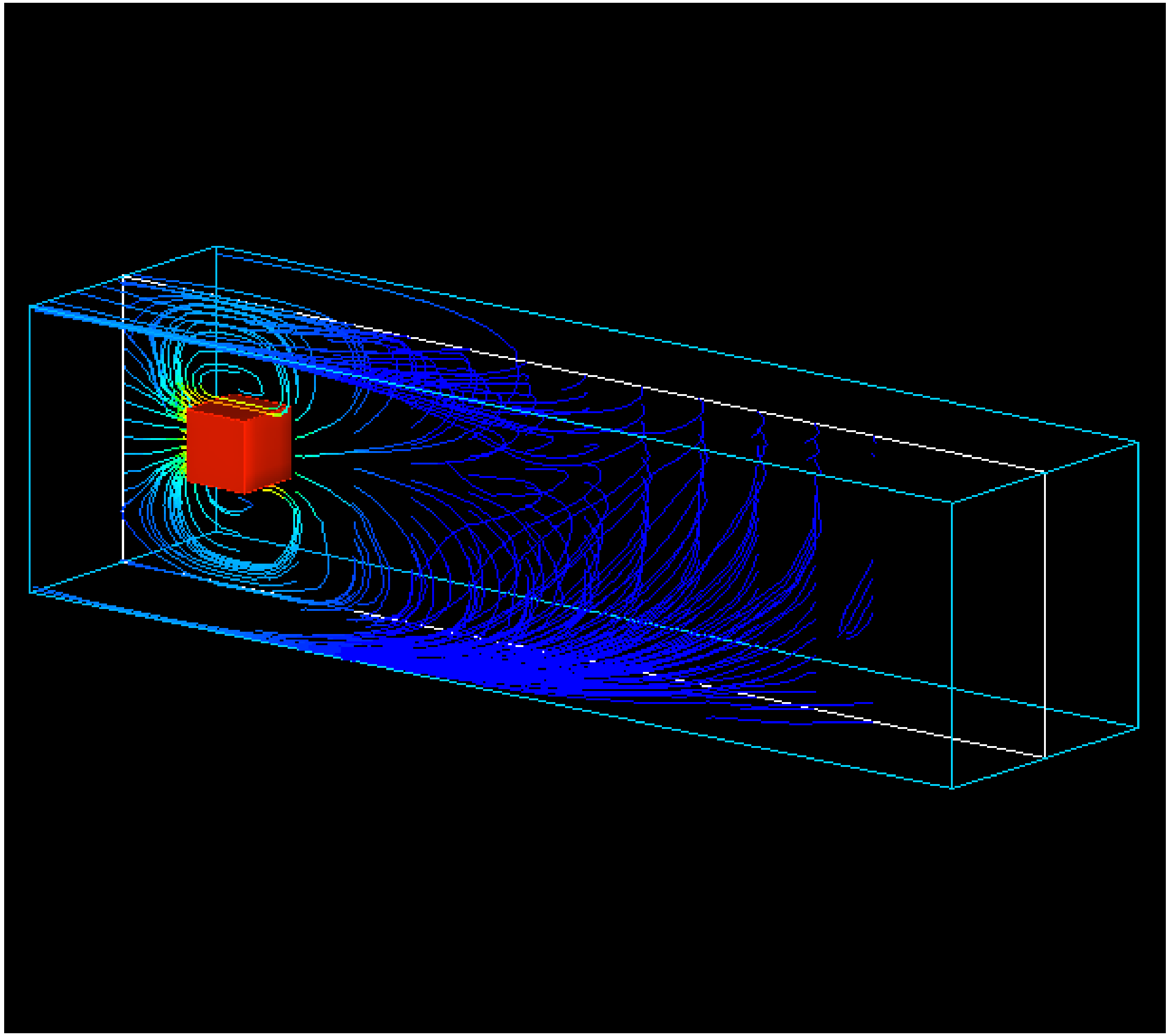
# Dual solution, $t = 1.5$



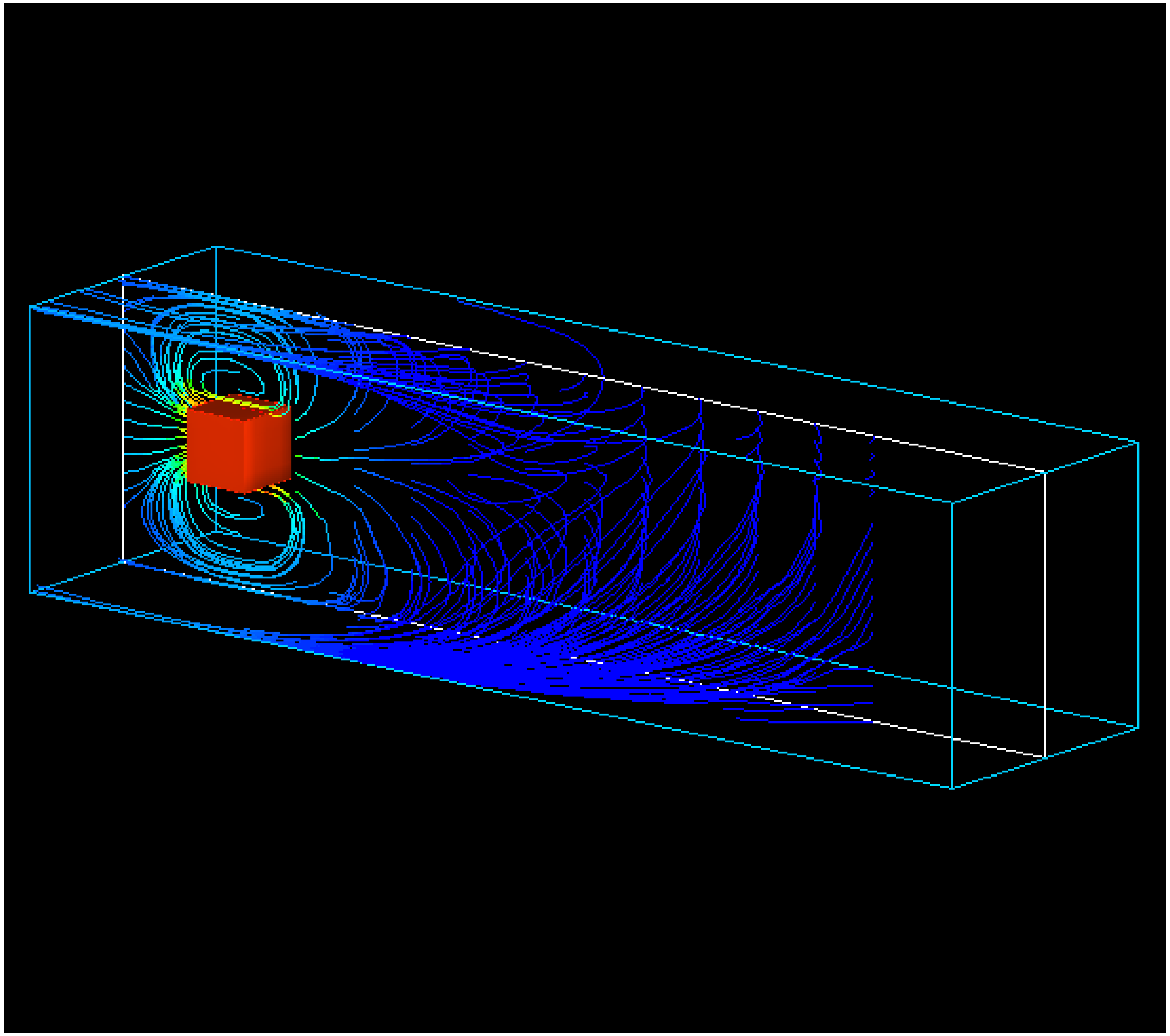
# Dual solution, $t = 1.25$



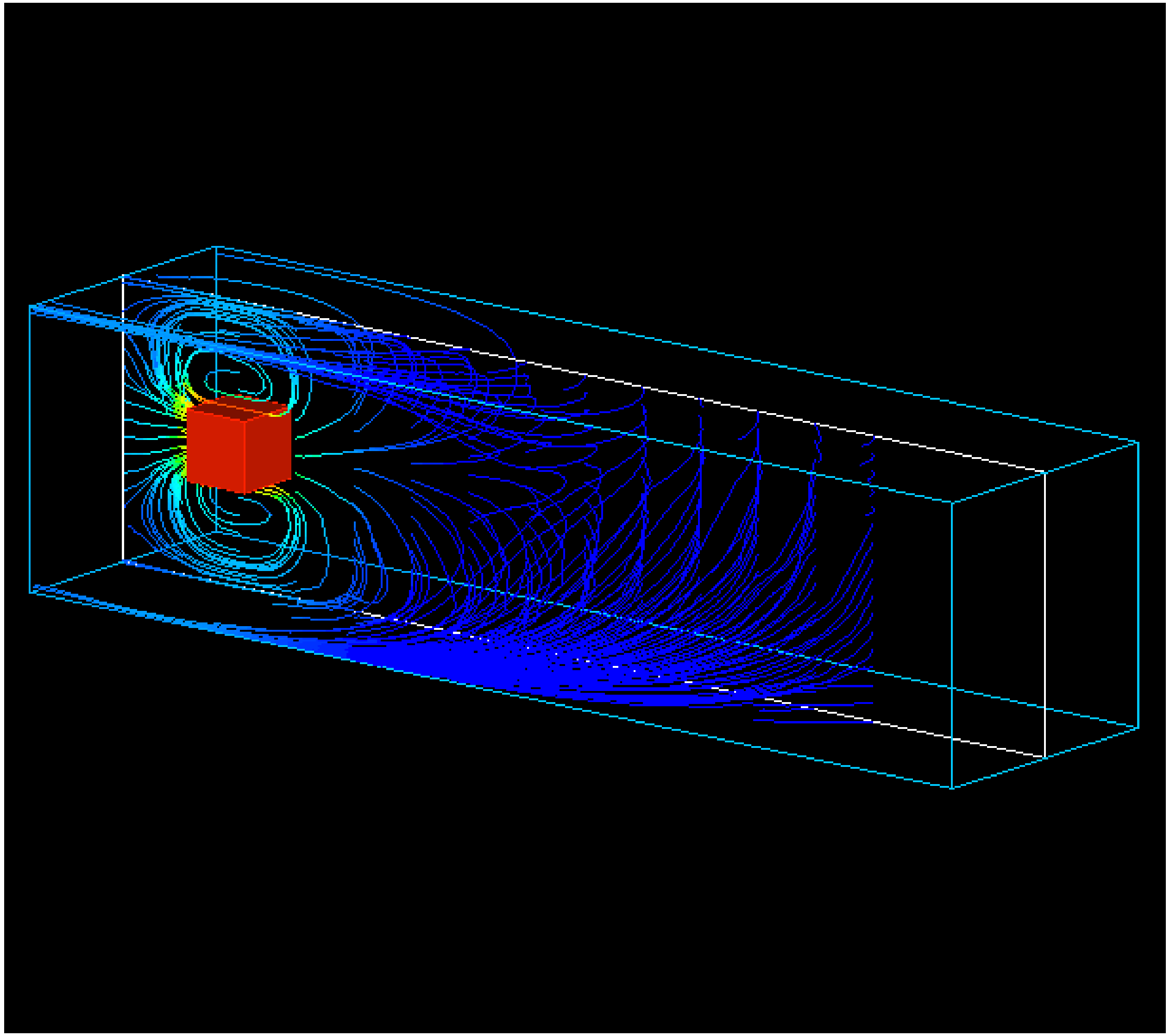
# Dual solution, $t = 1$



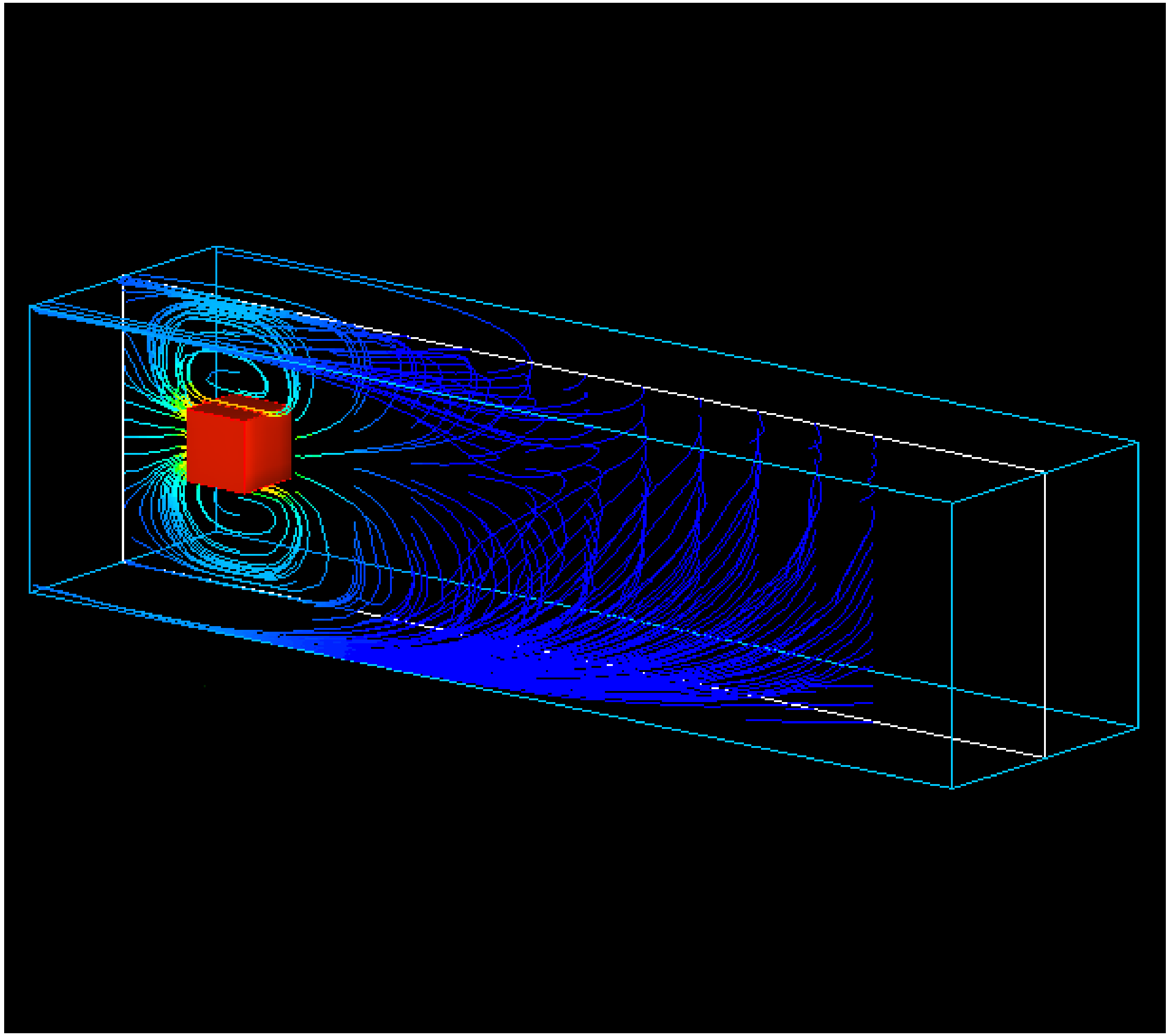
# Dual solution, $t = 0.75$



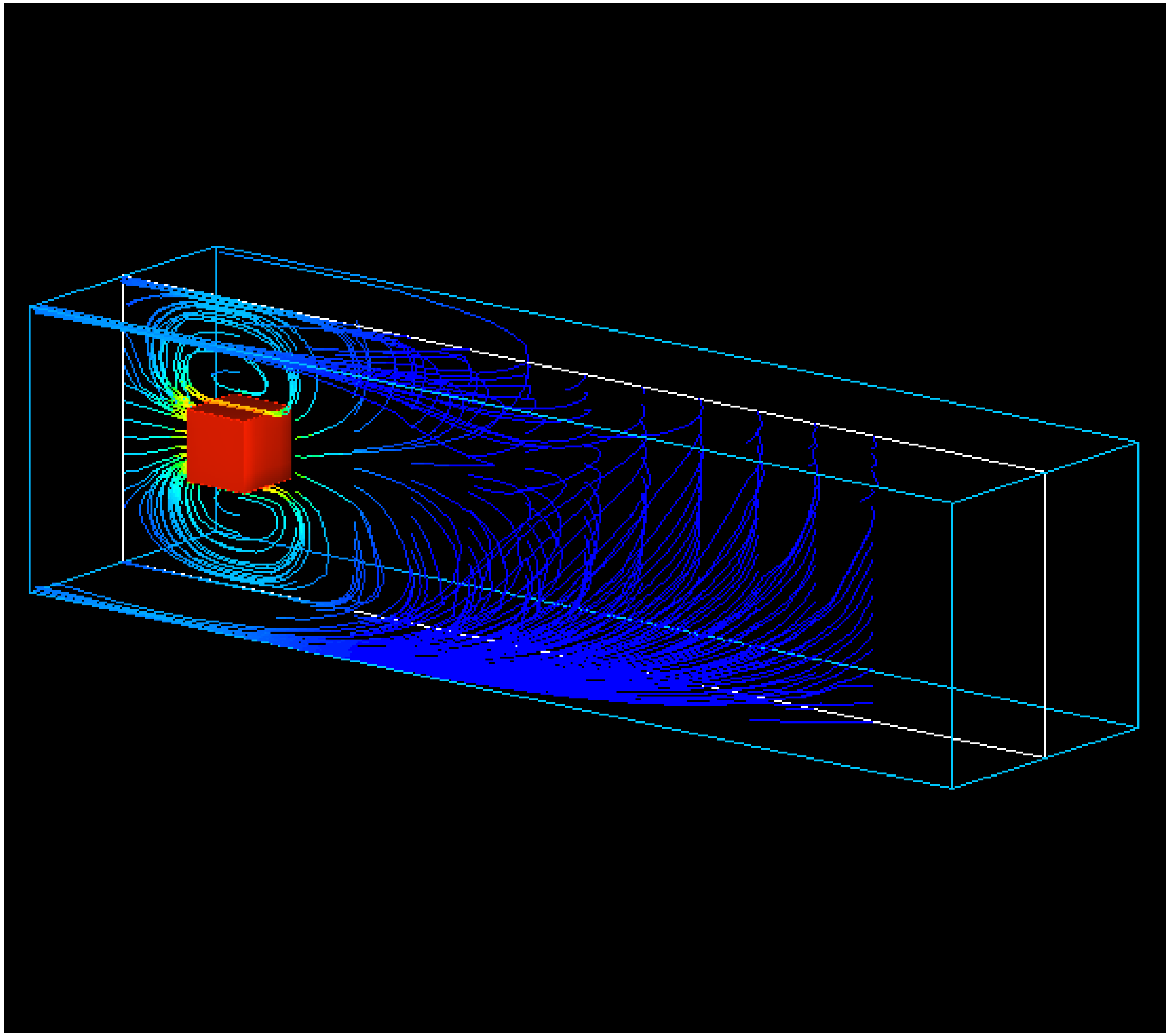
# Dual solution, $t = 0.5$



# Dual solution, $t = 0.25$



# Dual solution, $t = 0$



# Observation

For this problem, computing an average in time, the dual solution is very similar for the whole time interval $\Rightarrow$ mesh adaptation based on the dual solution does not have to update the mesh each time step

# Stability factors in $L_1(I; L_1(\Omega))$-norm

| $I$ | $S_{0,1}$ | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{2,1}$ |
|---|---|---|---|---|---|
| $[0, 2]$ | 0.019 | 0.39 | 0.13 | 0.079 | 8.8 |

Stability factors related to the error of the drag force over the time interval $I = [0, 2]$

