Adaptive Finite Element methods for incompressible fluid flow

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3 lectures

1. Adaptive FE methods for incompr. fluid flow
2. Hydrodynamic stability
3. Subgrid modeling & multi adaptivity

Subgrid modeling & Multi adaptivity

- General setting
- A posteriori error analysis - balancing numerical and modeling errors
- Dynamic large eddy simulations (DLES)
- Previous results on scale similarity subgrid modeling for convection-diffusion-reaction problems
- Multi adaptivity

Subgrid modeling

Turbulent flow is pointwise uncomputable on todays computers for most flows because of

1. unresolvable small-scale features

The finest scales in the flow are finer than the finest possible computational mesh (1024^3?), so the computed solution U cannot be pointwise close to the exact solution u

Subgrid modeling

Turbulent flow is pointwise uncomputable on todays computers for most flows because of

1. unresolvable small-scale features

Because of the non linearity of NSE, the unresolved scales also influence the resolved scales

Subgrid modeling

Turbulent flow is pointwise uncomputable on todays computers for most flows because of

1. unresolvable small-scale features
2. large stability factors

Pointwise quantities corresponds to local data in the linearized dual problem, giving large stability factors

Subgrid modeling

Turbulent flow is pointwise uncomputable on todays computers for most flows because of

1. unresolvable small-scale features
2. large stability factors

Linearizing the dual problem at the irregular turbulent flow u and the computed approximation U gives large stability factors

Subgrid modeling

A posteriori error estimation by duality of pointwise quantities in turbulent flow is hard on todays computers since

1. U cannot be pointwise close to $u \Rightarrow$ we get a large linearization error in the dual problem when we replace u by U
2. the dual problem has as fine scales as the exact solution u itself, making it expensive to solve

General setting

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

$u - \hat{u}$ = data/modeling error

$\hat{u} - U$ = discretization error

Total error = $u - U = u - \hat{u} + \hat{u} - U$

General setting

Mathematical model: $A(u) = f$ (MM)
Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

If h is the smallest computationally resolvable scale and the exact solution u contains smaller scales than h, then a pointwise accurate approximation of u is impossible.

General setting

Mathematical model: $A(u) = f$ (MM)
Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

Let $\hat{u} \equiv u^h$ be an approx. of the exact solution u corresponding to a local average of size h. We seek to compute a pointwise accurate approx. U of u^h, and we want to find an equation for u^h

General setting

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $\hat{A}(\hat{u}) = \hat{f}$ (PP)

We seek a pertubed equation $\hat{A}(u^h) = \hat{f}$ by making an Ansatz of the form

$$\hat{A}(u^h) = A(u^h) + A(u)^h - A(u^h) = f^h = \hat{f}$$

where we need to approximate

$F_h(u) \equiv A(u)^h - A(u^h)$ in terms of u^h

General setting

Mathematical model: $A(u) = f$ (MM)

Pertubed problem: $A(\hat{u}) + \hat{F}_h(\hat{u}) = f^h$ (PP)

$F_h(u) = A(u)^h - A(u^h)$ has the form of a generalized covariance

$\hat{F}_h(u^h) \approx F_h(u)$ is called a subgrid model

General setting

We solve the Galerkin equation: find $U \in V_h$ s.t.

$$(A(U) + \hat{F}_h(U), v) = (\hat{f}, v) \quad \forall v \in V_h$$

We now have

1. a discretization error from solving the Galerkin equation
2. a modeling error from the subgrid model $\hat{F}_h(U) \approx F_h(u)$

General setting

How to choose the averaging scale h?

We expect that by choosing finer h

- the modeling error decrease
- but the discretization error increase

and we expect that by choosing coarser h

- the discretization error decrease
- but the modeling error increase

General setting

How to choose the averaging scale h?

Examples:

- DNS = no averaging
- LES = averaging over the finest spatial scales
- RANS = coarse averaging (in space and time)

General setting

- We want to accurately balance the errors from modeling and discretization

General setting

- We want to accurately balance the errors from modeling and discretization
- We can achieve this balance using a posteriori error estimates

A posteriori error analysis

Galerkin equation: find $U \in V_h$ such that

$$(A(U) + \hat{F}_h(U), v) = (\hat{f}, v) \quad \forall v \in V_h \subset V$$

To estimate (e, ψ), $e = u^h - U$ and $\psi \in V$, write

$$A(u^h) - A(U) = \int_0^1 \frac{d}{ds} \hat{A}(su^h + (1-s)U)\, ds$$

$$\int_0^1 A'(su^h + (1-s)U)\, ds\; e \;\equiv\; A'(u^h, U)e$$

A posteriori error analysis

Let then $\phi \in V$ solve (the dual problem)

$$(A'(u^h, U)w, \phi) = (w, \psi), \quad \forall w \in V$$

Observe that

- the dual problem now is linearized at u^h and not u itself!
- The linearization error $U - u^h$ could be expected to be smaller than $u - U$, since u^h do not contain any subgrid scales
- the dual problem is indep. of $F_h(u)$ and $\hat{F}_h(U)$

