Adaptive Finite Element methods for incompressible fluid flow

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Chalmers Finite Element Center

3 lectures

1. Adaptive FE methods for incompr. fluid flow
2. Hydrodynamic stability
3. Subgrid modeling & multi adaptivity

Lecture 2: Hydrodynamic stability

Error propagation:

- A posteriori error estimates using duality
- Upper bound on solutions to the linearized dual NSE for computability

Pertubation growth:

- Bifurcation / Transition to turbulence
- Lower bound on pertubations for bifurcation / transition to take place

Computability

What quantities are computable? To what tolerance? To what cost?

That is: What quantities can be computed with a sufficiently small computational error?

Computability

Computational error $\leq SC\|R(U)\|$

Interpolation constant C, depends only on what finite elements used

The stability factor S depends on the solution of the linearized dual problem related to the problem and the quantity we are trying to approximate

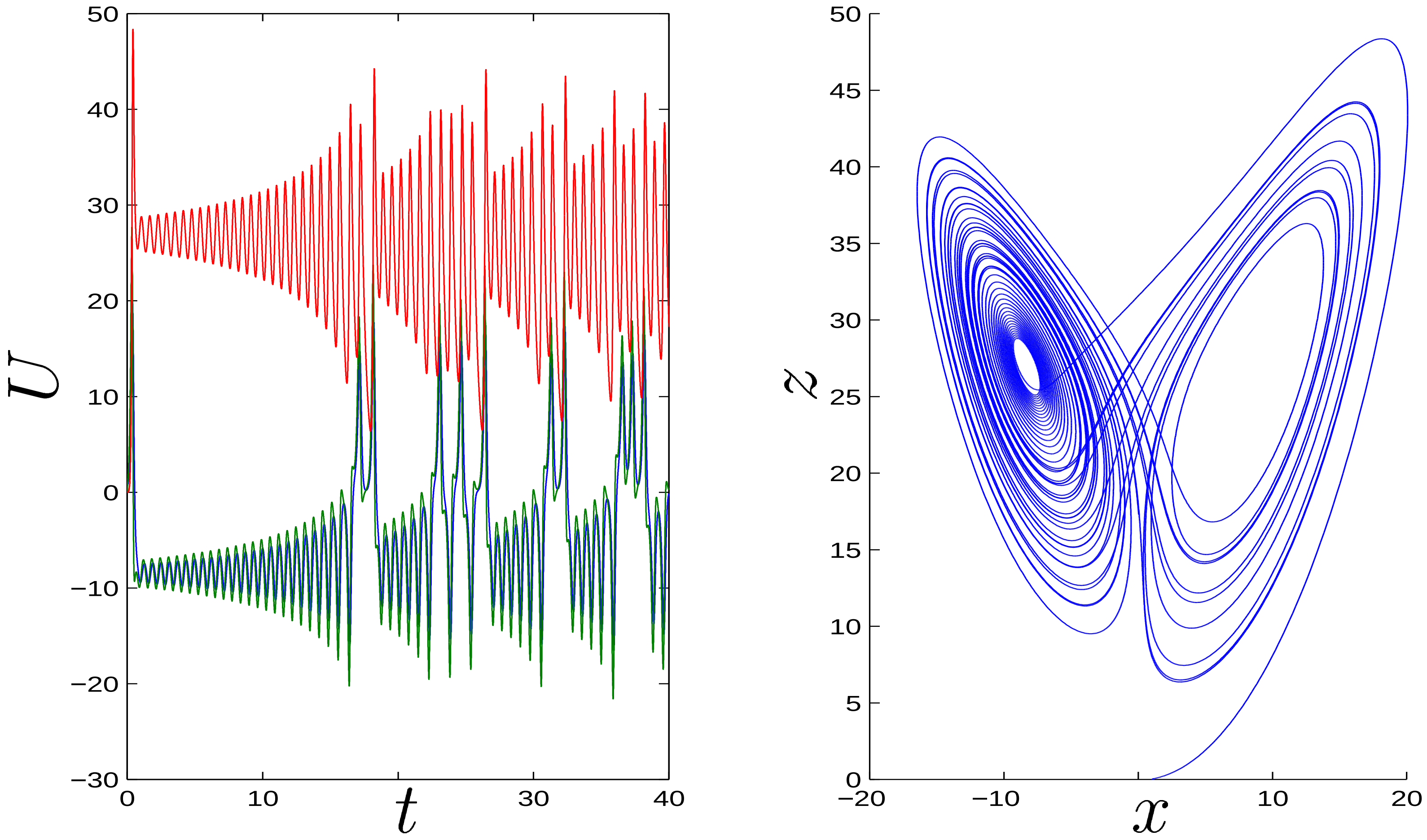
The Lorenz System (A.Logg)

$$\begin{cases} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz, \end{cases}$$

where we, as usual, take $(x_0, y_0, z_0) = (1, 0, 0)$, $\sigma = 10$, $b = 8/3$ and $r = 28$.

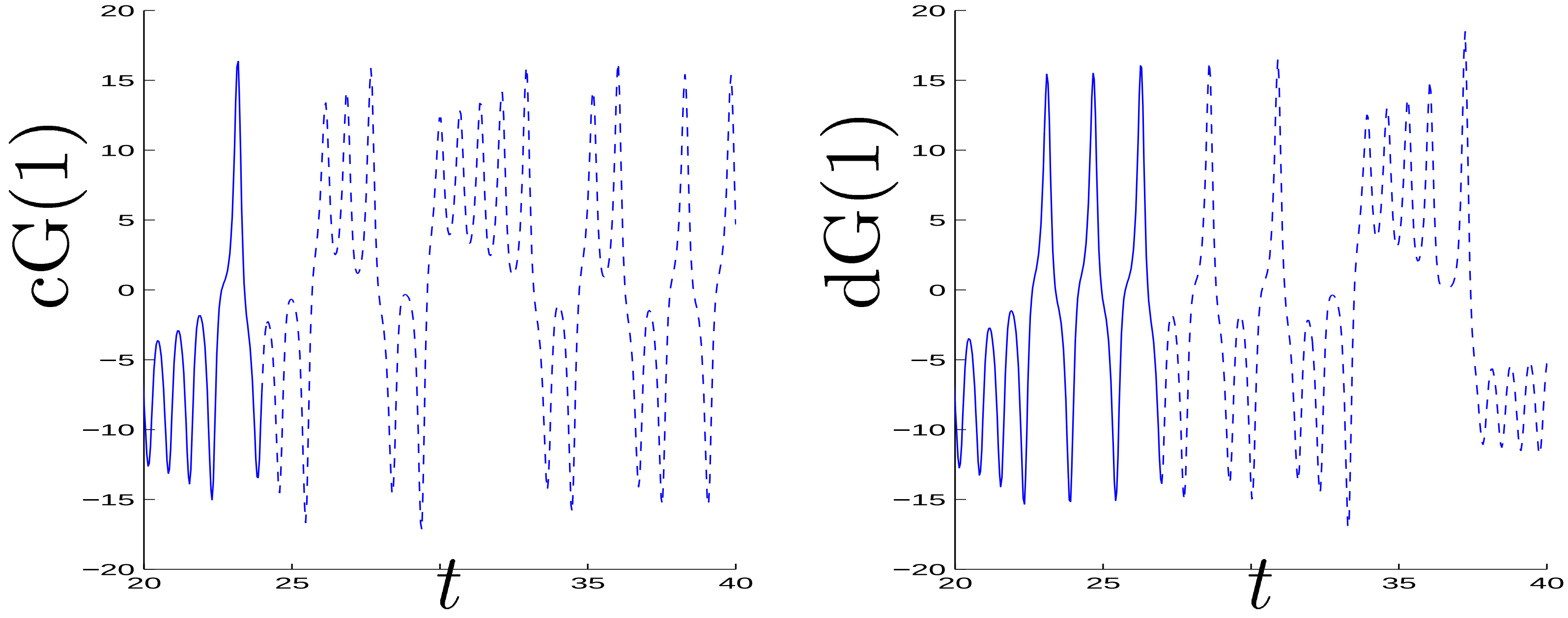
How far can we reach? (How long can we compute with small computational error?)

The Solution?



A Simple Experiment

- $T = 40$
- $k = 0.001$



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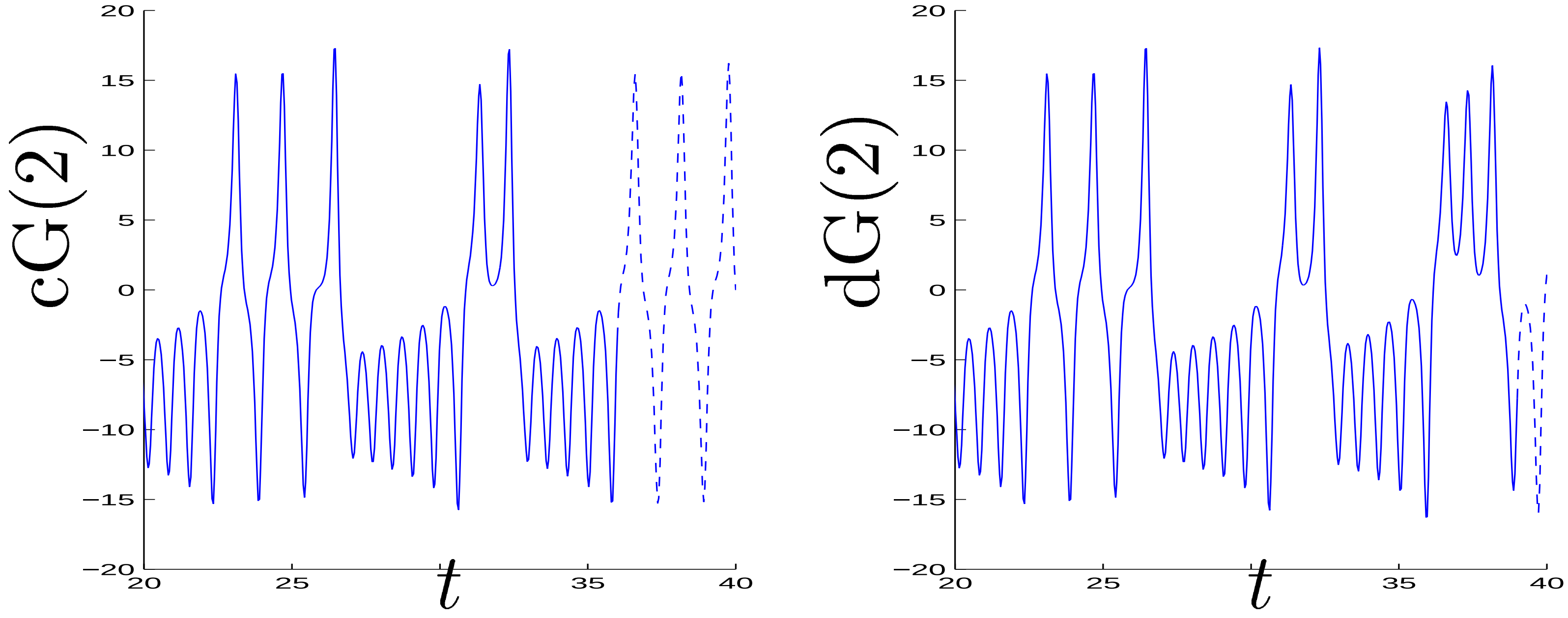

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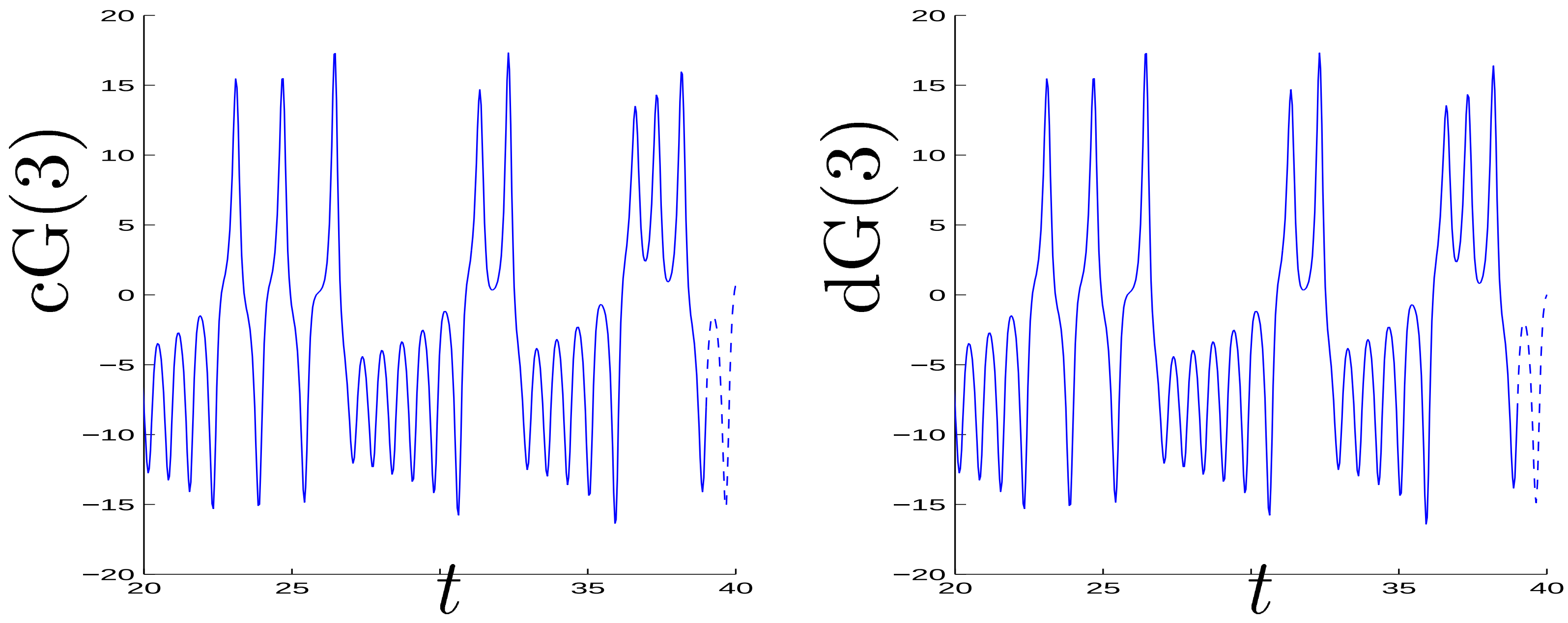

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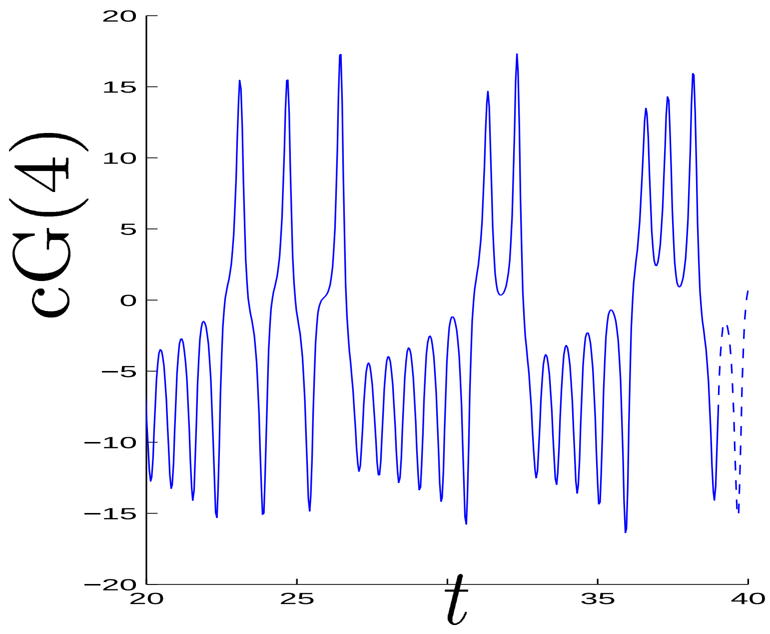

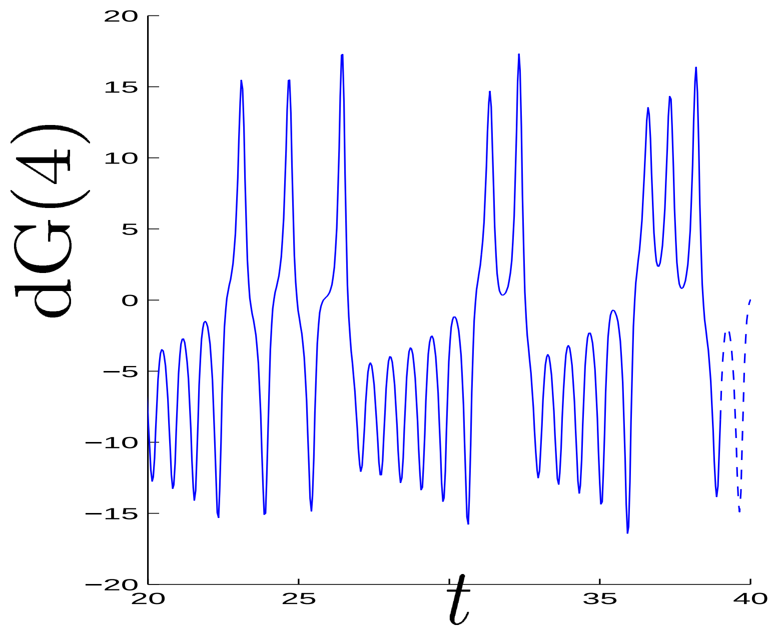

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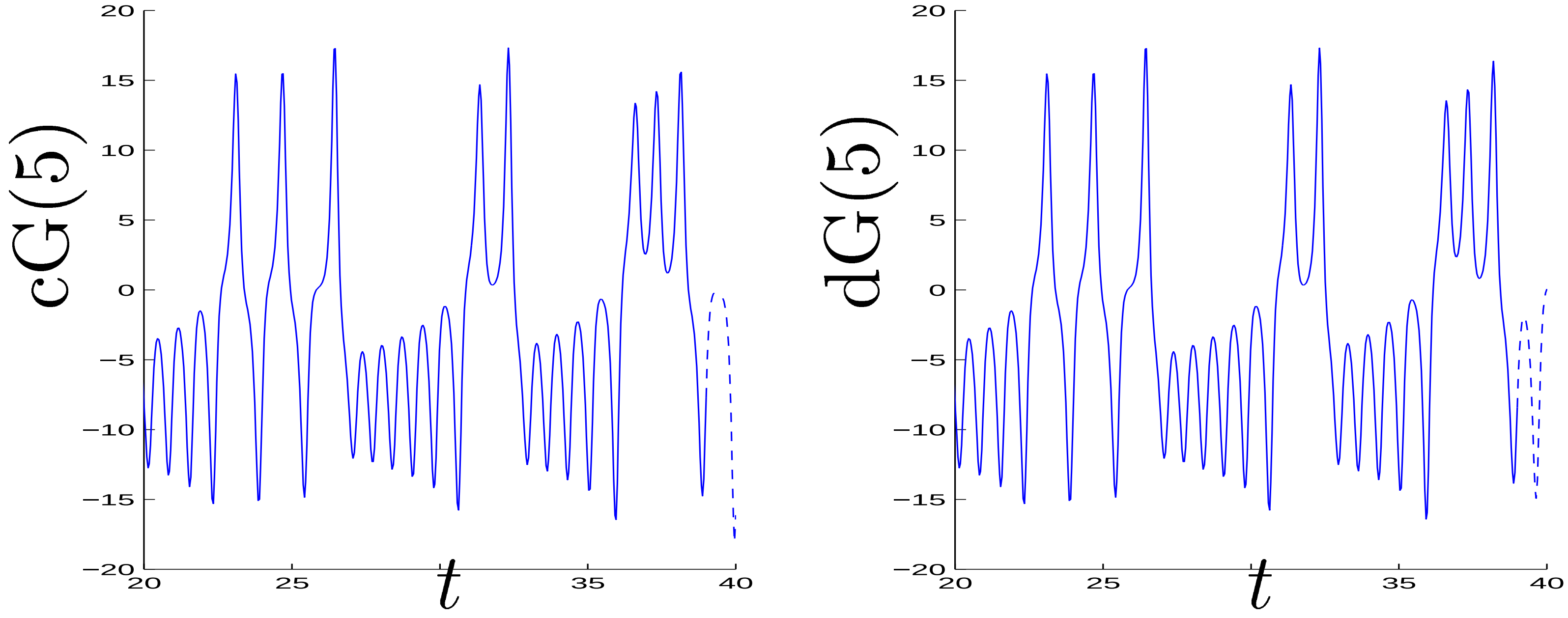

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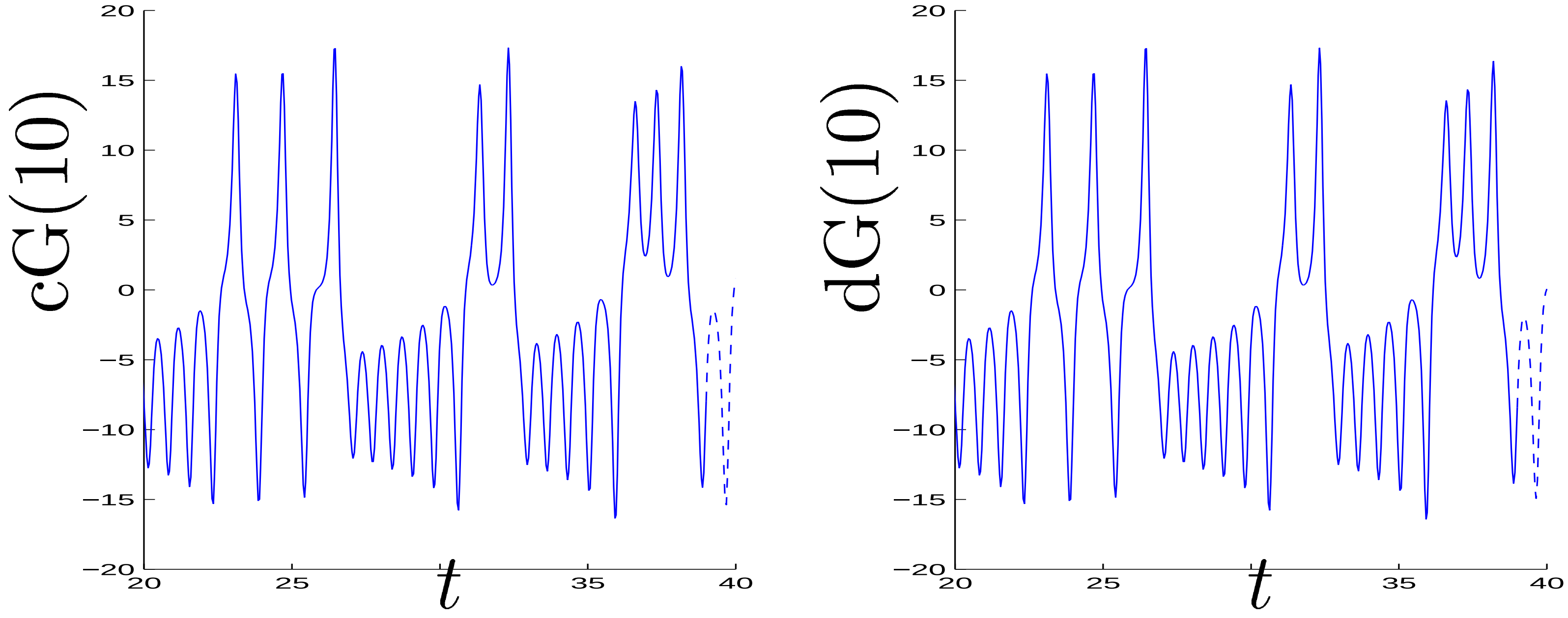
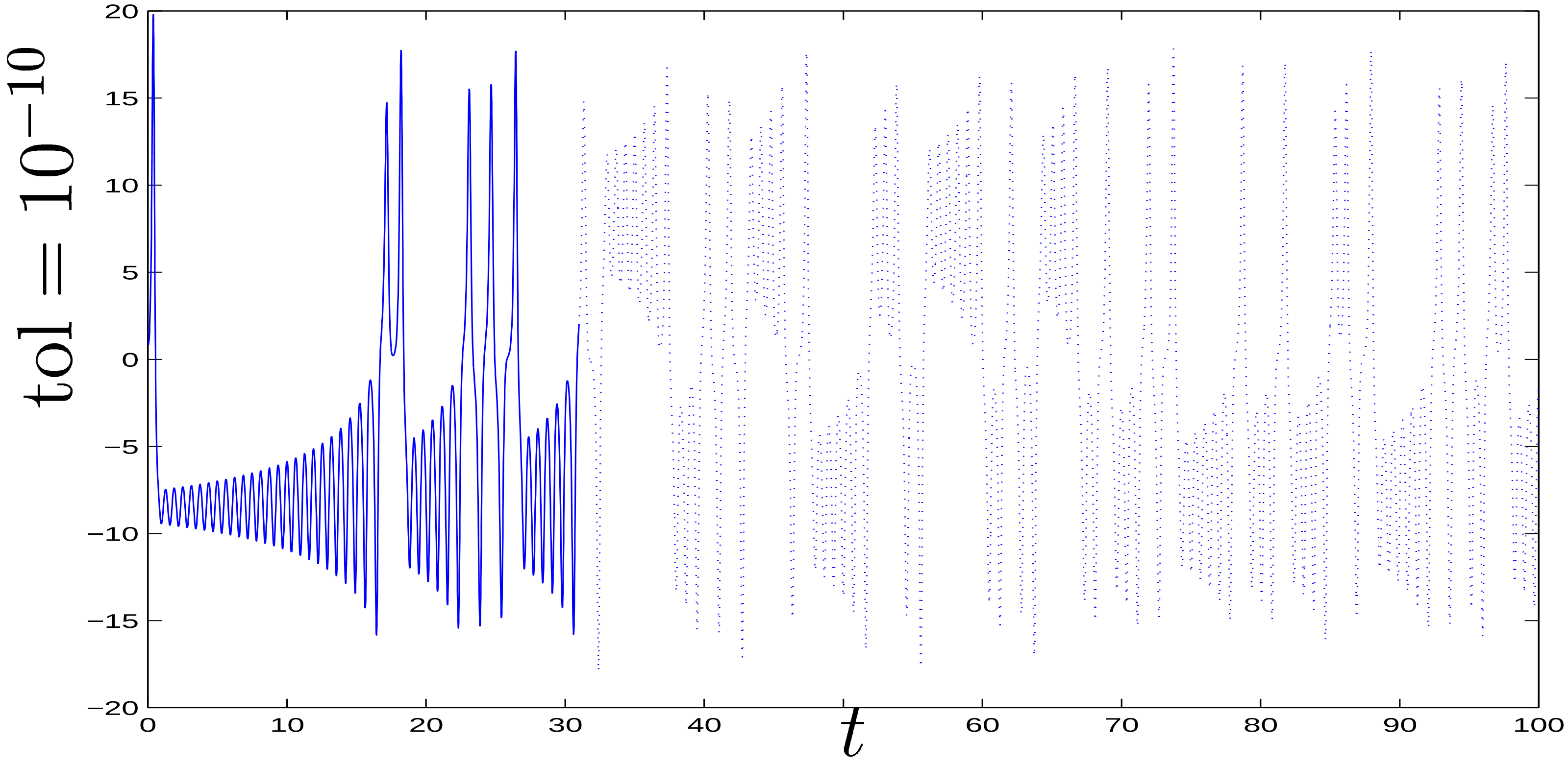


ode45

Trying the same thing with Matlabs ode45:



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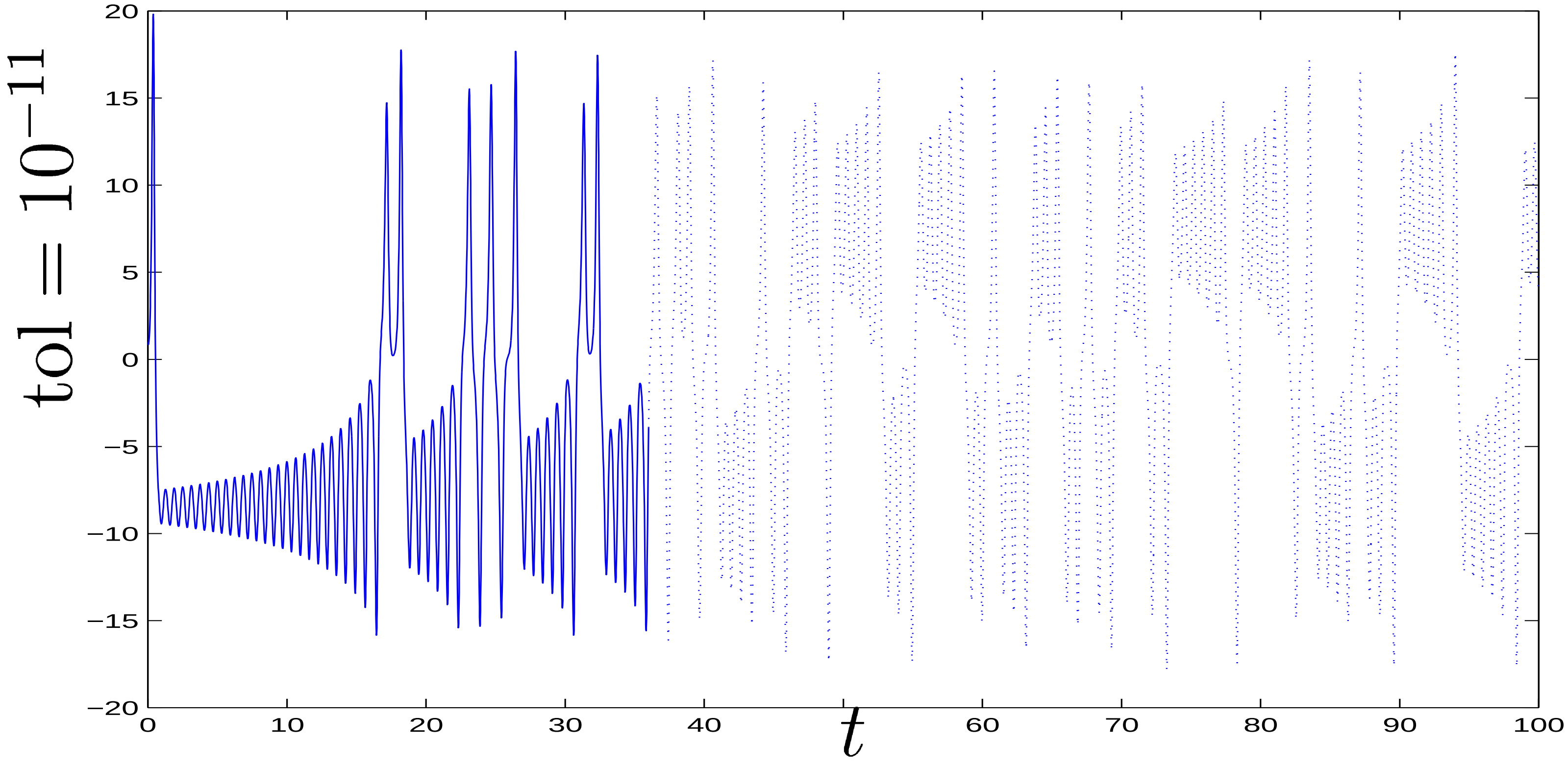

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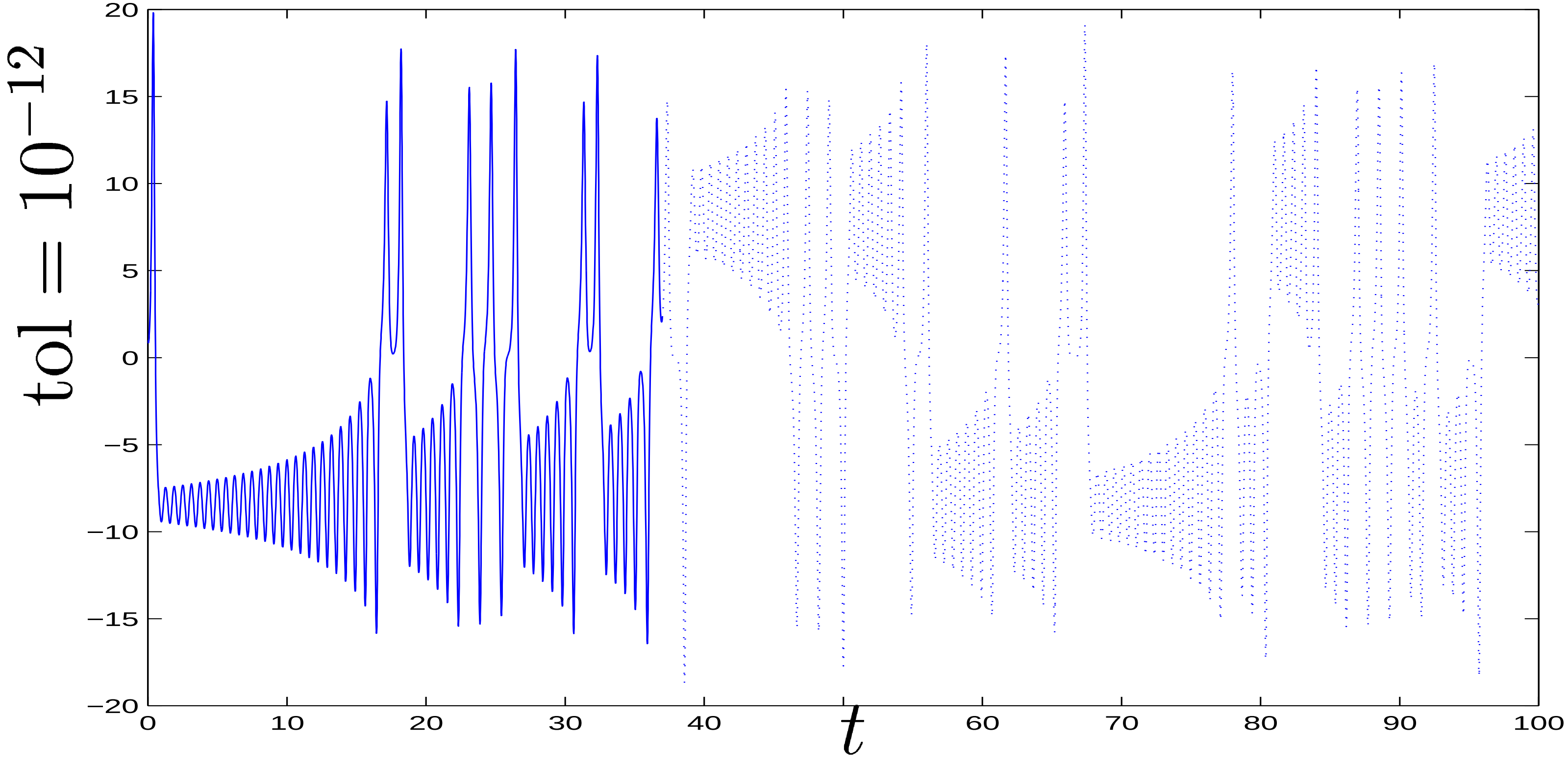