# Stability factors in $L_1(I; L_1(\Omega))$-norm

| $I$ | $S_{0,1}$ | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{2,1}$ |
|---|---|---|---|---|---|
| $[0, 2]$ | 0.019 | 0.39 | 0.13 | 0.079 | 8.8 |

We note that these stability factors are significally smaller than the stability factors related the momentary quantity computed for the step down problem

# Summary lecture 1

- The dual solution give valuable information for adaptive mesh refinement w.r.t different error measures
- Solutions to the linearized dual problem gives important information about computability of different quantities
- Pointwise quantities harder to compute than average quantities

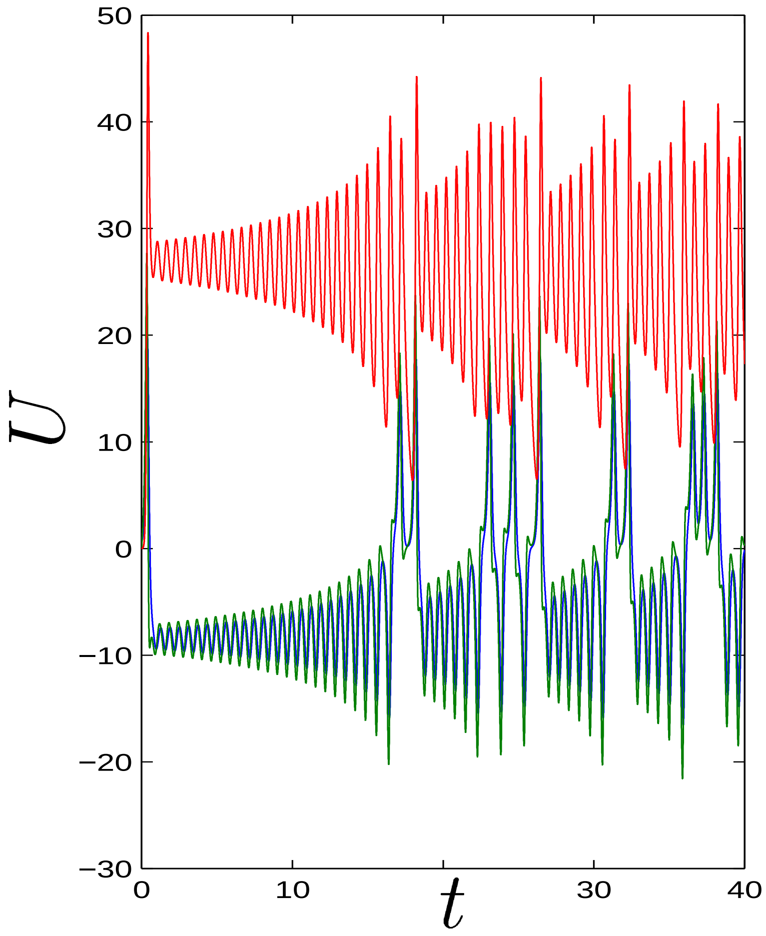
# The Lorenz System (A.Logg)

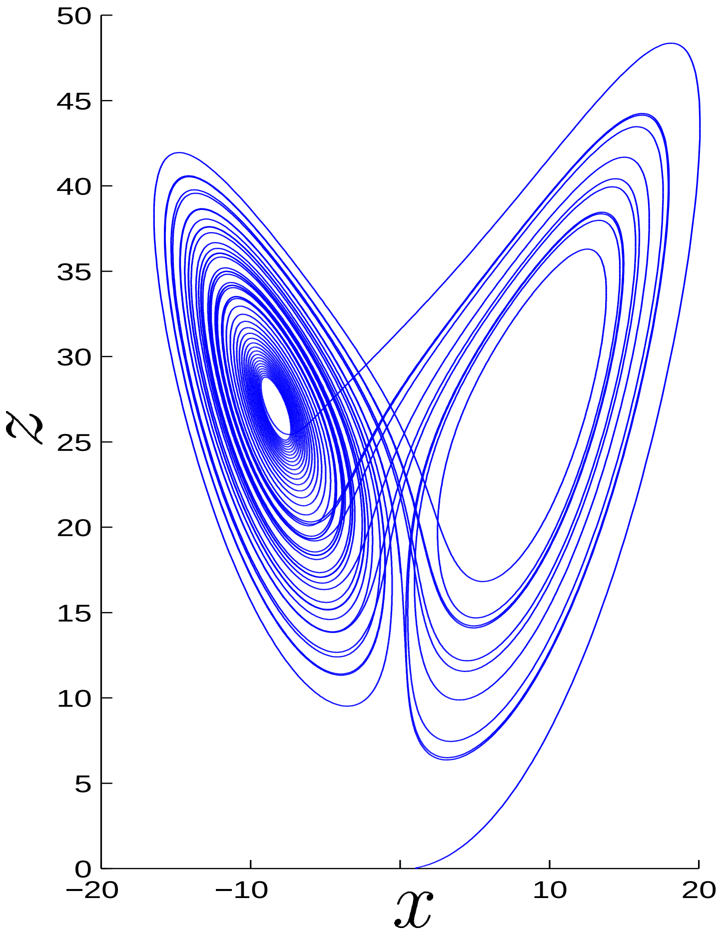
$$\begin{cases} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz, \end{cases}$$

where we, as usual, take $(x_0, y_0, z_0) = (1, 0, 0)$, $\sigma = 10$, $b = 8/3$ and $r = 28$.