A posteriori error analysis

Let then $\phi \in V$ solve (the dual problem)

$$(A'(u^h, U)w, \phi) = (w, \psi), \quad \forall w \in V$$

Setting $w = e$ gives the error representation

$$\begin{aligned}
(e, \psi) &= (A'(u^h, U)e, \phi) = (A(u^h) - A(U), \phi) \\
&= (\hat{f} - F_h(u) - A(U), \phi) \\
&\quad \text{(since } A(u^h) + F_h(u) = \hat{f}\text{)} \\
&= (\hat{f} - A(U) - \hat{F}_h(U), \phi) + (\hat{F}_h(U) - F_h(u), \phi) \\
&= (\hat{R}(U), \phi) + (\hat{F}_h(U) - F_h(u), \phi)
\end{aligned}$$

A posteriori error analysis

$$(e,\psi) \;=\; (\hat{R}(U),\phi) + (\hat{F}_h(U) - F_h(u),\phi)$$

- $\hat{R}(U) = \hat{f} - A(U) - \hat{F}_h(U)$ is the computable residual related to the discretization error from the Galerkin equation
- $\hat{F}_h(U) - F_h(u)$ is a residual related to the quality of the subgrid model $\hat{F}_h(U) \approx F_h(u)$, which has to be estimated

A posteriori error analysis

If we compute without subgrid model, we get

$$(e, \psi) \;=\; (\hat{R}(U), \phi) - (F_h(u), \phi)$$

We can then use the subgrid model $\hat{F}_h(U) \approx F_h(u)$ to estimate the modeling residual $F_h(u)$ in the a posteriori error estimate, to balance modeling and discretization errors

Simple example

- $$v^h(x) = \frac{1}{|\Box|}\int_{\Box} v(y)\, dy,$$

  $\Box$ centered in x, side length h
  (commutes with space & time differentiation)

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  ($\Rightarrow \ u(x) = \sin(20\pi x)$)

Simple example

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  ($\Rightarrow \; u(x) = \sin(20\pi x)$)