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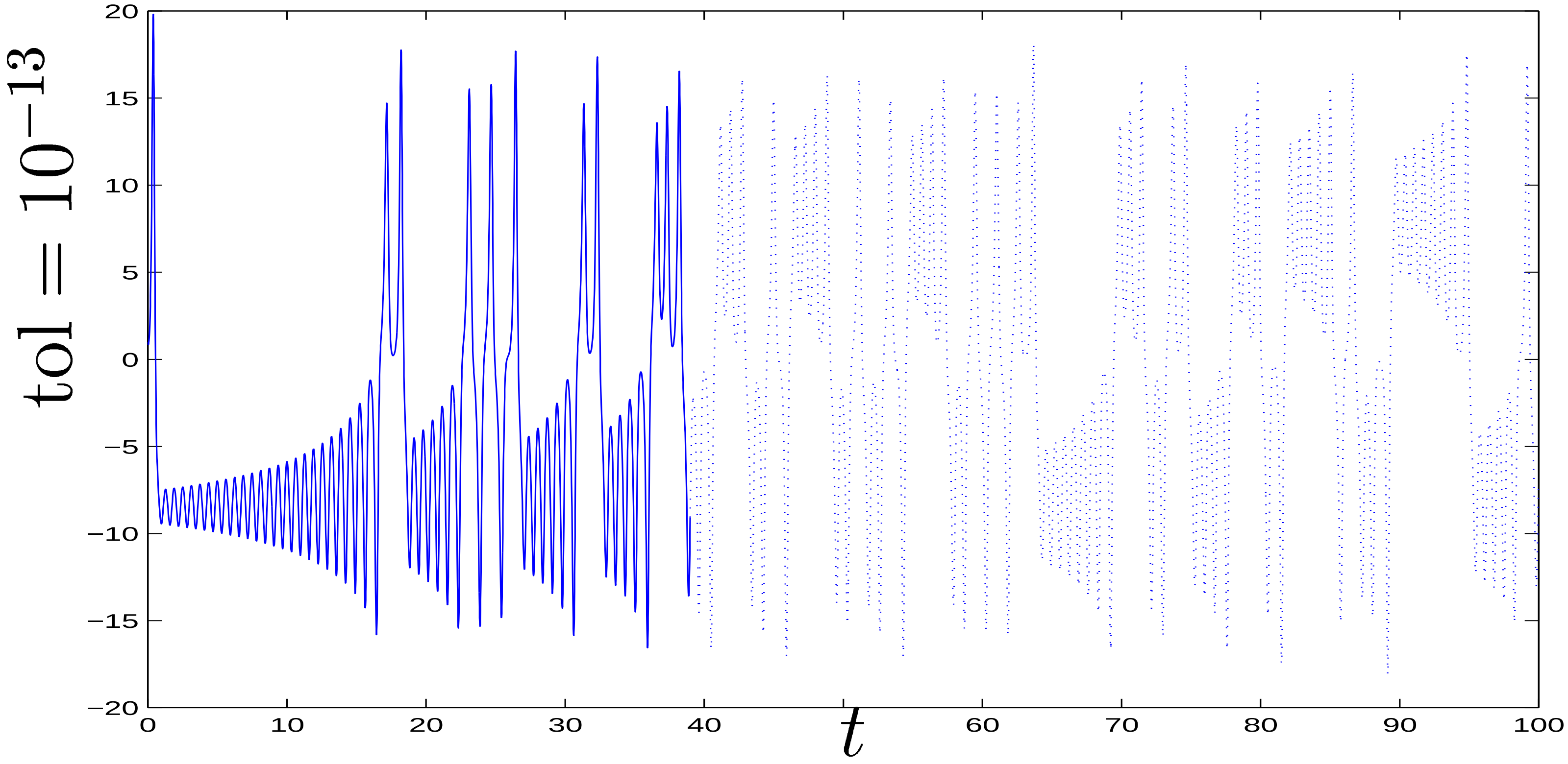

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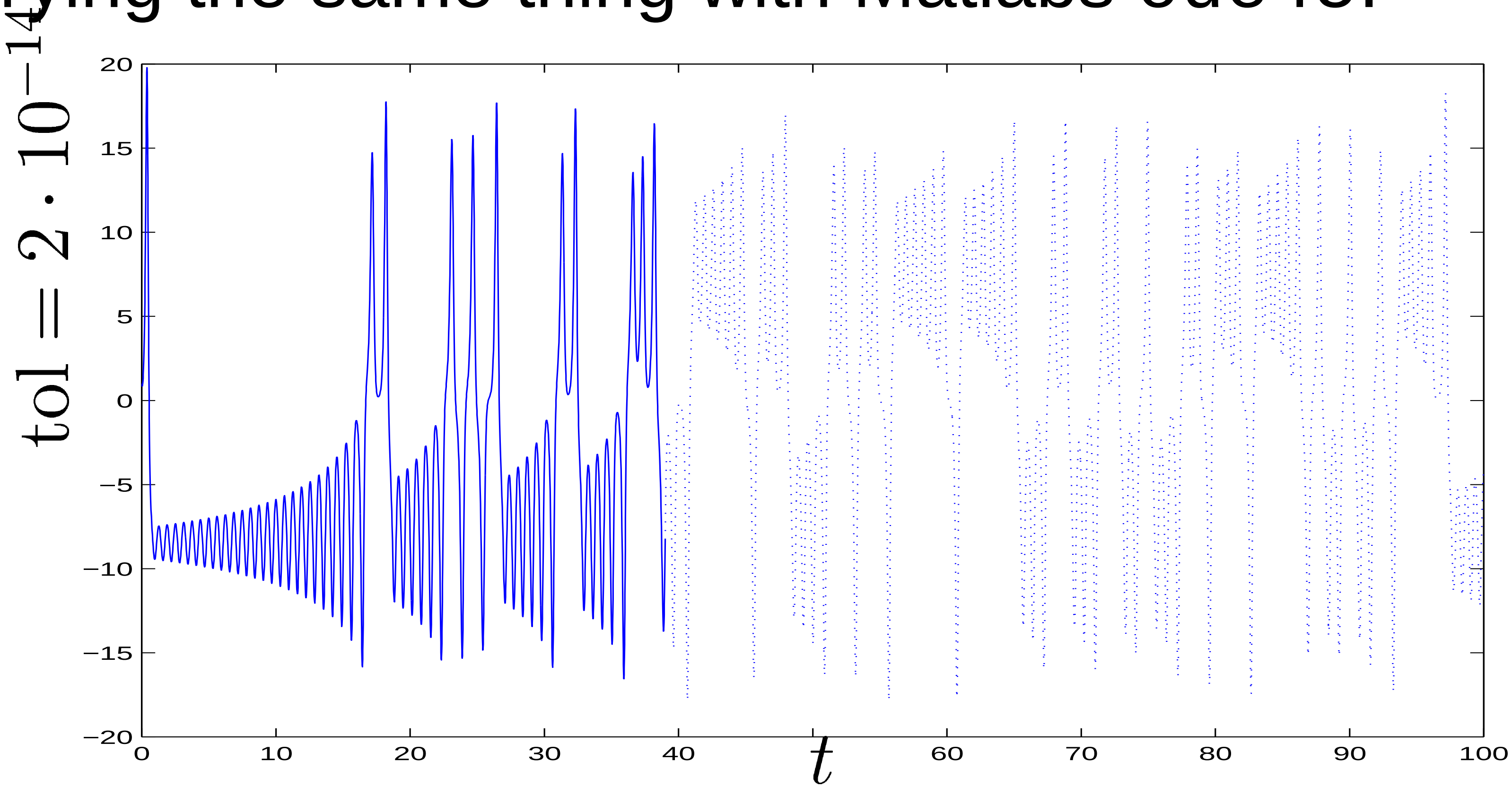


Getting Further

Computational error = $SC\|R(U)\|$

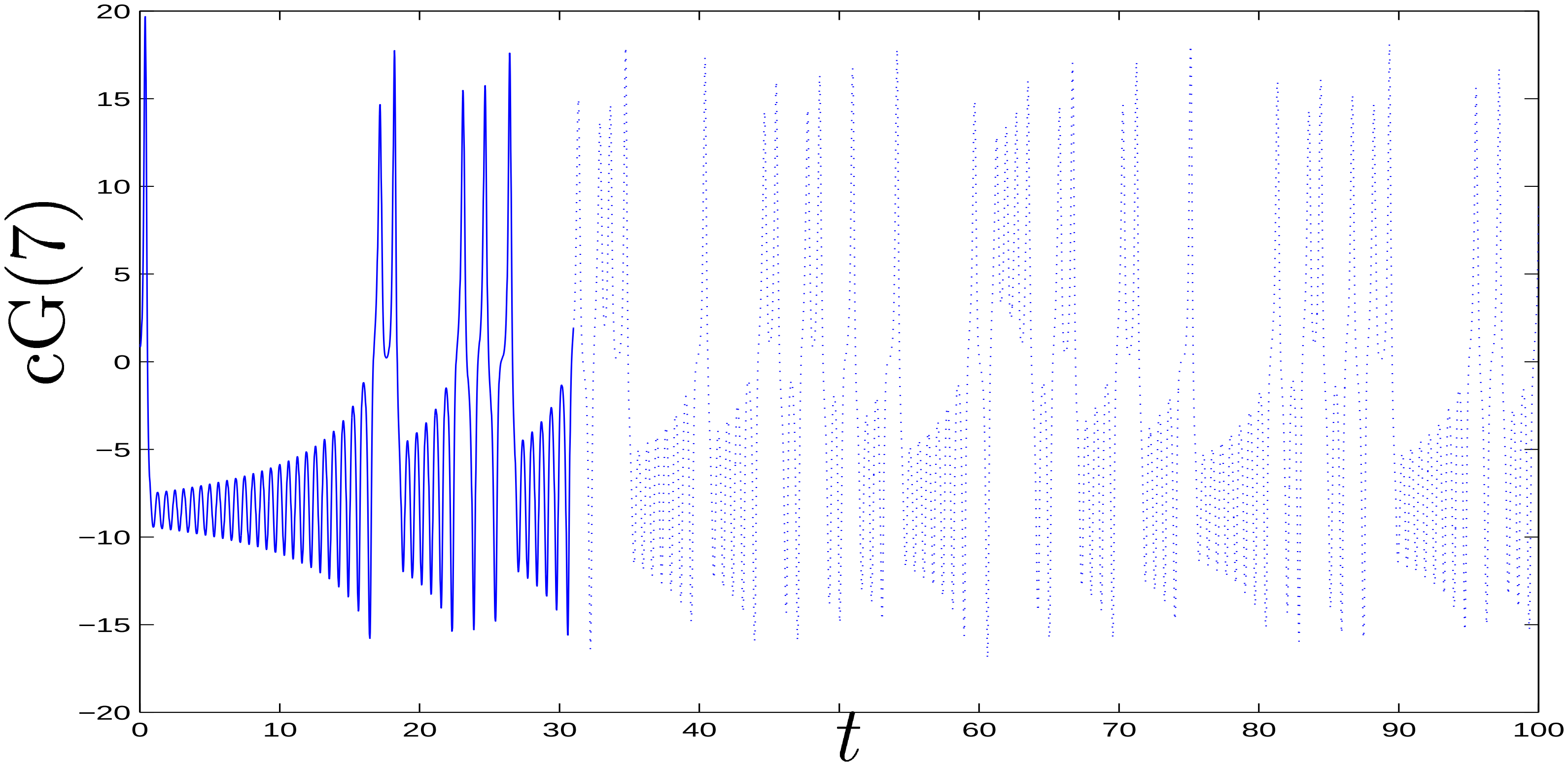
- Small residual
- Large stability factor

The round-off error is at least 10^{-16} in each time step using double precision

This means that we have to take fewer time steps, using a very high order method

A Simple Experiment — *continued*

Increasing the time-step to $k = 0.1$:



A Simple Experiment — *continued*

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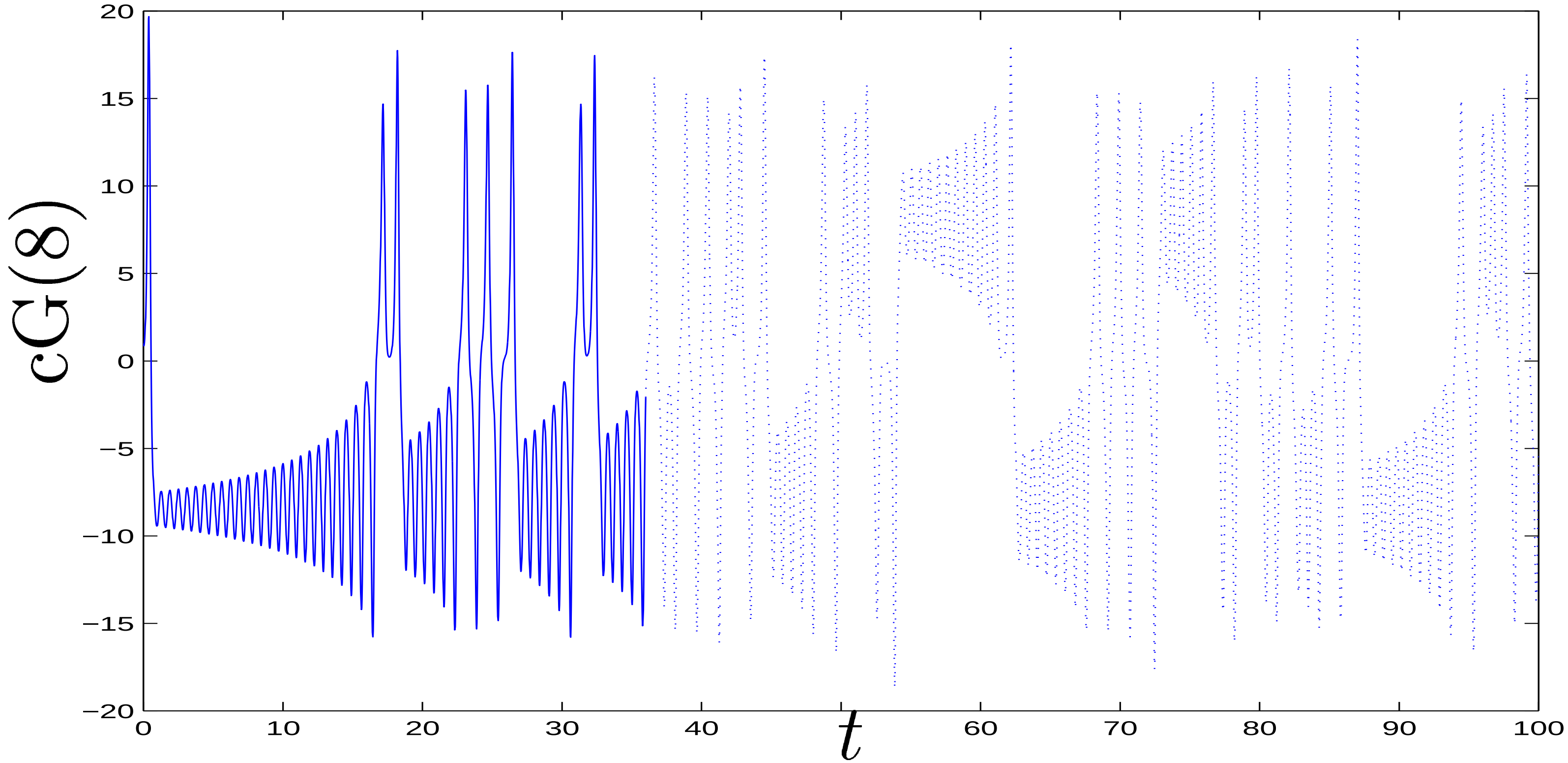

A Simple Experiment — *continued*

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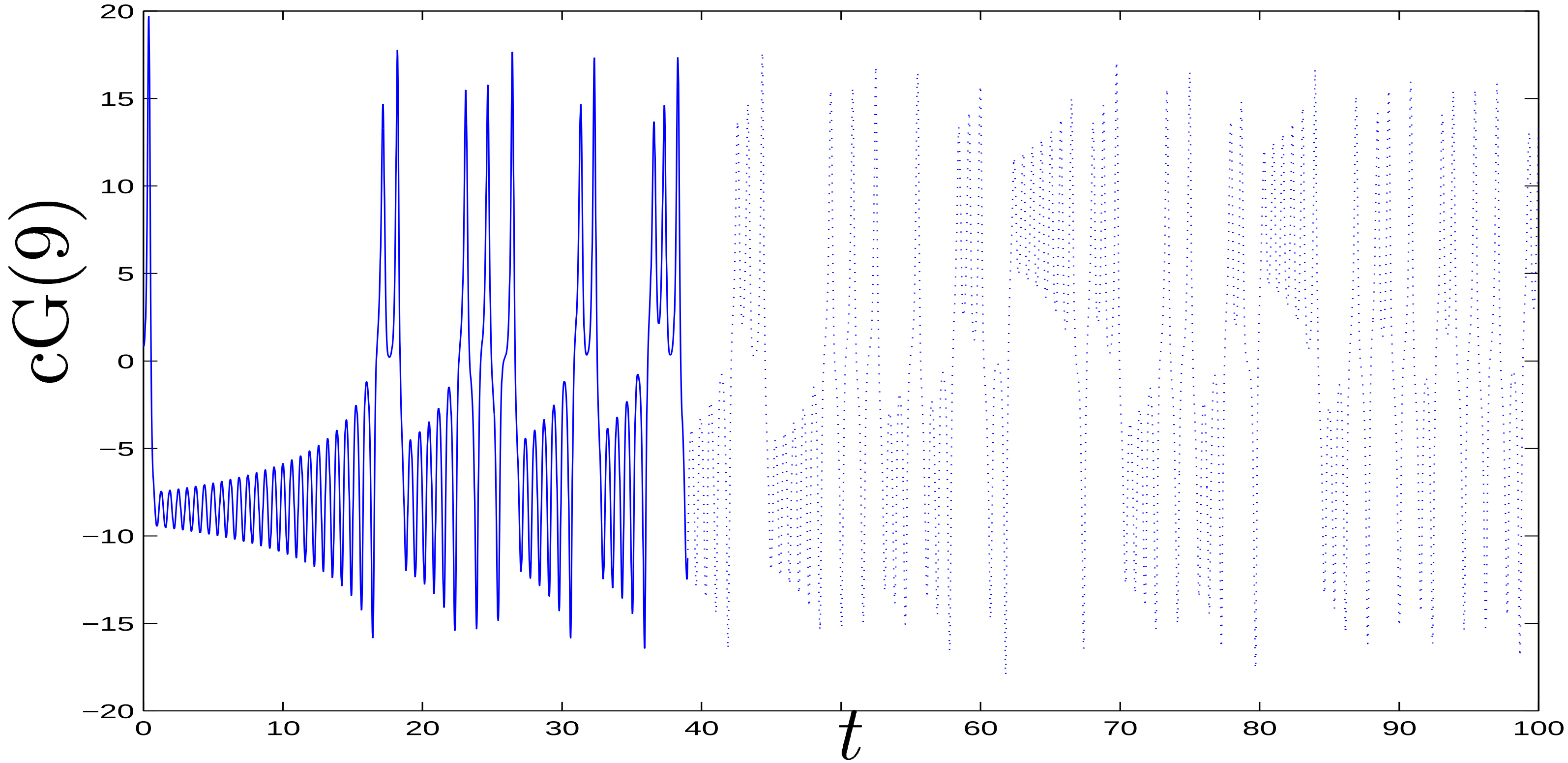

A Simple Experiment — *continued*

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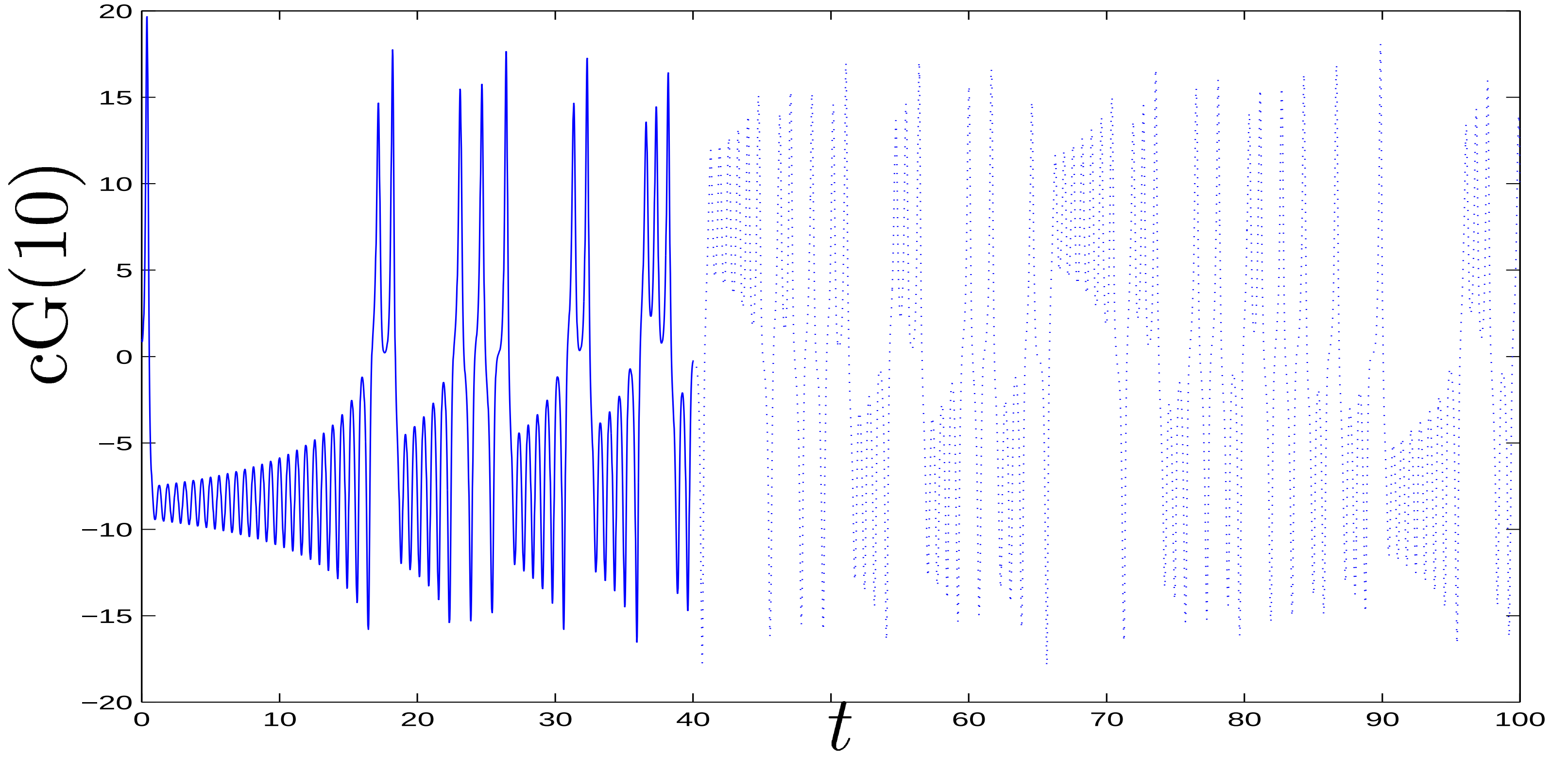

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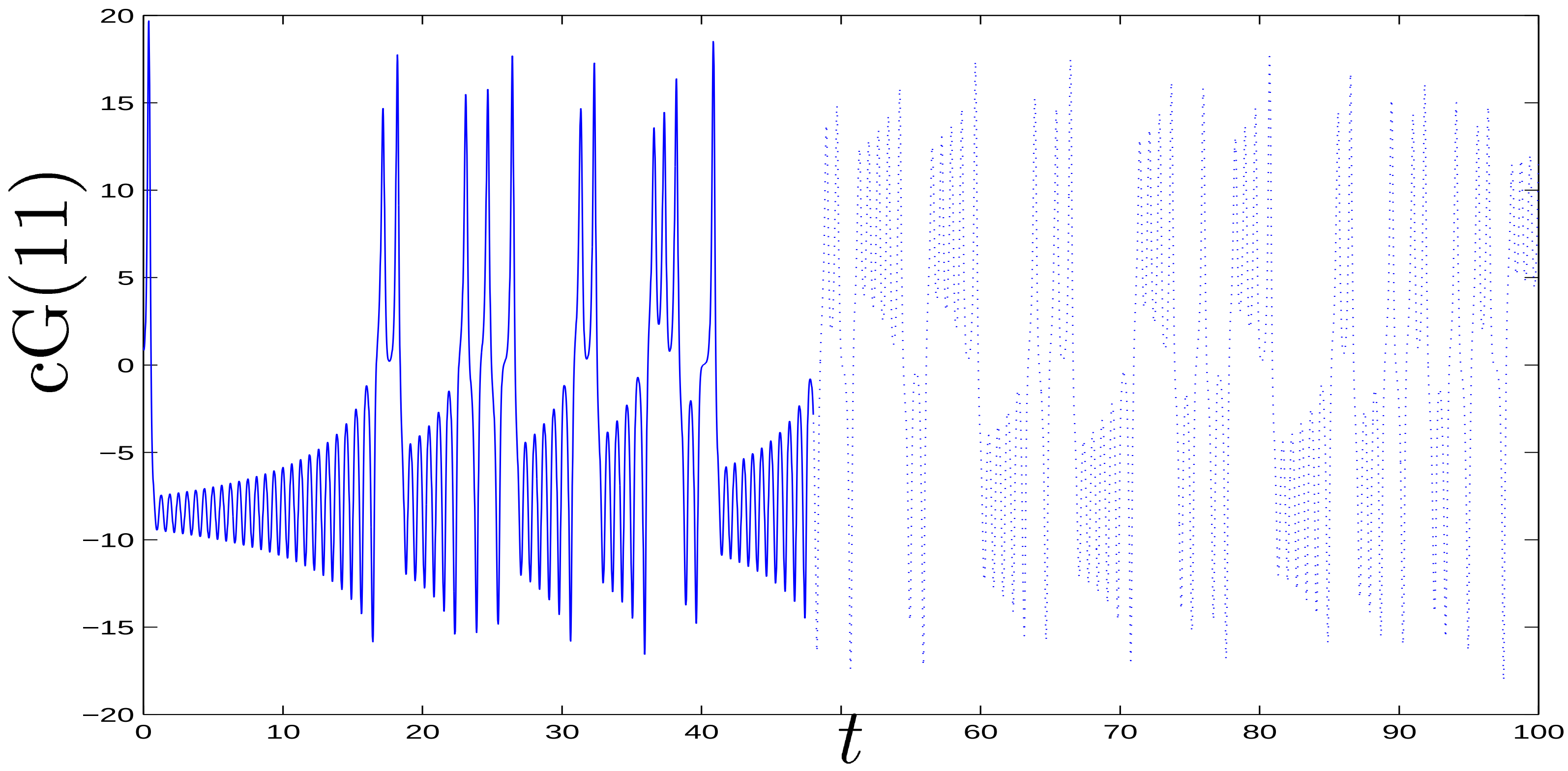

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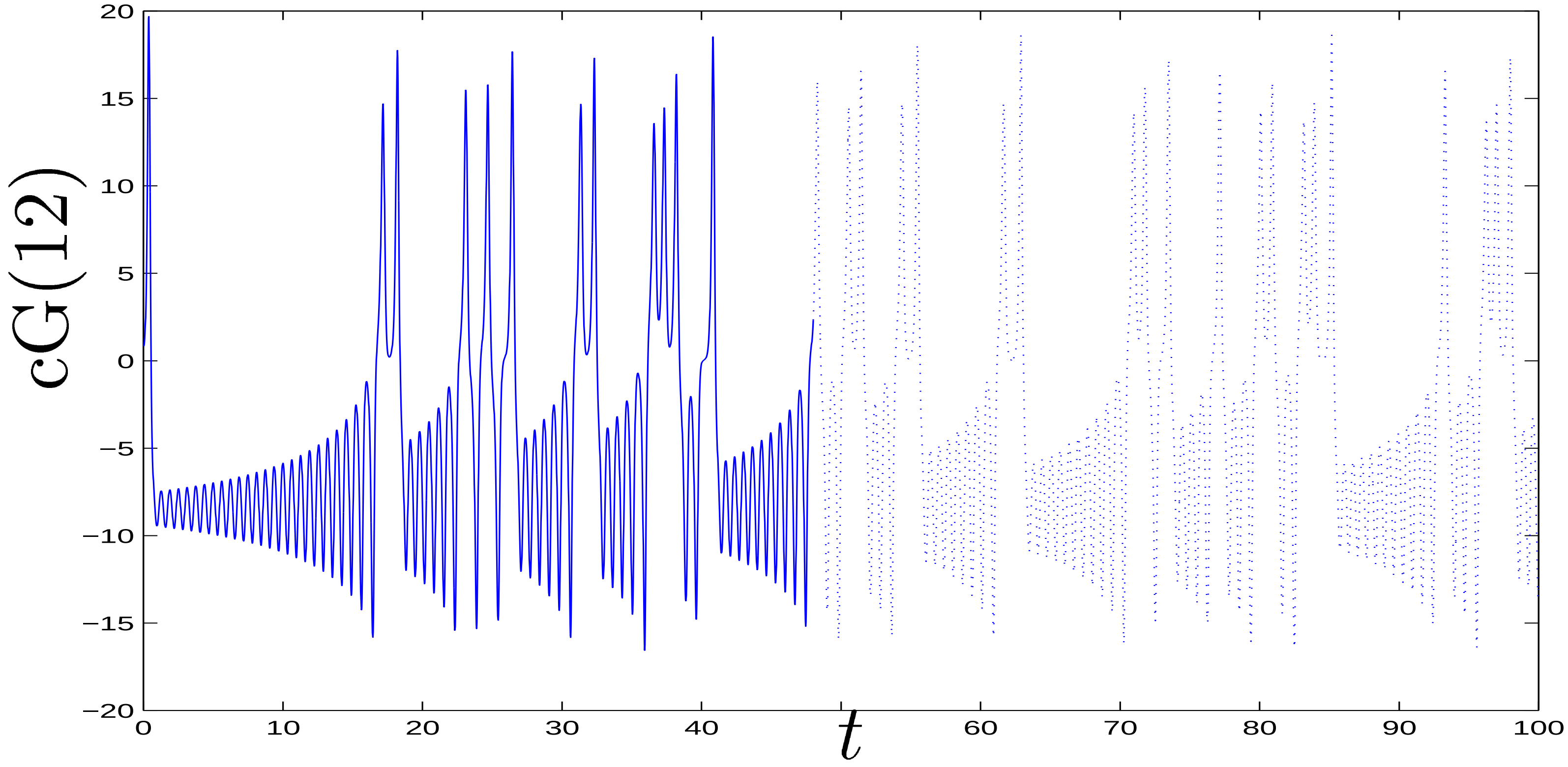

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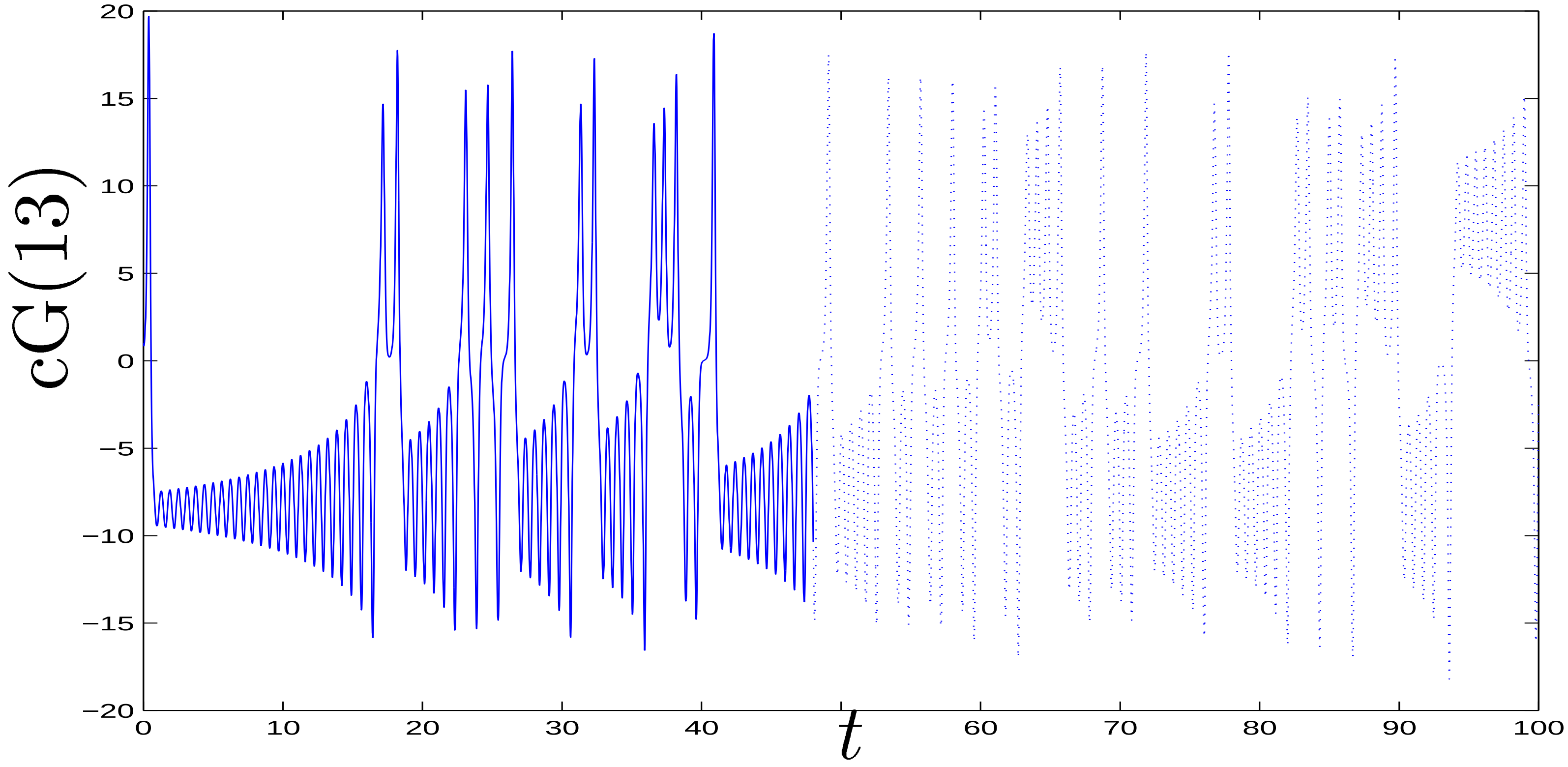

A Simple Experiment — *continued*

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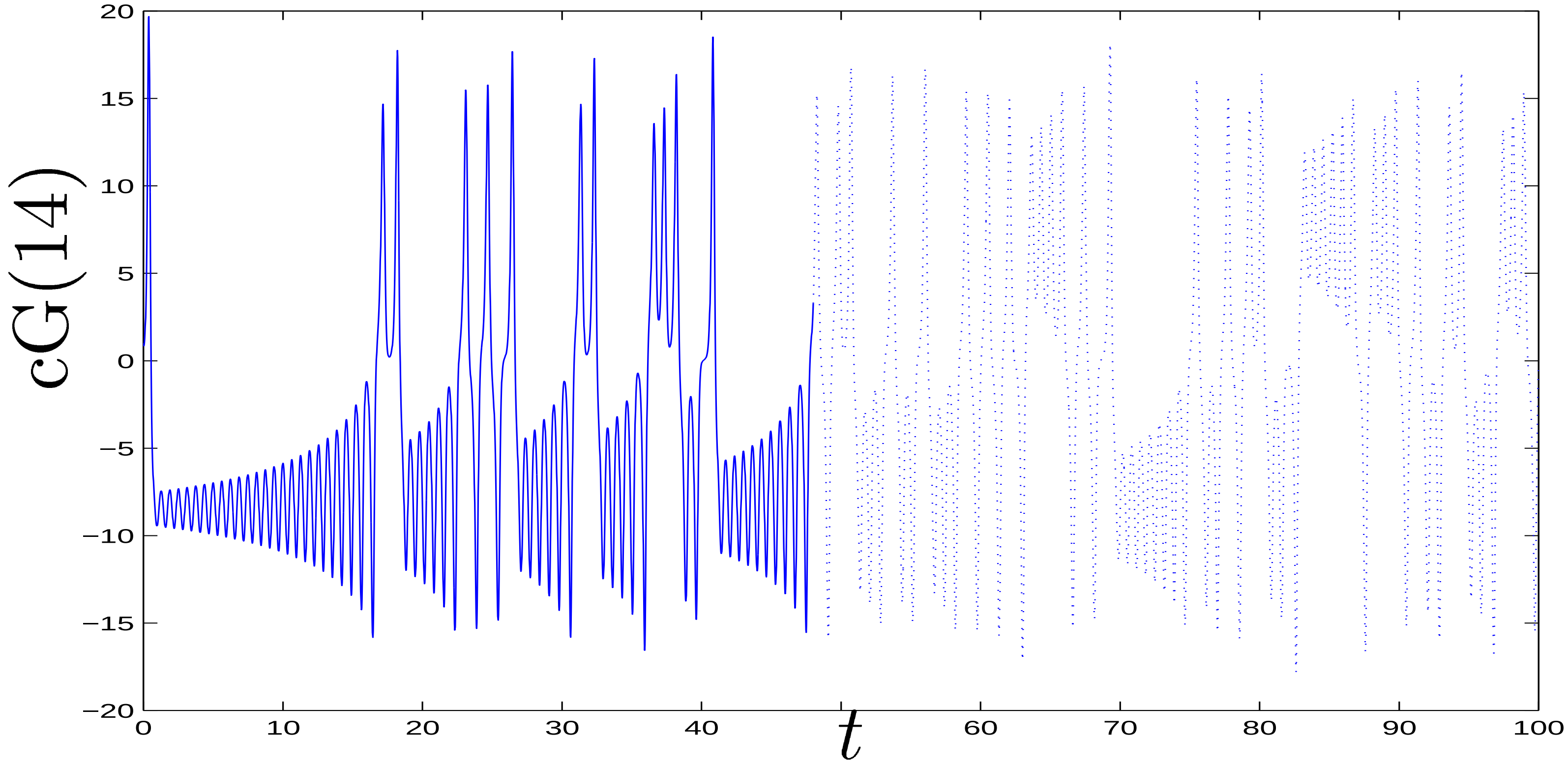