How far can we reach? (How long can we compute with small computational error?)

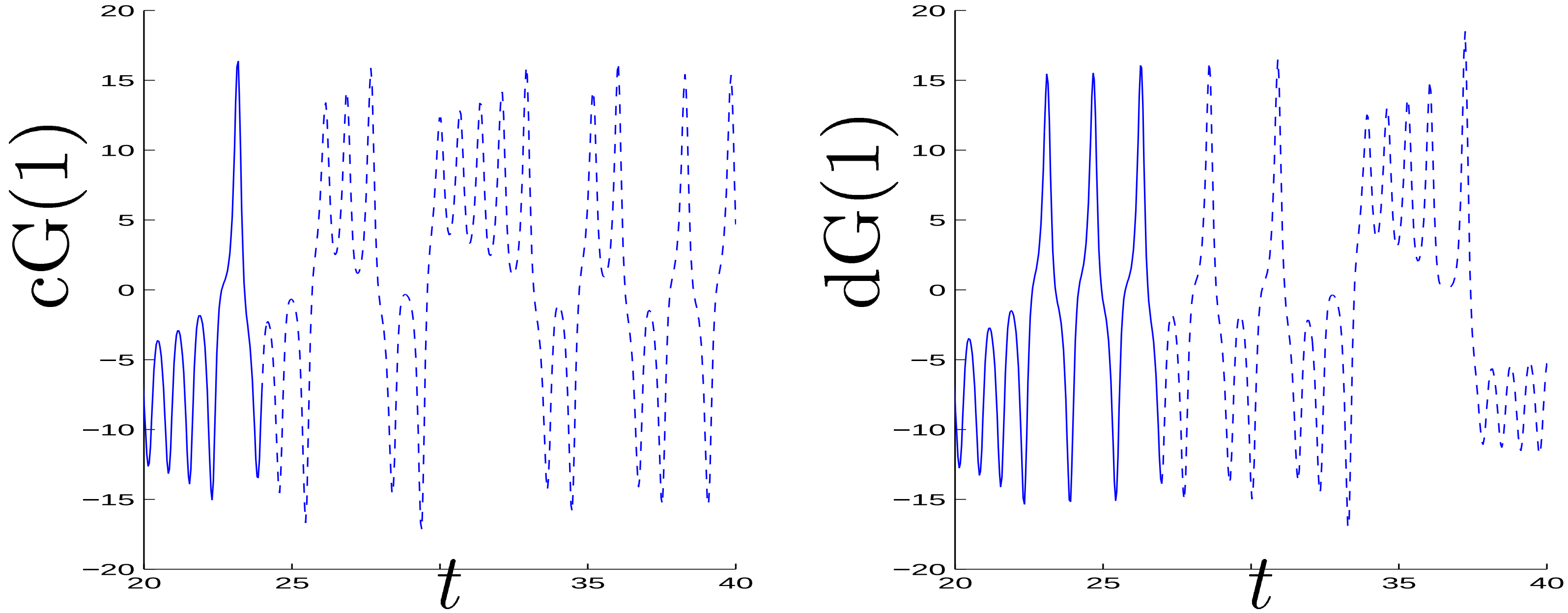
# The Solution?