- $(u^h)^2 \neq (u^2)^h$

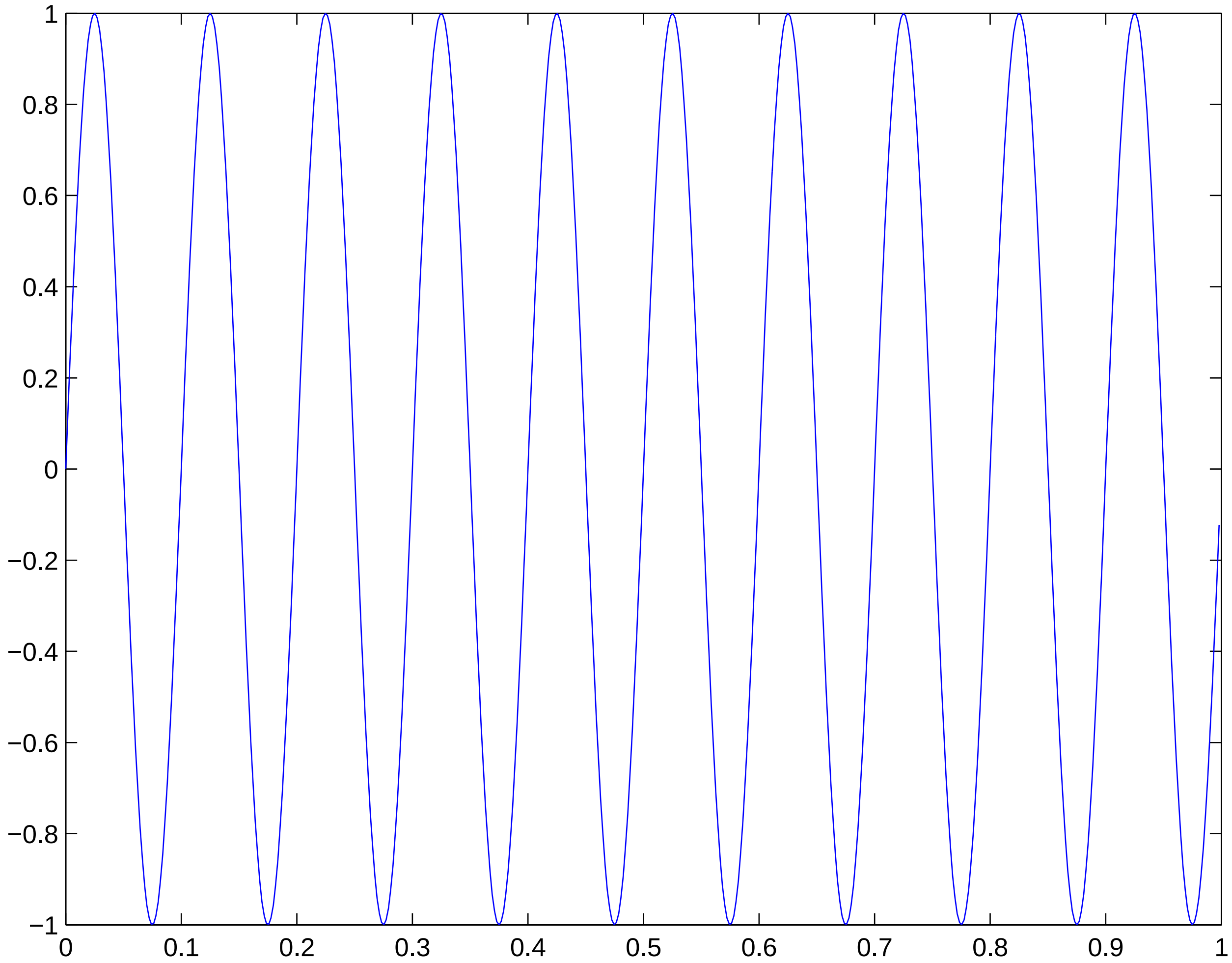
Simple example

- $v^h(x) = \frac{1}{|\square|}\int_{\square} v(y)\,dy,$

  $\square$ centered in x, side length h
  (commutes with space & time differentiation)
- $A(u) = u^2 = f, \quad f(x) = \sin^2(20\pi x)$

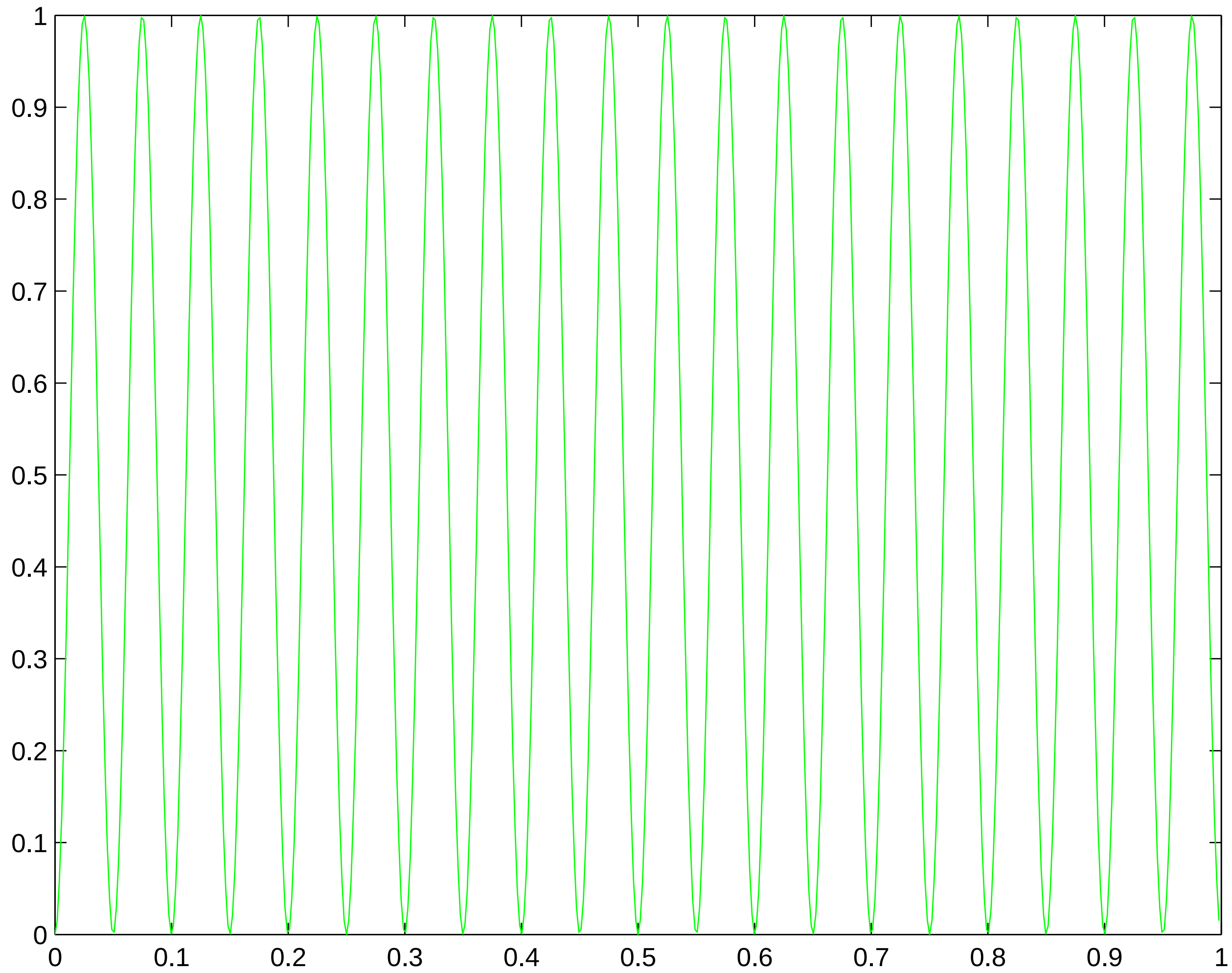
  $(\Rightarrow \; u(x) = \sin(20\pi x))$
- $\hat{A}(u^h) = (u^h)^2 + F_h(u) = f^h$
  $F_h(u) = (u^2)^h - (u^h)^2$