A Simple Experiment — *continued*

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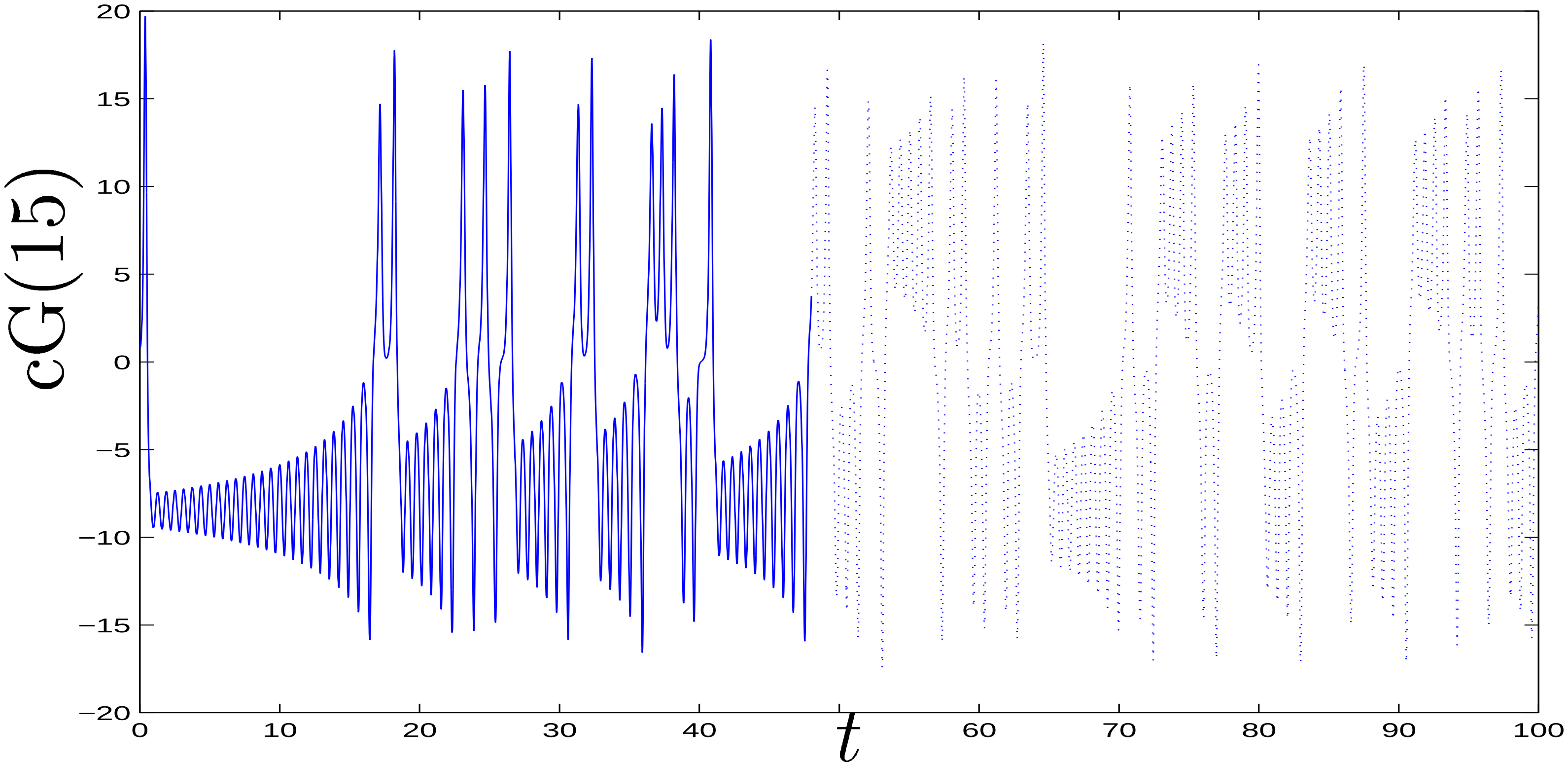
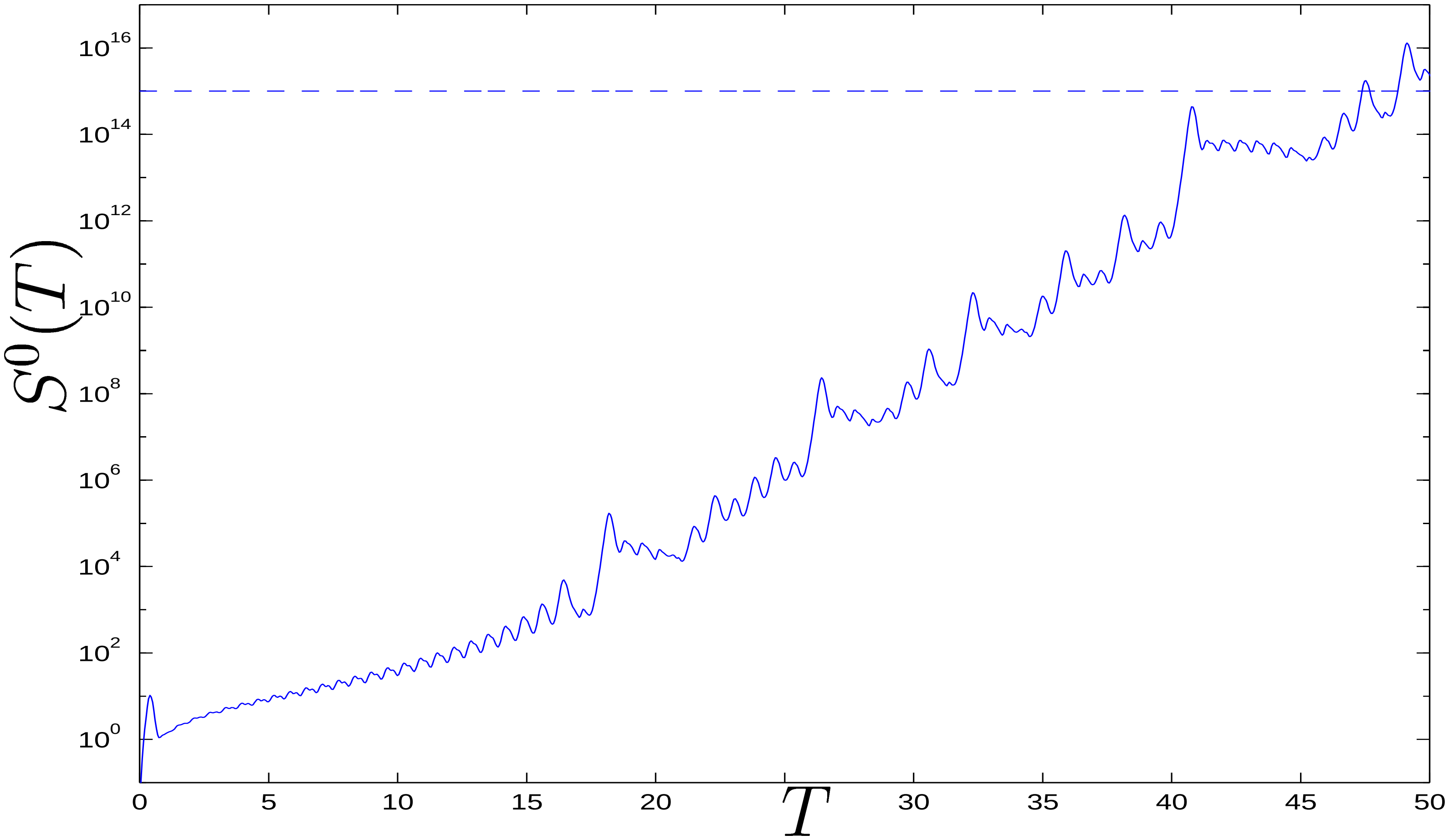


The Stability Factor

- We do not come further than 50 even if we take larger time steps
- The round-off error is at least 10^{-16} at every time step
- The stablity factor reaches 10^{16} at ~ 50

The Stability Factor



Linearized dual Navier-Stokes eqns.

$$\begin{aligned} -\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U \cdot \varphi + \nabla\theta - \epsilon\Delta\varphi &= 0 && \text{in } Q \\ \operatorname{div}\varphi &= 0 && \text{in } Q \\ \varphi &= 0 && \text{on } \Gamma \times I \\ \varphi(\cdot, T) &= \psi && \text{in } \Omega \end{aligned}$$

The interaction between transport $-(u \cdot \nabla)\varphi$, diffusion $-\nu\Delta\varphi$, and reaction $\nabla U \cdot \varphi$ is crucial

Linearized dual Navier-Stokes eqns.

$$
\begin{aligned}
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\varphi &= 0 \qquad \text{on } \Gamma \times I \\
\varphi(\cdot, T) &= \psi \qquad \text{in } \Omega
\end{aligned}
$$

The transport $-(u \cdot \nabla)\varphi$ depends on the base flow u acting as convecting velocity (backwards). An irregular base flow u gives an irreg. dual solution

Linearized dual Navier-Stokes eqns.

$$\begin{aligned} -\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U \cdot \varphi + \nabla\theta - \epsilon\Delta\varphi &= 0 && \text{in } Q \\ \text{div}\,\varphi &= 0 && \text{in } Q \\ \varphi &= 0 && \text{on } \Gamma \times I \\ \varphi(\cdot, T) &= \psi && \text{in } \Omega \end{aligned}$$

The reaction $\nabla U \cdot \varphi$ depends on the regularity of the computed flow through ∇U, causing more reaction for irregular flow (large ∇U)

Linearized dual Navier-Stokes eqns.

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Global data ψ may give cancellation in the reaction $\nabla U \cdot \varphi$, which would make global data to give smaller stability factors than local data

Example: Bluff body

- Channel flow with slip walls, 1×1 rectangular cross section and length 4.
- Cubic body of side length 0.25 centered at $(0.5, 0, 0)$
- $x_1 = 0$ inflow boundary with inflow condition $u = (1, 0, 0)$
- Transparent outflow condition
- cG(1)cG(1), tetrahedral mesh $h = 1/32$

Bluff body

start animation

Bluff body - dual problem

- Take the state at $t = 18$ as initial data at $t = 0$
- Compute mean error in u_1 over $\omega \subset \Omega$ at $t = 2$
- Final data $\psi = (\chi_\omega / |\omega|, 0, 0)$
- We compute for ω being a square behind the body with side length $d(\omega)$
- cG(1)cG(1), tetrahedral mesh $h = 1/32$
- $\nu = 1/1000$

Stability factors in $L_1(I; L_1(\Omega))$-norm

$$S_{0,1}(T) = \|\varphi\|_I$$

$$S_{1,1}(T) = \|\nabla\varphi\|_I$$

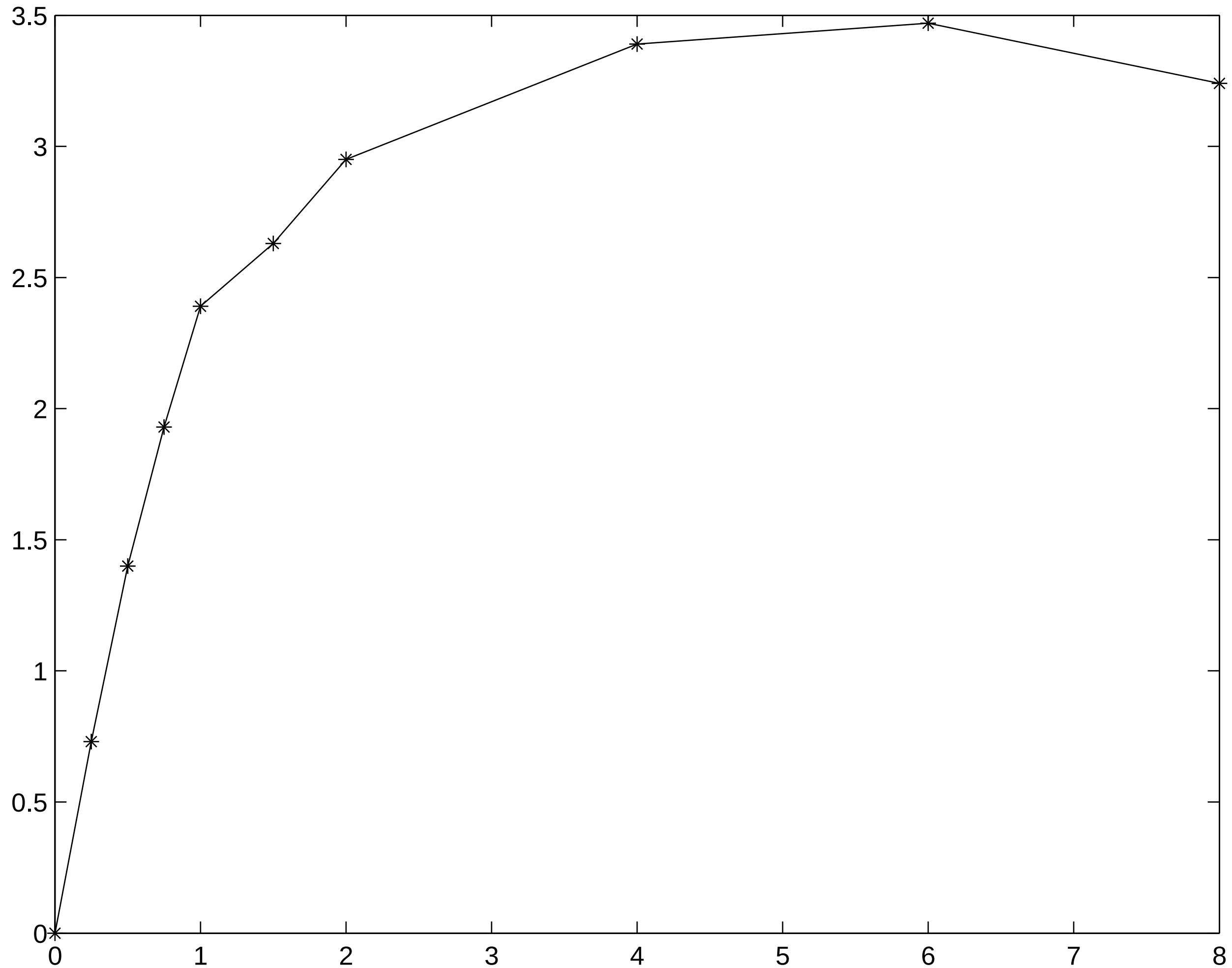
$$S_{1,2}(T) = \|\dot{\varphi}\|_I$$

$$S_{1,3}(T) = \|\nabla\theta\|_I$$

$$S_{2,1}(T) = \|\Delta\varphi\|_I$$

$\|\cdot\|_I$ is the $L_1(I; L1(\Omega))$-norm

$S_{1,1}(\mathbf{t})$



Stability factors in $L_1(I; L_1(\Omega))$-norm

$d(\omega)$	$S_{0,1}$	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{2,1}$
0.0625	11.3	304.2	45.3	18.5	7627.5
0.125	5.8	152.3	28.6	12.0	3593.0
0.25	3.1	66.2	14.6	6.4	1533.5
0.5	1.9	25.4	8.3	3.3	491.5

Stability factors proportional to $1/d(\omega)$, where $d(\omega)$ is the side length of ω.

Stability factors in $L_1(I; L_1(\Omega))$-norm

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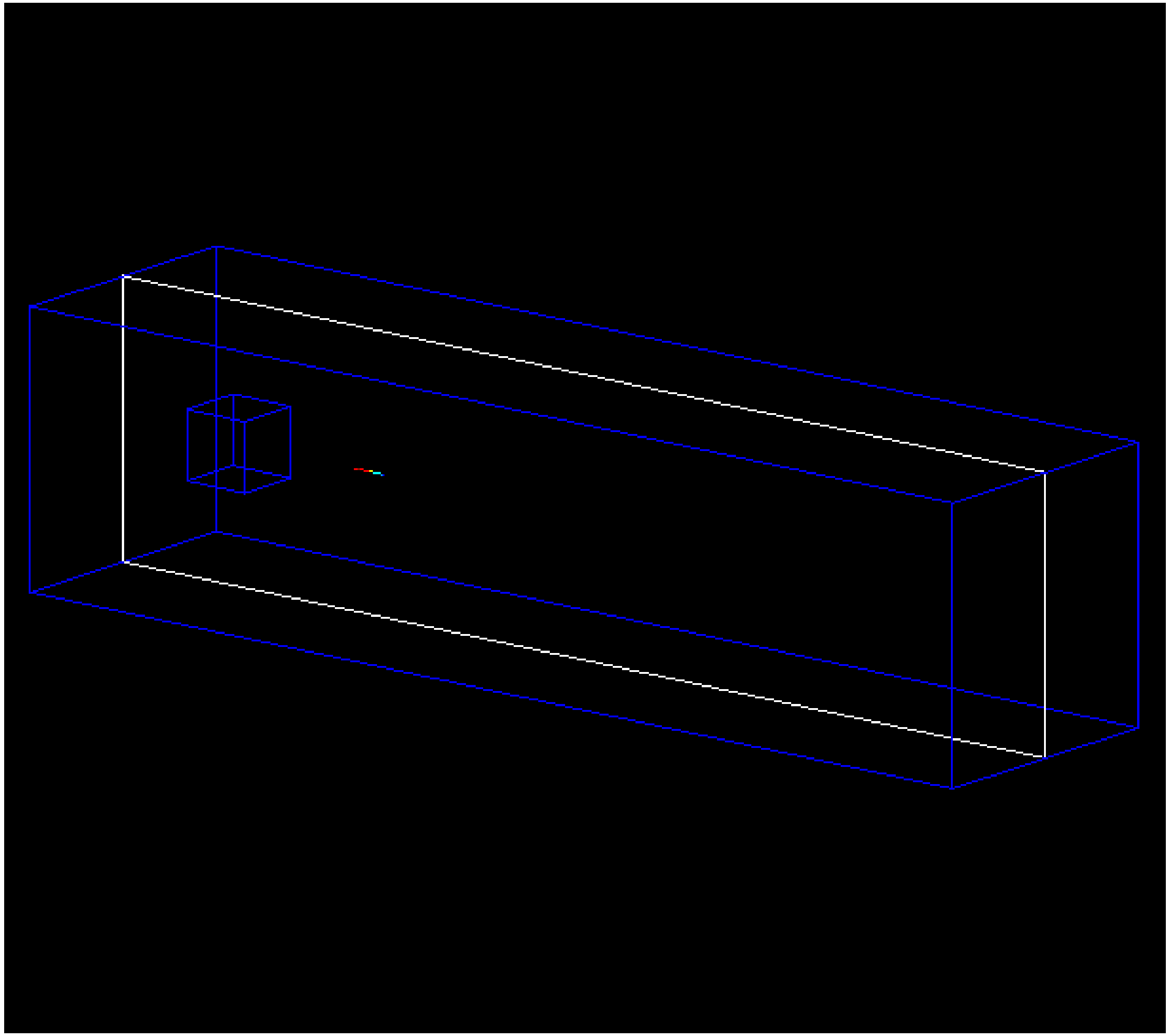
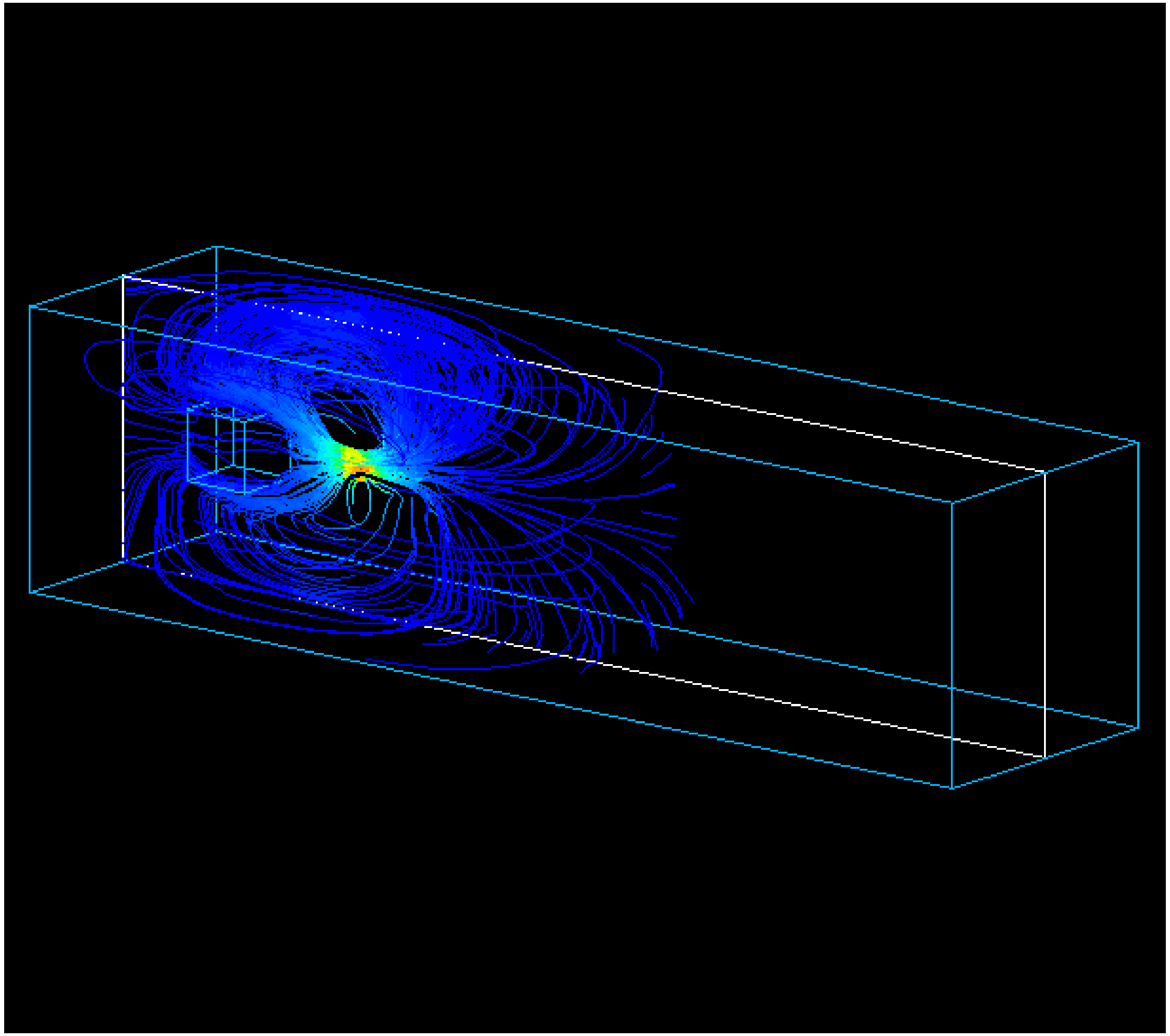
Residuals are of order 1, which makes a small average to appear uncomputable on the current mesh, while a large average may be computable

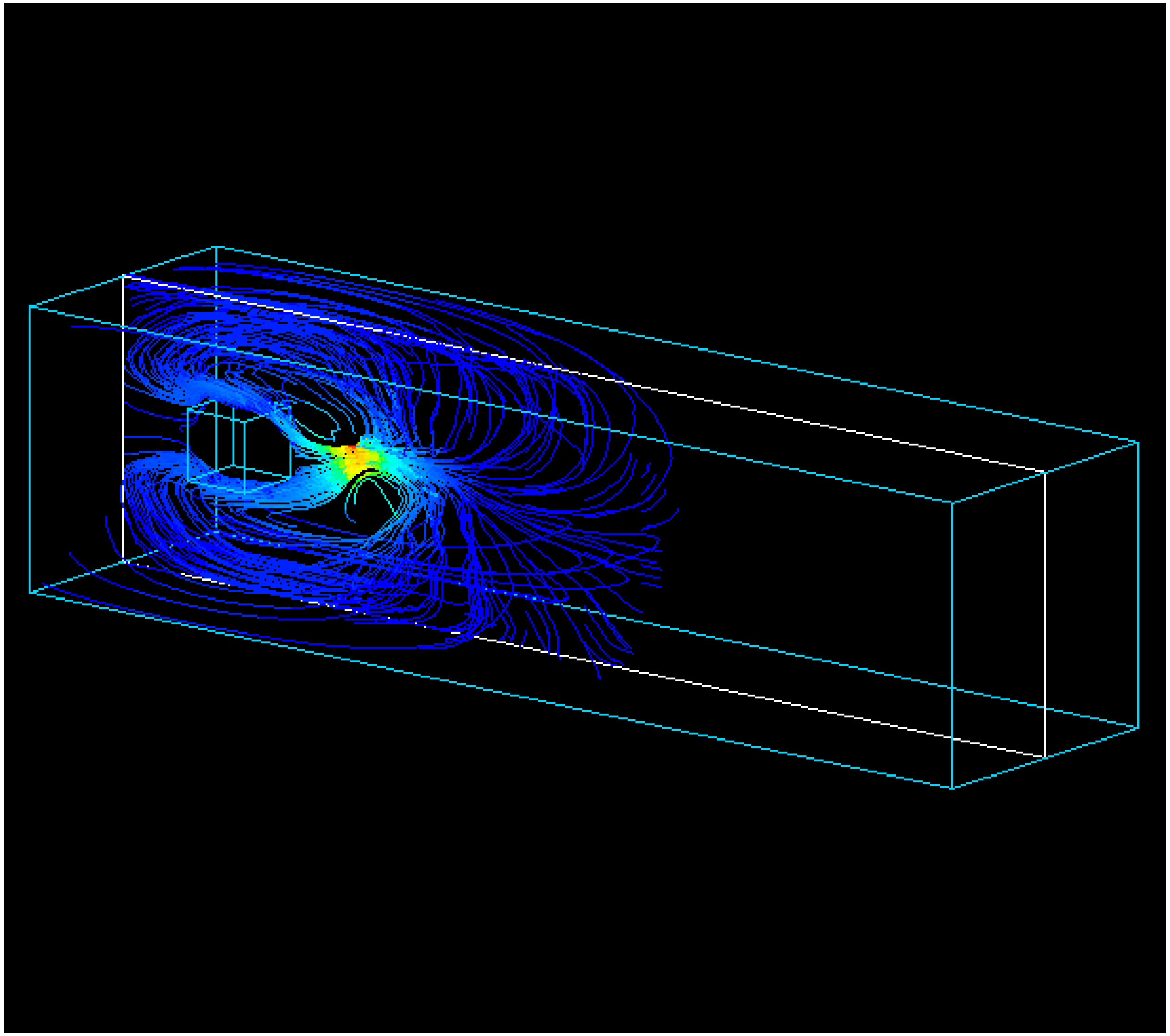
Dual solution, $d(\omega) = 0.0625$, $t = 2$



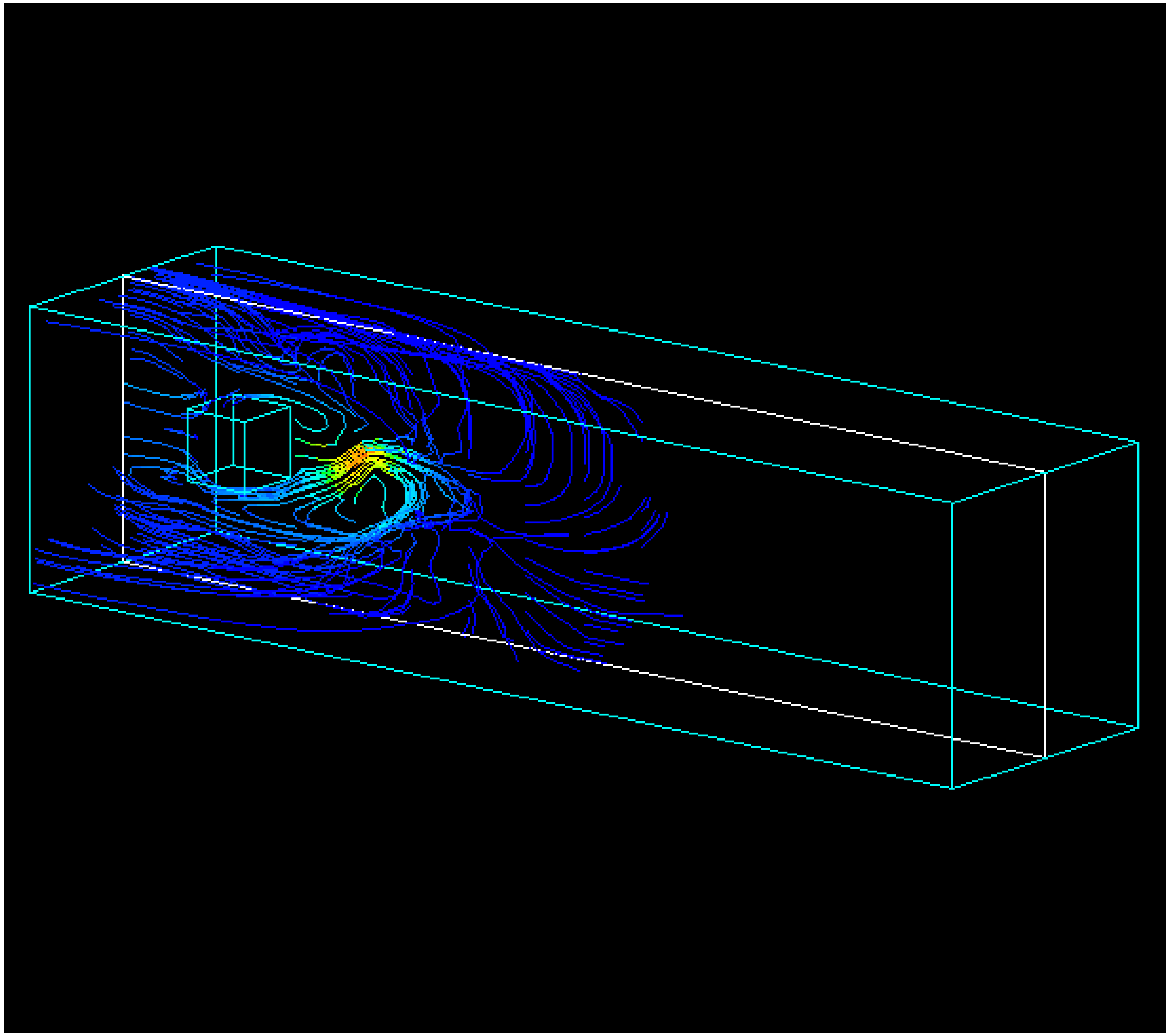
Dual solution, $d(\omega) = 0.0625$, $t = 1.75$



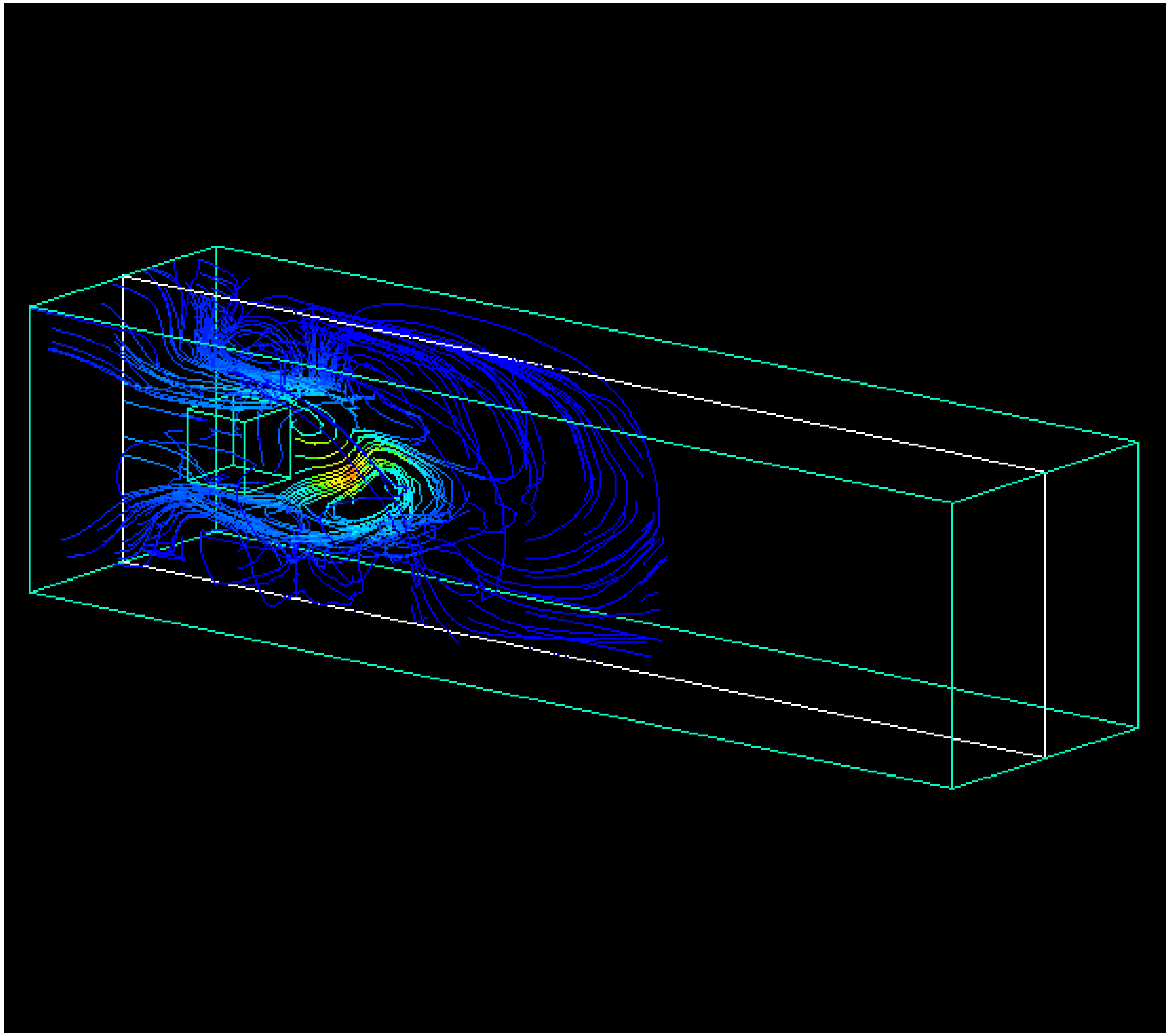
Dual solution, $d(\omega) = 0.0625$, $t = 1.5$