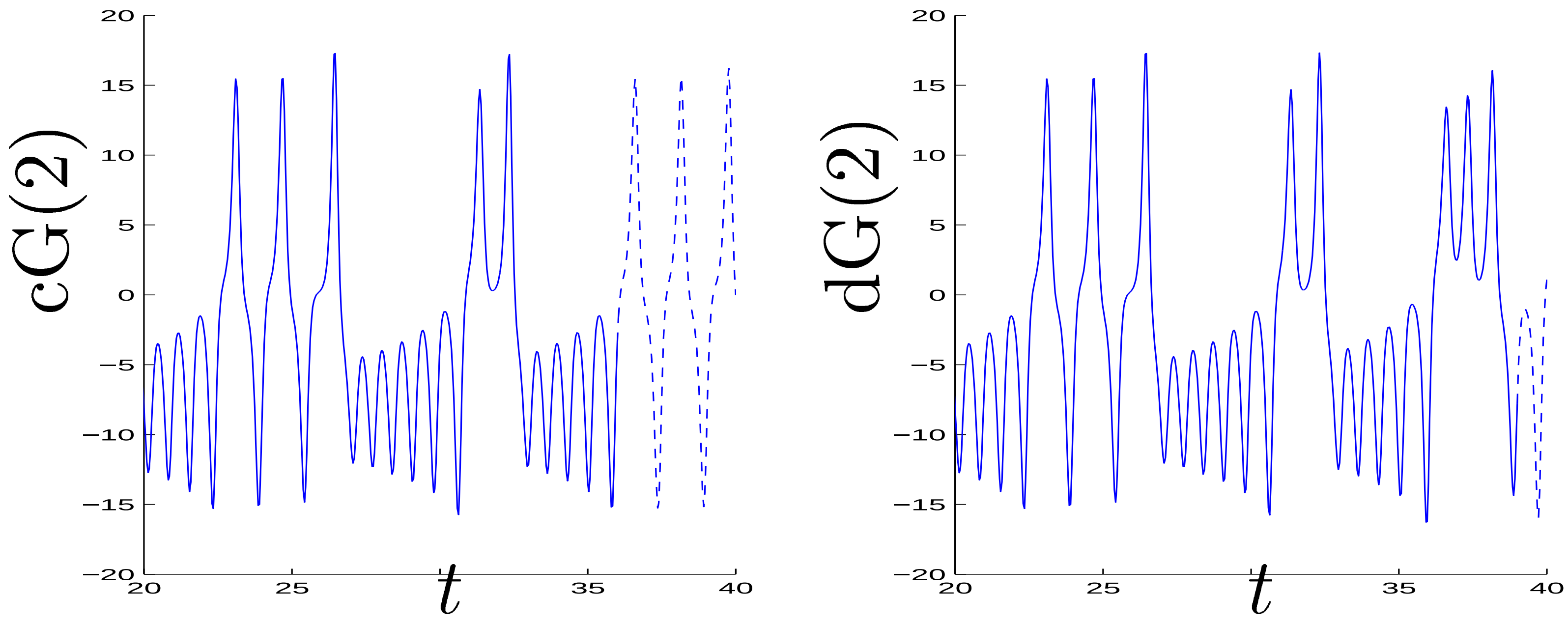
# A Simple Experiment

- $T = 40$
- $k = 0.001$



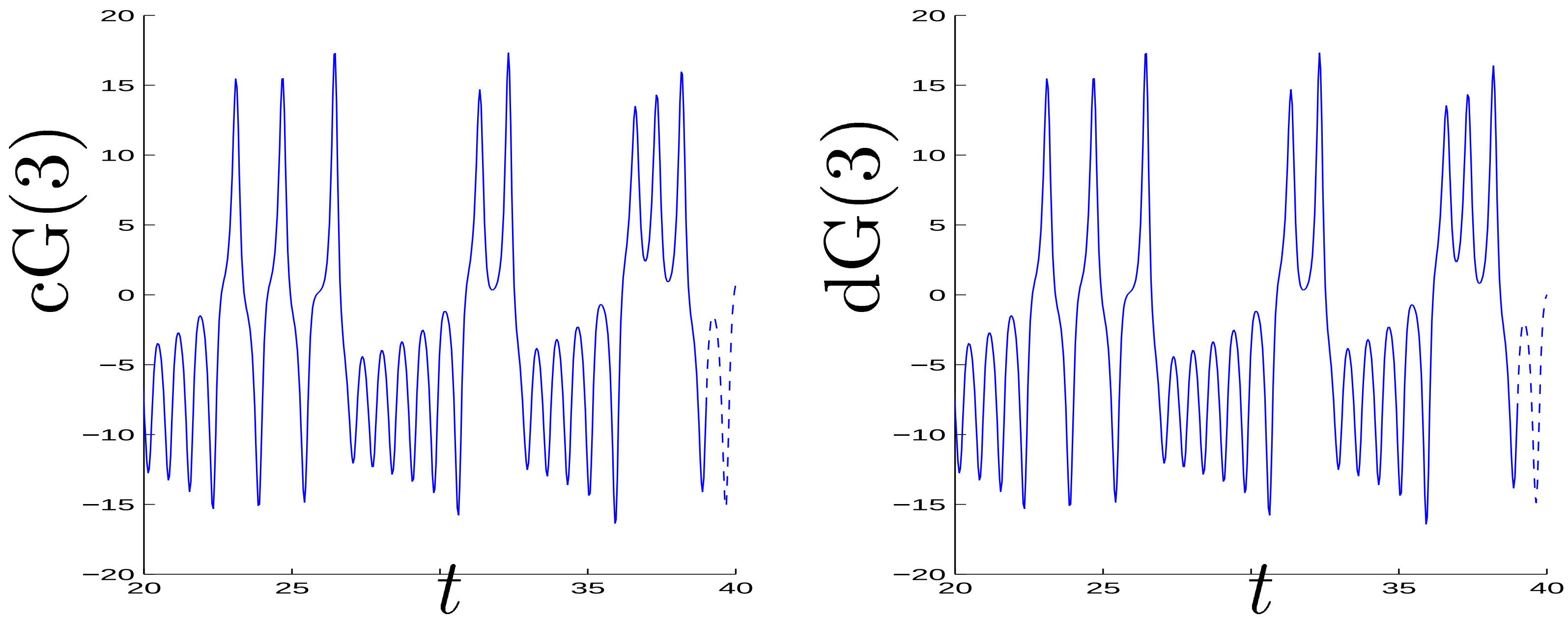
# A Simple Experiment

- $T = 40$
- $k = 0.001$



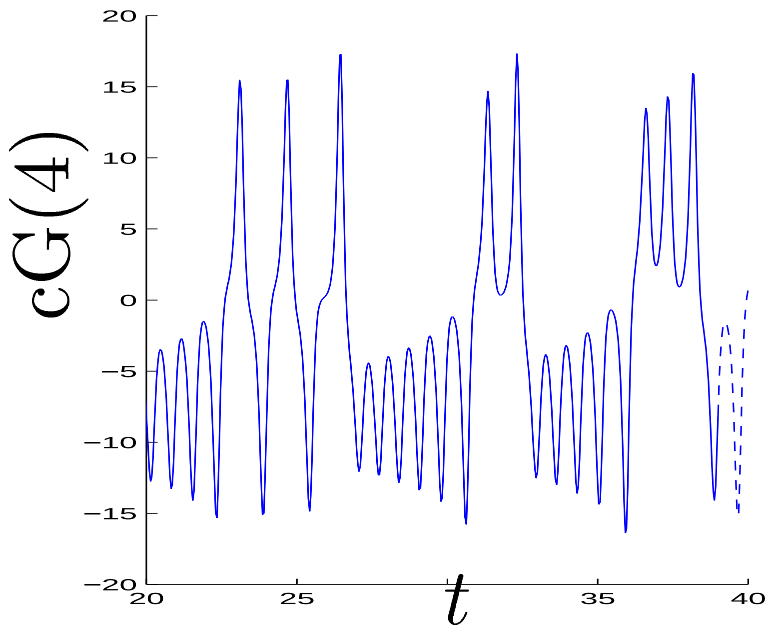
# A Simple Experiment

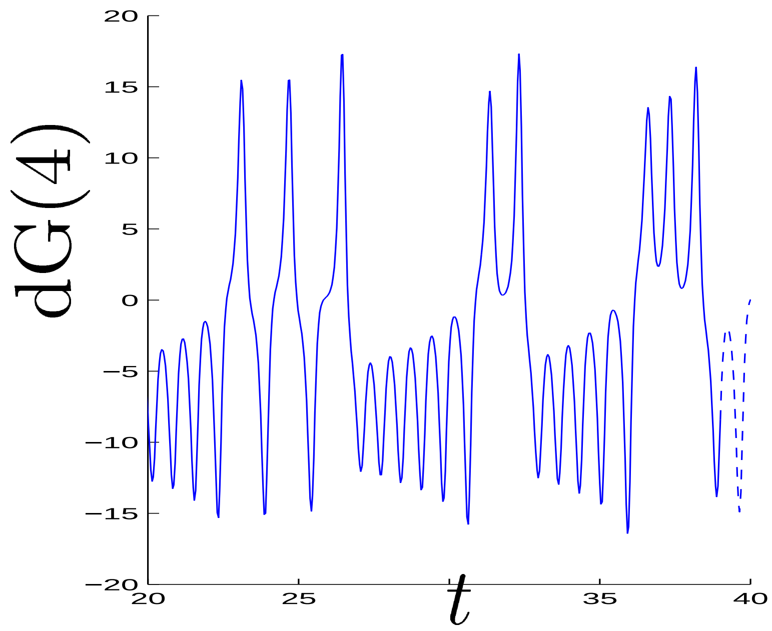
- $T = 40$
- $k = 0.001$



# A Simple Experiment

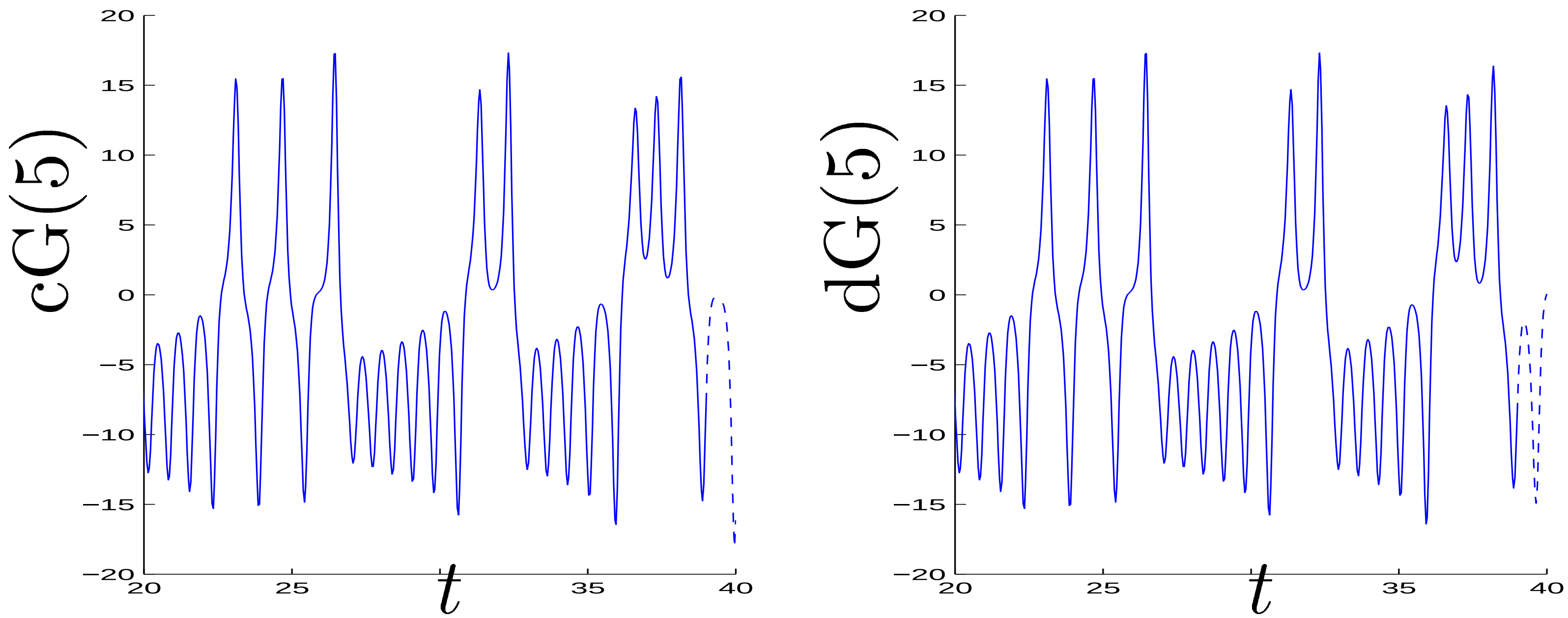
- $T = 40$
- $k = 0.001$



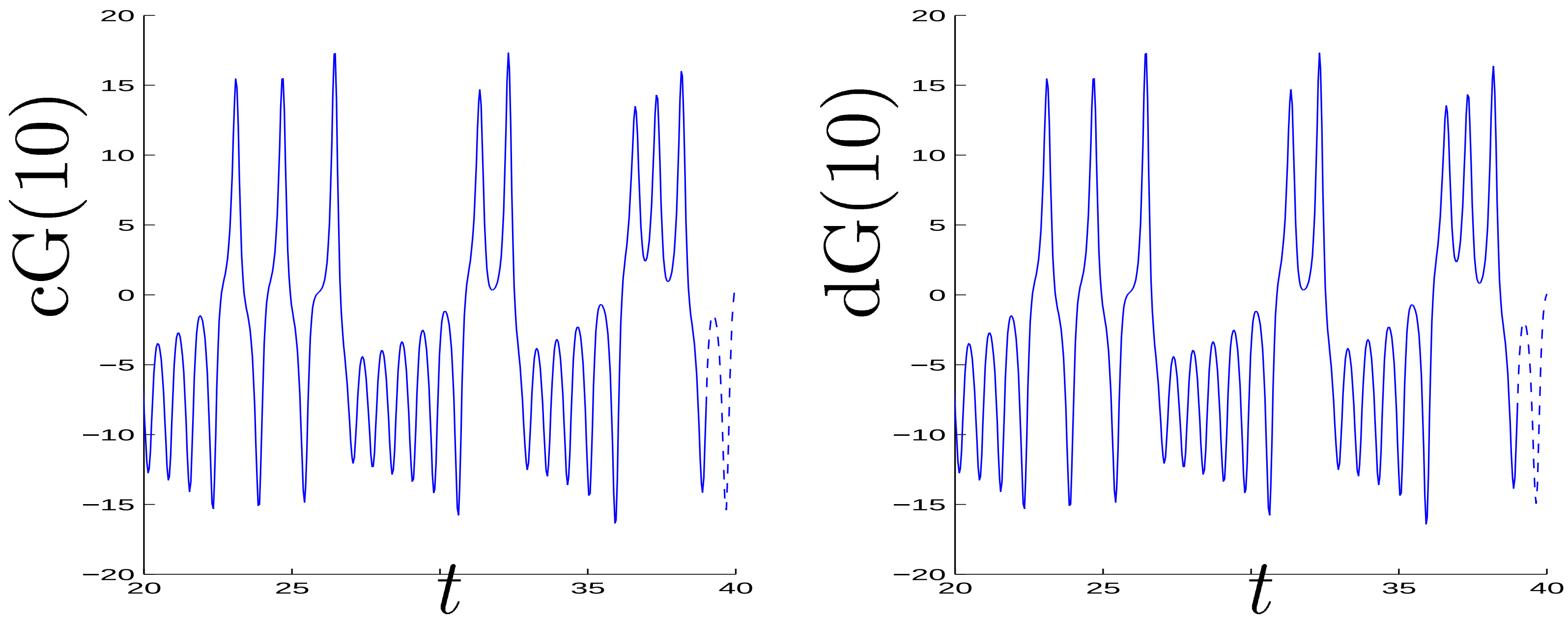