$u(x)$



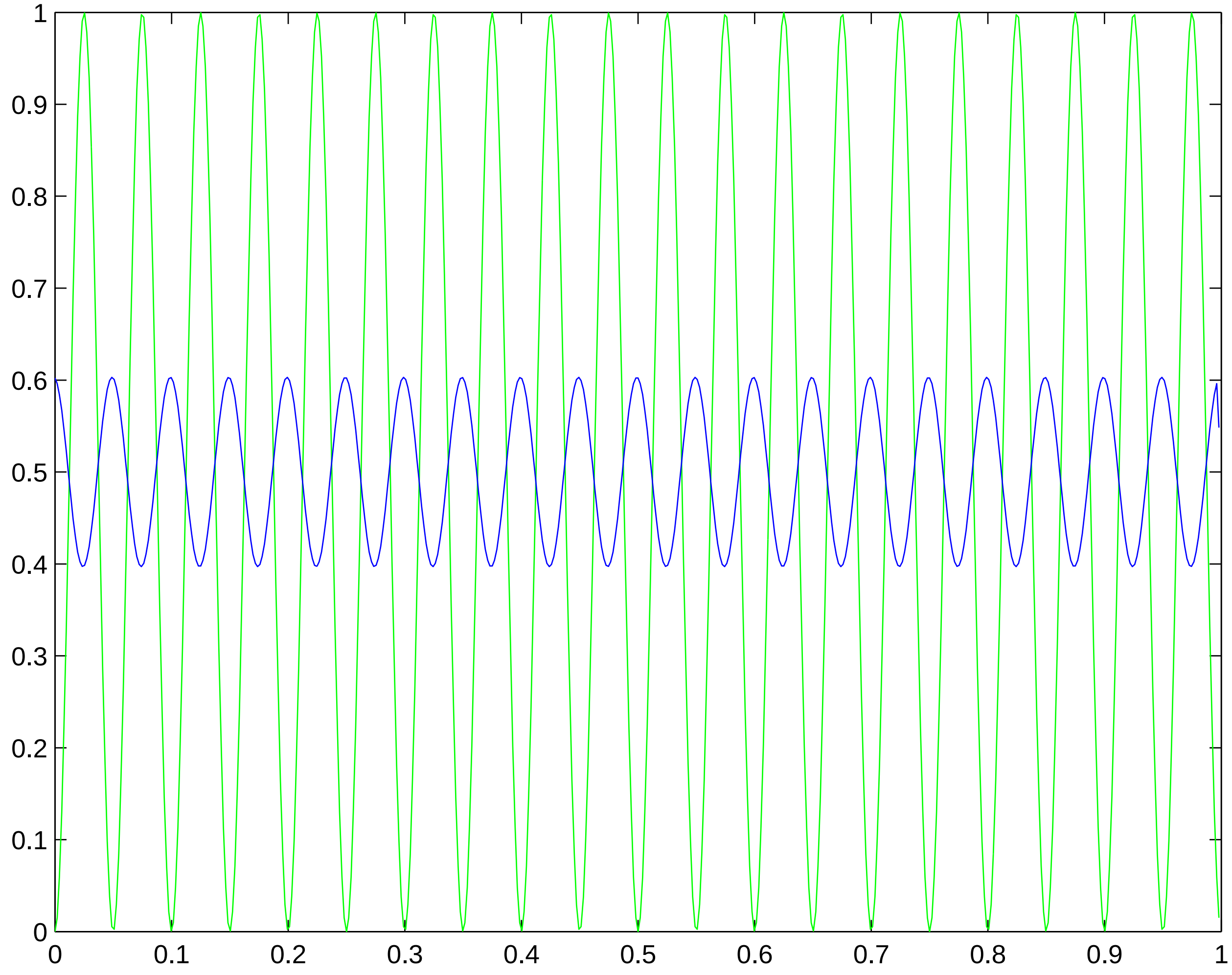
$$u(x) = \sin(20\pi x)$$

u^2



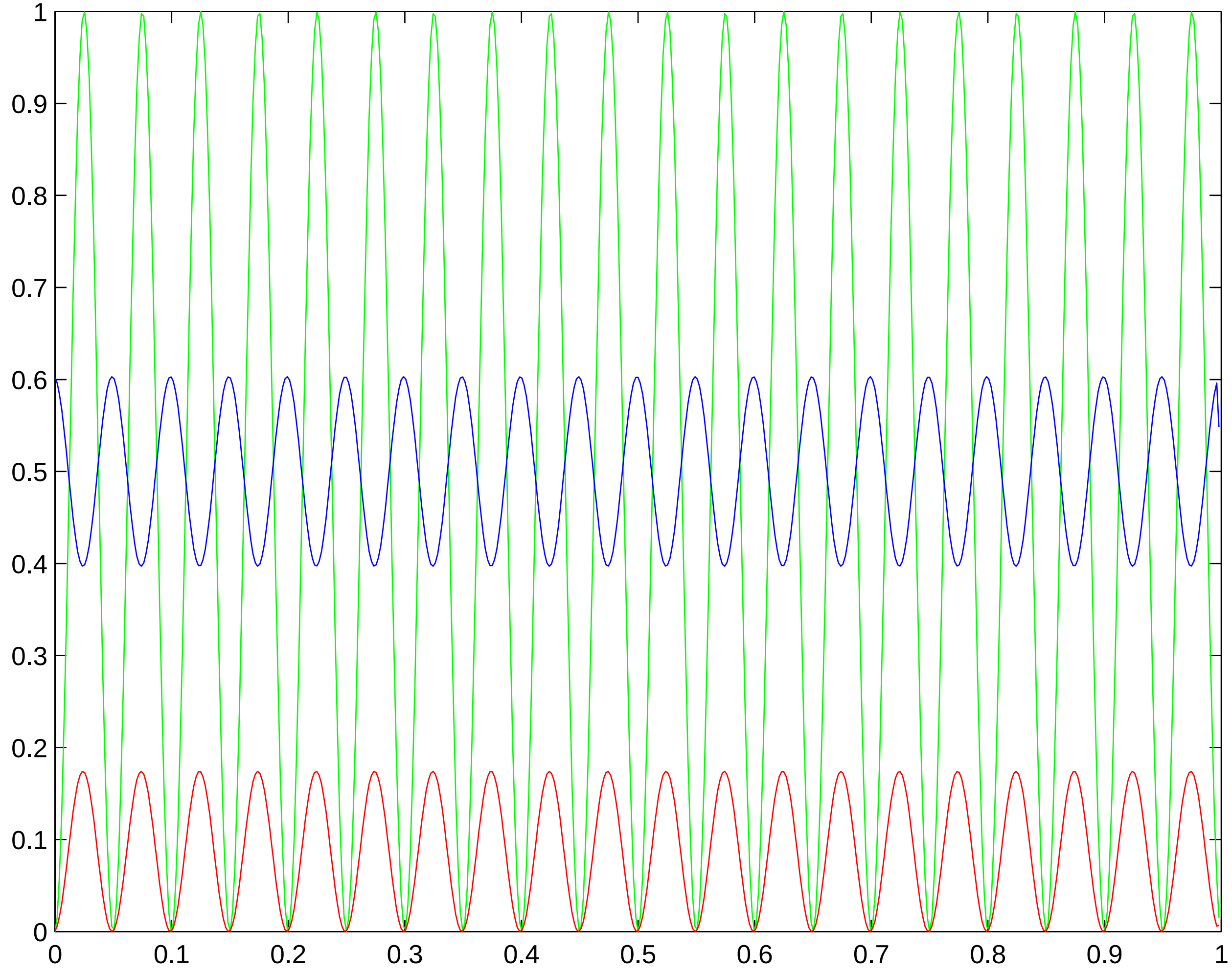