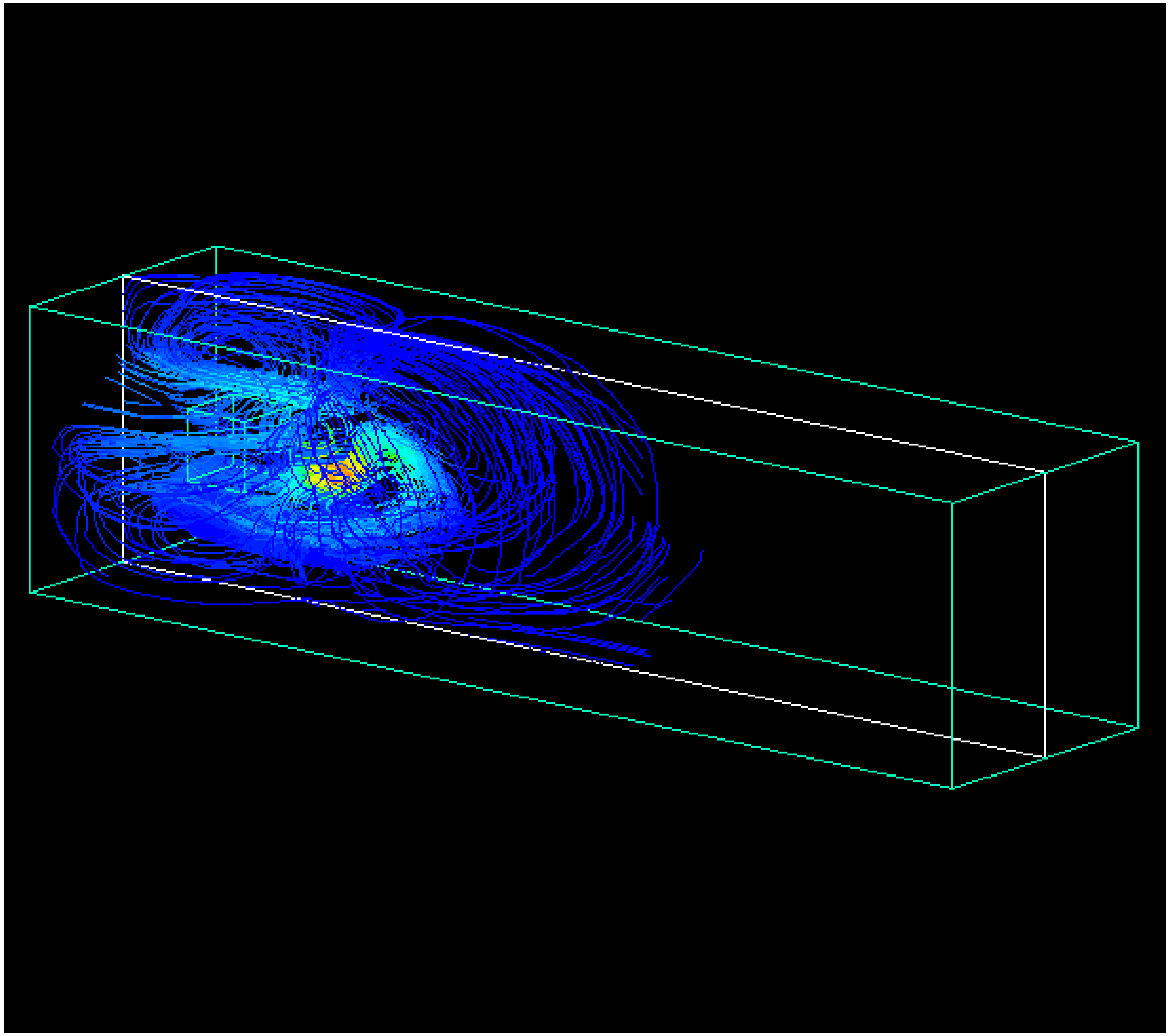
Dual solution, $d(\omega) = 0.0625$, $t = 1.25$



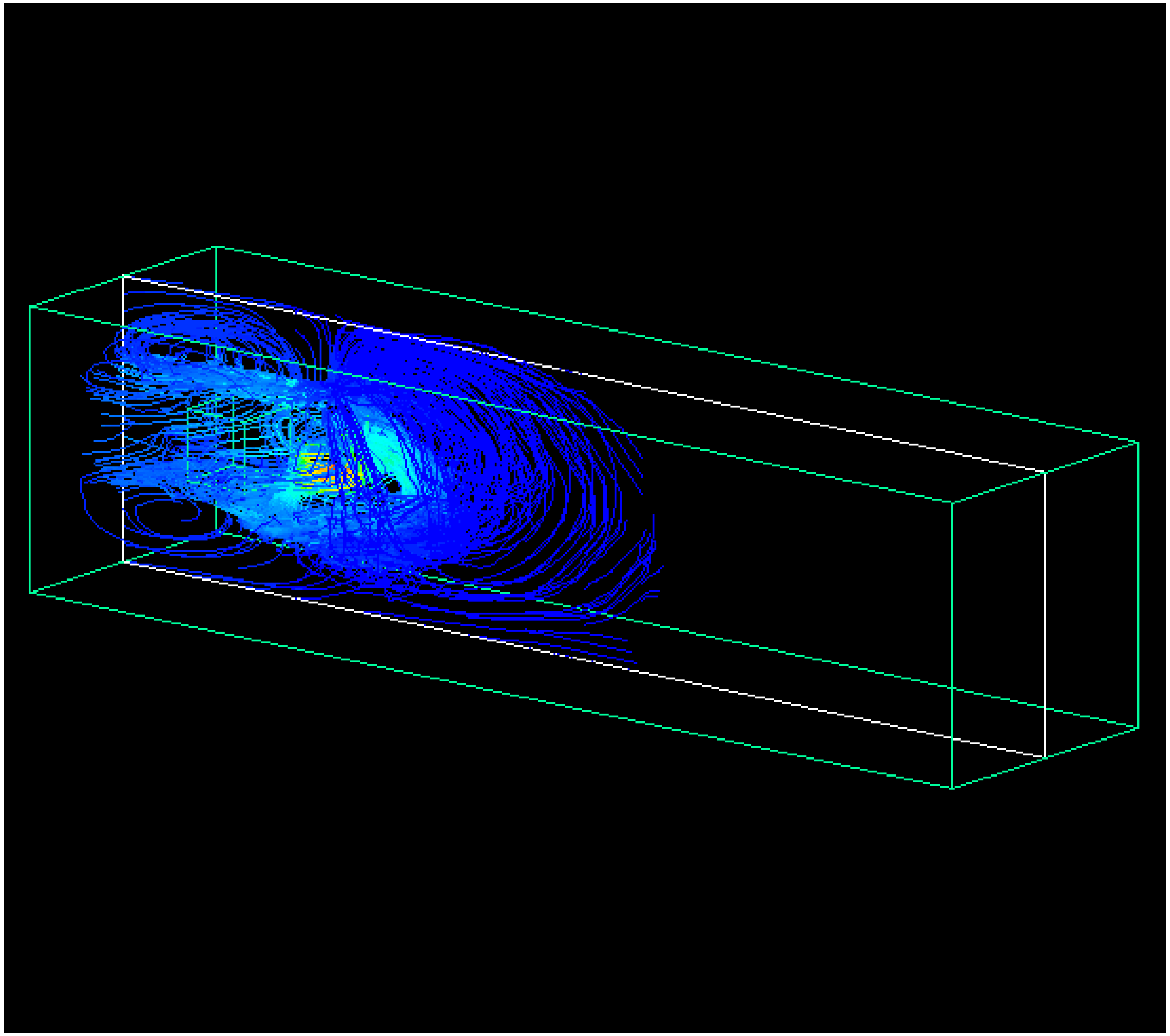
Dual solution, $d(\omega) = 0.0625$, $t = 1$



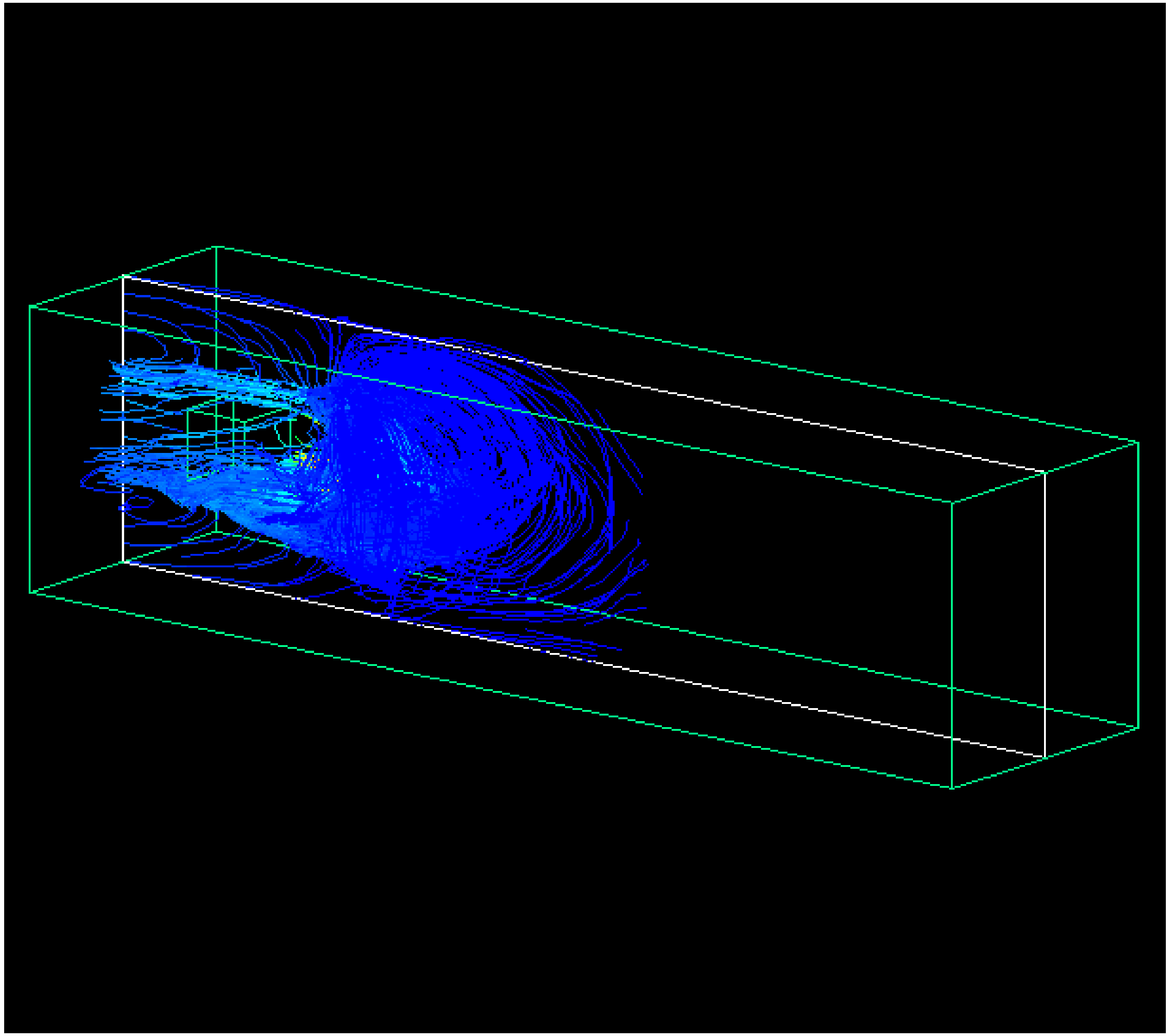
Dual solution, $d(\omega) = 0.0625$, $t = 0.75$



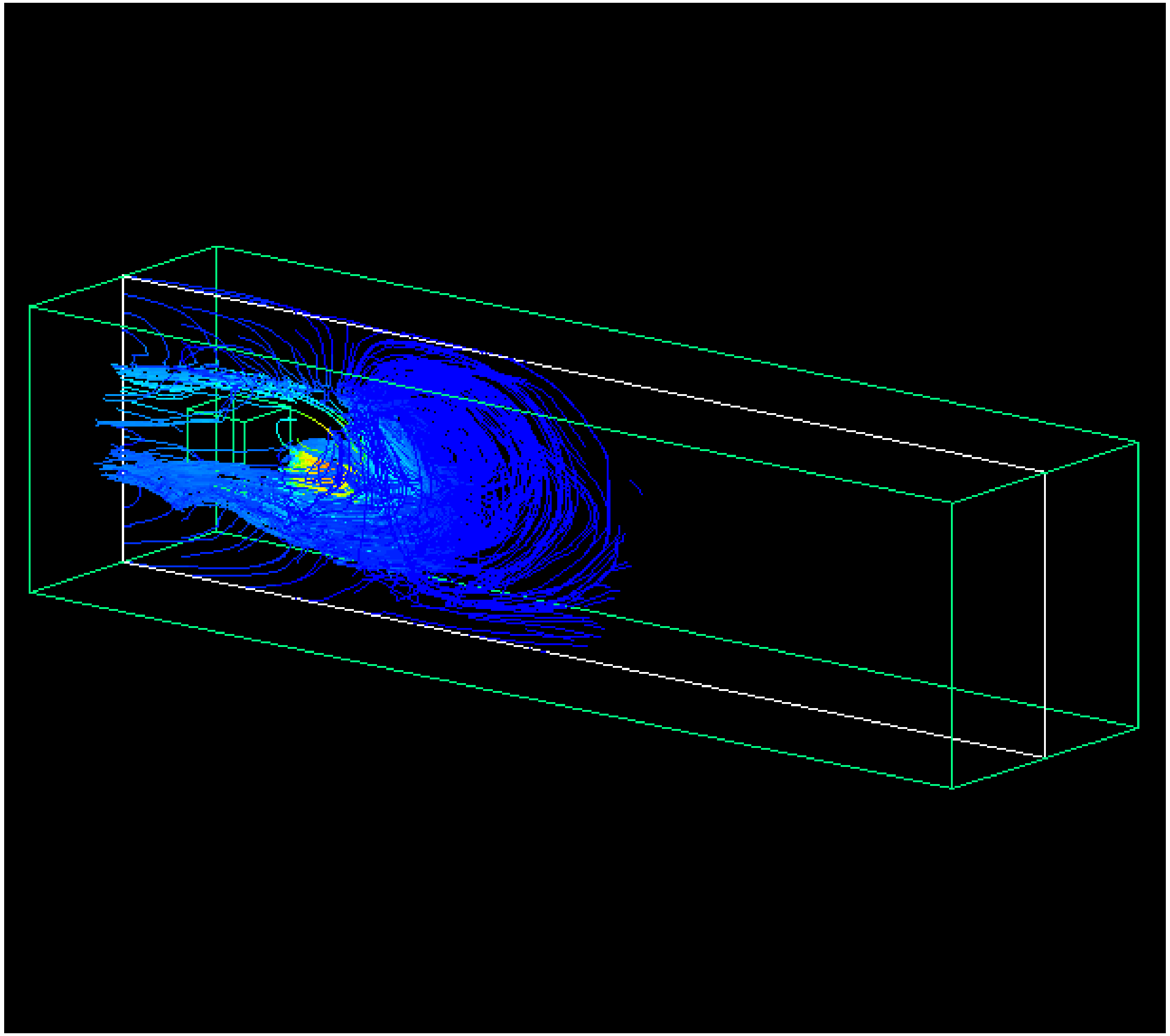
Dual solution, $d(\omega) = 0.0625$, $t = 0.5$



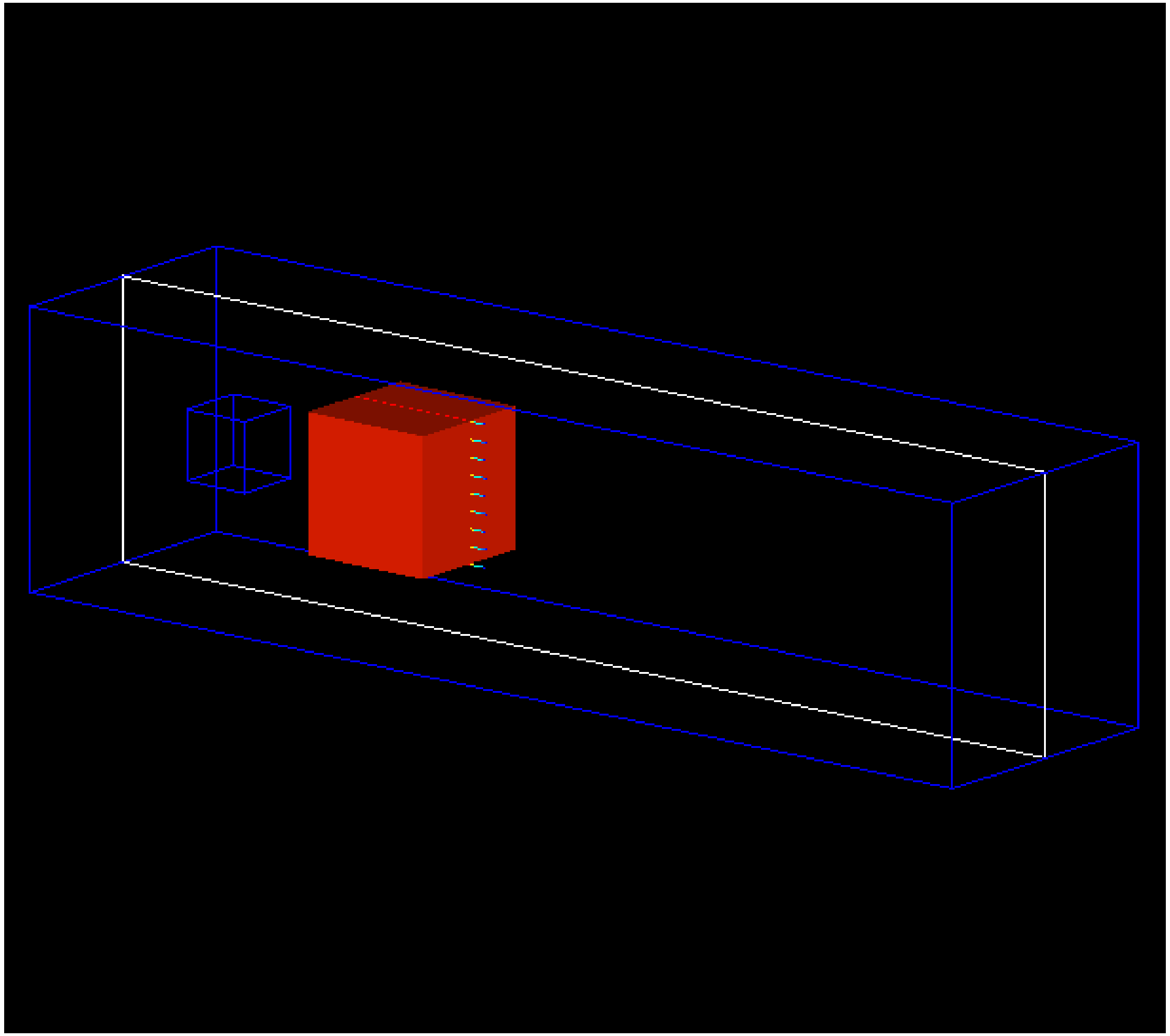
Dual solution, $d(\omega) = 0.0625$, $t = 0.25$



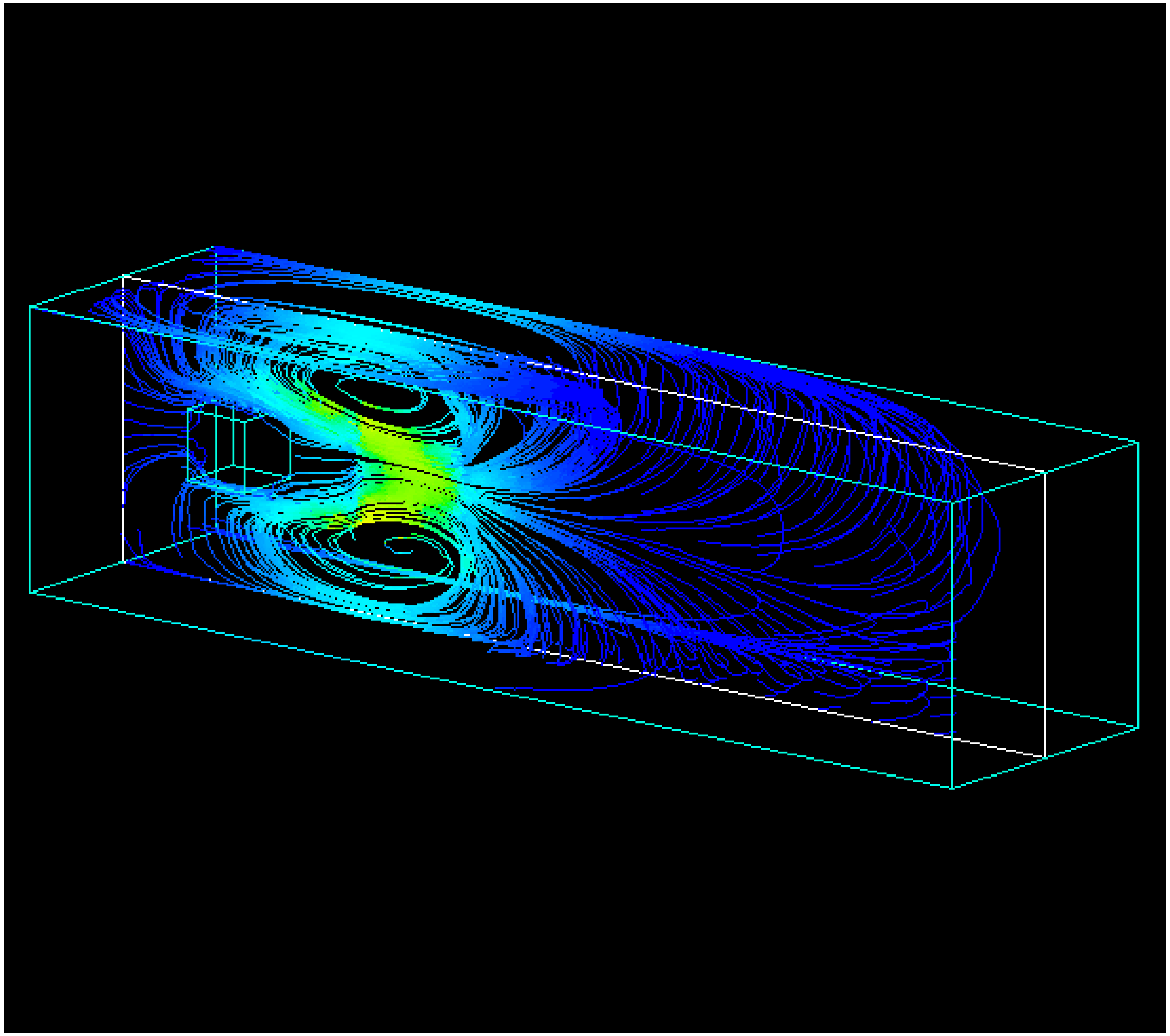
Dual solution, $d(\omega) = 0.0625$, $t = 0$



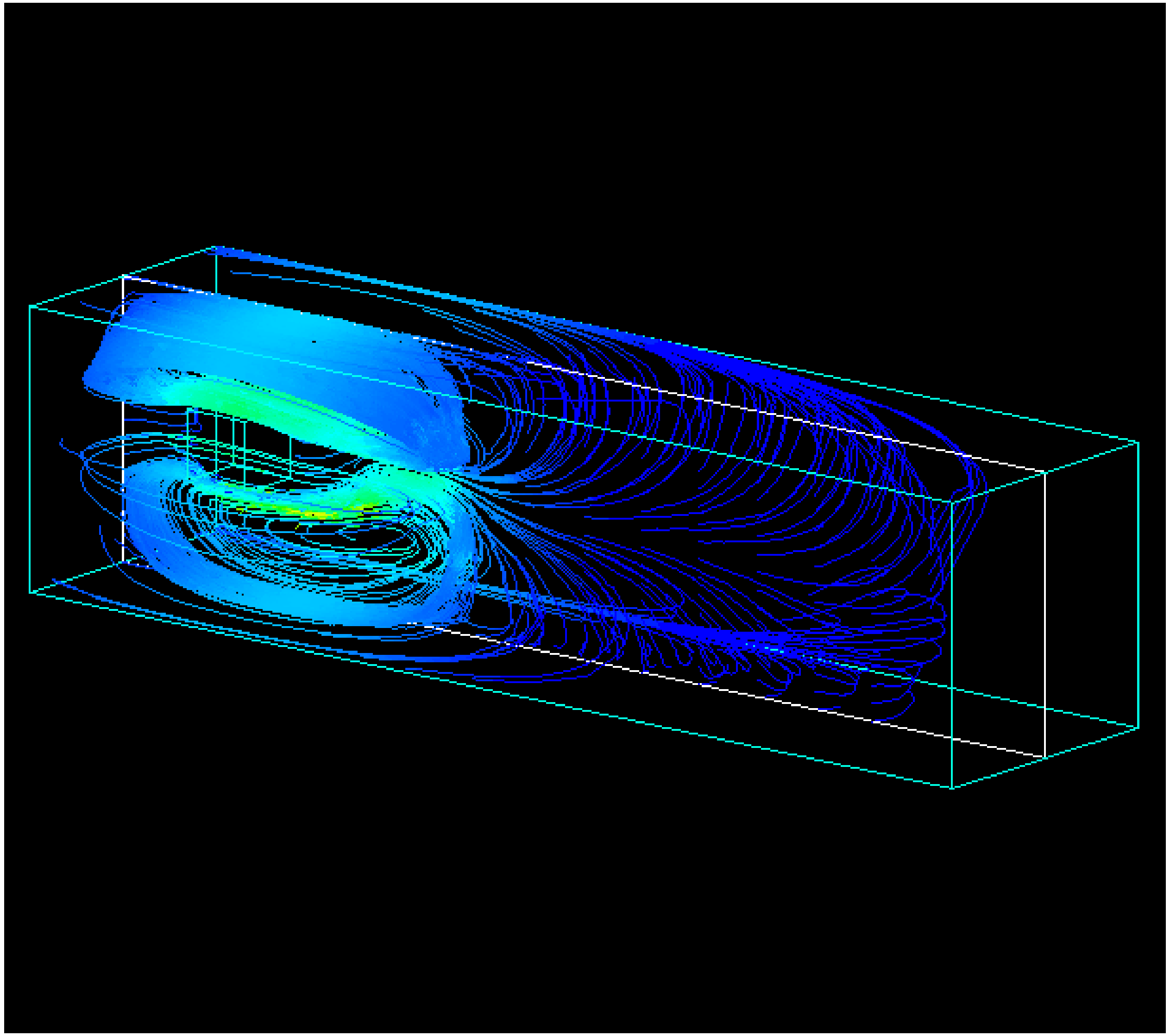
Dual solution, $d(\omega) = 0.5$, $t = 2$



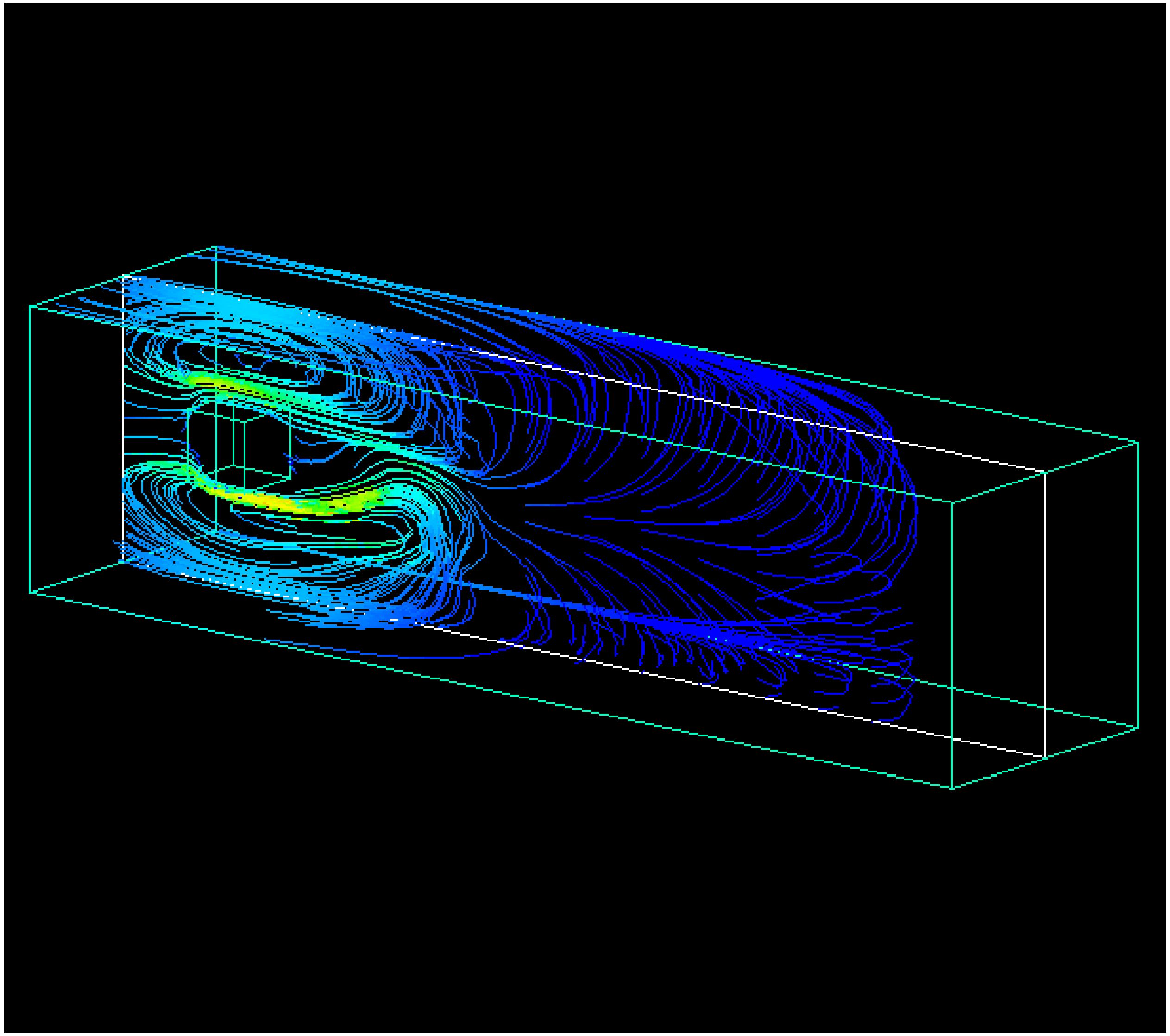
Dual solution, $d(\omega) = 0.5$, $t = 1.75$



Dual solution, $d(\omega) = 0.5$, $t = 1.5$



Dual solution, $d(\omega) = 0.5$, $t = 1.25$



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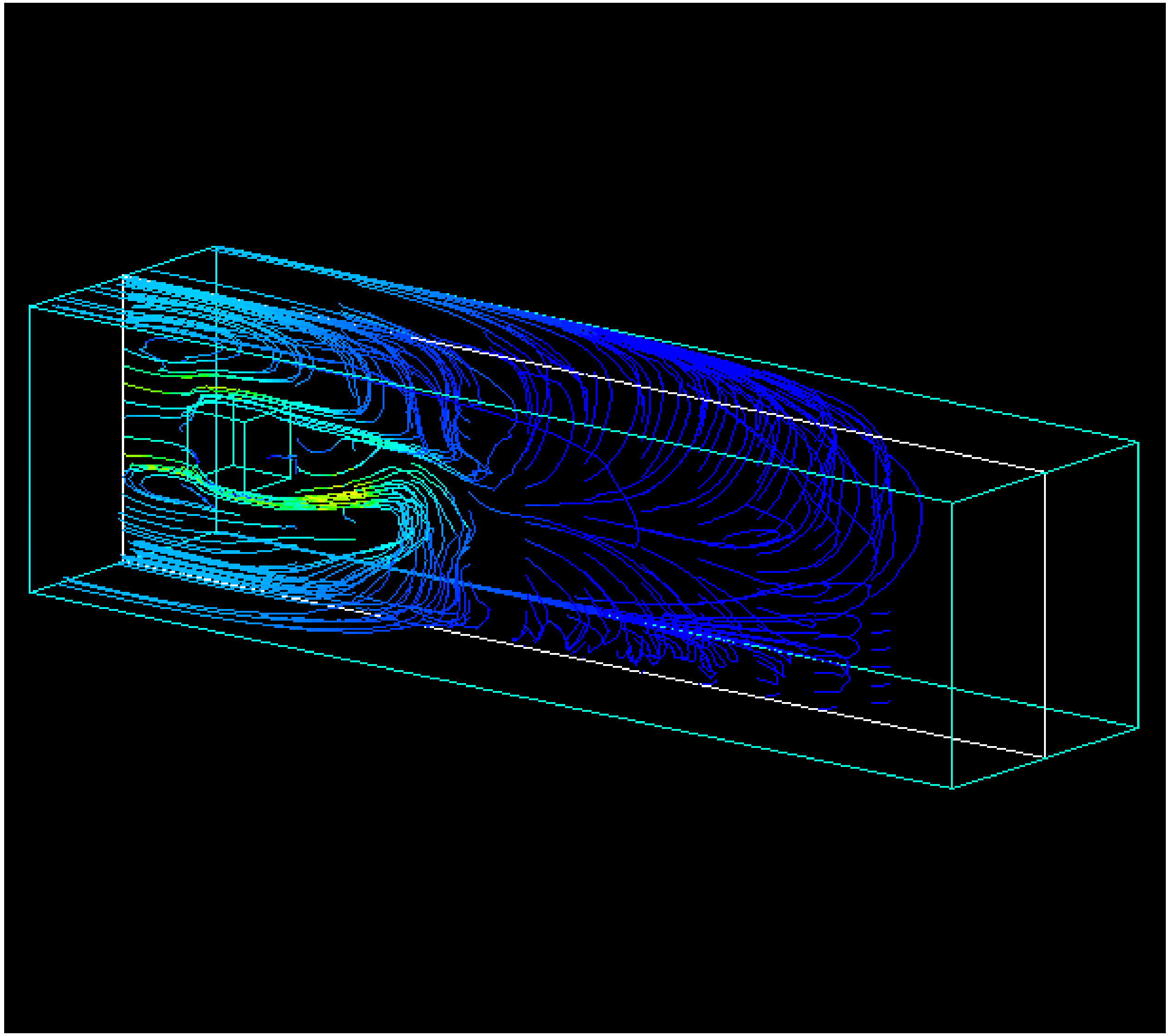

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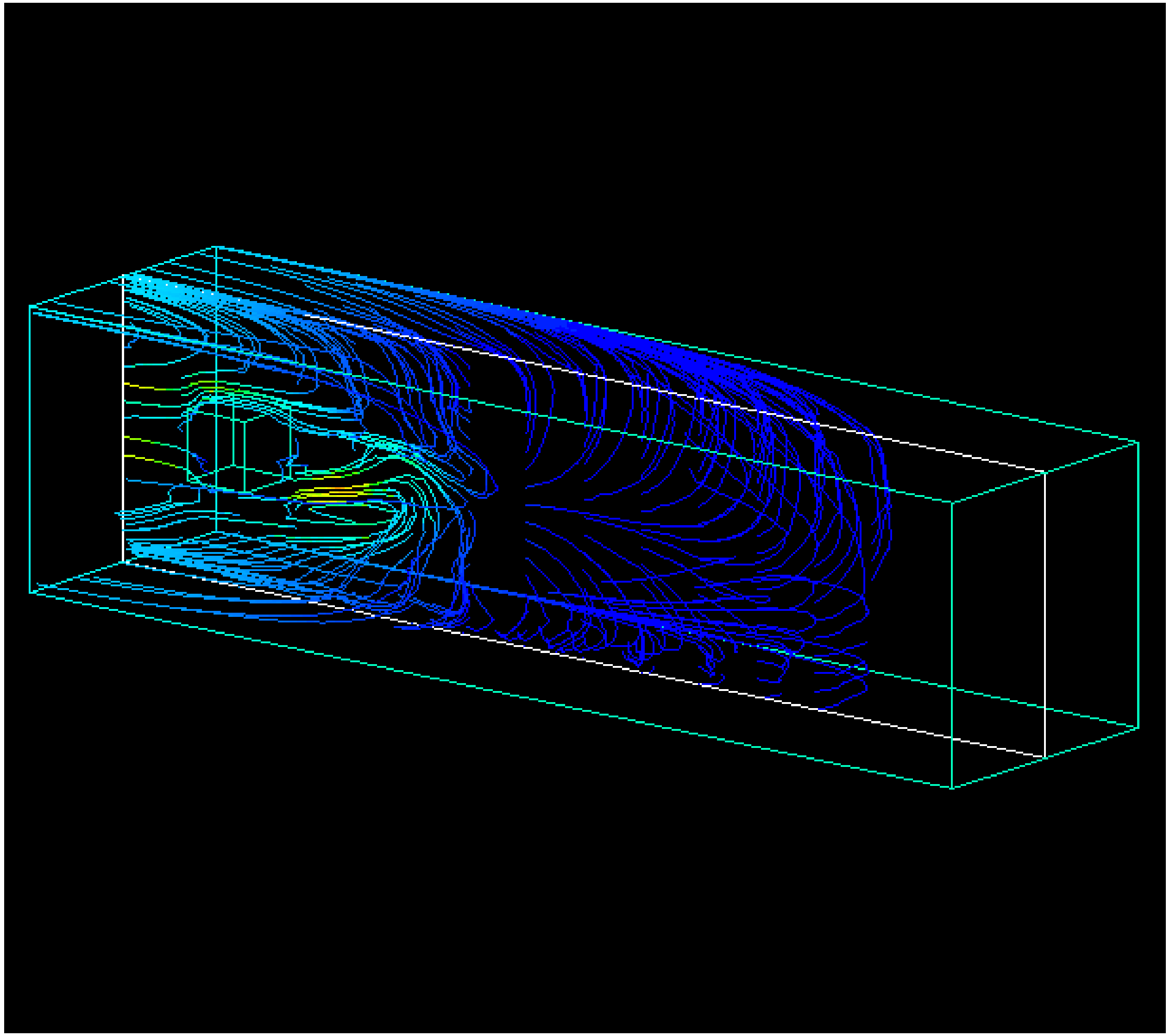