# A Simple Experiment

- $T = 40$
- $k = 0.001$



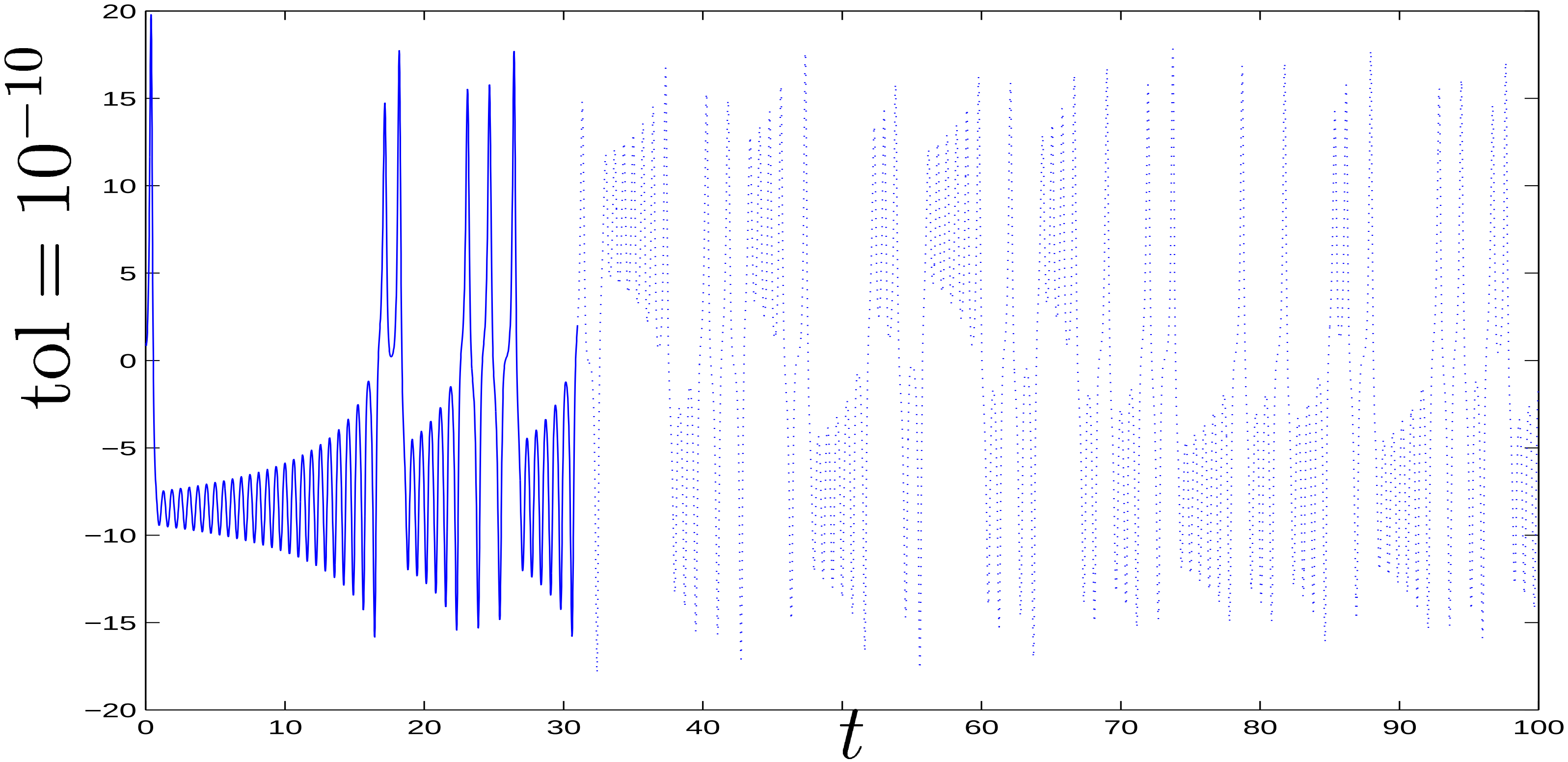
# A Simple Experiment

- $T = 40$
- $k = 0.001$



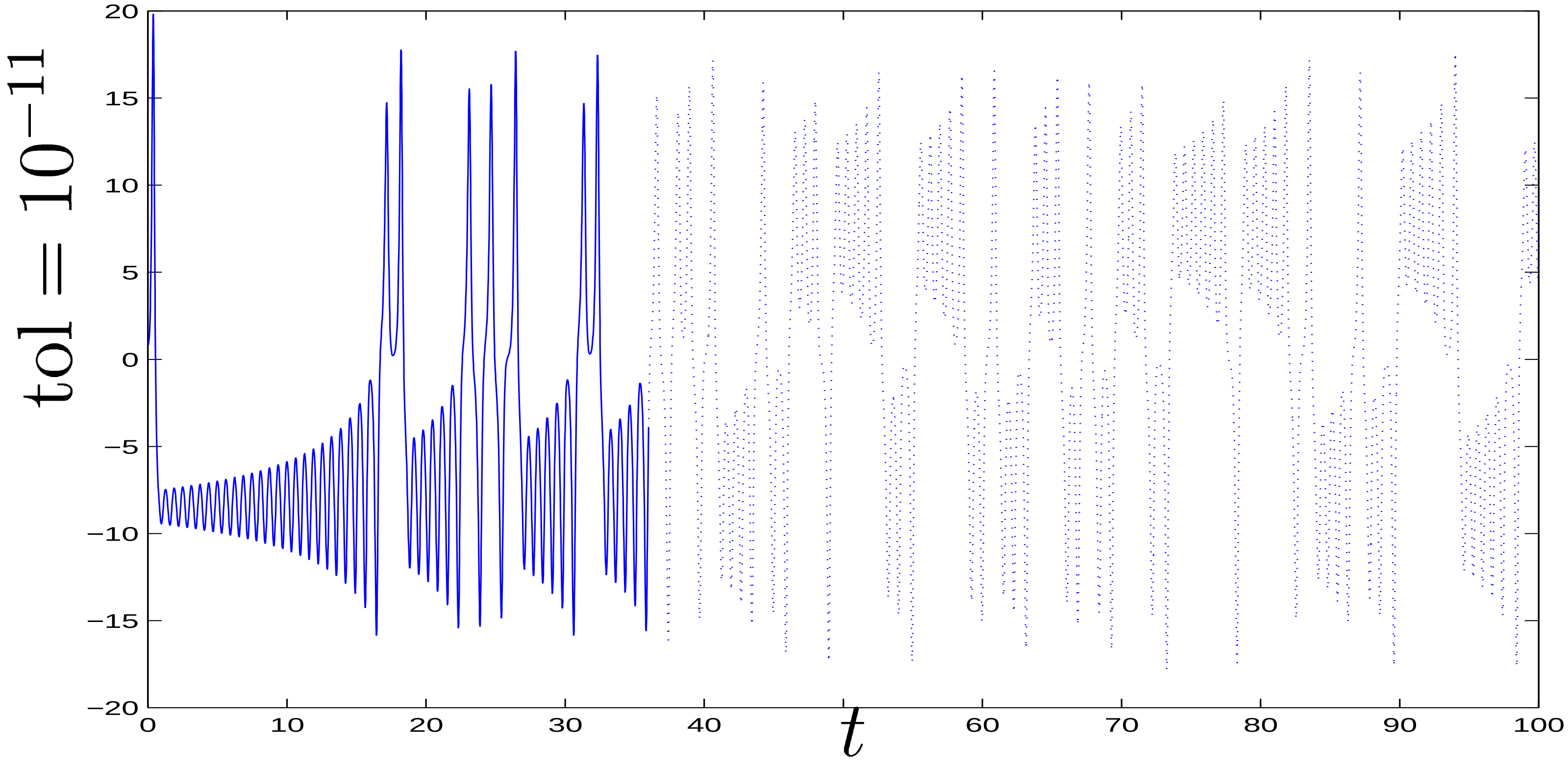
# ode45

Trying the same thing with Matlabs ode45:



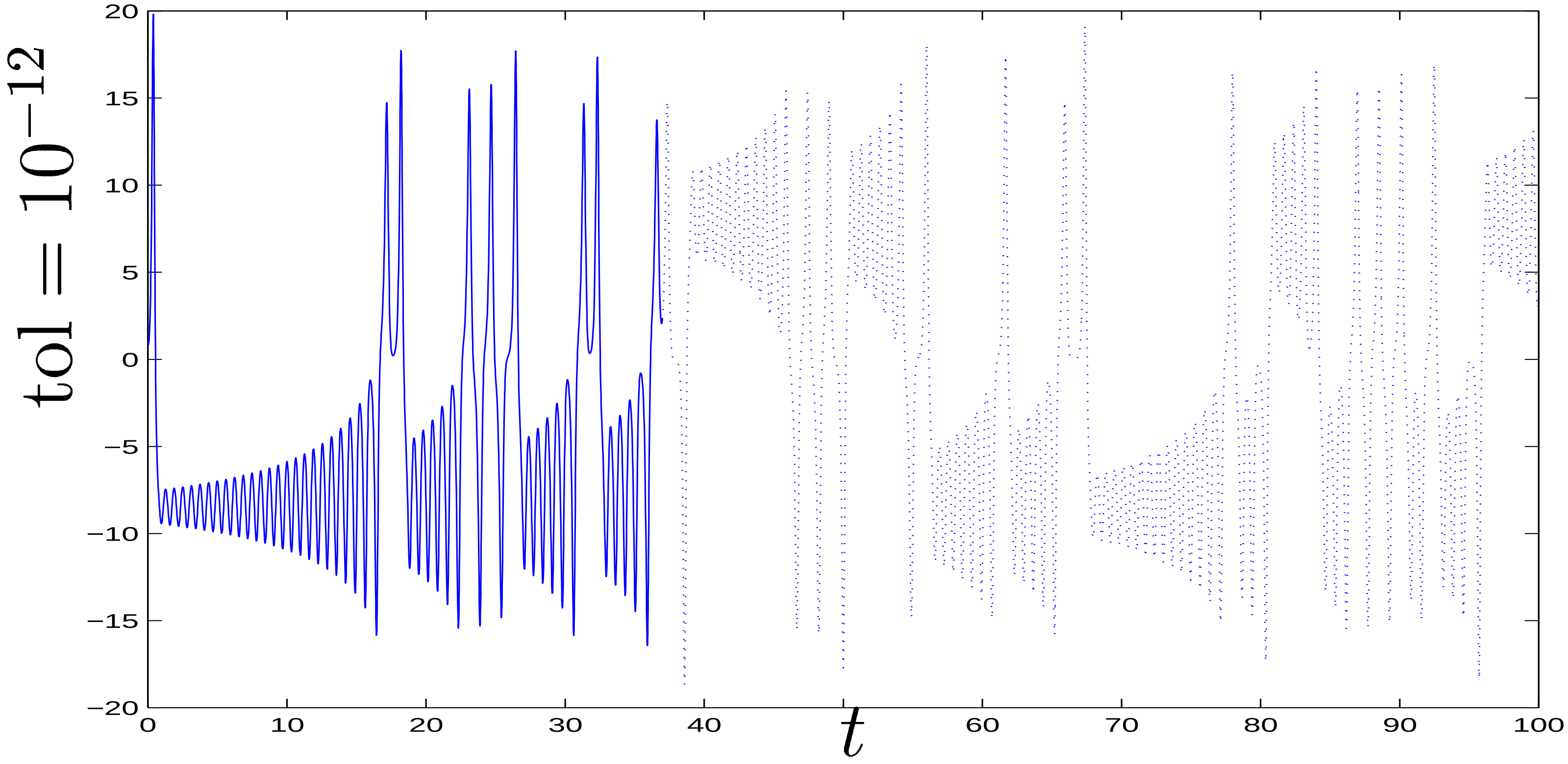
# ode45

Trying the same thing with Matlabs ode45:



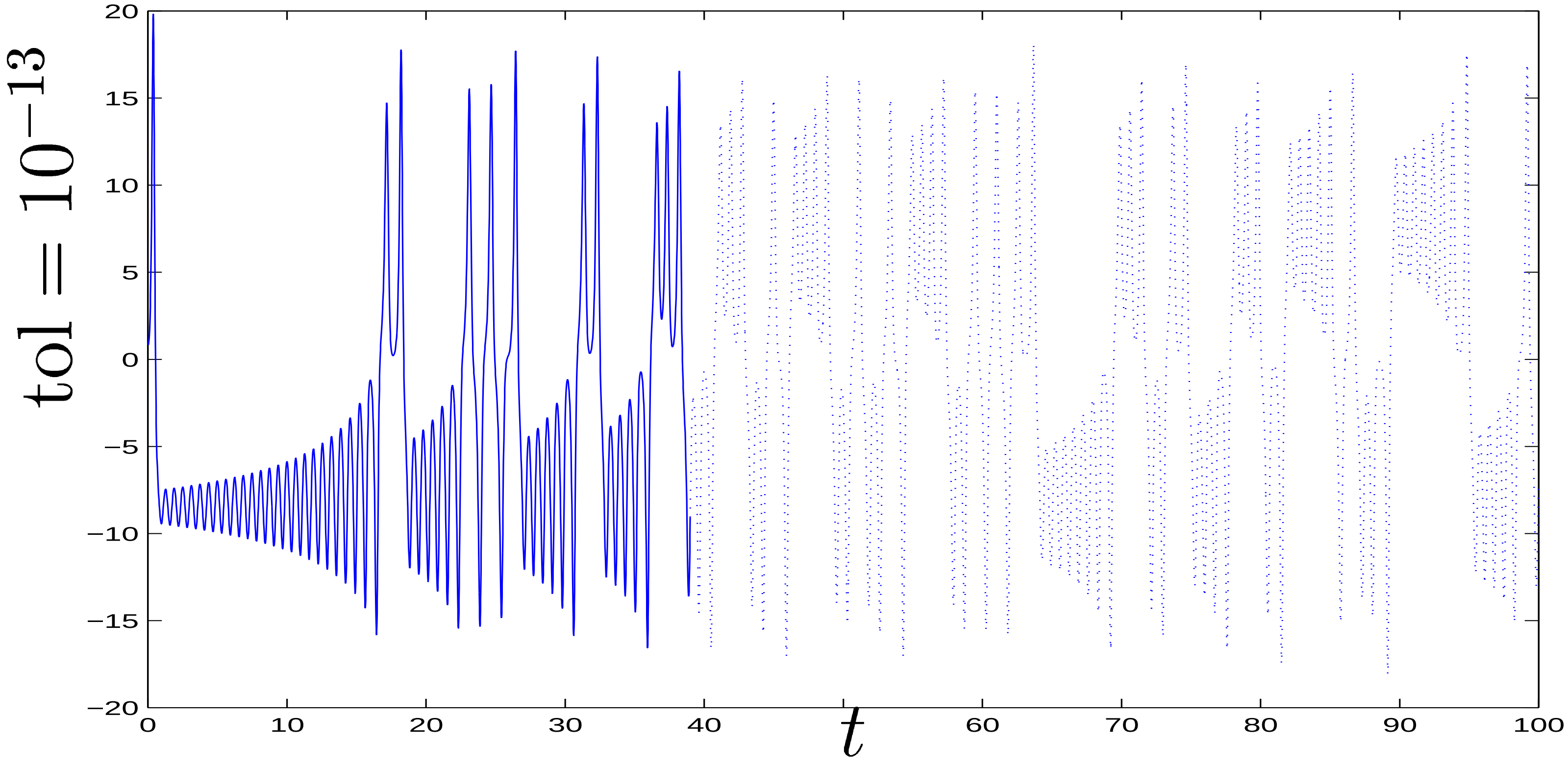
# ode45

Trying the same thing with Matlabs ode45:



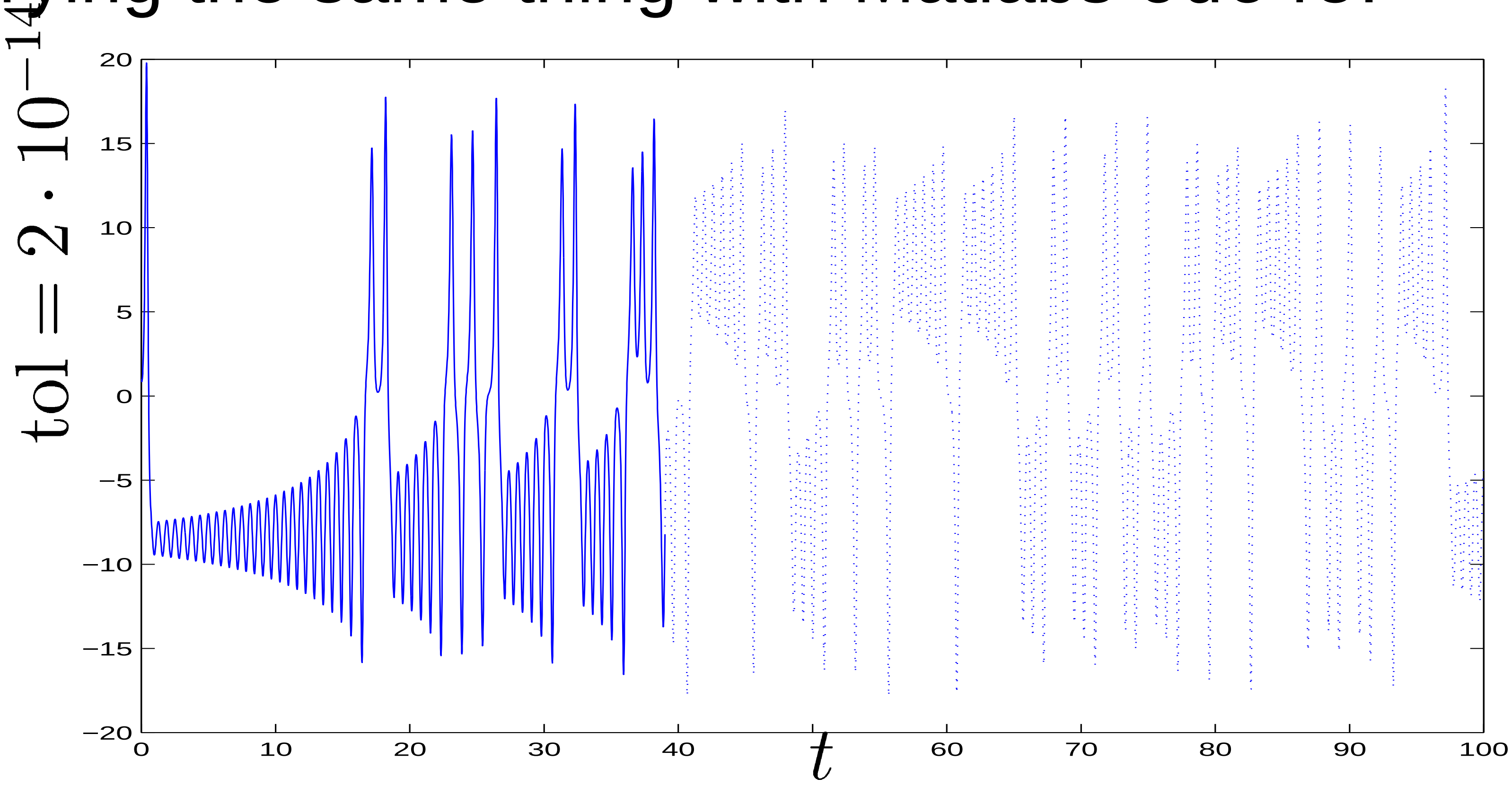
# ode45

Trying the same thing with Matlabs ode45:



# ode45

Trying the same thing with Matlabs ode45:



# Getting Further

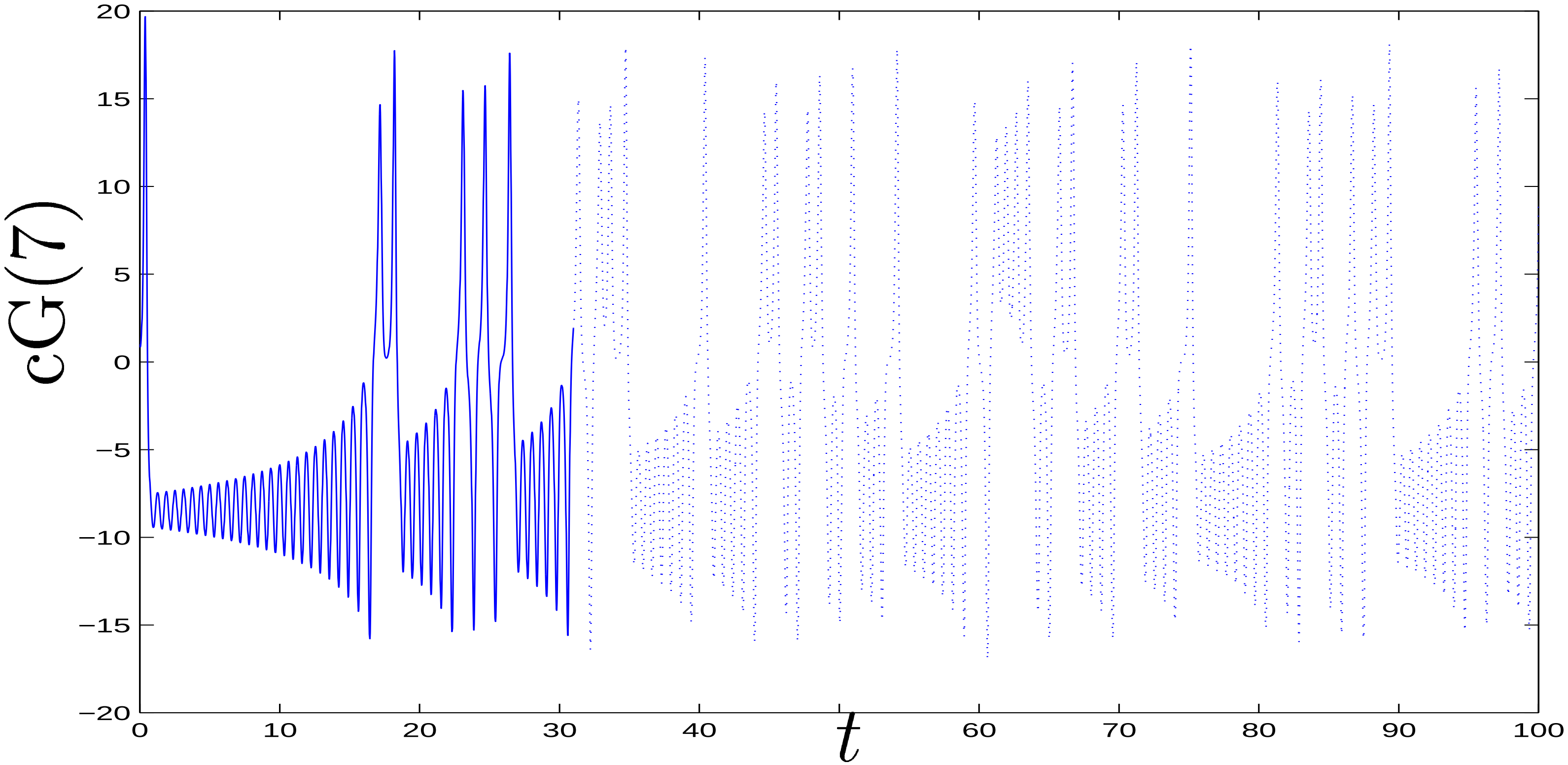
- Small residual
- Large stability factor

The round-off error is at least $10^{-16}$ in each time step using double precision

This means that we have to take fewer time steps, using a very high order method

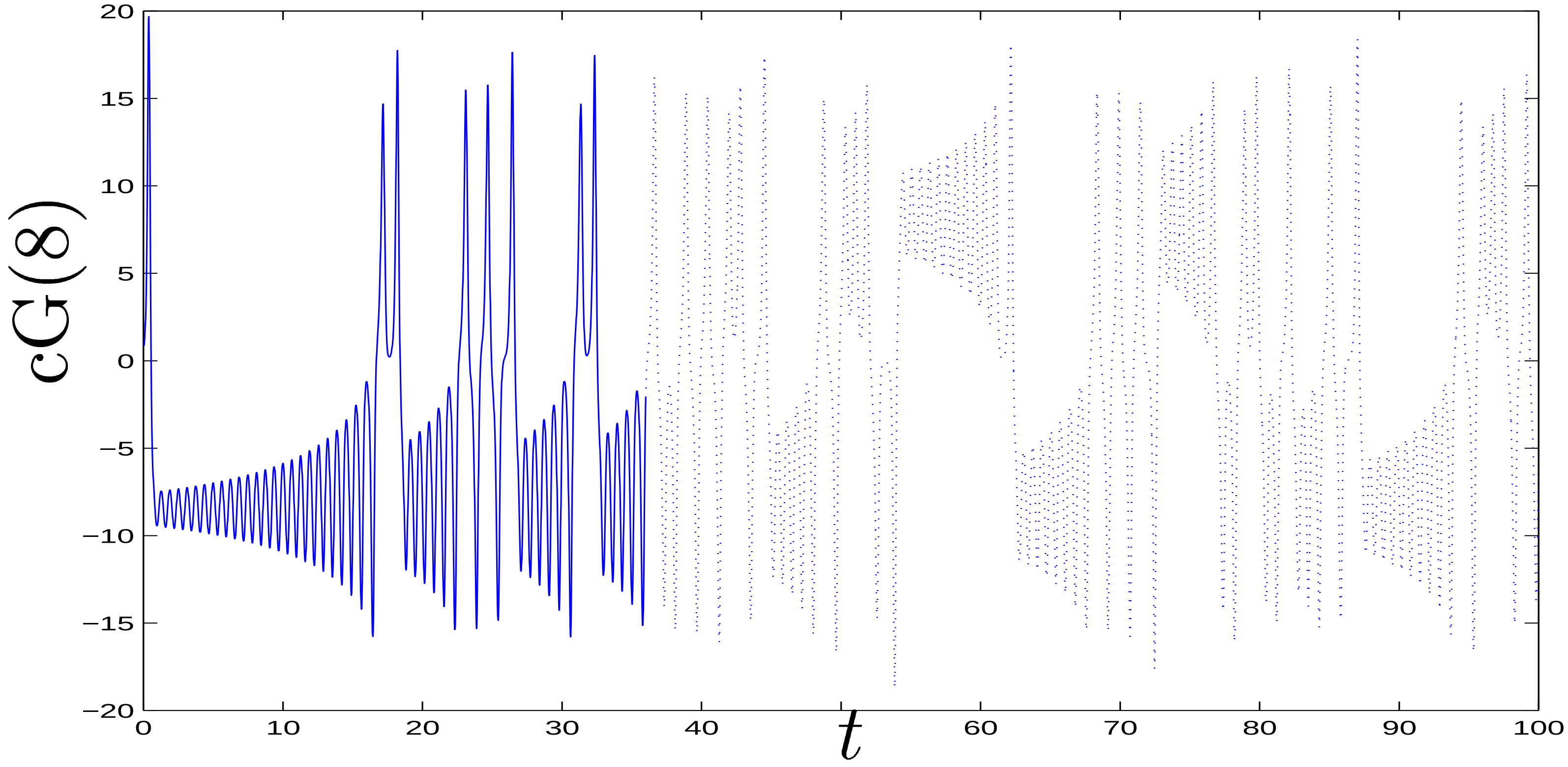
# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



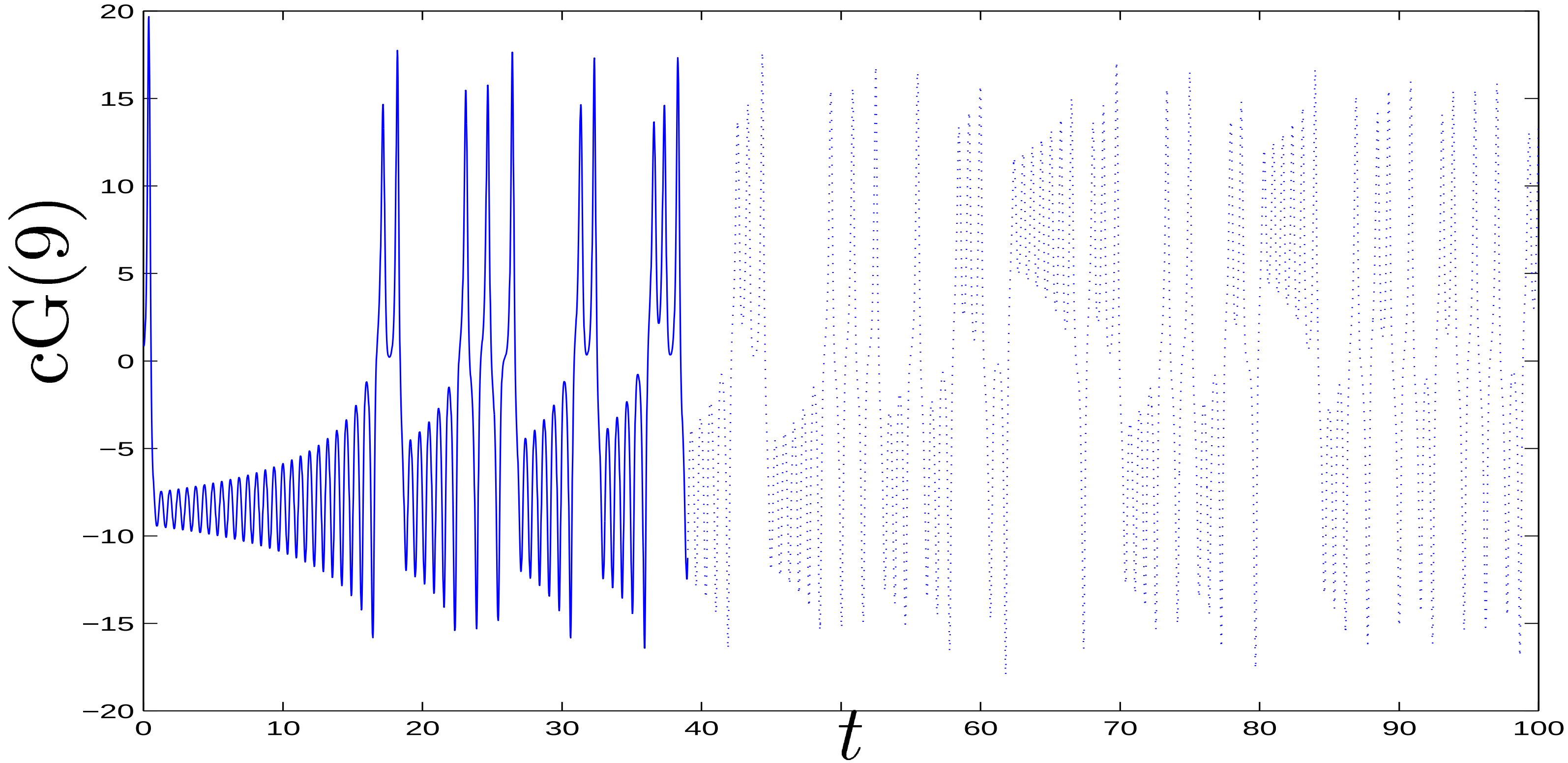
# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