$$u(x) = \sin(20\pi x)$$

$u^2, \quad (u^2)^h$



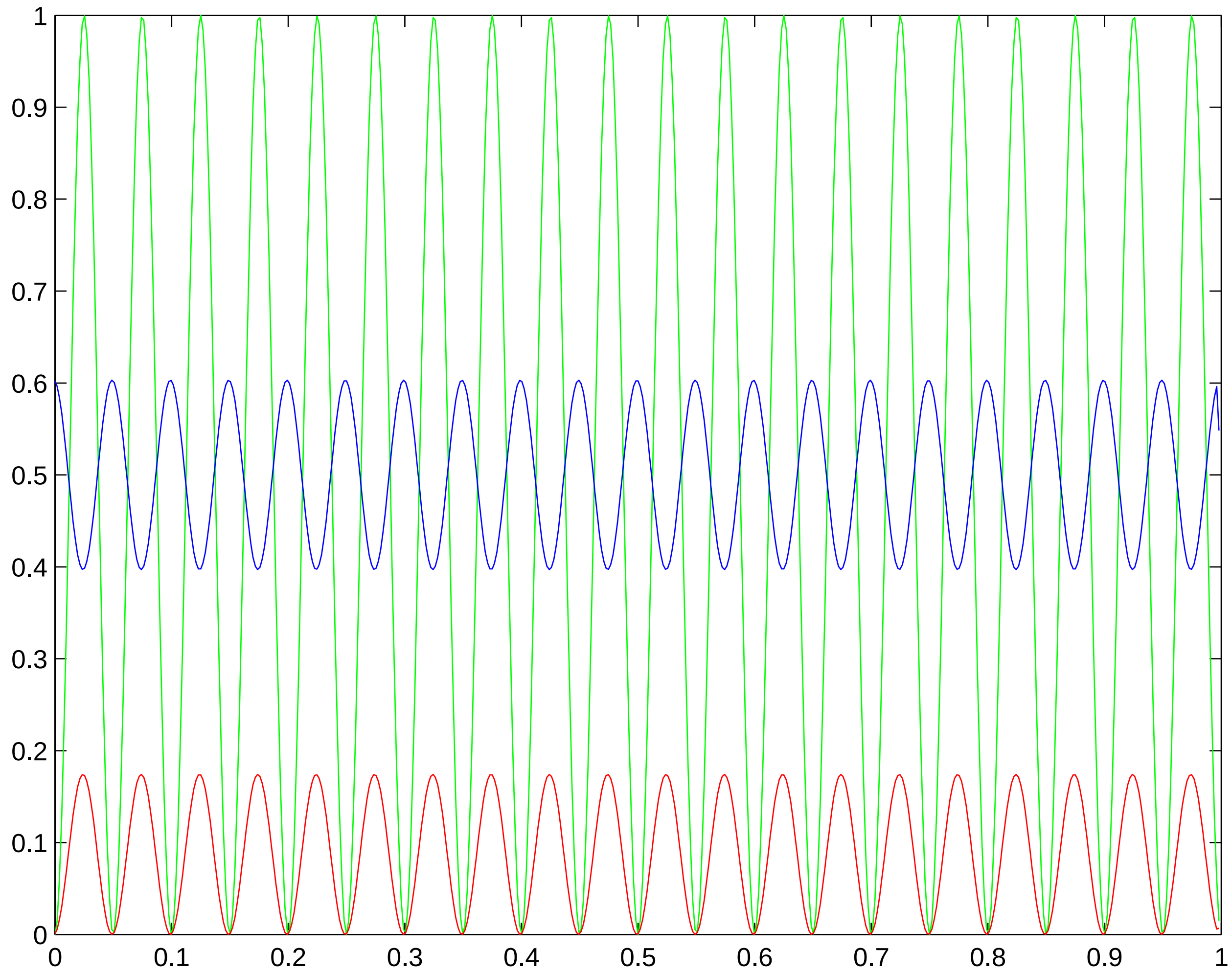
$$u(x) = \sin(20\pi x) \quad h = 2^{-4}$$