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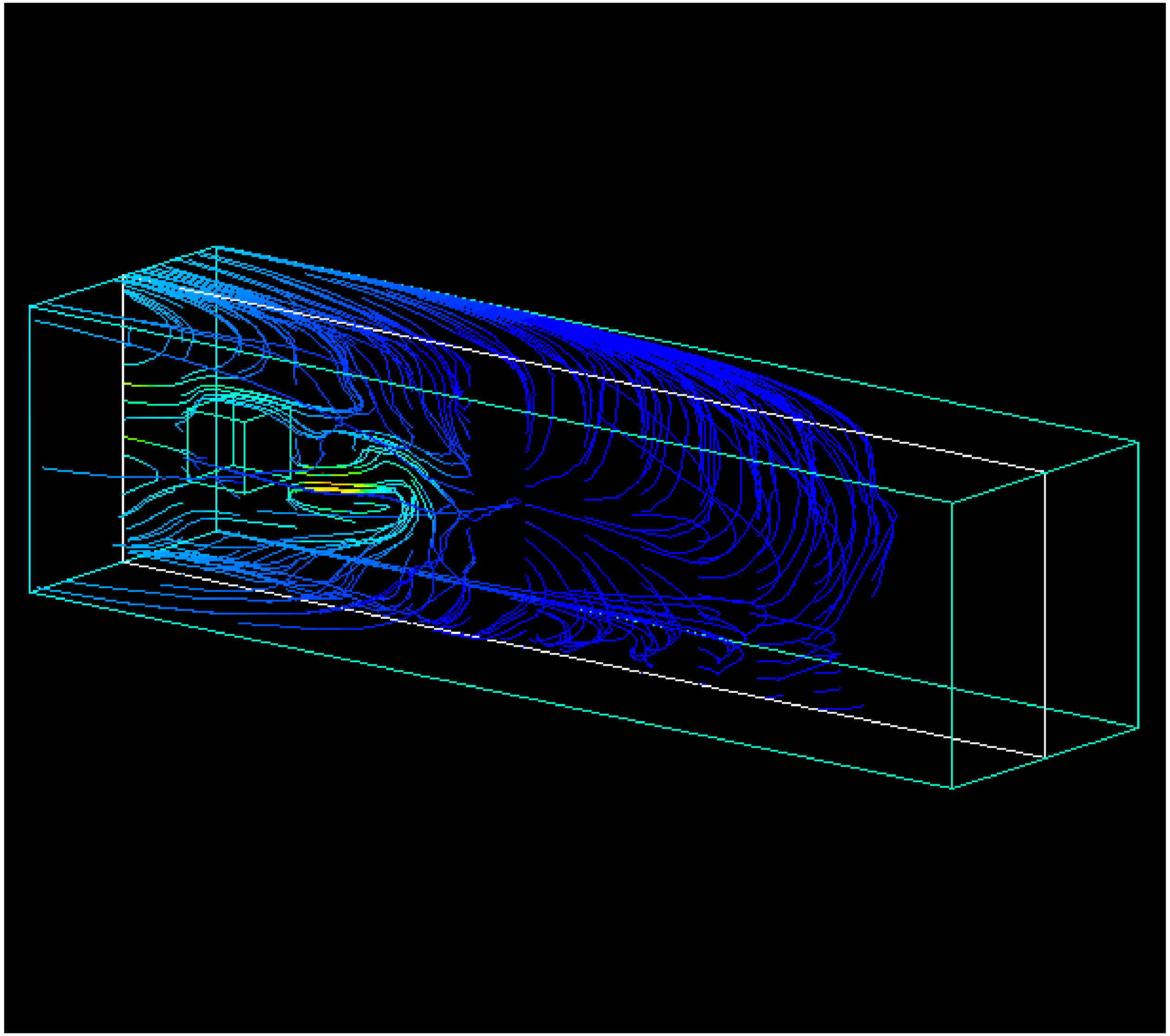

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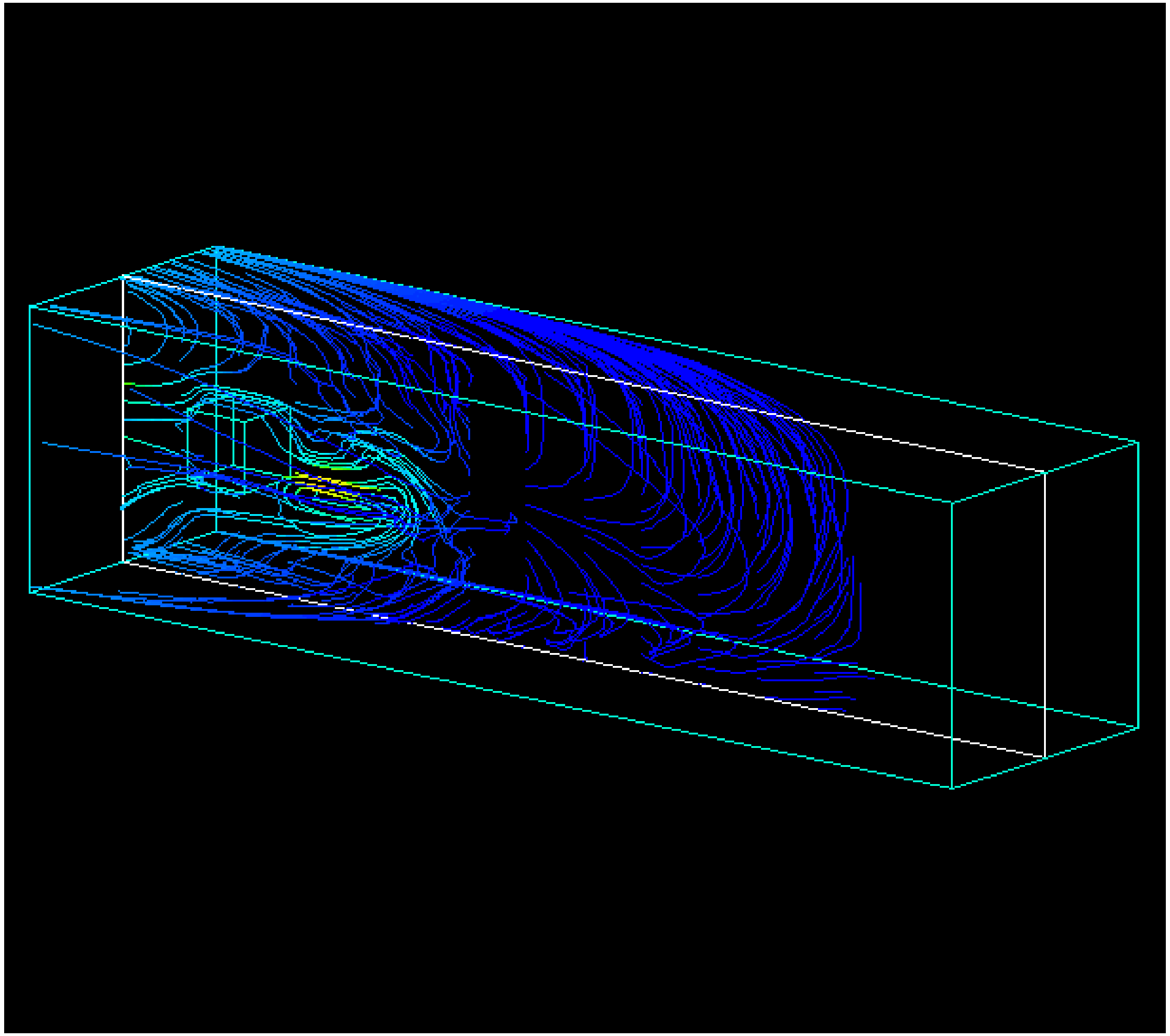

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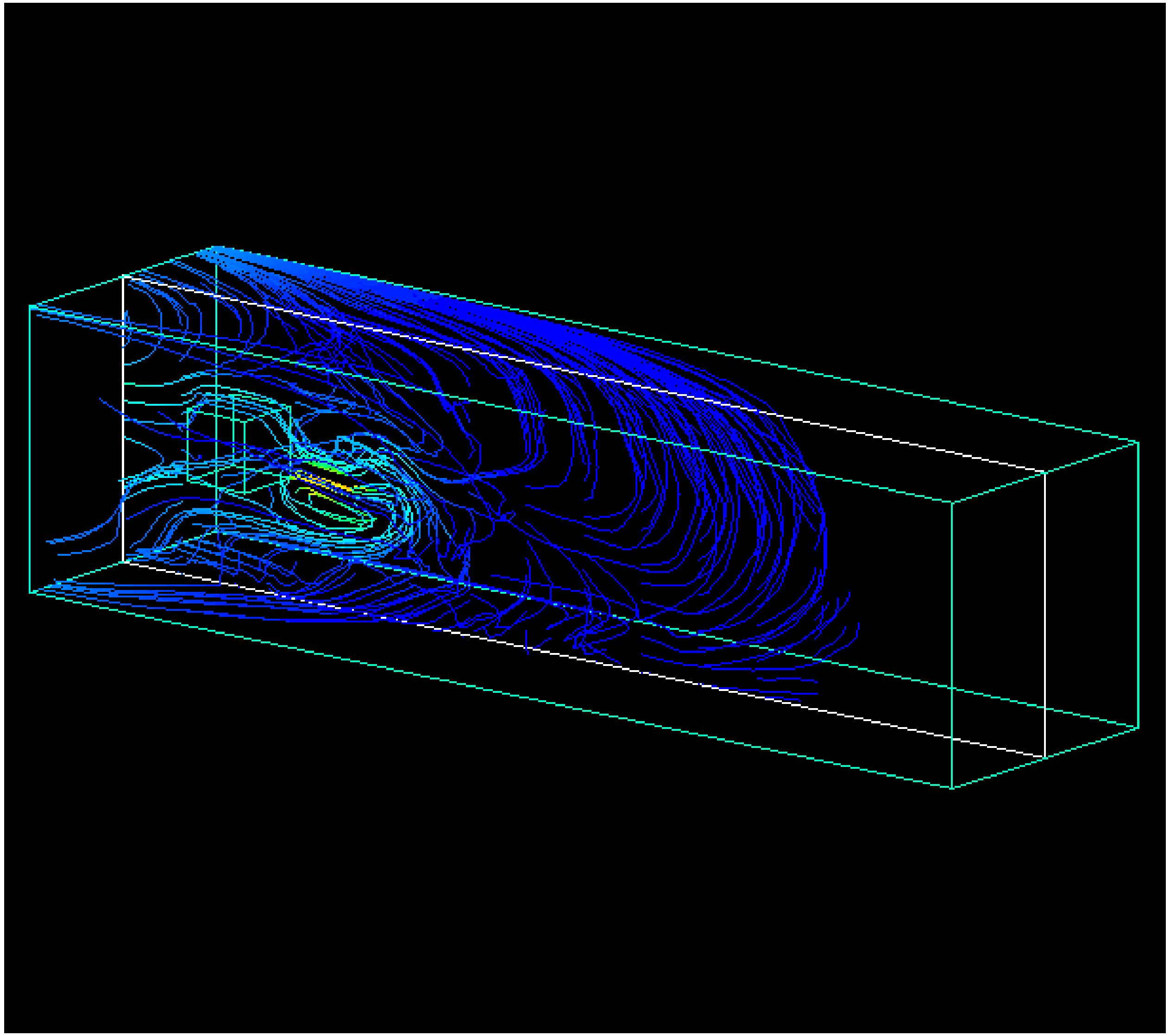
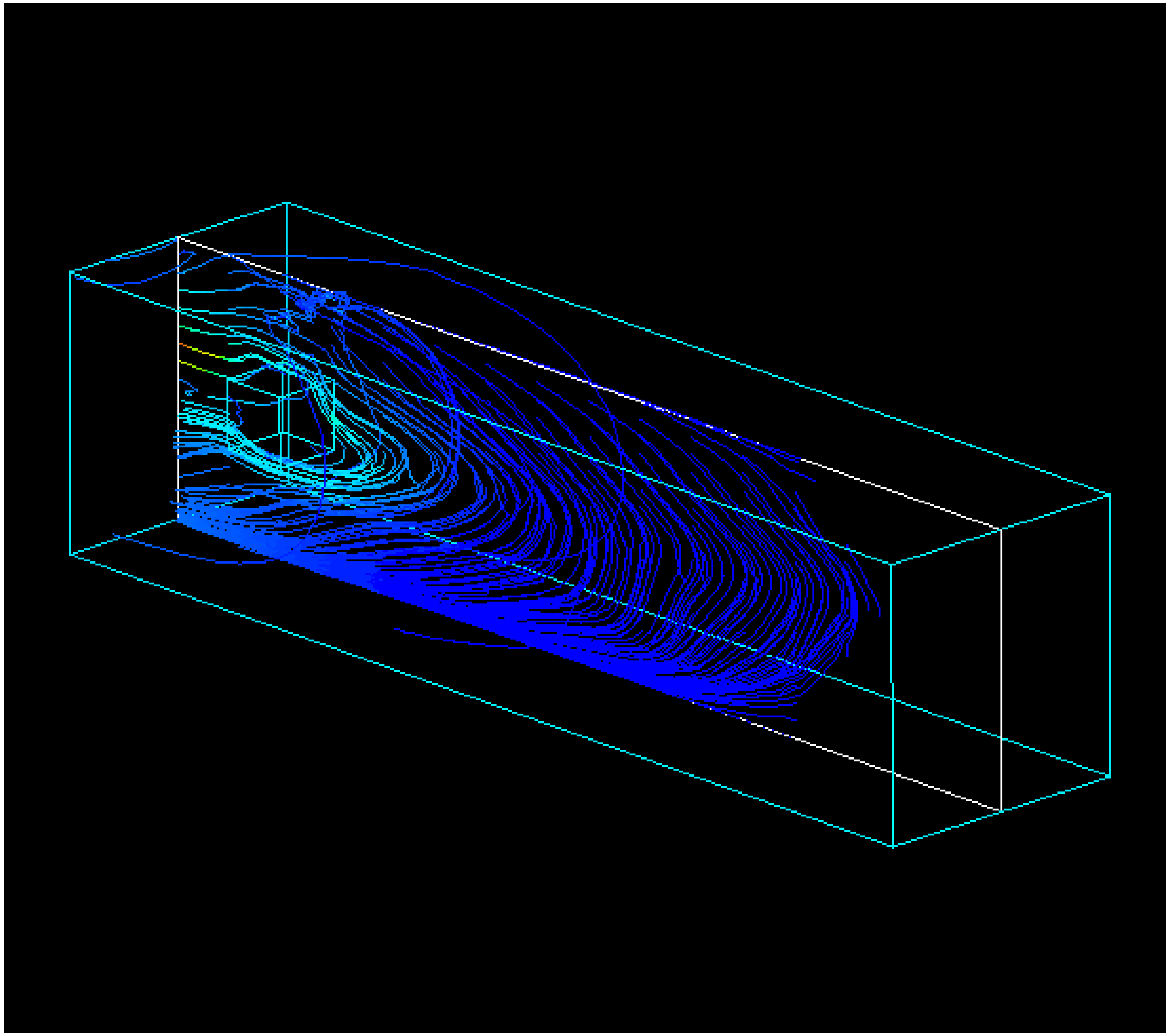


Stability factors with $d(\omega) = 0.25$

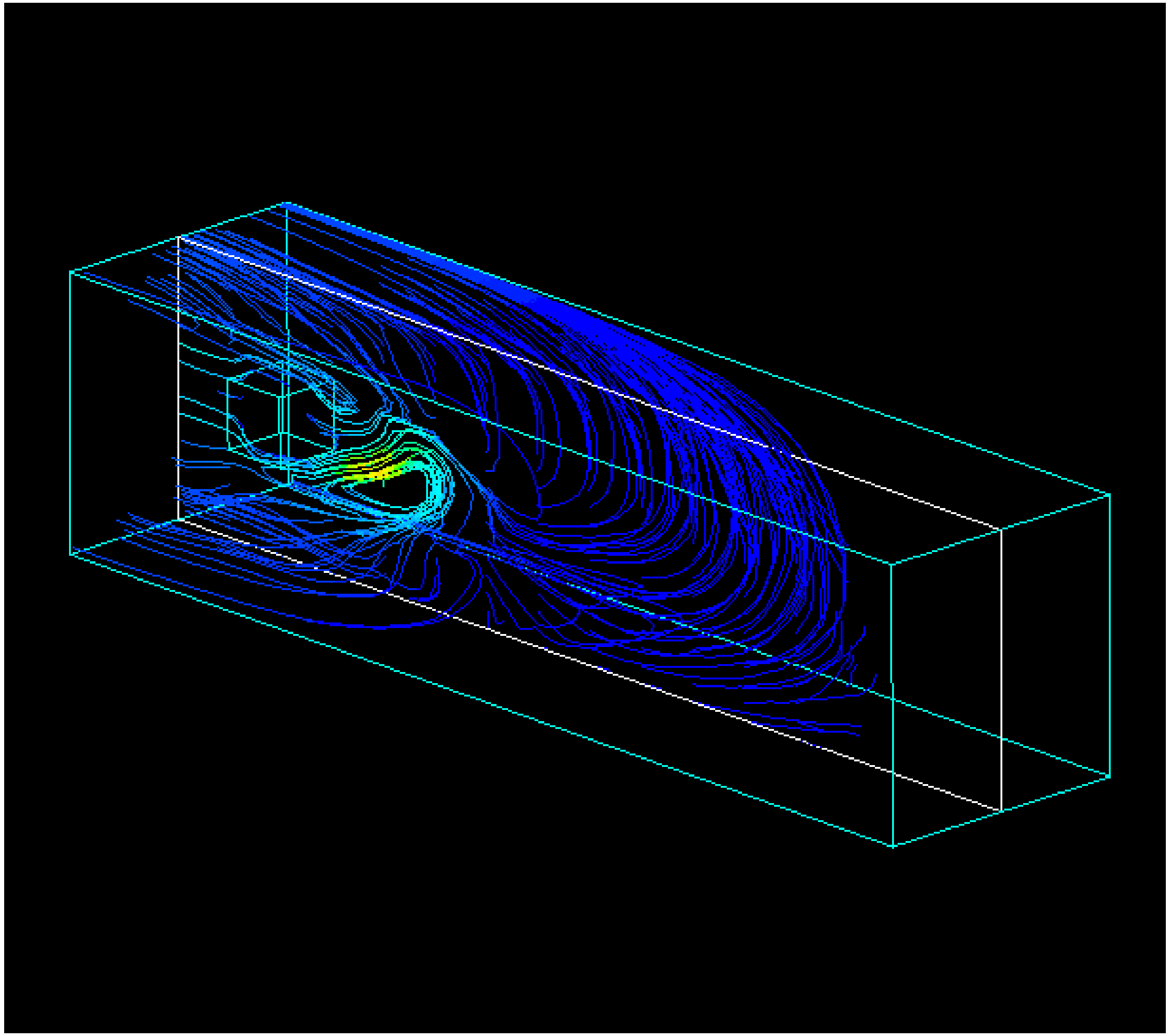
Area	$S_{0,1}$	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{2,1}$
Behind body	3.1	66.2	14.6	6.4	1533.5
Above body	1.9	32.0	12.3	5.5	663.9

Larger stability factors in the complex flow behind the body

Dual solution at $t = 1$, ψ above body



Dual solution at $t = 1$, ψ behind body



Bluff body - Computation of the drag

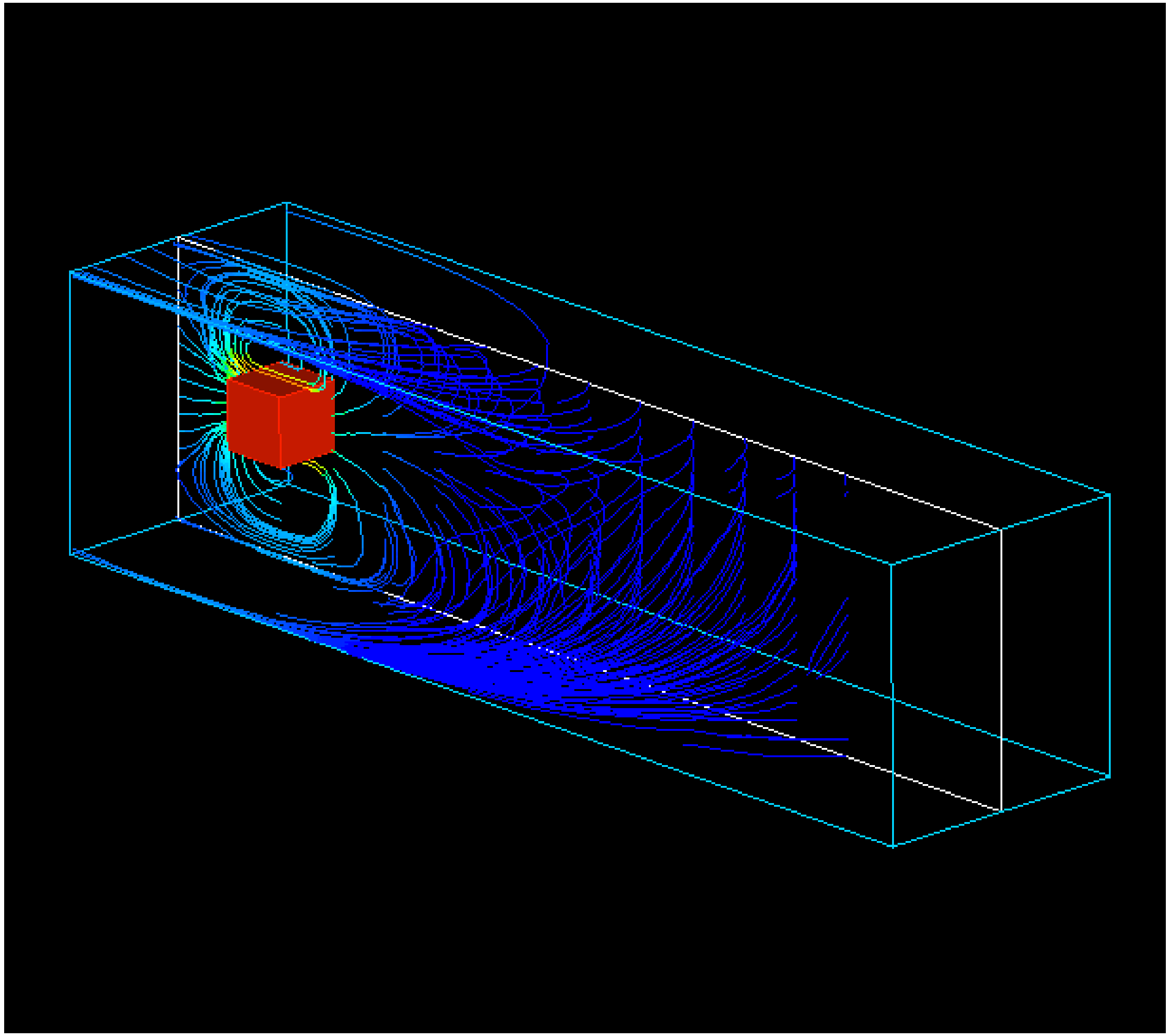
- Take the state at $t = 18$ as initial data at $t = 0$
- Want to compute the mean drag force over $I = [0, 2]$, which corresponds to a boundary condition $u = (1/|I|, 0, 0)$ on the surface of the body during $[0, 2]$
- We also compute the mean drag force over $I = [1.75, 2]$, which corresponds to a boundary condition $u = (1/|I|, 0, 0)$ on the surface of the body during $[1.75, 2]$

Stability factors for mean drag over I

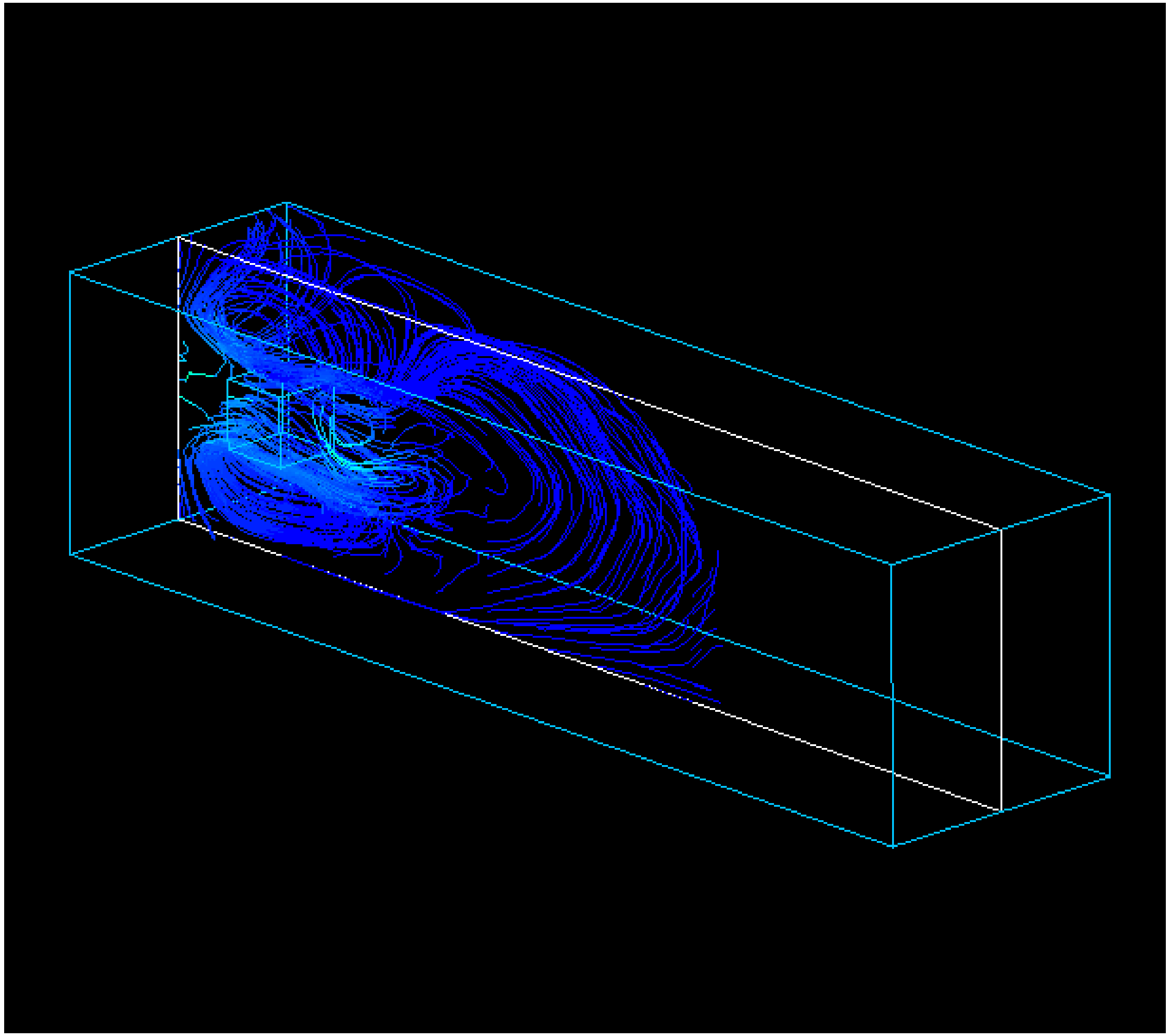
I	$S_{0,1}$	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{2,1}$
$[0, 2]$	0.019	0.39	0.13	0.079	8.8
$[1.75, 2]$	0.43	5.5	0.37	0.29	83.8

Smaller stability factors for averages over longer time intervals

Dual solution for drag, $I = [0, 2]$, $t = 1$



Dual solution for drag, $I = [1.75, 2]$, $t = 1$



Example: Jets

- Channel flow with slip walls, 1×1 rectangular cross section and length 4.
- Obstacle with 4 square holes of size 0.25×0.25 at $x_1 = 0.5$
- $x_1 = 0$ inflow boundary with inflow condition $u = (1, 0, 0)$
- Transparent outflow condition
- cG(1)cG(1), tetrahedral mesh $h = 1/32$
- $\nu = 1/10000$

Example: Jets

start animation

Example: Jets

- The mesh is too coarse for this problem $\Rightarrow$ the residuals are too large

Example: Jets

Stability factors related to the mean error over $\omega \subset \Omega$, $d(\omega) = 0.25$, centered at $(2, 0, 0)$

$S_{0,1}$	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{2,1}$
11.1	395.0	41.9	56.7	10374.5

This quantity is not computable on the current mesh since the residuals are too large!

Also the dual solution linearized at U should not be trusted!

Example: Jets

- As a test, compute the dual problem corresponding to the error in the global average over $\Omega \times [0, 2]$, linearized at (the wrong) solution U
- This corresponds to a force $(1, 0, 0)$ in the dual problem (and zero final data)

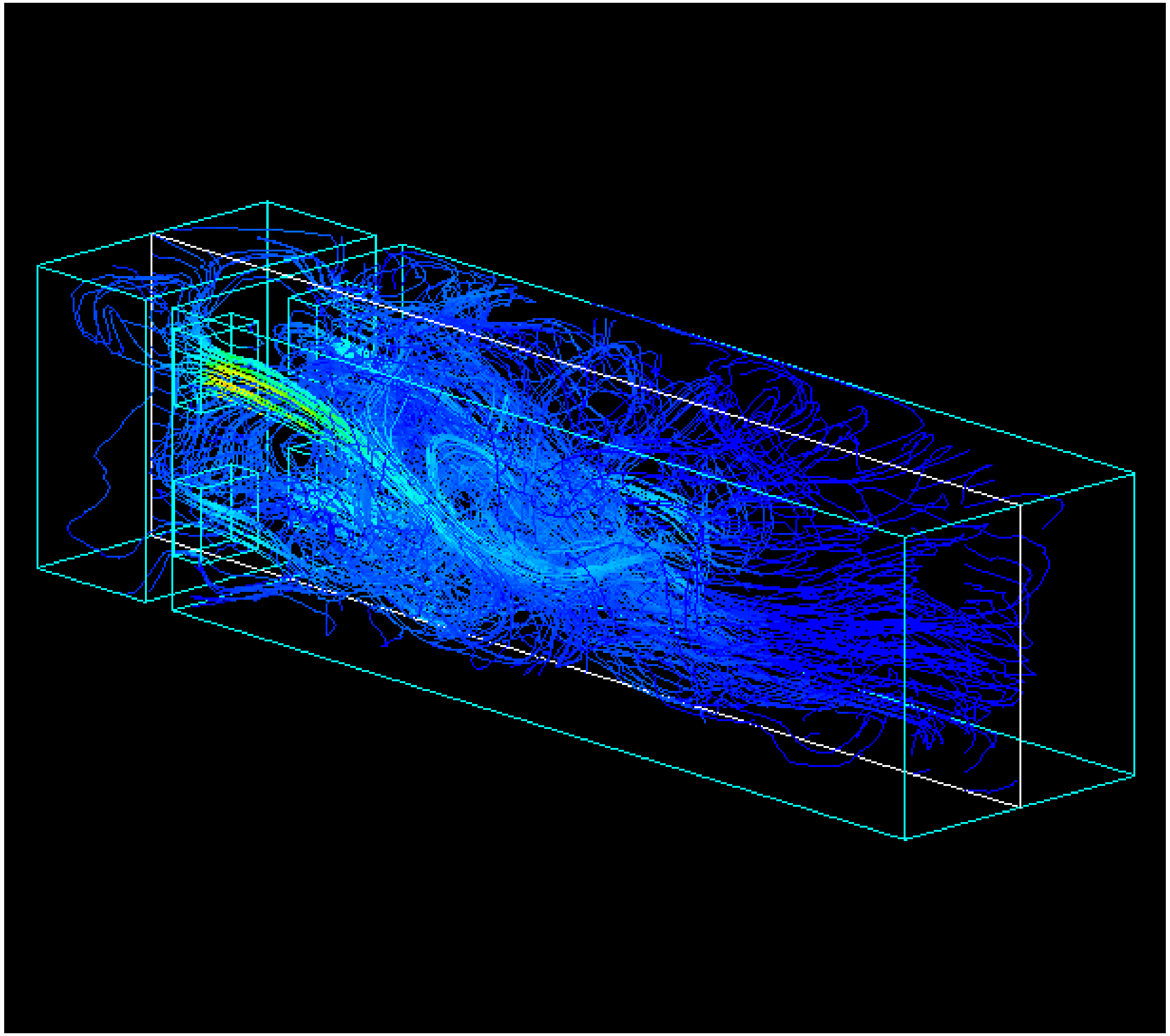
Stability factors for the jets

Stability factors related to the mean error over $\omega \subset \Omega$, with $d(\omega) = 0.25$ at $t = 2$ (local), and over $\Omega \times [0, 2]$ (global)

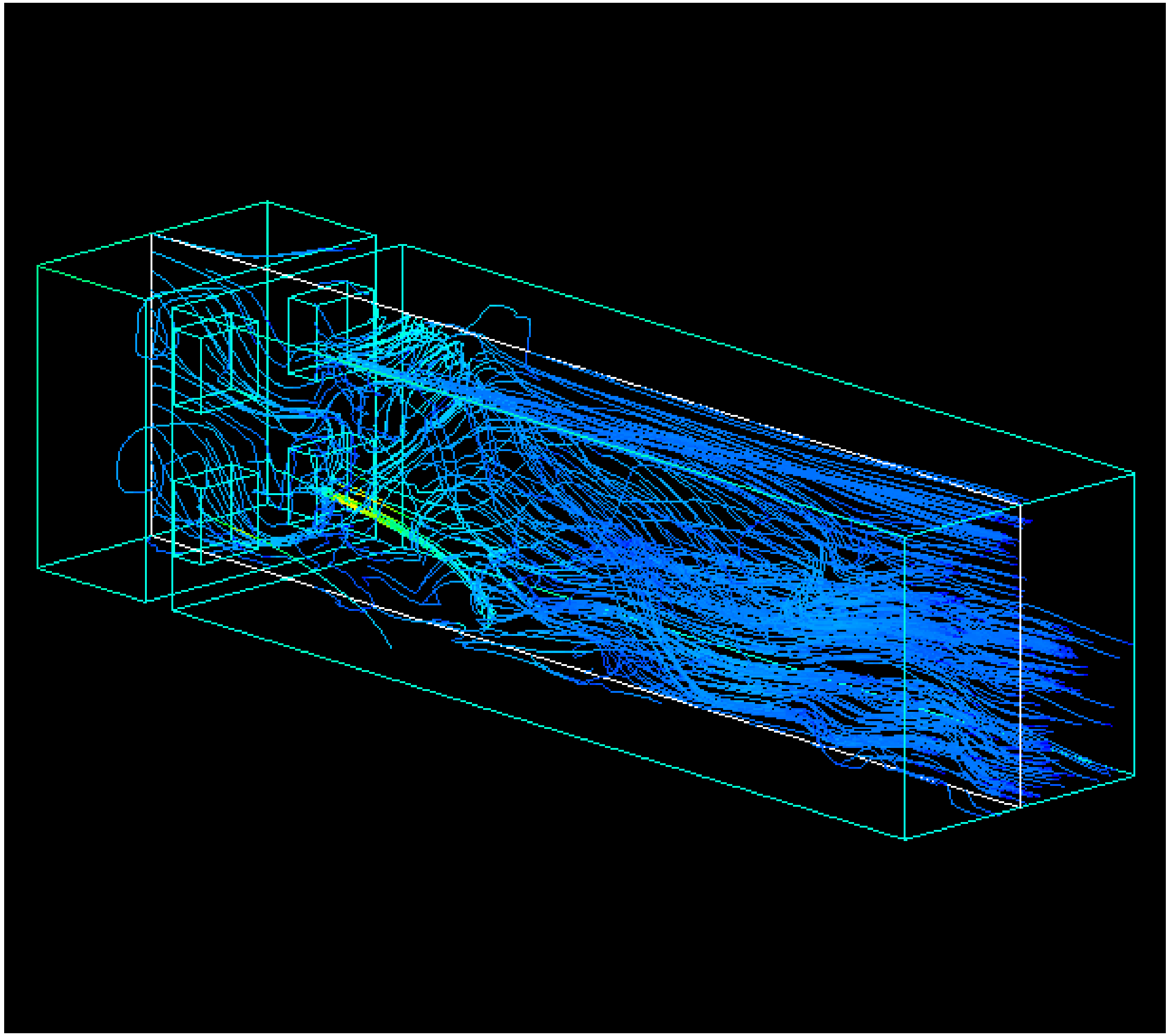
ω	$S_{0,1}$	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{2,1}$
local	11.1	395.0	41.9	56.7	10374.5
global	0.077	1.8	0.17	1.0	44.7

Global stability factors are considerably smaller, indicating that global quantities may be computed if the residuals were smaller $\Rightarrow$ Good news!