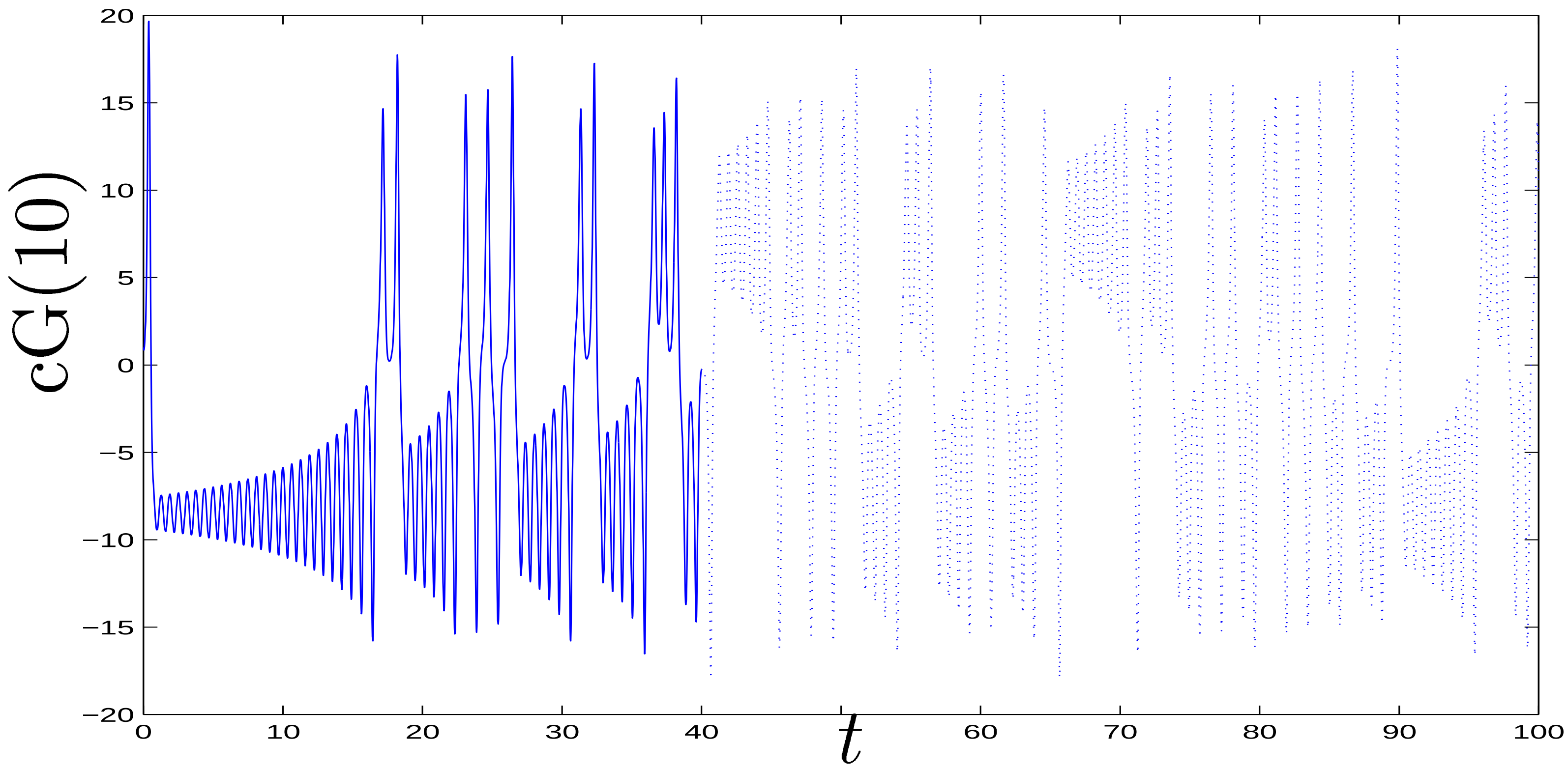
# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



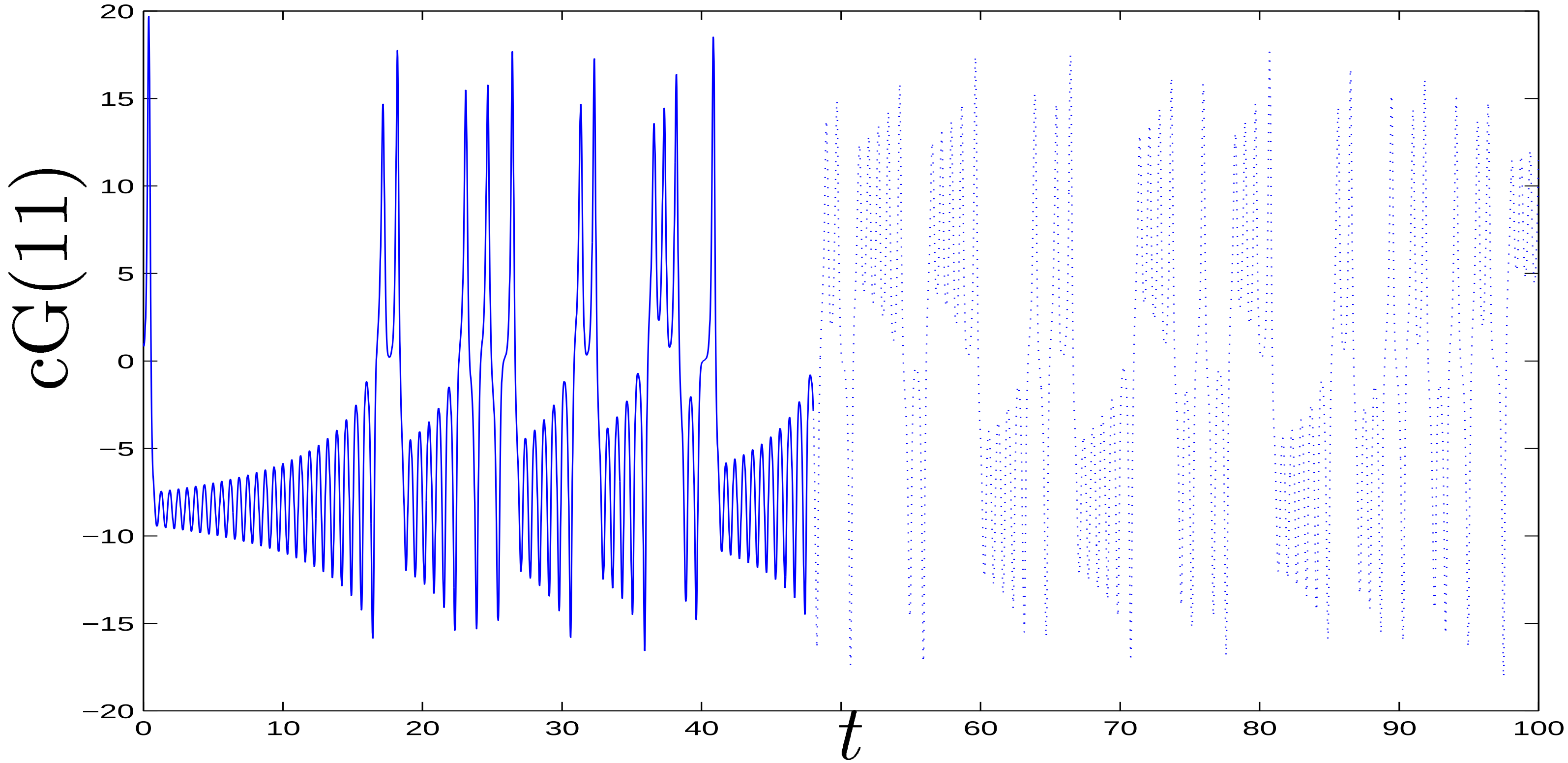
# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



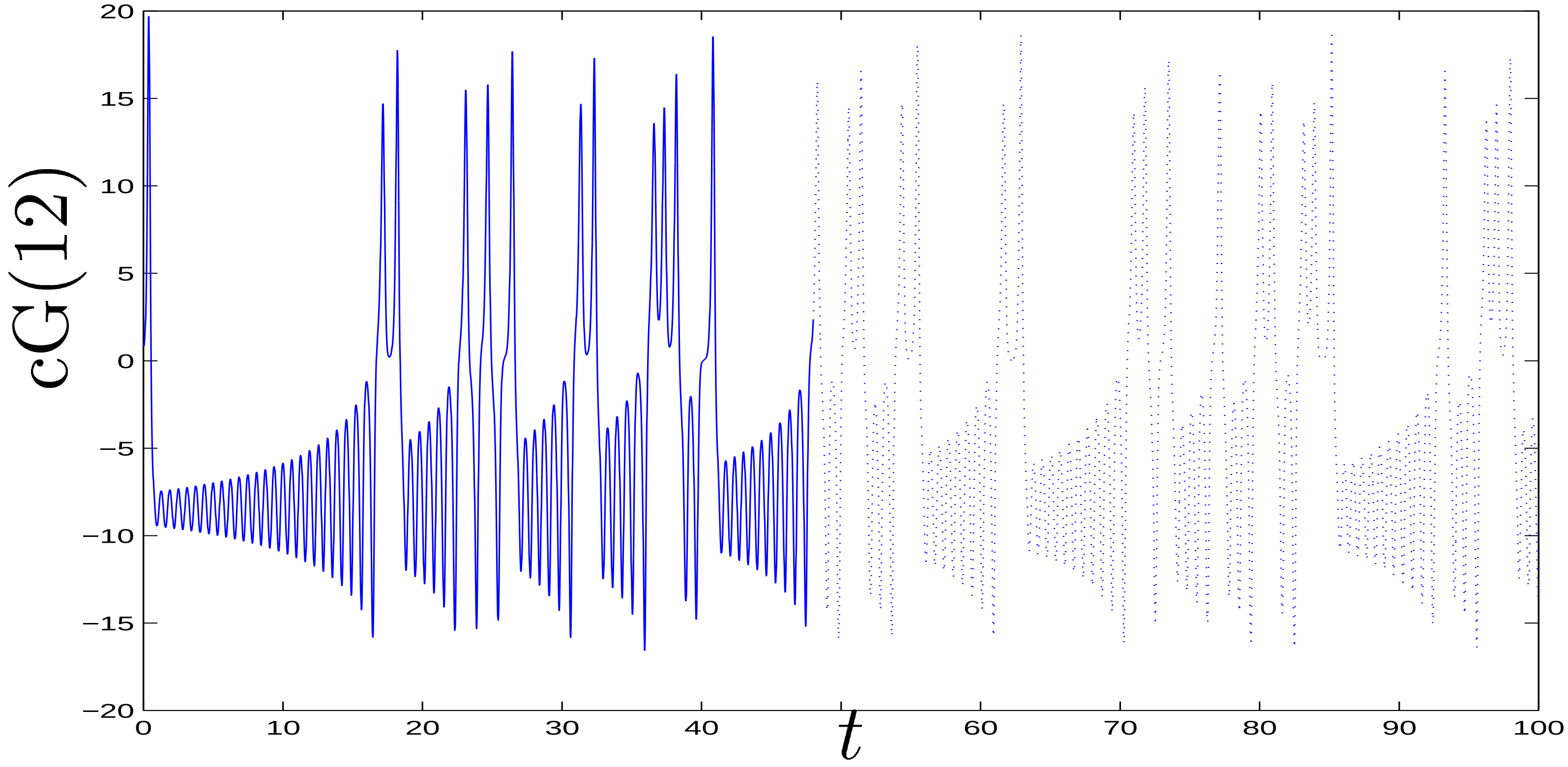
# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