$u^2, \quad (u^2)^h, \quad (u^h)^2$



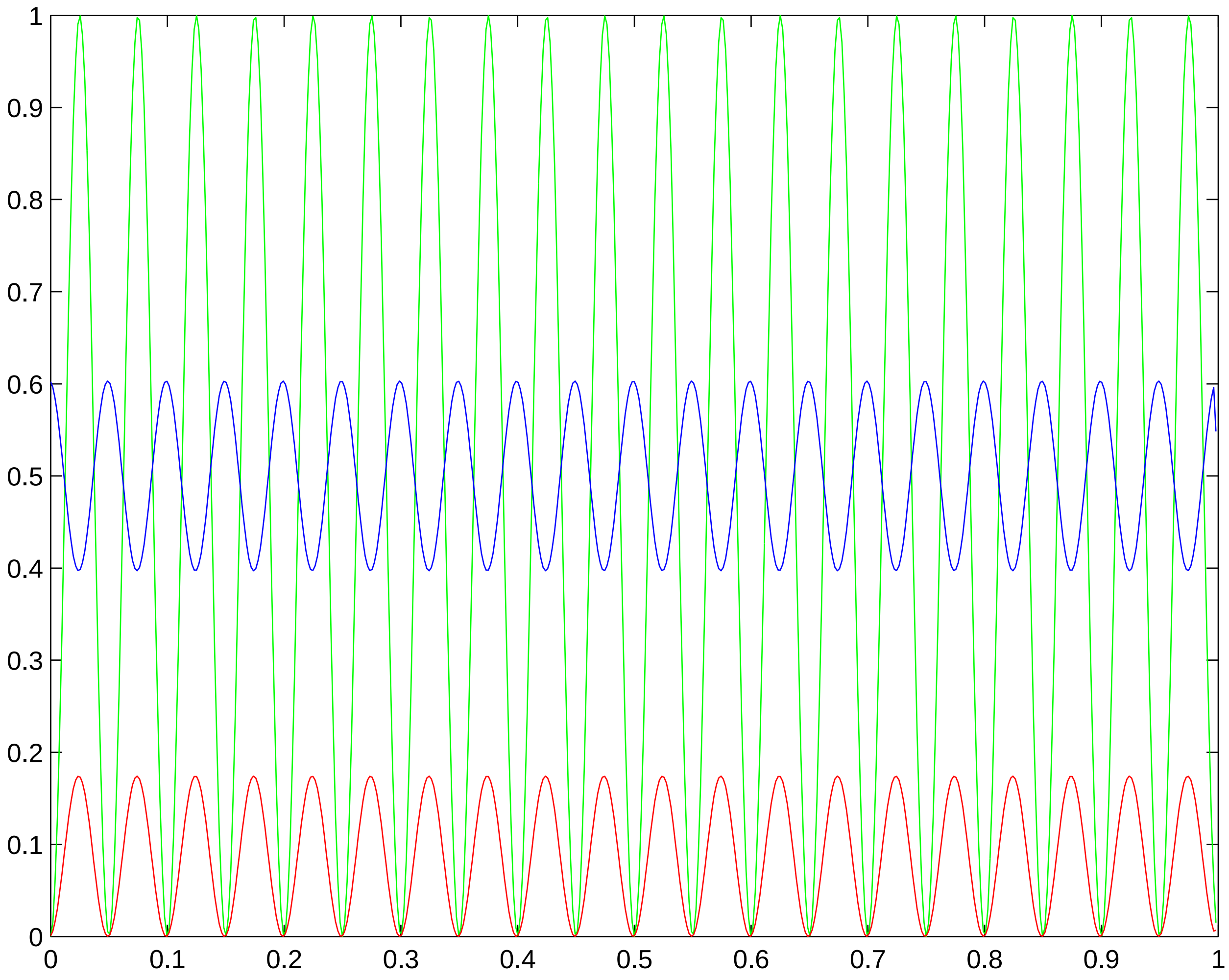
$$u(x) = \sin(20\pi x) \quad h = 2^{-4}$$