Dual solution, jet with local data, $t = 1$



Dual solution, jet with global data, $t = 1$



Pertubation growth in NSE

- Linear/exponential pertubation growth?
- Transition to turbulence

Linearized Navier-Stokes eqns. (LNSE)

$$\begin{aligned} \varphi_t + u\cdot\nabla\varphi + (\varphi\cdot\nabla)u - \nu\Delta\varphi + \nabla q &= g \\ \nabla\cdot\varphi &= 0 \\ \varphi|_{\partial\Omega} = 0, \quad \varphi(\cdot,0) &= \varphi^0 \end{aligned}$$

(φ, q) pertubation of given solution (u, p) to NSE, obtained by subtracting (u, p) from $(u+\varphi, p+q)$ corresponding to pertubed initial data $u^0+\varphi^0$ and right hand side $f+g$, omitting $(\varphi\cdot\nabla)\varphi$

Linearized Navier-Stokes eqns. (LNSE)

$$\begin{aligned} \varphi_t + u \cdot \nabla\varphi + (\varphi \cdot \nabla)u - \nu\Delta\varphi + \nabla q &= g \\ \nabla \cdot \varphi &= 0 \\ \varphi|_{\partial\Omega} = 0, \quad \varphi(\cdot, 0) &= \varphi^0 \end{aligned}$$

Again the interaction between the transport $u \cdot \nabla\varphi$, the diffusion $-\nu\Delta\varphi$, and the reaction $(\varphi \cdot \nabla)u$ is crucial

Linearized Navier-Stokes eqns. (LNSE)

$$\begin{aligned} \varphi_t + u \cdot \nabla\varphi + (\varphi \cdot \nabla)u - \nu\Delta\varphi + \nabla q &= g \\ \nabla \cdot \varphi &= 0 \\ \varphi|_{\partial\Omega} = 0, \quad \varphi(\cdot, 0) &= \varphi^0 \end{aligned}$$

The basic question is to estimate (φ, q) in terms of the data (g, φ_0)

Linearized Navier-Stokes eqns. (LNSE)

$$\begin{aligned} \varphi_t + u \cdot \nabla\varphi + (\varphi \cdot \nabla)u - \nu\Delta\varphi + \nabla q &= g \\ \nabla \cdot \varphi &= 0 \\ \varphi|_{\partial\Omega} = 0, \quad \varphi(\cdot, 0) &= \varphi^0 \end{aligned}$$

Weak stability factor $S_0(u, T, \varphi_0) = \dfrac{||\varphi||_I}{||\varphi^0||}$

where φ is the sol. to LNSE with $g = 0$ and $\varphi_0 \neq 0$

Linearized Navier-Stokes eqns. (LNSE)

$$\begin{aligned} \varphi_t + u\cdot\nabla\varphi + (\varphi\cdot\nabla)u - \nu\Delta\varphi + \nabla q &= g \\ \nabla\cdot\varphi &= 0 \\ \varphi|_{\partial\Omega} = 0, \quad \varphi(\cdot,0) &= \varphi^0 \end{aligned}$$

Weak stability factor $S_0(u,T) = \sup_{\varphi^0\in L_2} \frac{\|\varphi\|_I}{\|\varphi^0\|}$

($\|v\|_I = \sup_{0<t<T} \|v(\cdot,t)\|$)

Worst case exponential pertub. growth

$$\|\varphi(\cdot,t)\|^2 \leq -2\int_0^t \int_\Omega (\varphi\cdot\nabla)u\cdot\varphi\, dxds + \|\varphi^0\|^2$$

from multiplying LNSE by φ and integrating in space and time

Worst case exponential pertub. growth

$$\|\varphi(\cdot,t)\|^2 \leq -2\int_0^t \int_\Omega (\varphi\cdot\nabla)u\cdot\varphi\, dxds + \|\varphi^0\|^2$$

Grönwall's inequality $\Rightarrow S_0(u,T) \leq \exp(CKT)$,

where $C \approx 1$ and $K = \|\nabla u\|_\infty$

Exp. growth due to the zero order (reaction) term $(\varphi\cdot\nabla)u$

Worst case exponential pertub. growth

Worst case exponential pertubation growth:

$$\|\varphi(\cdot,t)\|^2 \leq -2\int_0^t \int_\Omega (\varphi\cdot\nabla)u\cdot\varphi \, dxds + \|\varphi^0\|^2$$

A flow with exponential pertubation growth cannot exist as a stable flow $\Rightarrow$

Must be possible to obtain reduced growth rates by using special features of the zero order term $(\varphi\cdot\nabla)u$ in some cases

Nearly parallell flow/shear flow

$$\|u_1\| \approx 1, \quad \|\bar{\nabla} u_1\|_\infty = C, \quad \|\bar{u}\|_\infty + \|\bar{\nabla}\bar{u}\|_\infty \leq c\nu,$$

where $\bar{u} = (u_2, u_3), \quad \bar{\nabla} = (\partial/\partial x_2, \partial/\partial x_3)$

$u_1 \approx 1, \quad u_{1,1} \sim \nu, \quad u_{2,1}, u_{3,1} \sim \nu^2, \quad T \sim 1/\nu = Re$

Assume that the pertubations (φ, q) are independent of x_1

Nearly parallell flow/shear flow

$$\begin{aligned}
\varphi_{1,t} + u\cdot\nabla\varphi_1 + (\bar\varphi\cdot\bar\nabla)u_1 - \nu\Delta\varphi_1 &= 0\\
\bar\varphi_t + u\cdot\nabla\bar\varphi + (\bar\varphi\cdot\bar\nabla)\bar u - \nu\Delta\bar\varphi + \bar\nabla q &= 0\\
\bar\nabla\cdot\bar\varphi \equiv \varphi_{2,2} + \varphi_{3,3} &= 0\\
\varphi|_{\partial\Omega} = 0, \quad \varphi(\cdot,0) &= \varphi^0
\end{aligned}$$

Observe that $\bar\varphi$ is fully decoupled from φ_1!

Nearly parallell flow/shear flow

$$
\begin{aligned}
\varphi_{1,t} + u\cdot\nabla\varphi_1 + (\bar\varphi\cdot\bar\nabla)u_1 - \nu\Delta\varphi_1 &= 0\\
\bar\varphi_t + u\cdot\nabla\bar\varphi + (\bar\varphi\cdot\bar\nabla)\bar u - \nu\Delta\bar\varphi + \bar\nabla q &= 0\\
\bar\nabla\cdot\bar\varphi \equiv \varphi_{2,2} + \varphi_{3,3} &= 0\\
\varphi|_{\partial\Omega} = 0, \quad \varphi(\cdot,0) &= \varphi^0
\end{aligned}
$$

And that no zero order (reaction) term is not present in the equation for φ_1

Nearly parallell flow/shear flow

Compare with the ODE

$$
\begin{aligned}
\dot{\varphi}_1 - \varphi_2 &= 0 \\
\dot{\varphi}_2 &= 0 \\
\varphi^0 &= (0, \varphi_2^0)
\end{aligned}
$$

with solution $\varphi_1(t) = t\ \varphi_2^0, \quad \varphi_2(t) = \varphi_2^0$, showing linear growth for φ_1

Nearly parallell flow/shear flow

Compare with the ODE

$$\begin{aligned}\dot{\varphi}_1 - \varphi_2 &= 0 \\ \dot{\varphi}_2 &= 0 \\ \varphi^0 &= (0, \varphi_2^0)\end{aligned}$$

with solution $\varphi_1(t) = t\ \varphi_2^0$ and $\varphi_2(t) = \varphi_2^0$, showing linear growth for φ_1, which is very different from

$$\dot{\varphi}_1 - \varphi_1 = 0, \quad \varphi_1^0 \neq 0$$

with solution $\varphi_1(t) = \exp(t)\ \varphi_1^0$

Nearly parallell flow/shear flow

THEOREM: For nearly parallell flow and $T = \nu^{-1}$:

$$S_0(u, T) \leq C\nu^{-1}$$

We refer to the physical mechanism corresponding to this linear pertubation growth as the Taylor-Görtler mechanism

Nearly parallell flow/shear flow

Consider laminar pipe flow in a pipe directed along the x_1 axis:

- cross section ω
- velocity $u = (u_1, 0, 0)$
- computed velocity $U = (U_1, 0, 0)$
- u and U depends on (x_2, x_3) and time t

Corresponding linearized dual problem

$$\begin{aligned}
-\dot{\varphi}_1 - \nu\Delta\varphi_1 &= 0 \quad \text{in } \omega \times I \\
-\dot{\varphi}_2 - \nu\Delta\varphi_2 + \theta_{,2} + U_{1,2}\varphi_1 &= 0 \quad \text{in } \omega \times I \\
-\dot{\varphi}_3 - \nu\Delta\varphi_3 + \theta_{,3} + U_{1,3}\varphi_1 &= 0 \quad \text{in } \omega \times I \\
\varphi_{2,2} + \varphi_{3,3} &= 0 \quad \text{in } \omega \times I \\
\varphi &= 0 \quad \text{on } \partial\omega \times I \\
\varphi(\cdot, T) &= \varphi^T \quad \text{in } \omega
\end{aligned}$$

φ_1 decoupled from $\bar{\varphi} = (\varphi_2, \varphi_3)$, so pertubations (stability factors) grow linear with T or Re

ODE-model for transition

Find $w(t) = (w_1(t), w_2(t))$ such that

$$\begin{aligned} \dot{w}_1 + \nu w_1 - \lambda w_1 w_2 &= \nu \quad t > 0 \\ \dot{w}_2 + 2\nu w_2 - \nu w_2 w_1 &= 0 \quad t > 0 \\ w_1(0) = 1, \quad w_2(0) &= \kappa\nu \end{aligned}$$

where ν is a small positive parameter, and λ and κ are positive parameters of moderate size

w_1 represents flow in the main direction, and w_2 small transversal velocities

ODE-model for transition

The ODE models the terms

$$\dot{u}_1 + \nu\Delta u_1 - u_{1,2}u_2 = \nu \quad t > 0$$
$$\dot{u}_2 + \nu\Delta u_2 - u_{2,1}u_1 = 0 \quad t > 0$$

from the momentum equations of NSE, assuming the non linear coupling terms are modelled $u_{1,2} = -\lambda u_1$ and $u_{2,1} = -\nu u_2$

ODE-model for transition

Find $w(t) = (w_1(t), w_2(t))$ such that

$$\begin{aligned} \dot{w}_1 + \nu w_1 - \lambda w_1 w_2 &= \nu \quad t > 0 \\ \dot{w}_2 + 2\nu w_2 - \nu w_2 w_1 &= 0 \quad t > 0 \\ w_1(0) = 1, \quad w_2(0) &= \kappa\nu \end{aligned}$$

Two stationary points: $(1, 0)$ and $(2, \nu/(2\lambda))$

$w = (1, 0)$ is stable, corresponding to Couette flow between two parallel plates or Poiseuille flow in a pipe, and $(2, \nu/(2\lambda))$ is unstable

ODE-model for transition

Find $w(t) = (w_1(t), w_2(t))$ such that

$$\begin{aligned} \dot{w}_1 + \nu w_1 - \lambda w_1 w_2 &= \nu \quad t > 0 \\ \dot{w}_2 + 2\nu w_2 - \nu w_2 w_1 &= 0 \quad t > 0 \\ w_1(0) = 1, \quad w_2(0) &= \kappa\nu \end{aligned}$$

We shall show how a small pertubation $\kappa\nu$ may take the flow from the stable base flow $(1,0)$ to the unstable flow $(2, \nu/(2\lambda))$ if $\lambda\kappa$ is larger than some critical value of moderate size

ODE-model for transition

Find $w(t) = (w_1(t), w_2(t))$ such that

$$\begin{aligned} \dot{w}_1 + \nu w_1 - \lambda w_1 w_2 &= \nu \quad t > 0 \\ \dot{w}_2 + 2\nu w_2 - \nu w_2 w_1 &= 0 \quad t > 0 \\ w_1(0) = 1, \quad w_2(0) &= \kappa\nu \end{aligned}$$

We see that w_1 will grow as long as $\lambda w_2 > \nu$, and w_2 starts to grow when $w_1 > 2$

ODE-model for transition

Find $w(t) = (w_1(t), w_2(t))$ such that

$$\begin{aligned} \dot{w}_1 + \nu w_1 - \lambda w_1 w_2 &= \nu \quad t > 0 \\ \dot{w}_2 + 2\nu w_2 - \nu w_2 w_1 &= 0 \quad t > 0 \\ w_1(0) = 1, \quad w_2(0) &= \kappa\nu \end{aligned}$$

If w_1 reaches the value 2 before w_2 gets below ν/λ, then (w_1, w_2) will blow up, corresponding to instability

Linearized equations at $(1, 0)$

$$
\begin{aligned}
\dot{\varphi}_1 + \nu\varphi_1 - \lambda\varphi_2 &= \nu \quad t > 0 \\
\dot{\varphi}_2 - \nu\varphi_2 &= 0 \quad t > 0 \\
\varphi_1(0) = 0, \quad \varphi_2(0) &= \kappa\nu
\end{aligned}
$$

Linearized equations at $(1,0)$

$$\begin{aligned} \dot\varphi_1 + \nu\varphi_1 - \lambda\varphi_2 &= \nu \quad t>0 \\ \dot\varphi_2 - \nu\varphi_2 &= 0 \quad t>0 \\ \varphi_1(0)=0, \quad \varphi_2(0) &= \kappa\nu \end{aligned}$$

We have

$$w_1(t) \approx 1 + \lambda\kappa t\nu \exp(-t\nu), \quad w_2(t) \approx \kappa\nu \exp(-t\nu)$$

which shows slow growth in w_1 and slow decay in w_2 over a long time scale prior to the blow up, occuring if $\lambda\kappa$ is above a certain threshold

Computational experiment 1

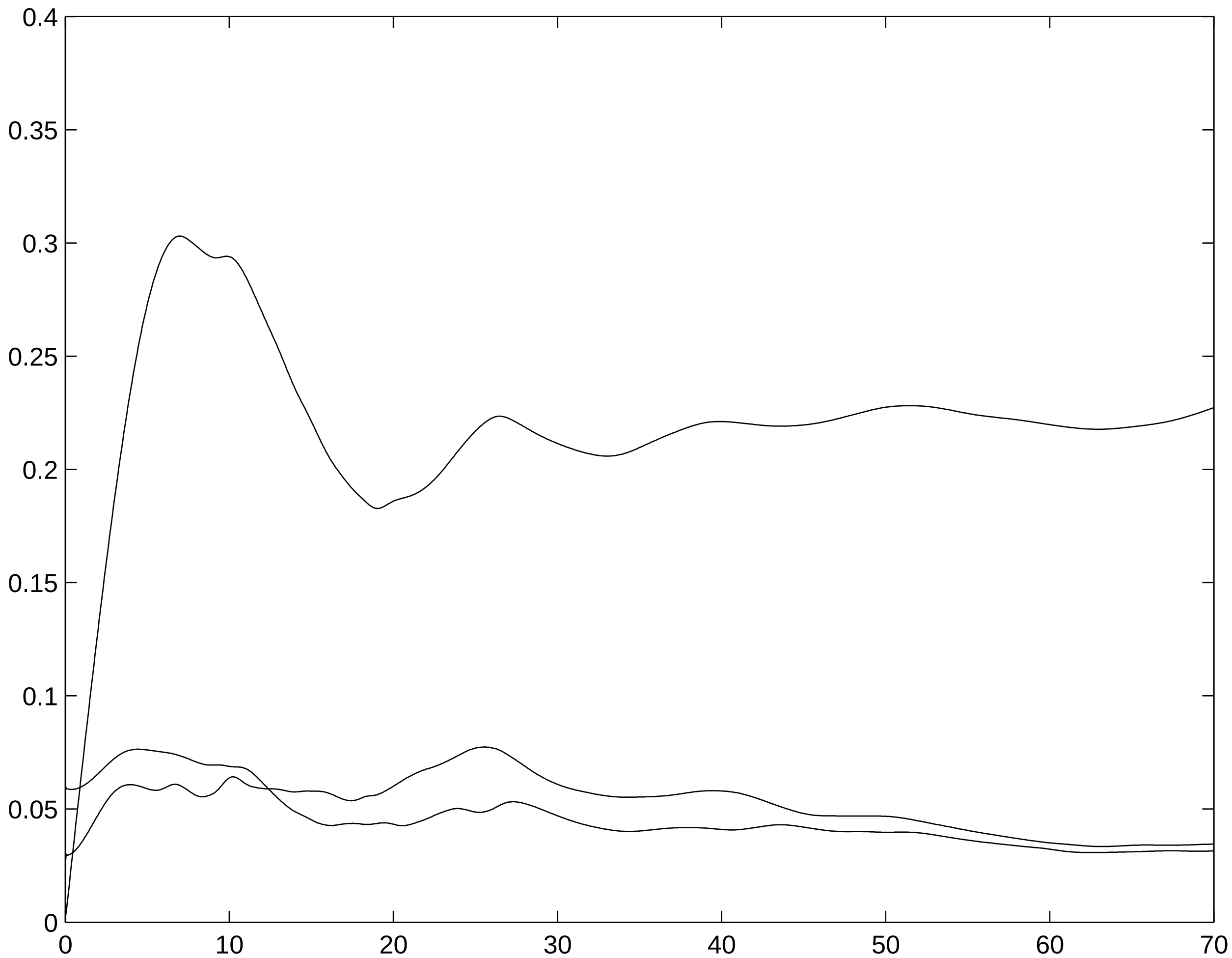
'Poiseuille flow' driven by a force
$f = (32\nu(x_2(1-x_2) + x_3(1-x_3)), 0, 0)$

- rectangular cross section 1×1
- no slip b.c. at walls
- periodic b.c. in streamwise direction
- cG(1)cG(1) method, regular tetrahedral mesh with $h = 1/64$

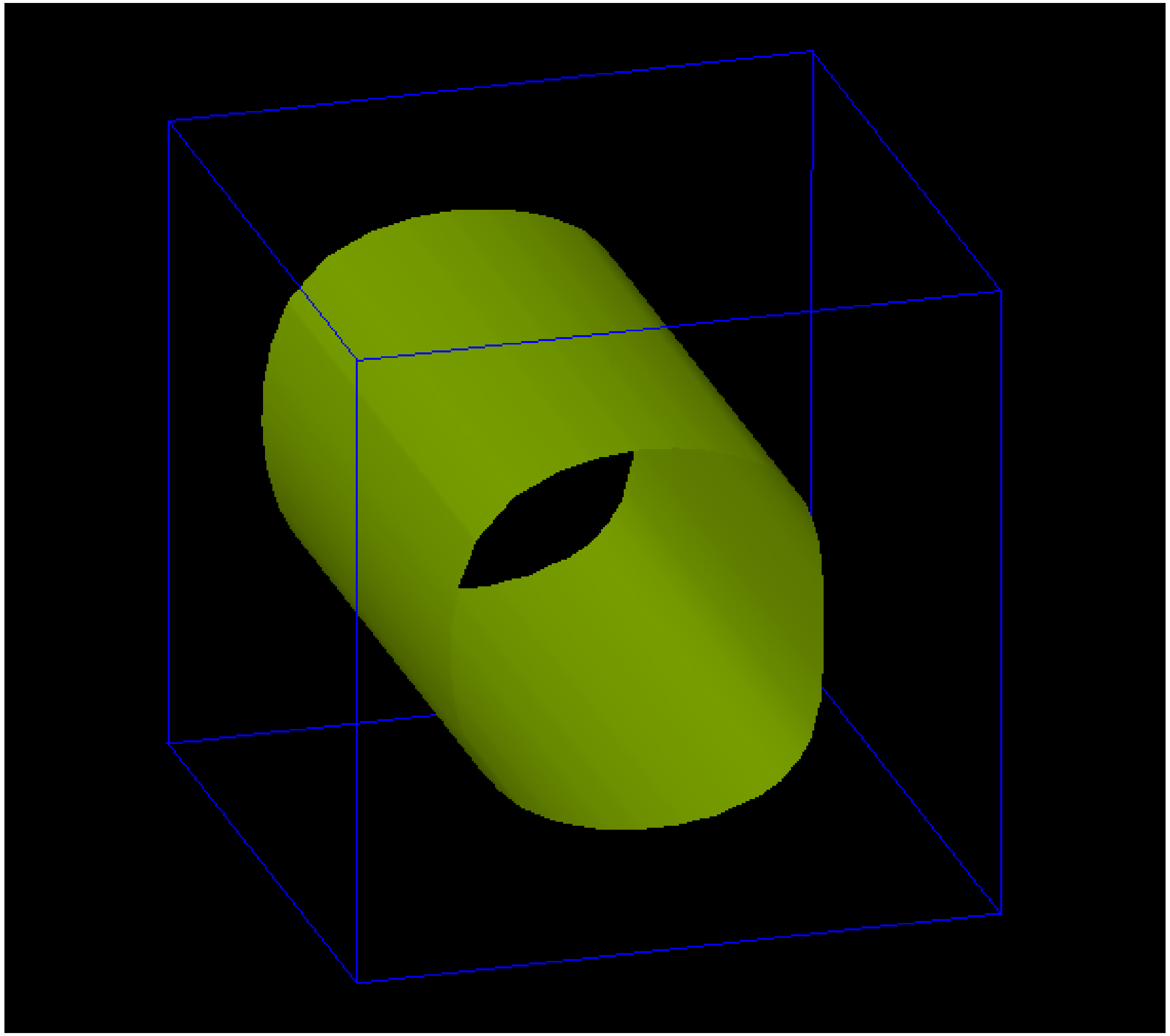
Computational experiment 1

- small initial transversal pertubation $\bar{\varphi}(x_2, x_3)$ of order 0.1
- very small x_1-dependent driving force of order 10^{-3} creating and sustaining very small streamwise variation

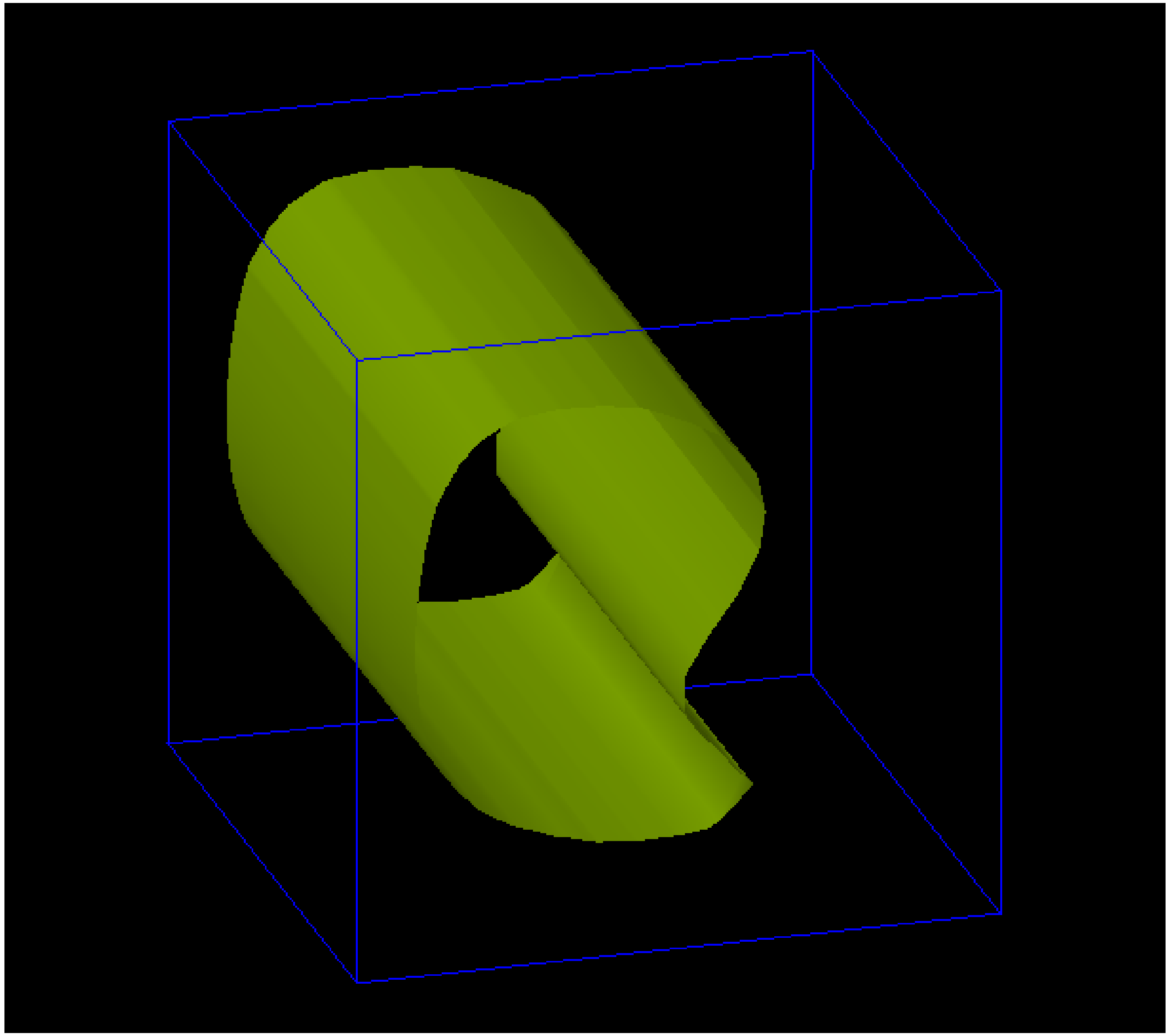
$\|\varphi_1\|$, $\|\varphi_2\|$, $\|\varphi_3\|$, linear growth in $\|\varphi_1\|$



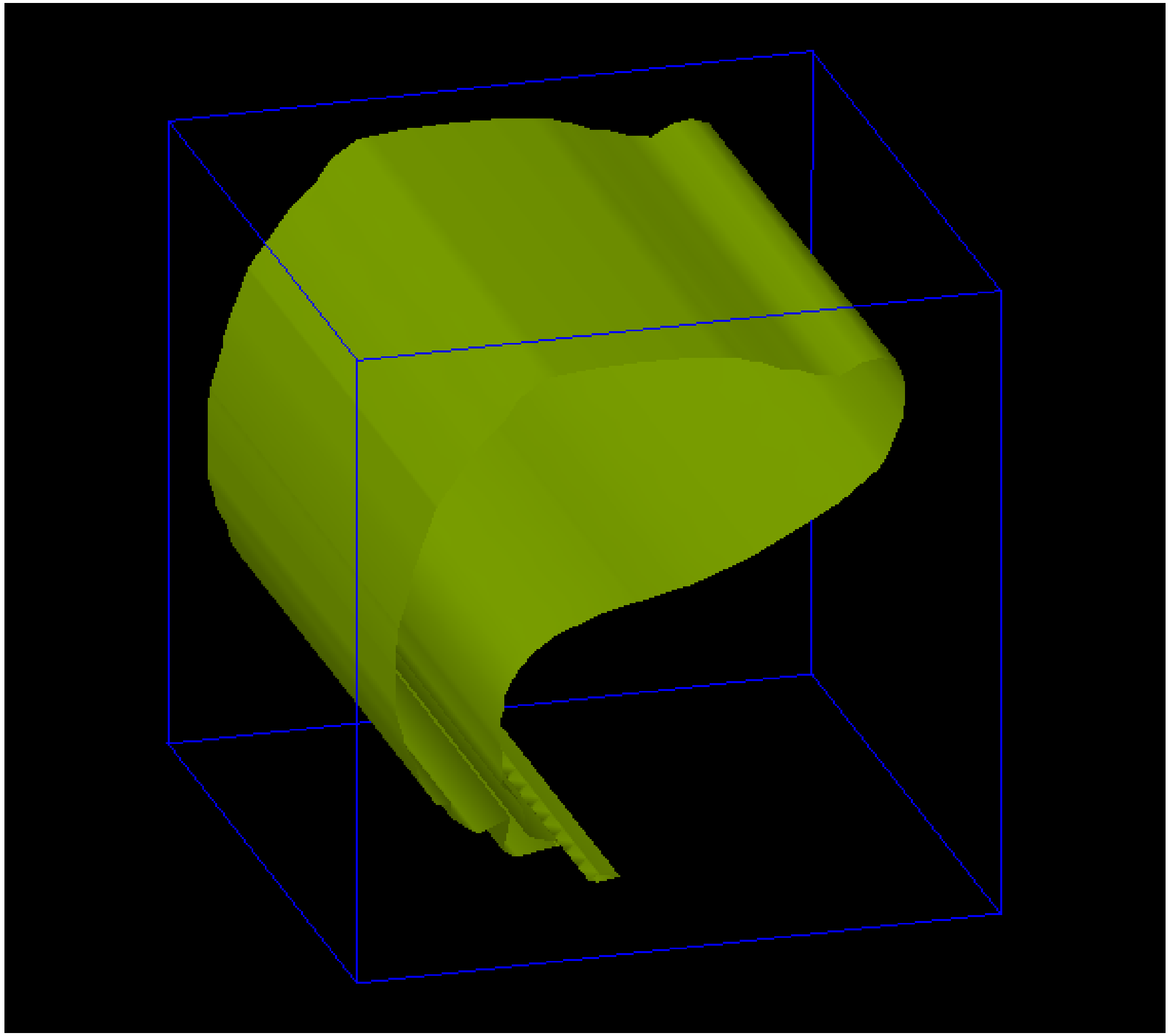
$\varphi_1, \quad t = 1$ (Taylor-Görtler mech.)



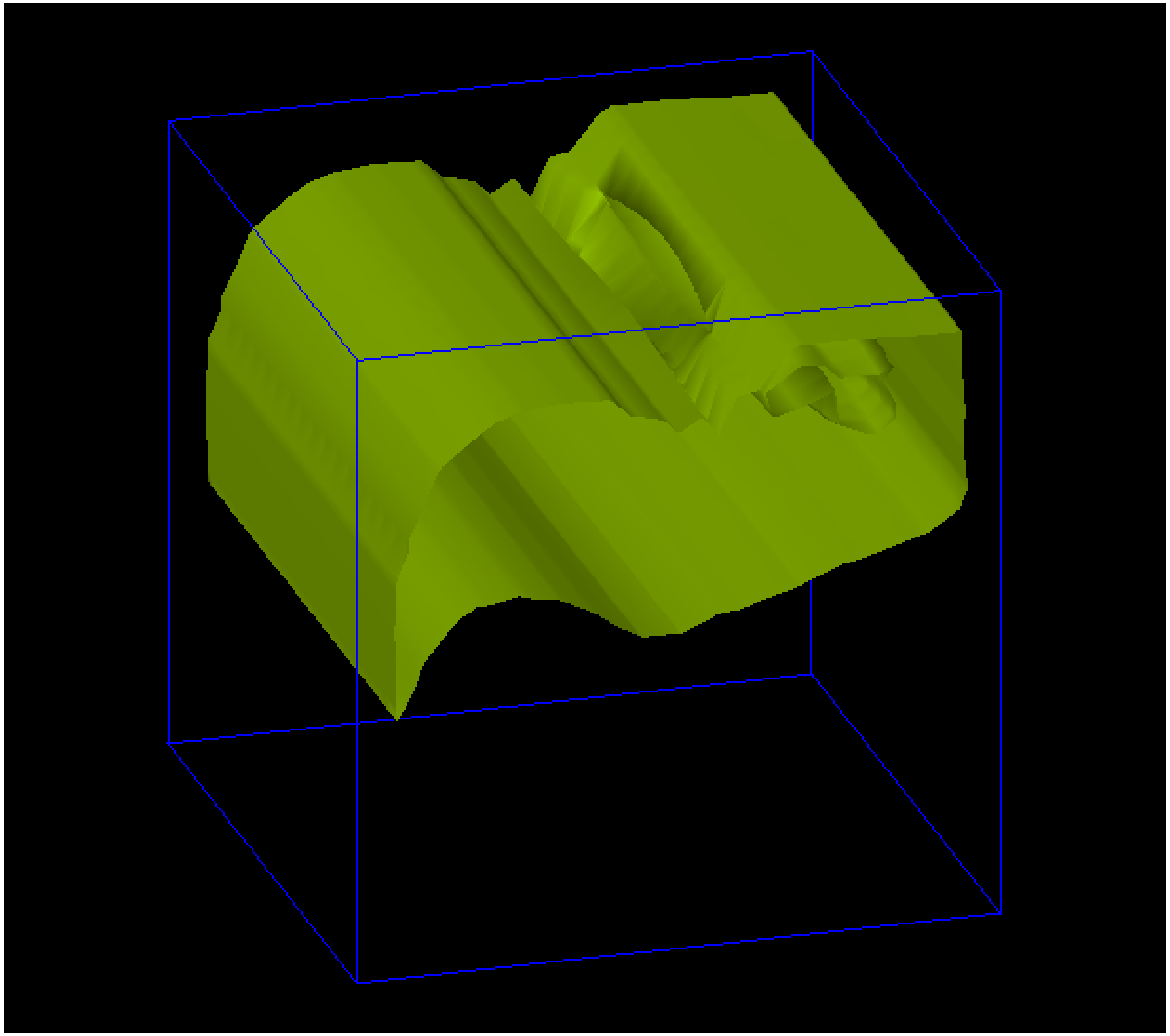
$\varphi_1, \quad t = 3$ (Taylor-Görtler mech.)