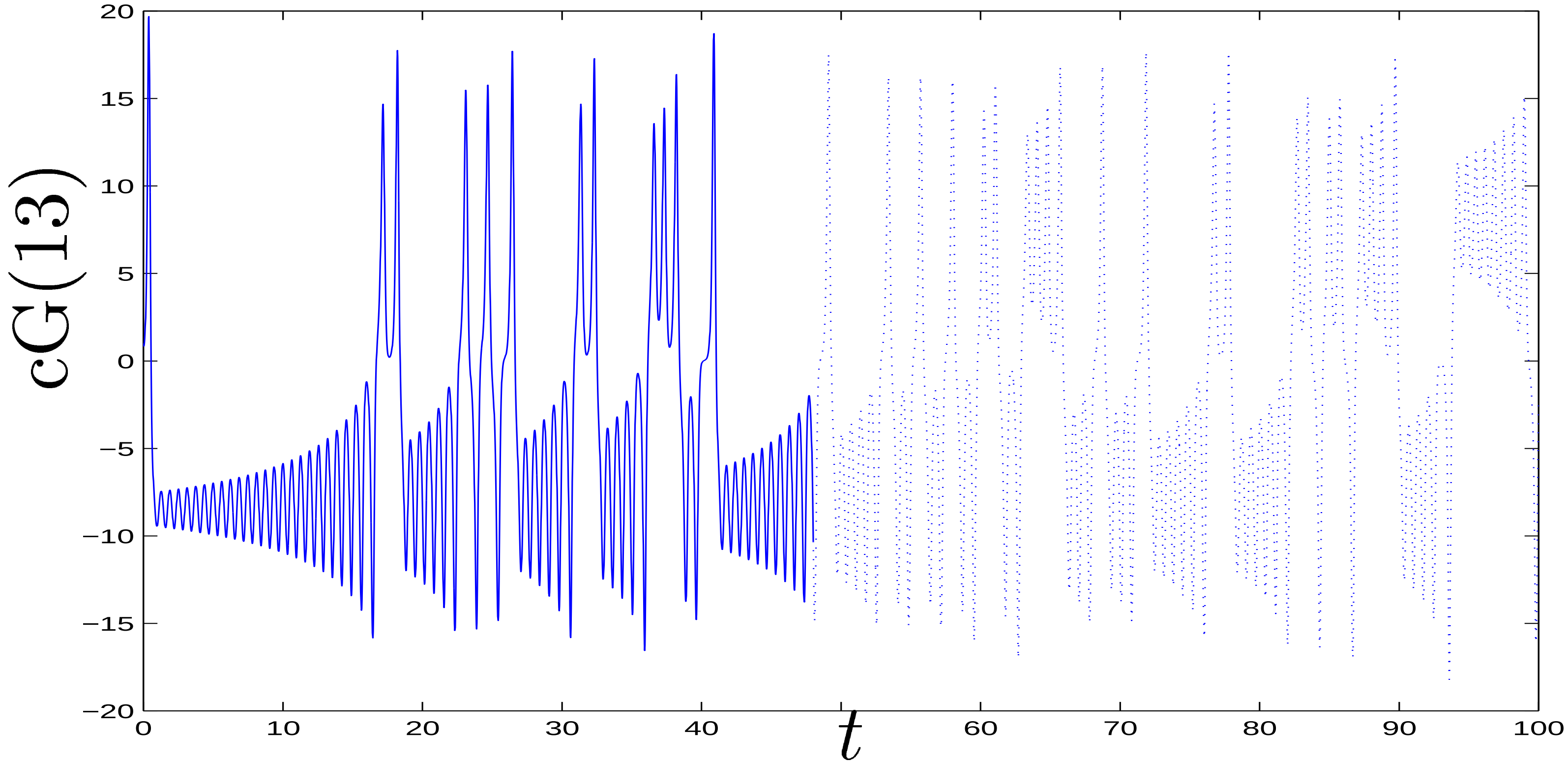
# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



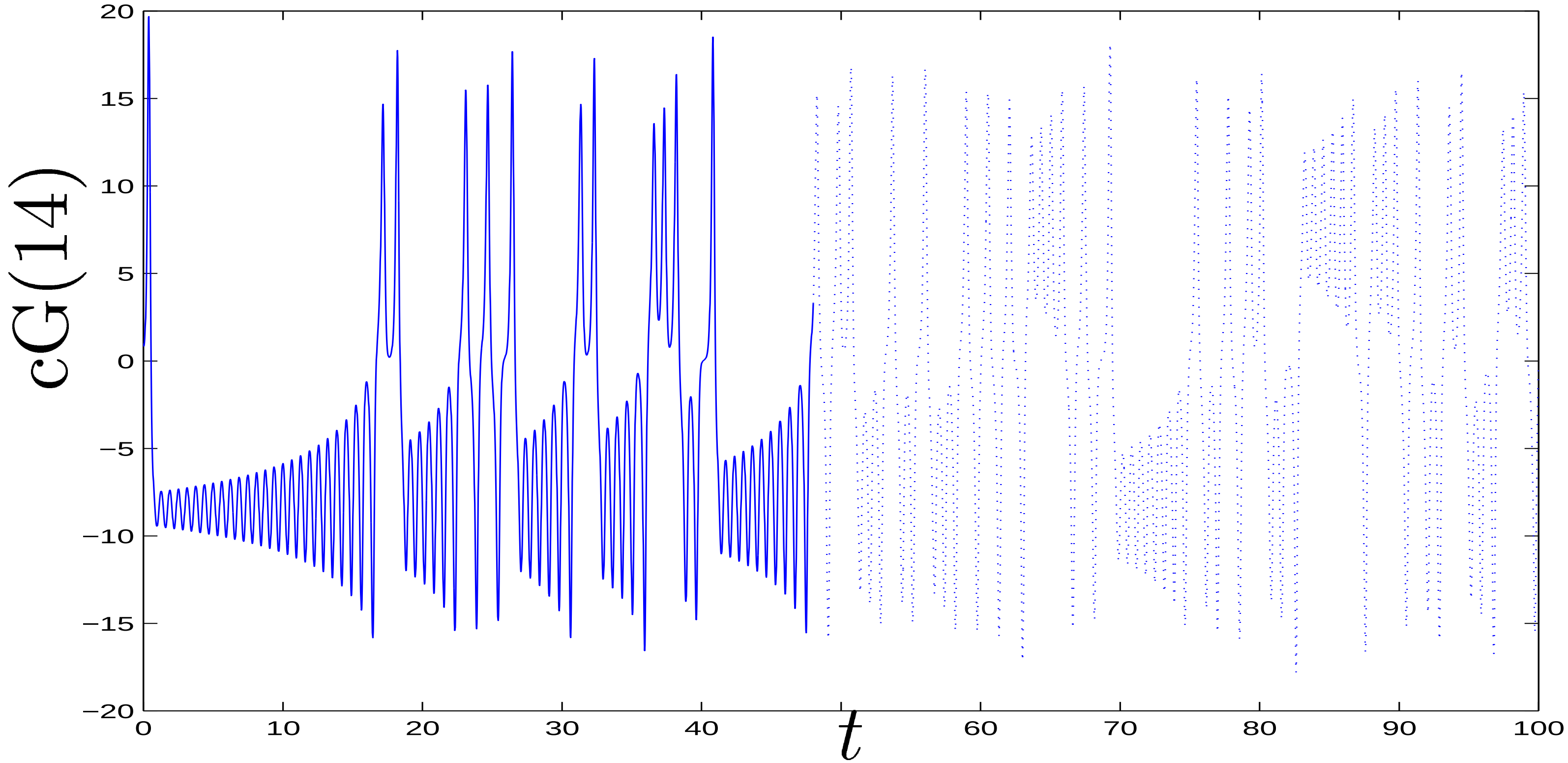
# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



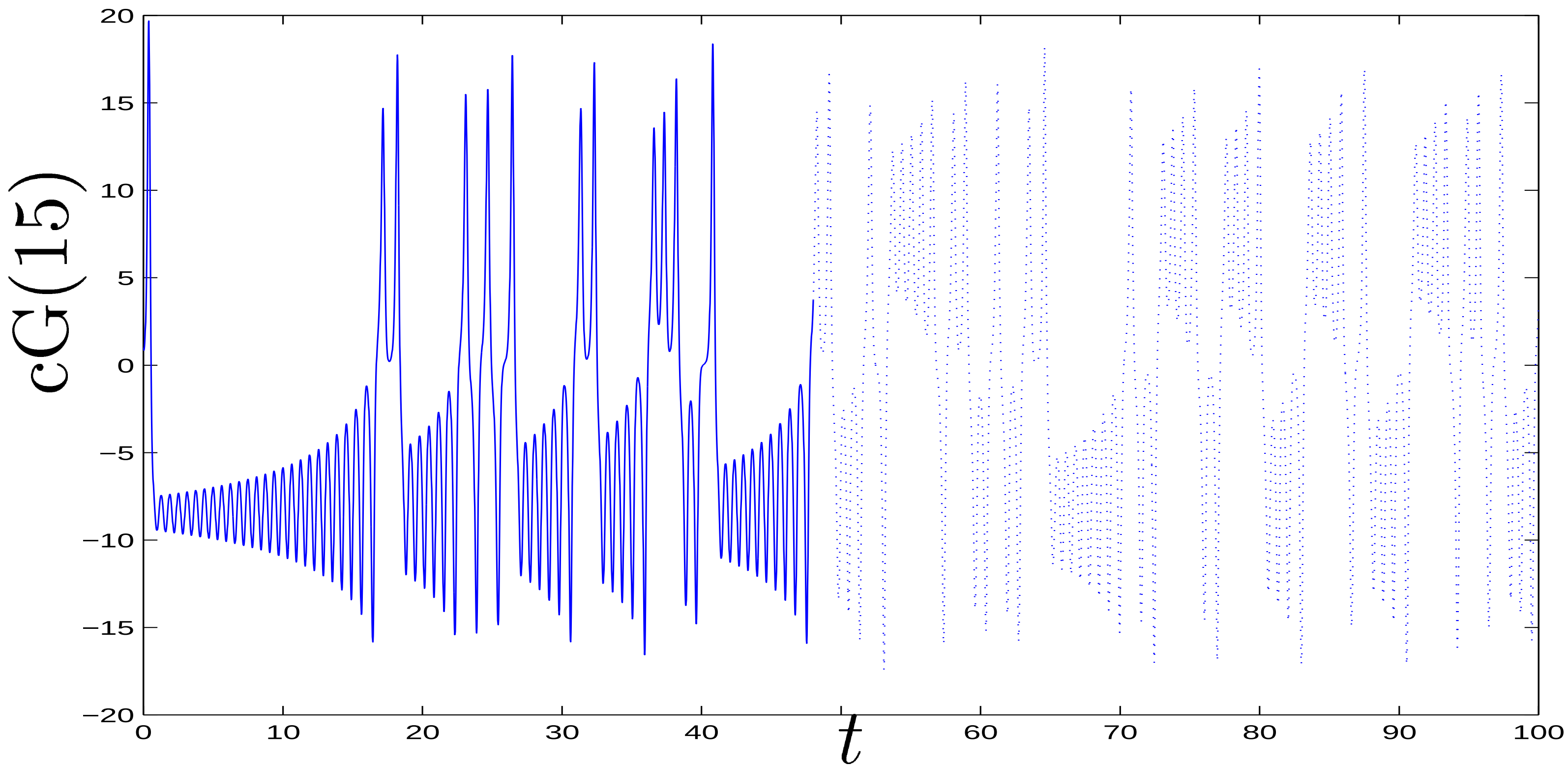
# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



# A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



# The Stability Factor

- We do not come further than 50 even if we take larger time steps
- The round-off error is at least $10^{-16}$ at every time step
- The stablity factor reaches $10^{16}$ at $\sim 50$

# The Stability Factor