$\Rightarrow F_h(u) = (u^h)^2 - (u^2)^h$ large!



$$u(x) = \sin(20\pi x) \quad h = 2^{-4}$$

⇒ Large modeling error!



$$u(x) = \sin(20\pi x) \quad h = 2^{-4}$$

Large Eddy Simulation (LES)

$$N(u,p) = \dot{u} + u \cdot \nabla u - \frac{1}{Re}\Delta u + \nabla p = f$$
$$\nabla \cdot u = 0$$

$$u(x,0) = u_0(x)$$

Large Eddy Simulation (LES)

$$N(u,p) = \dot{u} + u \cdot \nabla u - \frac{1}{Re}\Delta u + \nabla p = f$$
$$\nabla \cdot u = 0$$
$$u(x,0) = u_0(x)$$

$$N(u^h,p^h) + \nabla \cdot F_h(u) = f$$
$$\nabla \cdot u^h = 0$$
$$u^h(x,0) = u_0^h(x)$$

$$\nabla \cdot F_h(u) = \nabla \cdot \tau^h$$
$$\tau_{ij}^h = (u_i u_j)^h - u_i^h u_j^h$$ (Reynolds stresses)

How to choose the subgrid model $\hat{F}_h$?

- $F_h(u)$ contains the effect of unresolvable scales on resolvable scales

How to choose the subgrid model $\hat{F}_h$?

- $F_h(u)$ contains the effect of unresolvable scales on resolvable scales
- We want to choose $\hat{F}_h$ only based on resolvable scales

Subgrid models in turbulence

- Eddy viscosity models (EVM)
  (τ^h modelled as an extra viscosity)

The classical Smagorinsky model:

$$\tau_{ij} - \frac{1}{3}\tau_{kk} = -2\nu_T \epsilon_{ij}(u^h), \quad \nu_T = (C_S h)^2 |\epsilon(u^h)|$$

where C_S is the Smagorinsky constant.

Subgrid models in turbulence

- Eddy viscosity models (EVM)
  (τ^h modelled as an extra viscosity)

In dynamic variants the constant C_S is computed by fitting the model on a coarser mesh using a fine mesh solution as reference

Subgrid models in turbulence

- Eddy viscosity models (EVM)
  (τ^h modelled as an extra viscosity)
- Scale Similarity Models (SSM)
  (τ^h prop. to τ^h of the resolved field)

Here one seeks to extrapolate $\tau^h(u)$ from $\tau^H(U_h)$ with $H > h$ and U_h a solution computed on mesh h from a scale similarity Ansatz

Subgrid models in turbulence

- Eddy viscosity models (EVM)
  (τ^h modelled as an extra viscosity)
- Scale Similarity Models (SSM)
  (τ^h prop. to τ^h of the resolved field)
- Mixed Models = EVM + SSM

Scale similarity in turbulent flow?

- Kolmogorov (1941): "$v(r+l) - v(r) \sim l^{1/3}$"
- Scotti, Meneveau & Saddoughi (1995): "Experimental findings of fractal scaling of velocity signals in turbulent flow"
- Papanicolaou (1999): "Experimental aerothermal data scale similar with respect to wavelet (Haar) analysis"
- ...