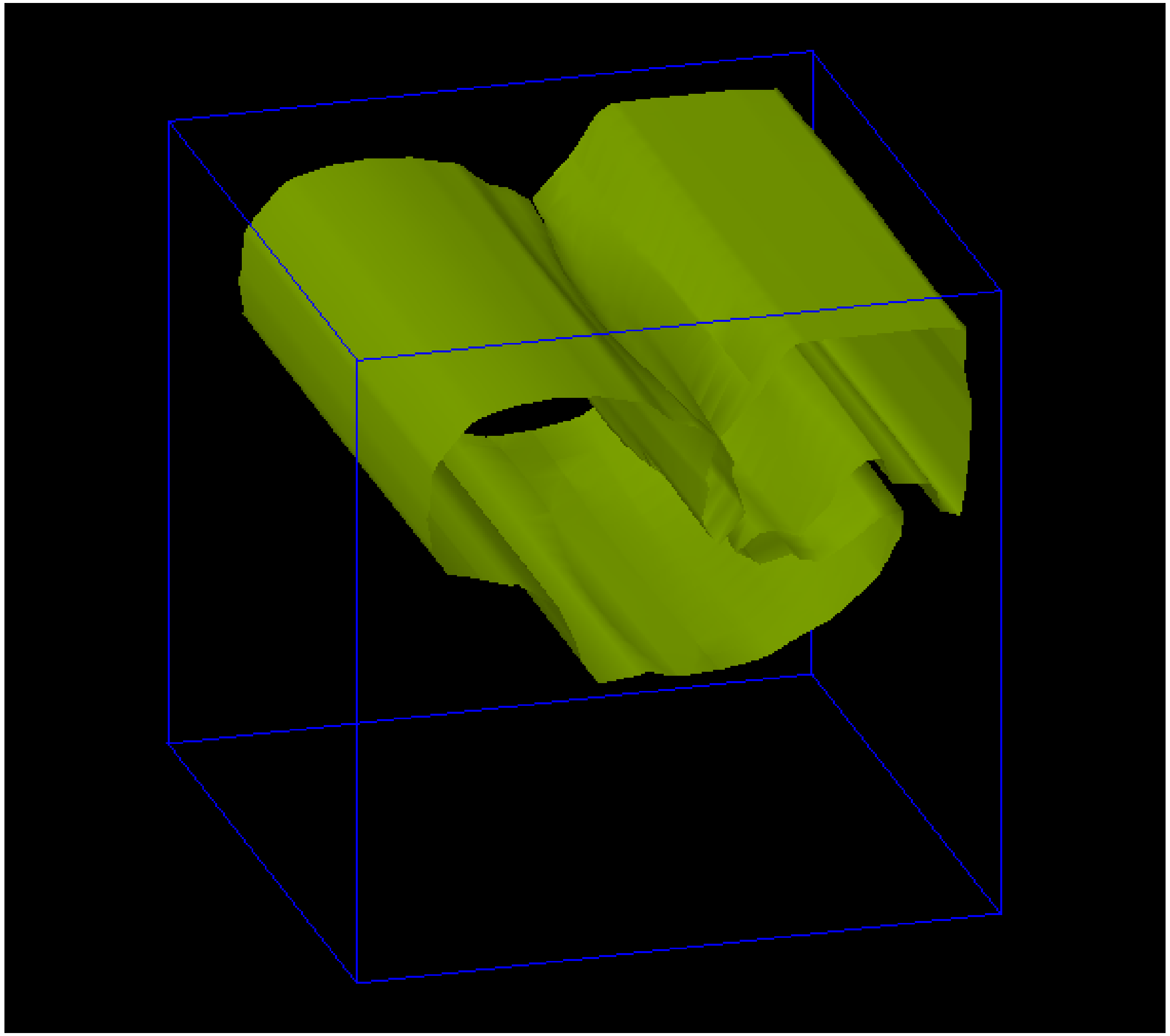
$\varphi_1, \quad t = 6$ (Taylor-Görtler mech.)



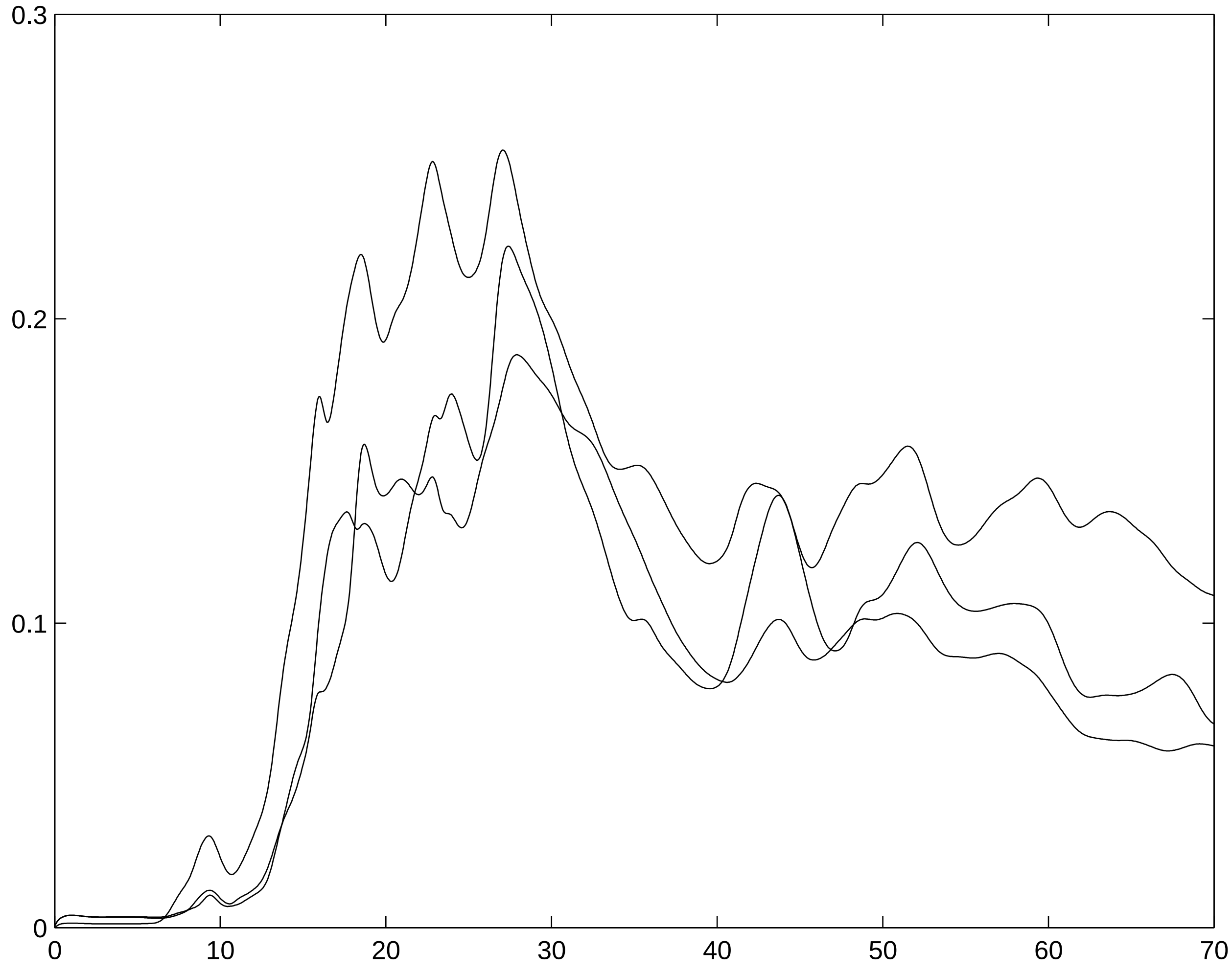
$\varphi_1, \quad t = 9$ (Taylor-Görtler mech.)



$\varphi_1, \quad t = 12$ (Taylor-Görtler mech.)



$||du_i/dx_1||$



Computational experiment 2

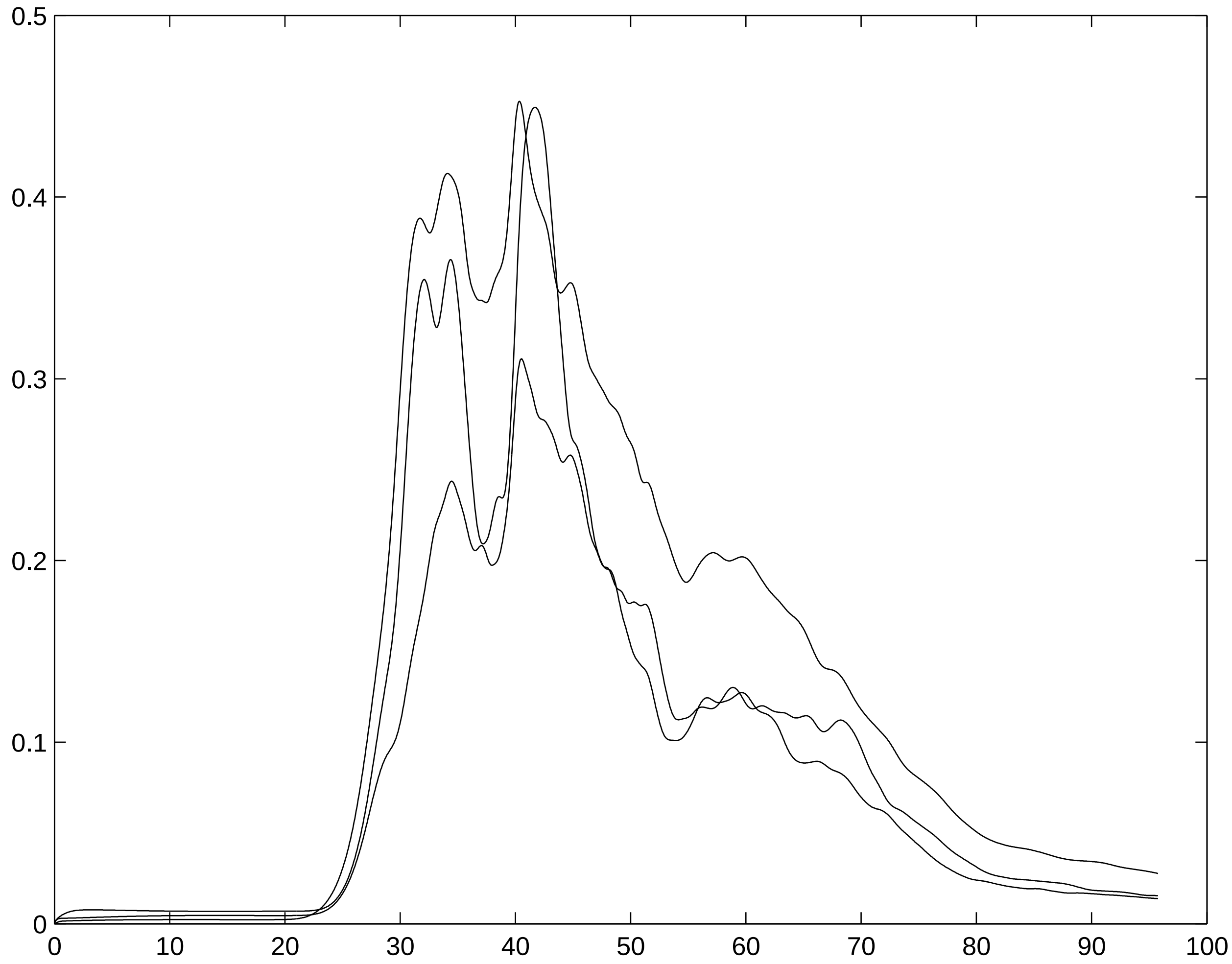
Couette flow driven by ± 1 on the top and bottom

- rectangular cross section 1×1
- slip b.c. at spanwise walls
- periodic b.c. in streamwise direction
- cG(1)cG(1) method, regular tetrahedral mesh with $h = 1/64$

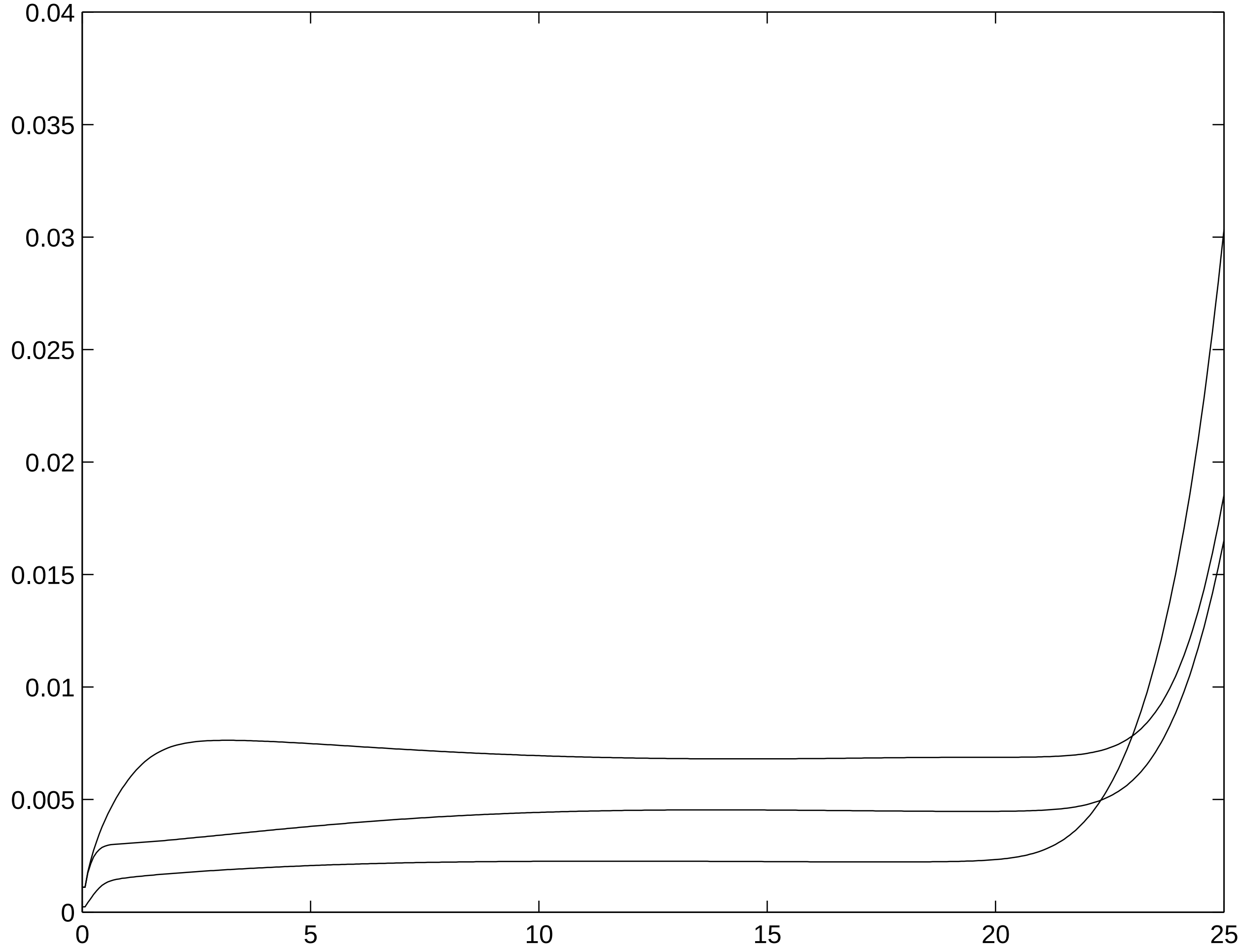
Computational experiment 2

- small initial pertubation $\bar{\varphi}(x_2, x_3)$ of order 0.1
- very small x_1-dependent driving force of order 10^{-3} creating and sustaining very small streamwise variation

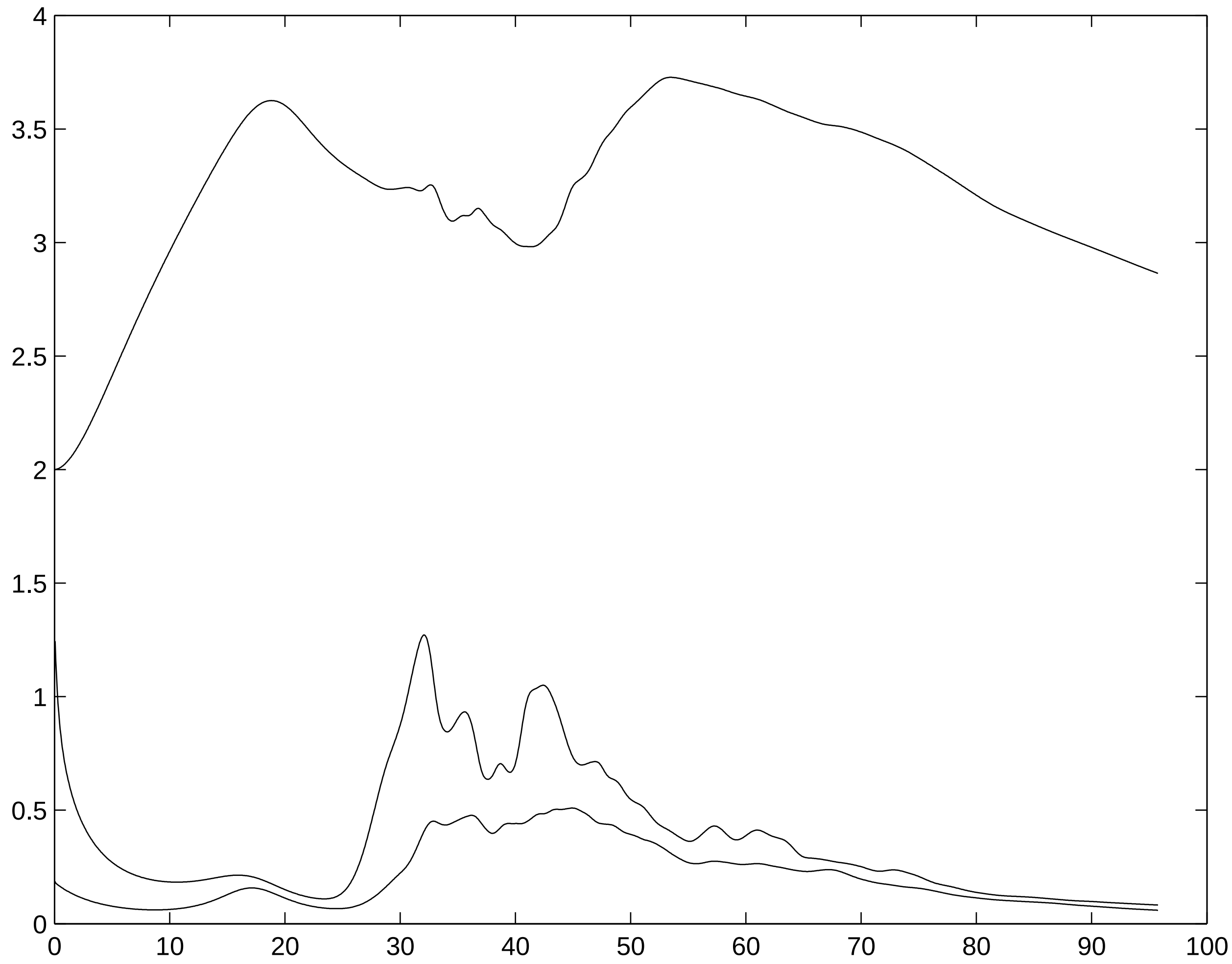
$\|du_i/dx_1\|$



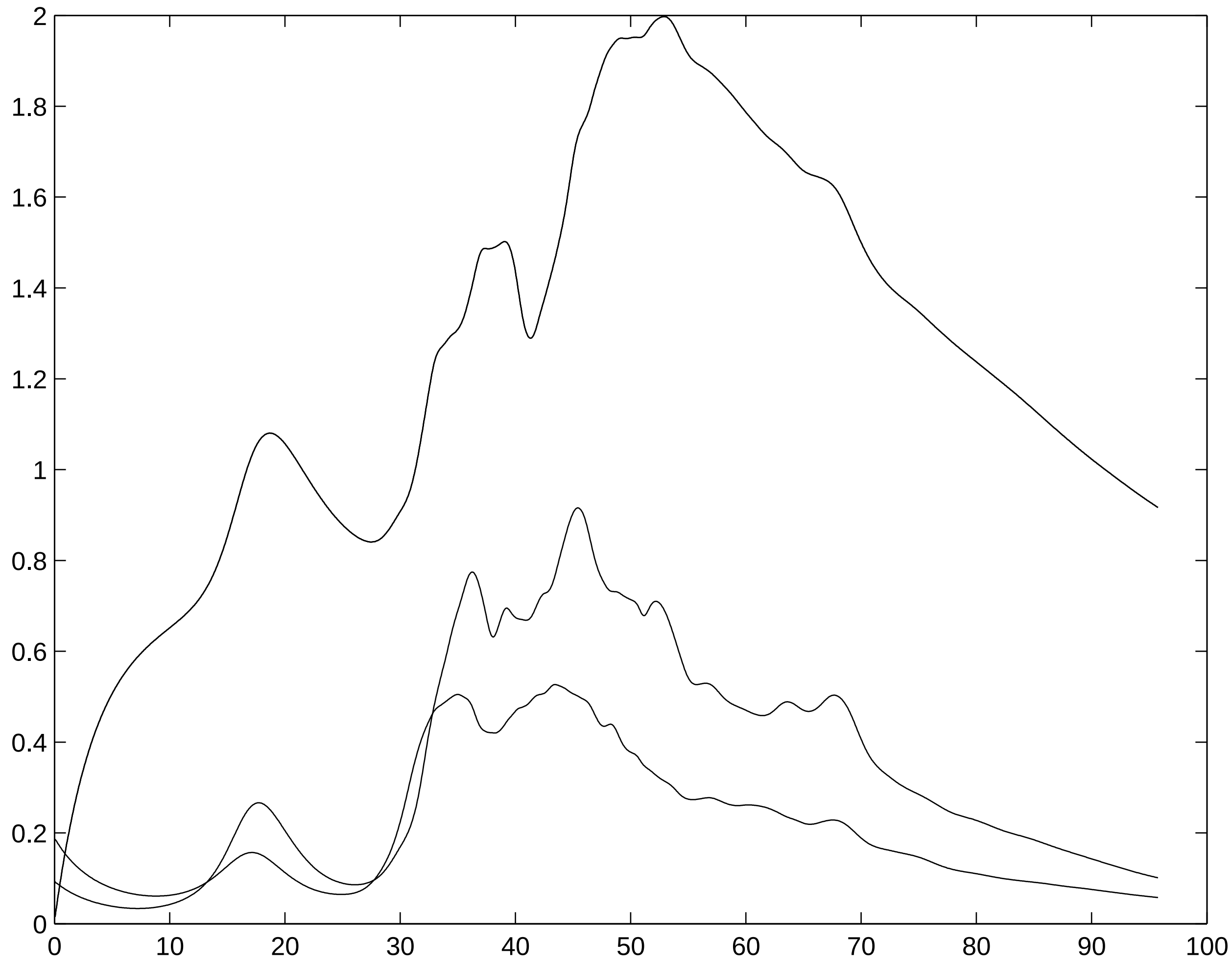
$||du_i/dx_1||$



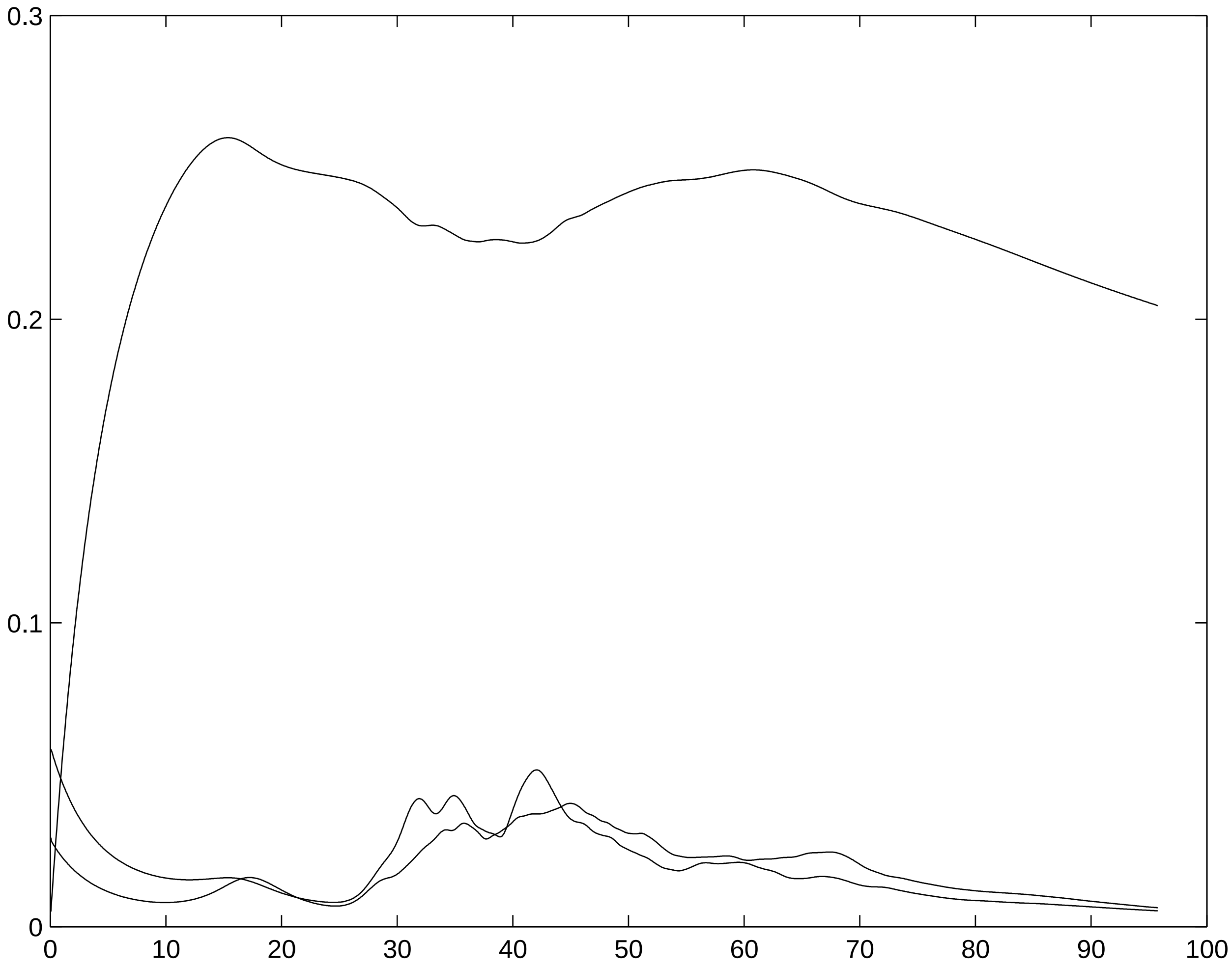
$||du_i/dx_2||$



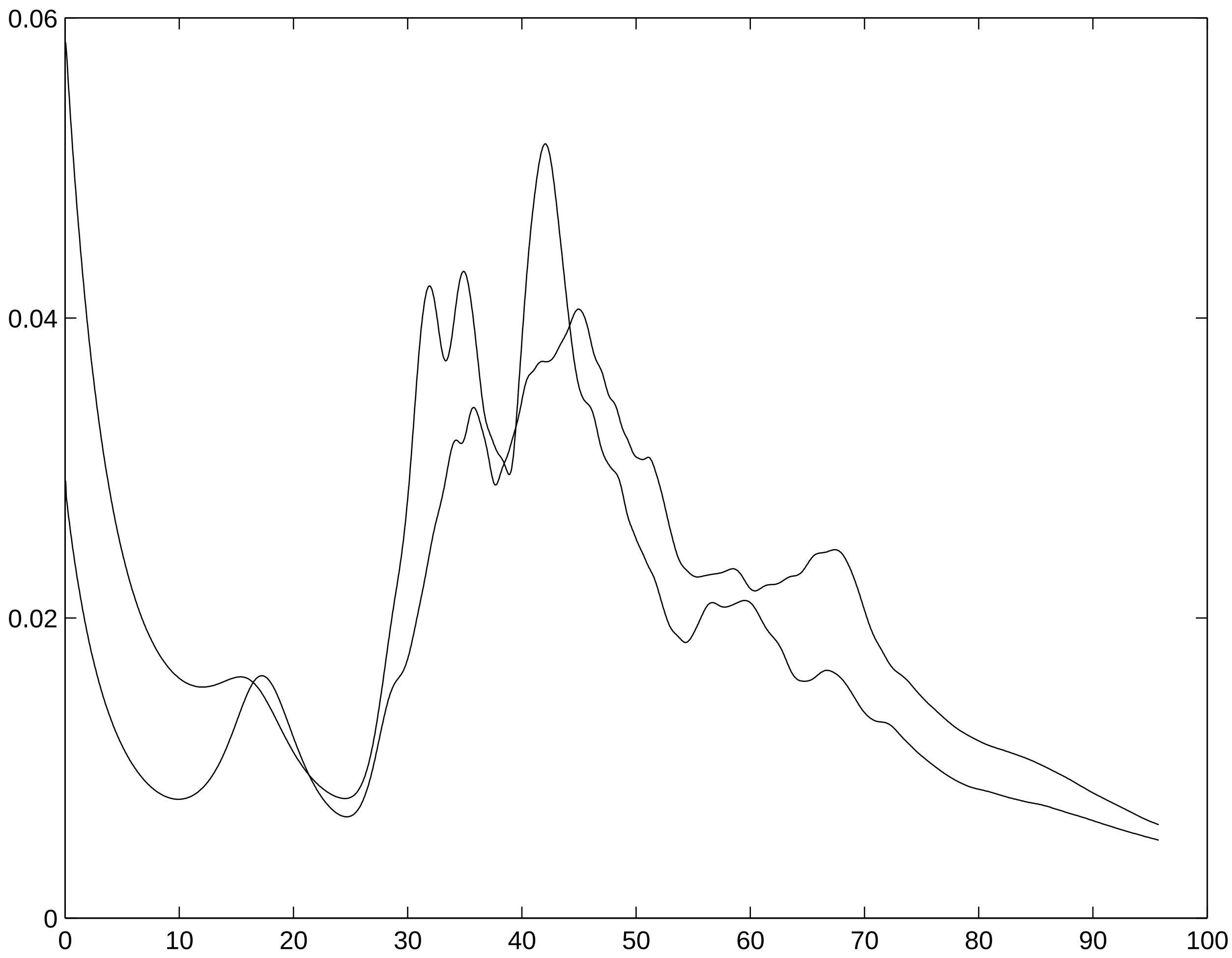
$\|du_i/dx_3\|$



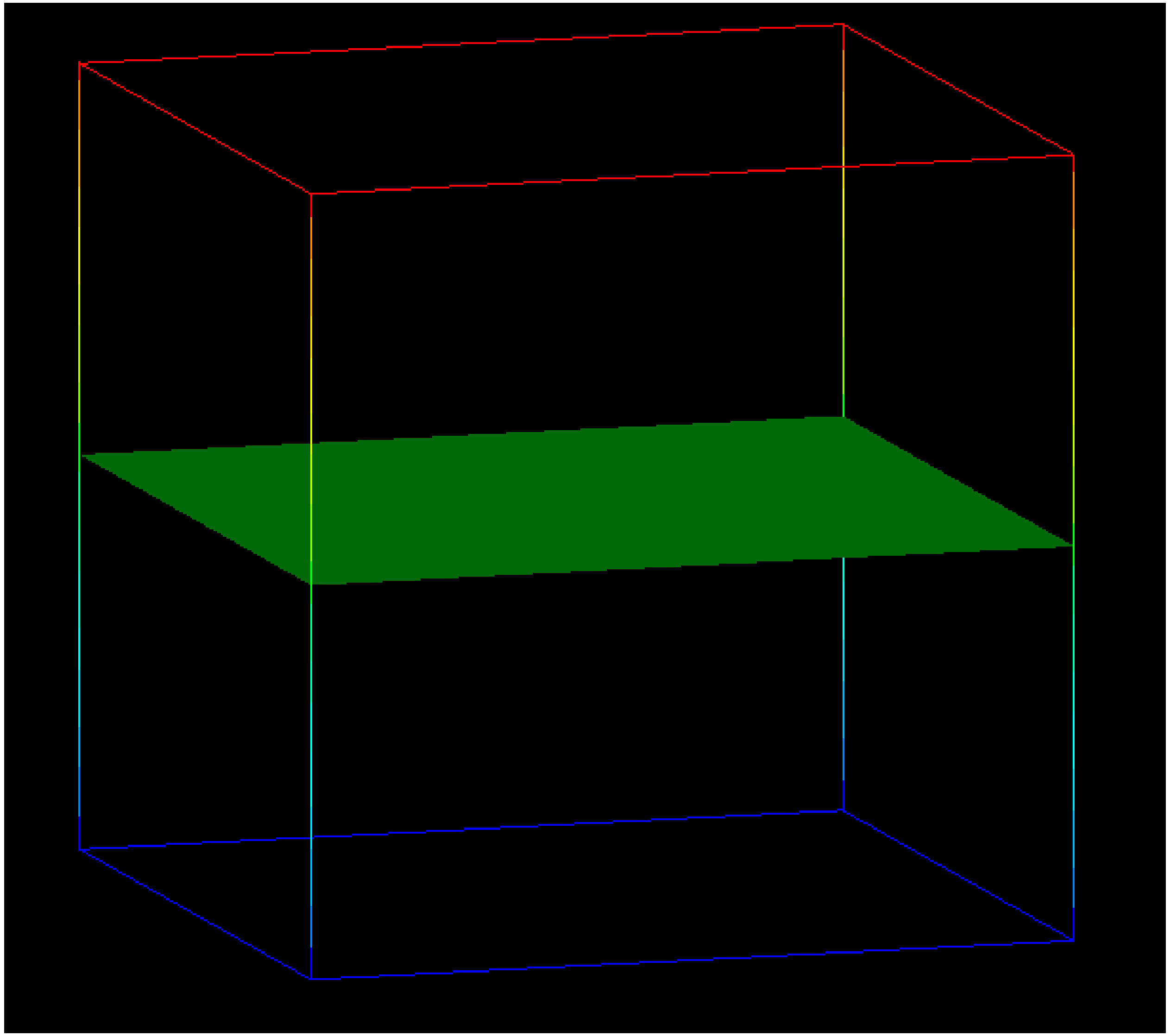
$\|\varphi_1\|$, $\|\varphi_2\|$, $\|\varphi_3\|$, linear growth in $\|\varphi_1\|$



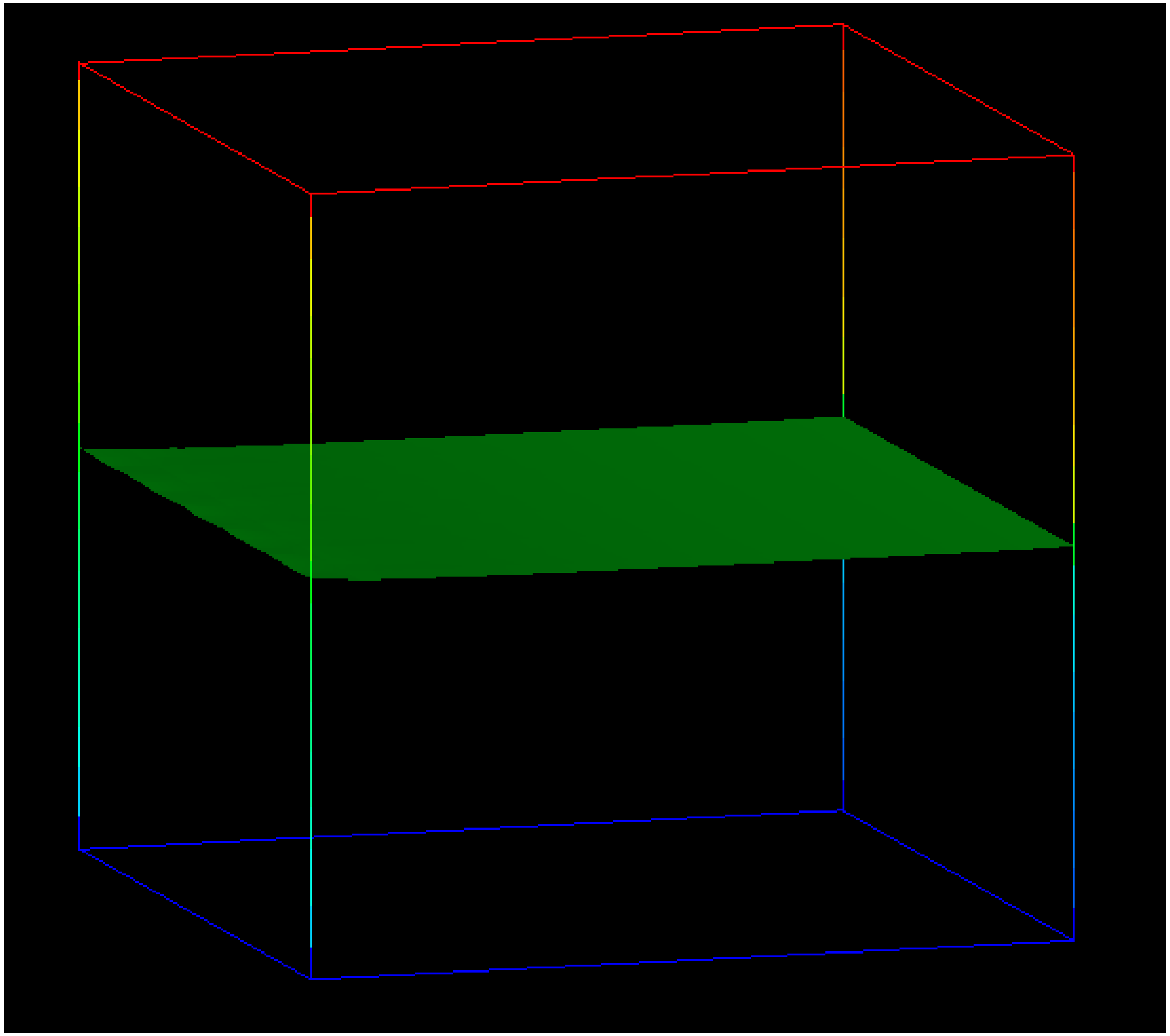
$\|\varphi_2\|, \|\varphi_3\|$



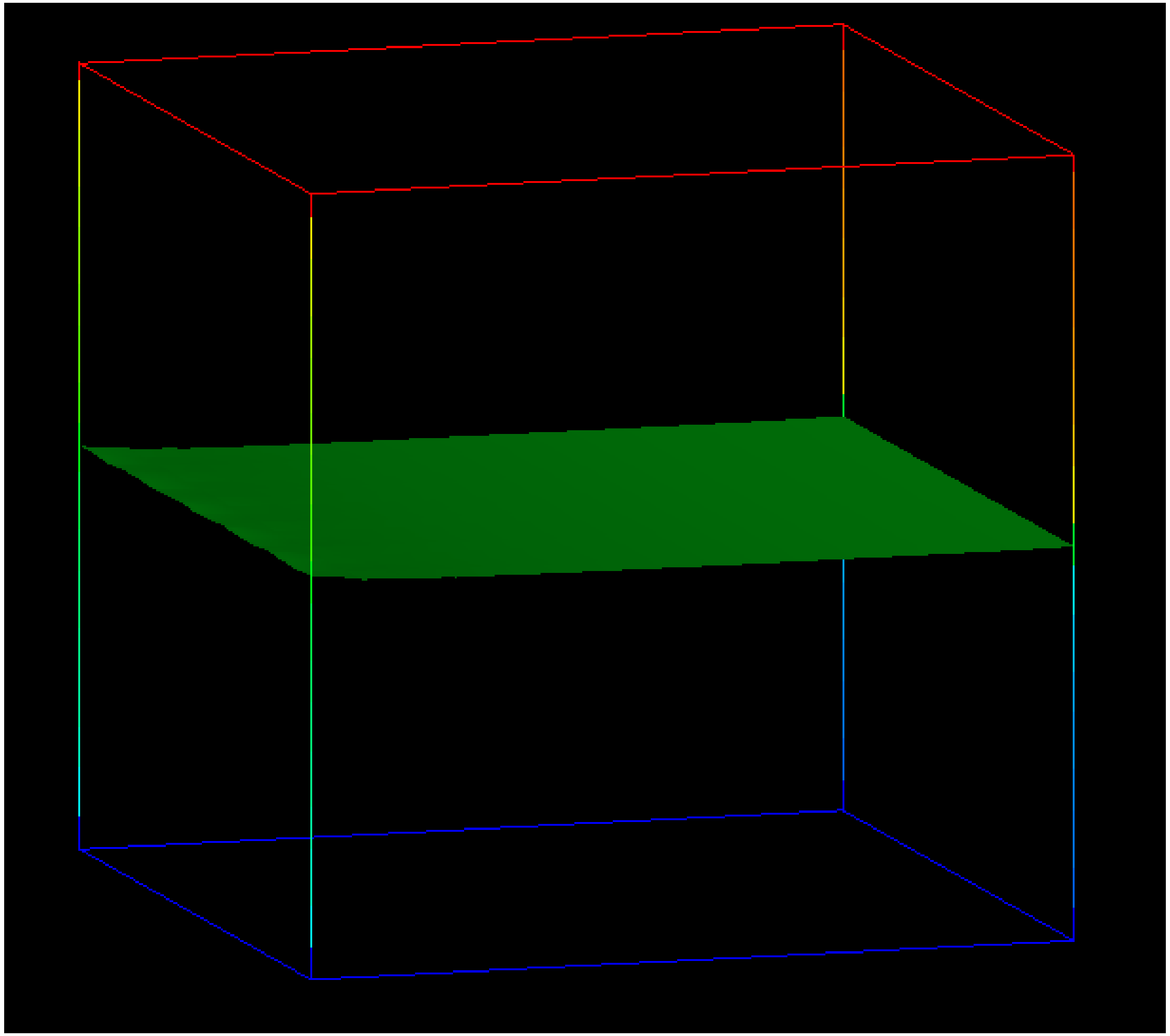
$\varphi_1, \quad t = 0,$ isosurface for $u_1 = 0$



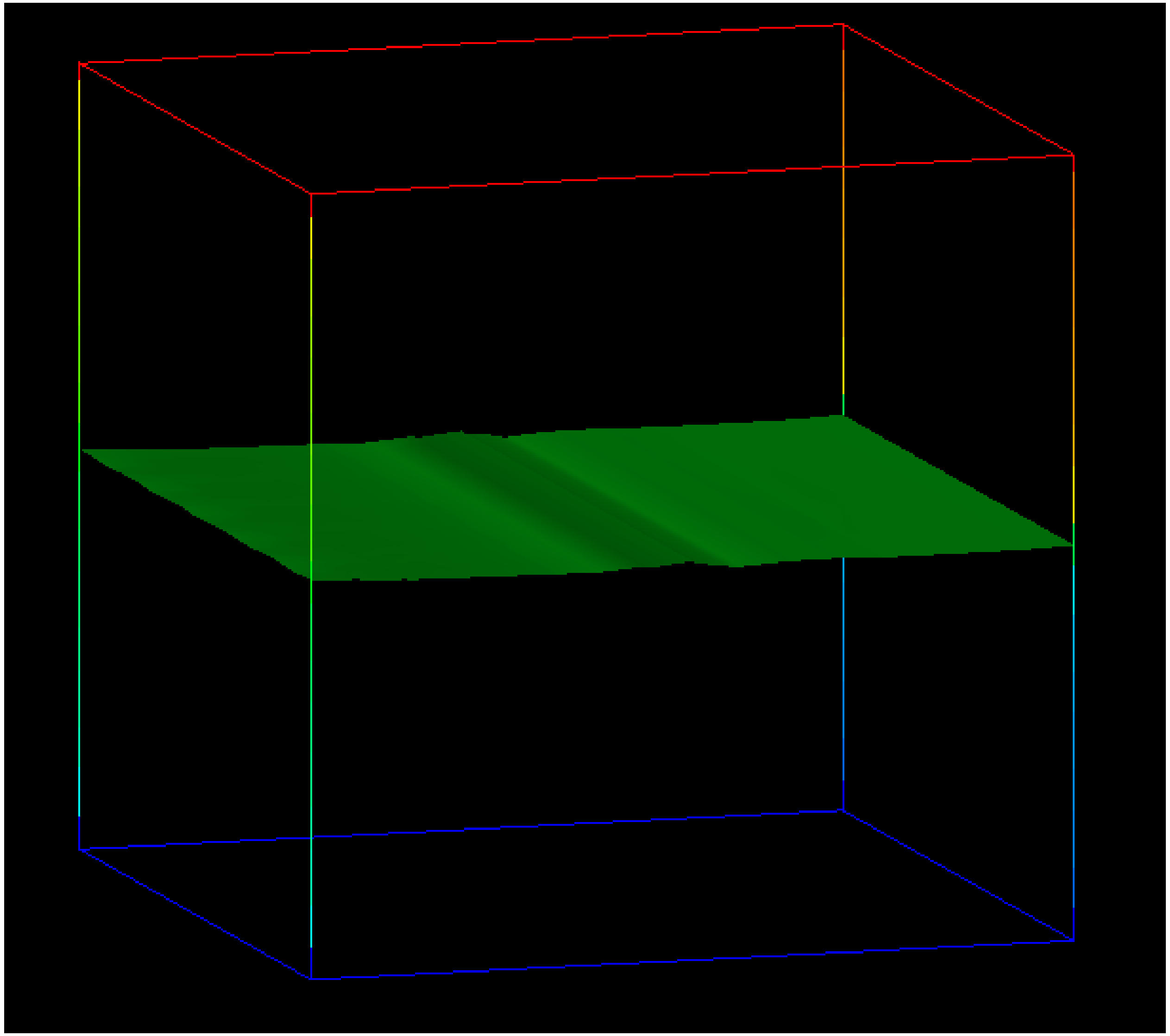
$\varphi_1, \quad t = 5, \quad$ isosurface for $u_1 = 0$



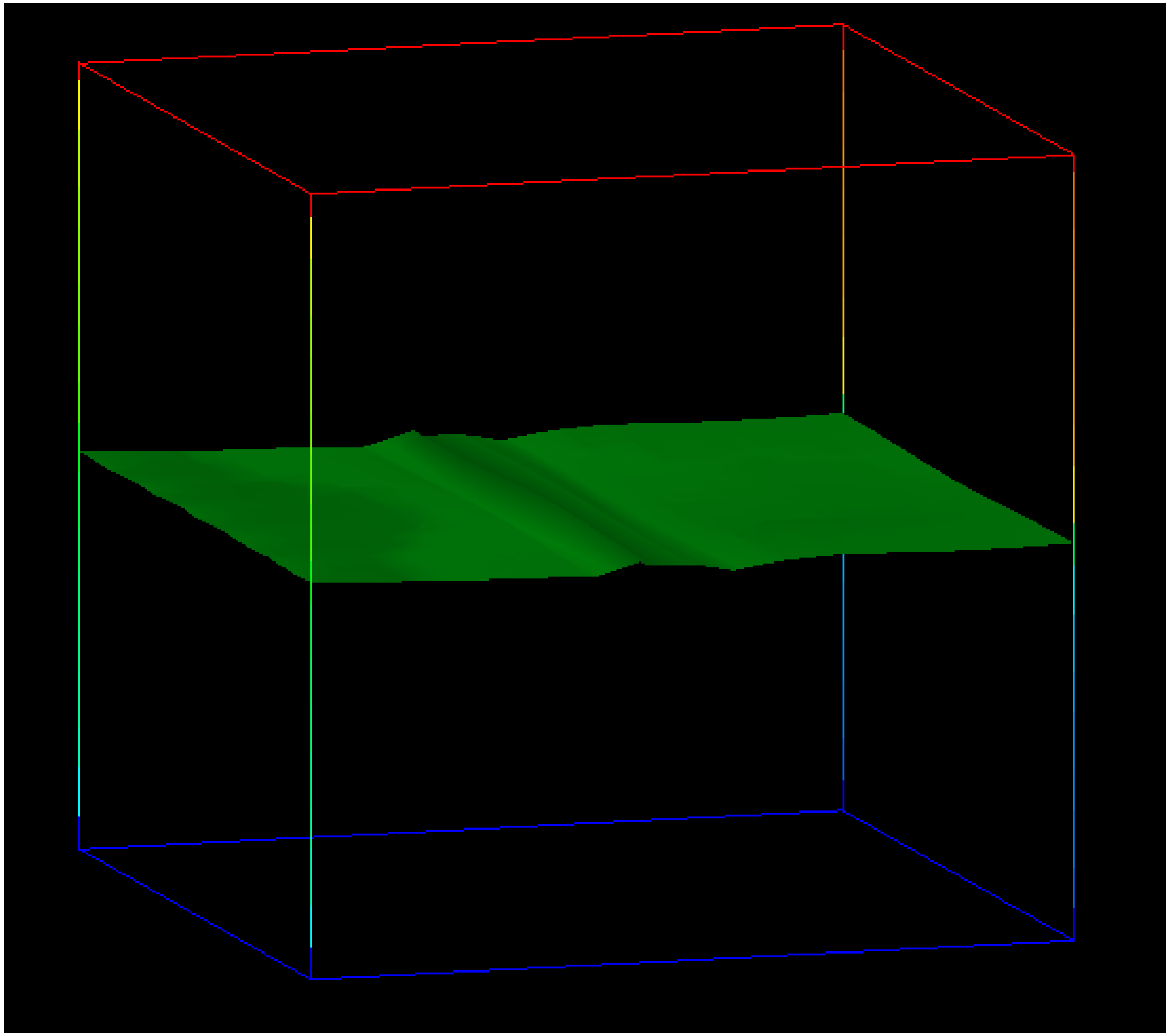
$\varphi_1, \quad t = 10,$ **isosurface for** $u_1 = 0$



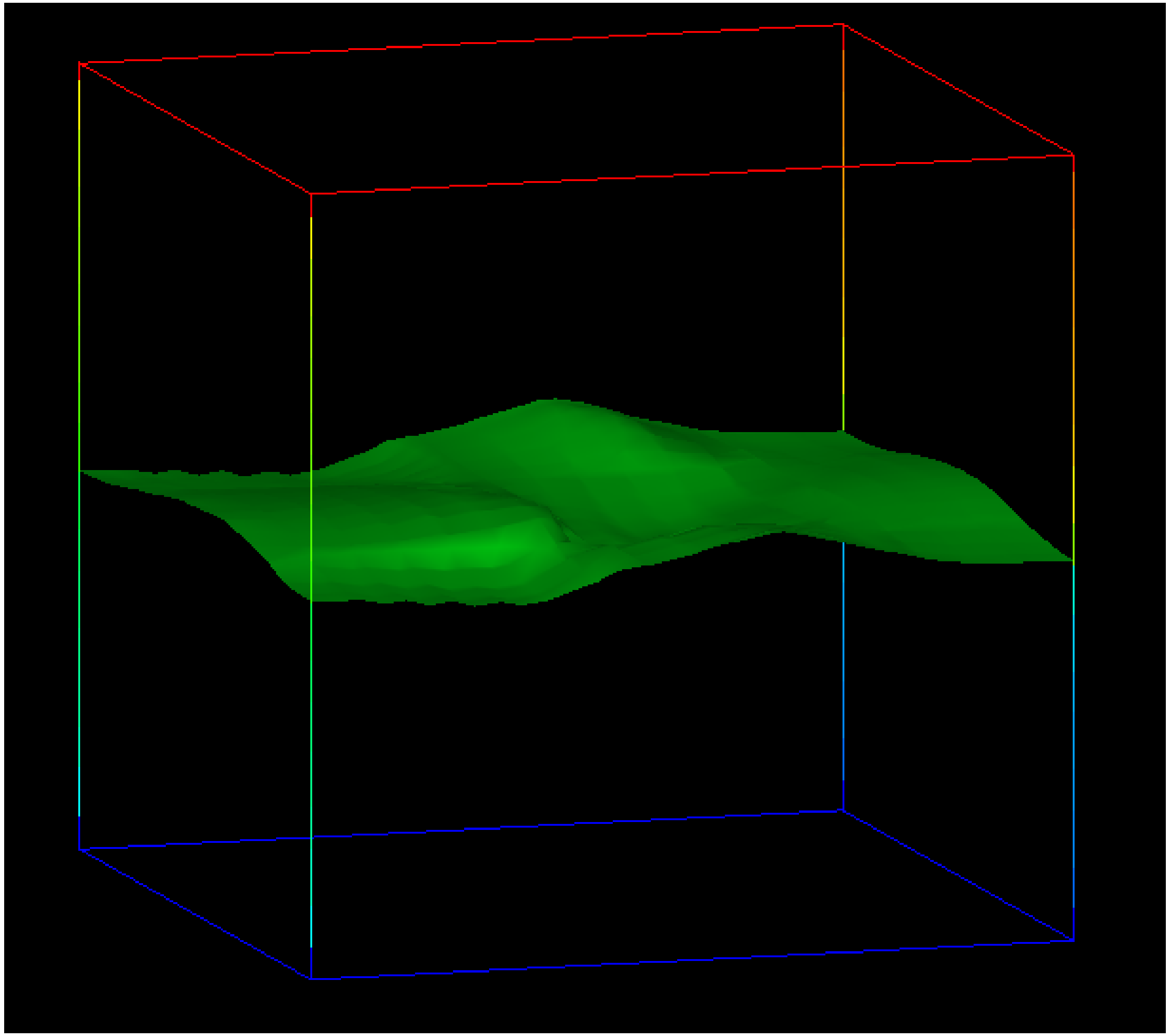
$\varphi_1, \quad t = 20, \quad$ isosurface for $u_1 = 0$



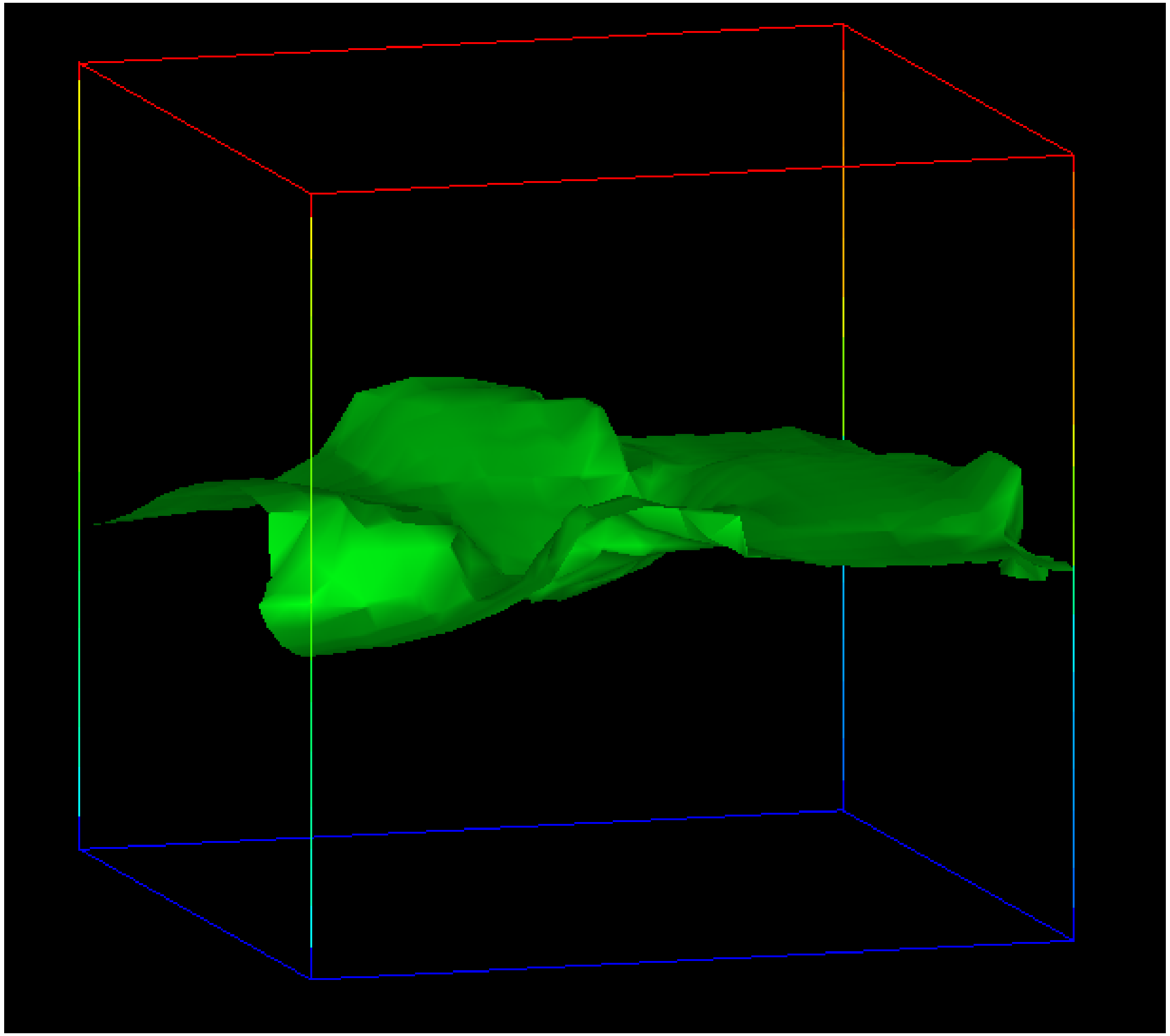
$\varphi_1, \quad t = 25, \quad$ isosurface for $u_1 = 0$



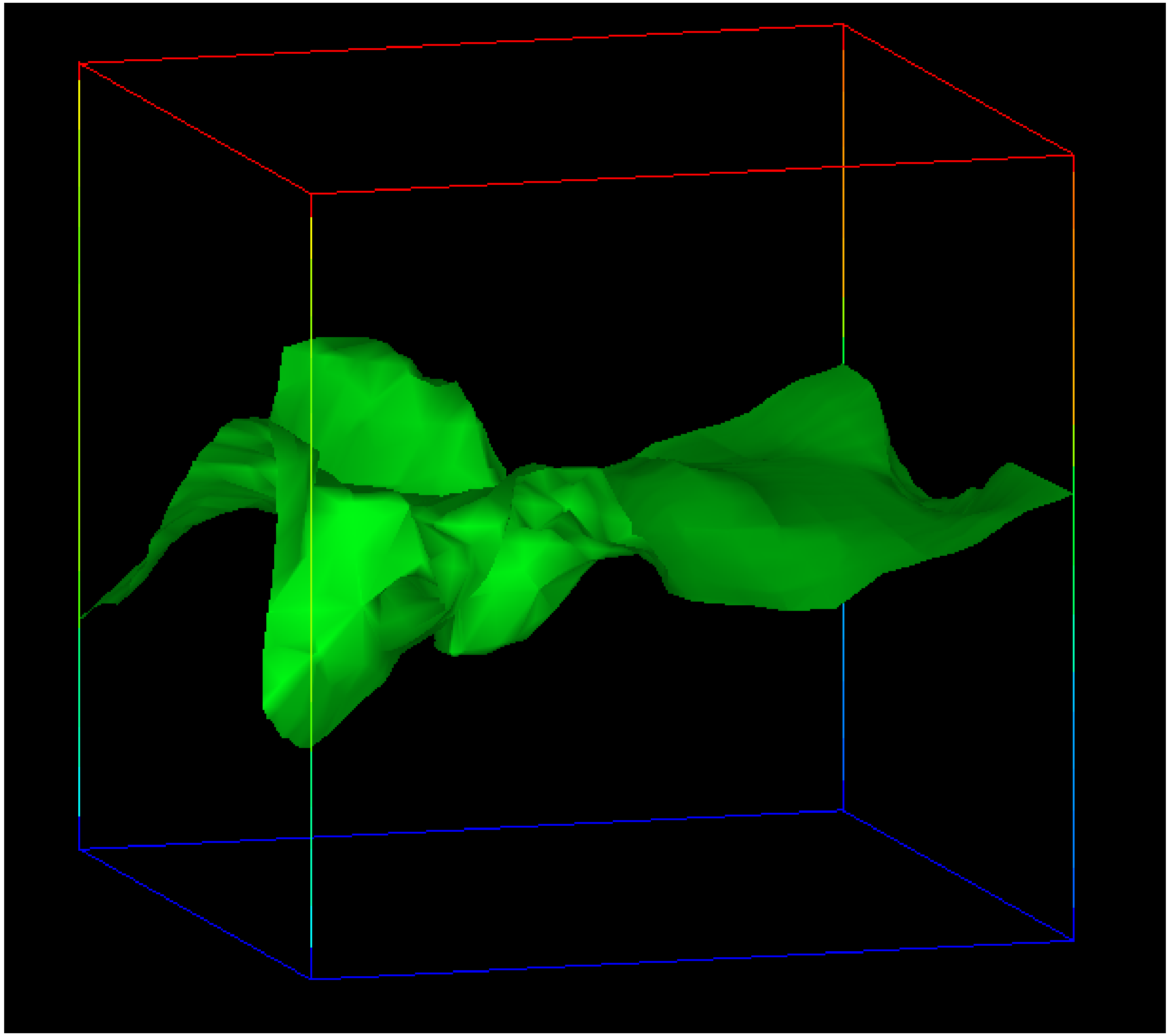
$\varphi_1, \quad t = 30,$ **isosurface for** $u_1 = 0$



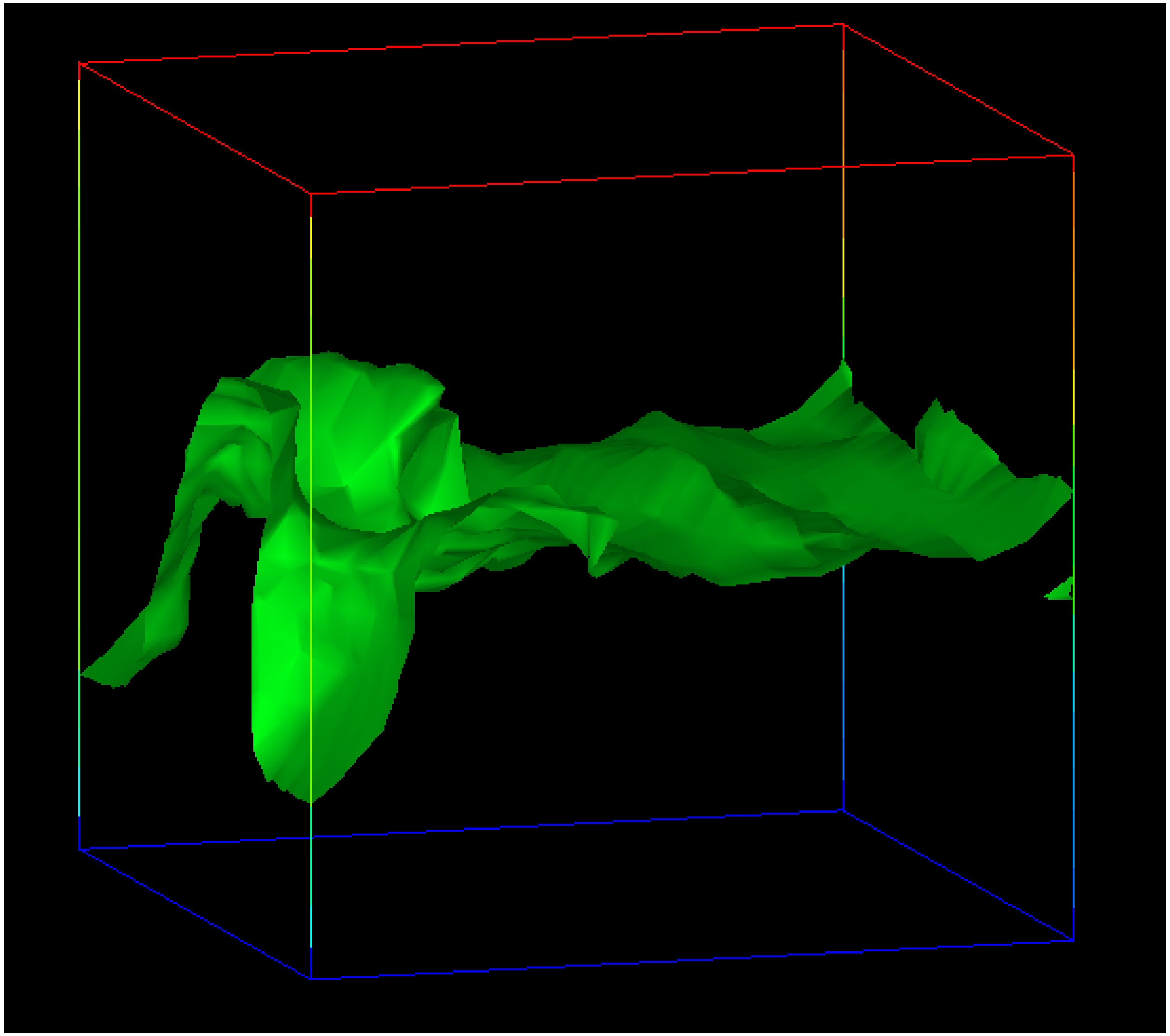
$\varphi_1, \quad t = 32, \quad$ isosurface for $u_1 = 0$



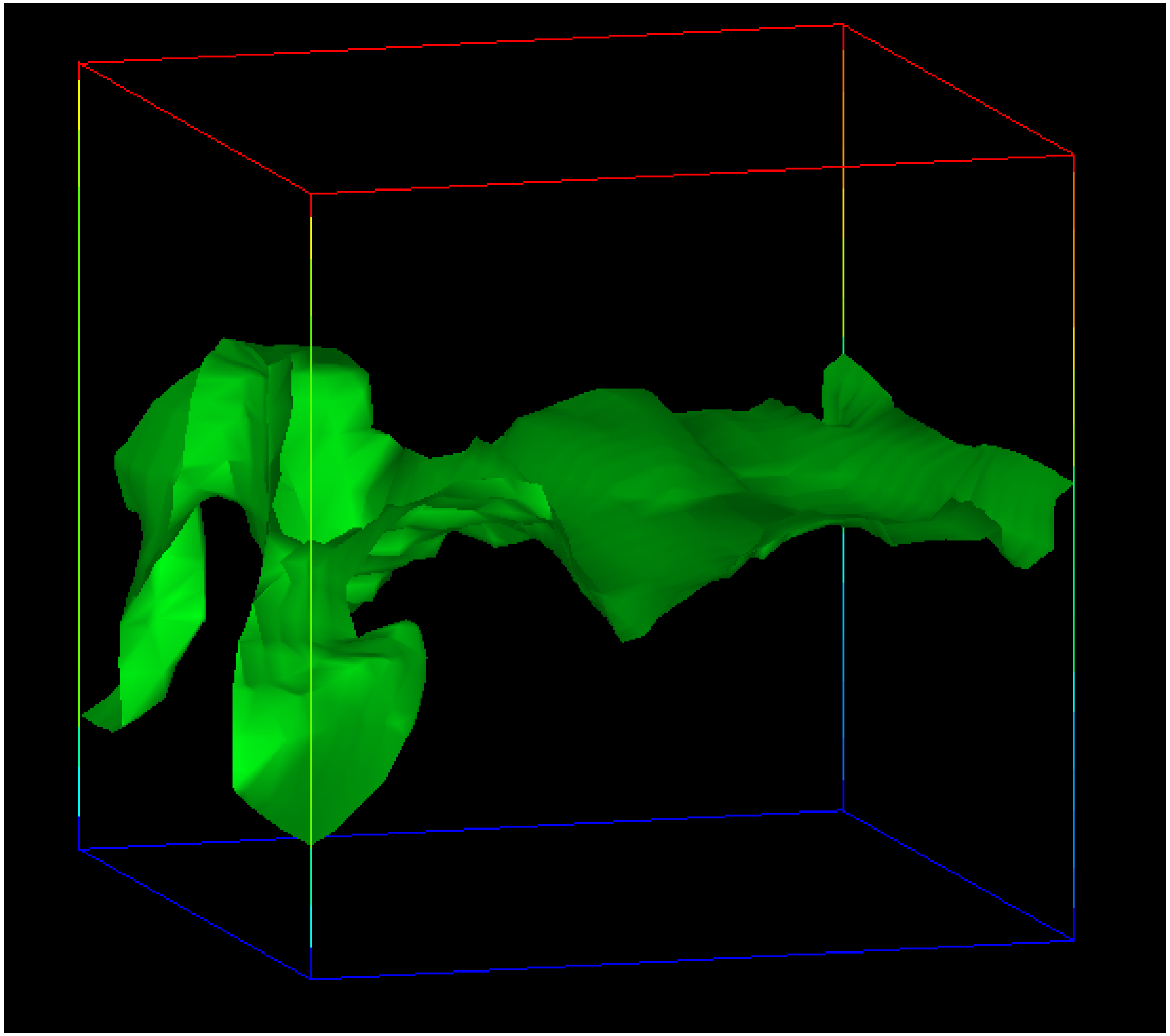
$\varphi_1, \quad t = 34, \quad$ isosurface for $u_1 = 0$



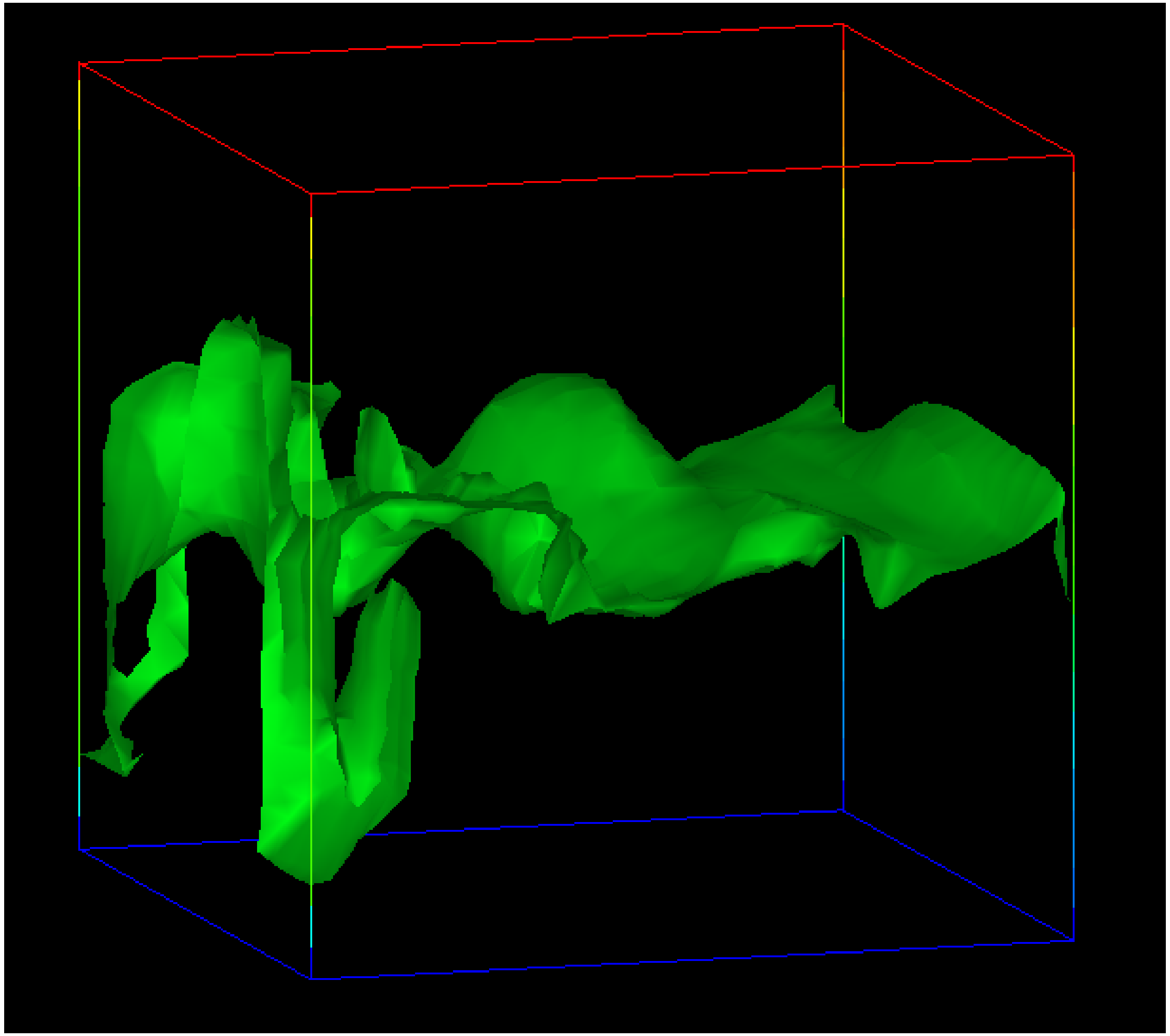
$\varphi_1, \quad t = 36, \quad$ isosurface for $u_1 = 0$



$\varphi_1, \quad t = 38,$ **isosurface for** $u_1 = 0$



$\varphi_1, \quad t = 40, \quad$ isosurface for $u_1 = 0$



Transition scenario

1. Lin. growth in φ_1 (Taylor-Görtler mechanism), du_1/dx_2 and du_1/dx_3
2. Initial decrease in φ_2, φ_3, and du_i/dx_j for $i, j = 1, 2$
3. Suddenly (Couette $t \approx 22$, Poiseuille $t \approx 6$) all derivatives grows exponentially (x_1-derivatives increases by a factor 100)

Comments

- The predictions of the ODE-model are verified by the numerical experiments
- Both computational experiments are questionable after the transition, due to coarse meshes and large stability factors

Summary, lecture 2

- Linearized dual problems contain information about the error propagation for different error measures
- In general are global quantities easier to compute than local quantities
- Even for dual problems linearized at a very irregular primal solution, global quantities may be computable (due to cancelations in the dual solution)

Summary, lecture 2

- Linear pertubation growth in streamwise velocity possible from small initial transversal pertubations (Taylor-Görtler mechanism)
- ODE-model explaining transition to turbulence from small initial pertubations