Scale Similarity Models

$$\tau_{ij}^h(u) = (u_i u_j)^h - u_i^h u_j^h \approx C\,\tau_{ij}^H(u^h)$$

- $H = h$, $\quad C = 1 \quad$ (Bardina,...)
- $H > h$, $\quad C \sim 1 \quad$ (Liu,...)
- $H > h$, $\quad C = C(u^h) \quad$ (Dynamic model)

τ_{ij}^h has the form of a covariance $(vw)^h - v^h w^h$

Scale Similarity Ansatz

We base a subgrid model on the Ansatz

$$F_h(u) = (vw)^h - v^h w^h(x) \approx C(x) h^{\mu(x)}$$

where the coefficients $C(x)$ and $\mu(x)$ have to be extrapolated from coarser scales

Scale Similarity Ansatz

We base a subgrid model on the Ansatz

$$F_h(u) = (vw)^h - v^h w^h(x) \approx C(x)h^{\mu(x)}$$

$$\Rightarrow \; F_h(u) \approx g(F_h(u^h), F_{2h}(u^h), F_{4h}(u^h))$$

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$$\Rightarrow \; F_h(u) \approx g(F_h(u^h), F_{2h}(u^h), F_{4h}(u^h))$$

$$g(a,b,c) = \left(1 - \left(\frac{c - b^{4h}}{b^{4h} - a^{4h}}\right)^{-n}\right) \frac{b^{4h} - a^{4h}}{\frac{c - b^{4h}}{b^{4h} - a^{4h}} - 1}$$

(n corresponds to the finest scale in u)

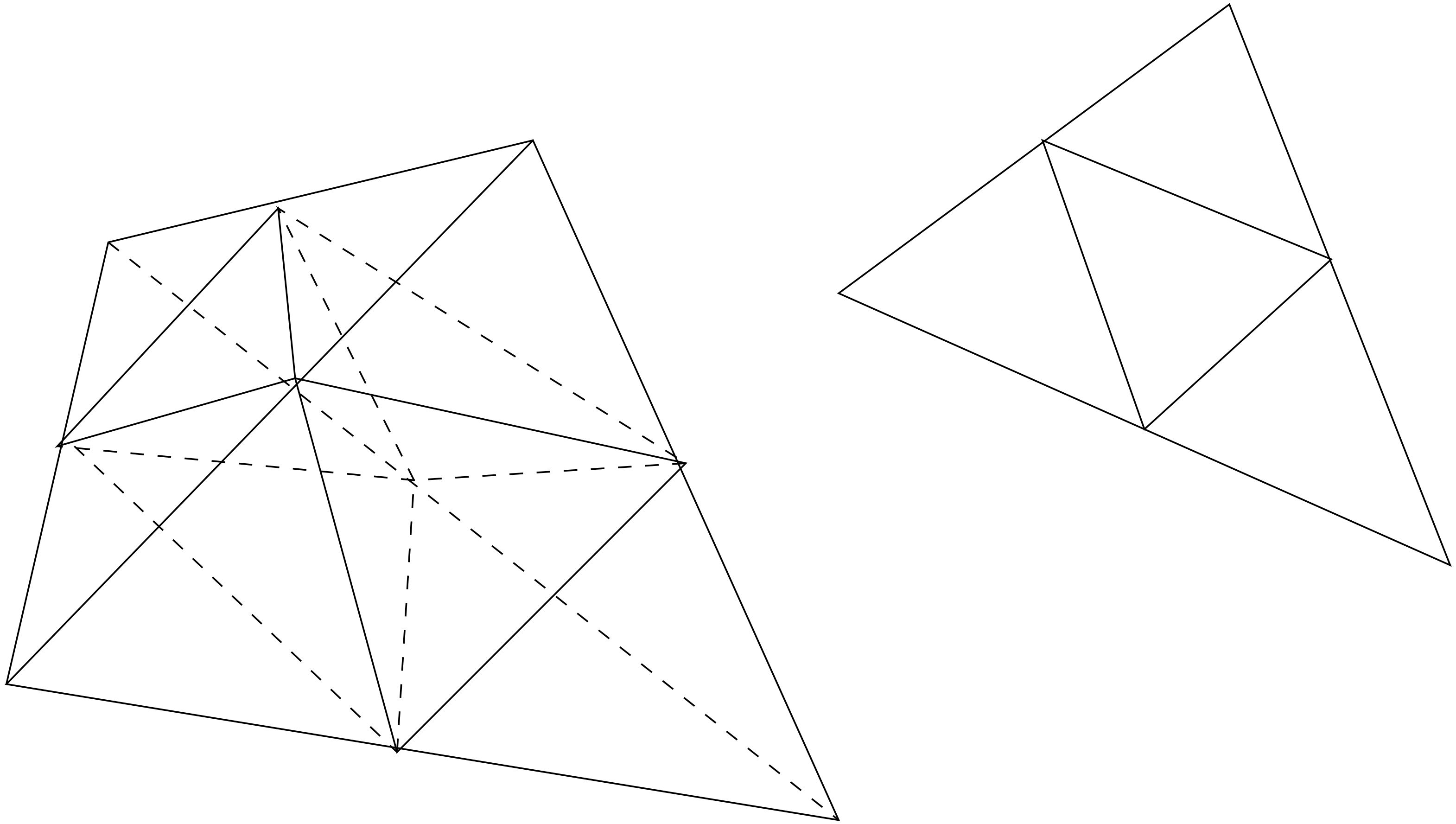
Scale Similarity Ansatz

We base a subgrid model on the Ansatz

$$F_h(u) = (vw)^h - v^h w^h(x) \approx C(x) h^{\mu(x)}$$

In particular, we have used the p.w. constant Haar basis as filter (averaging operator), which is generated from mesh refinements

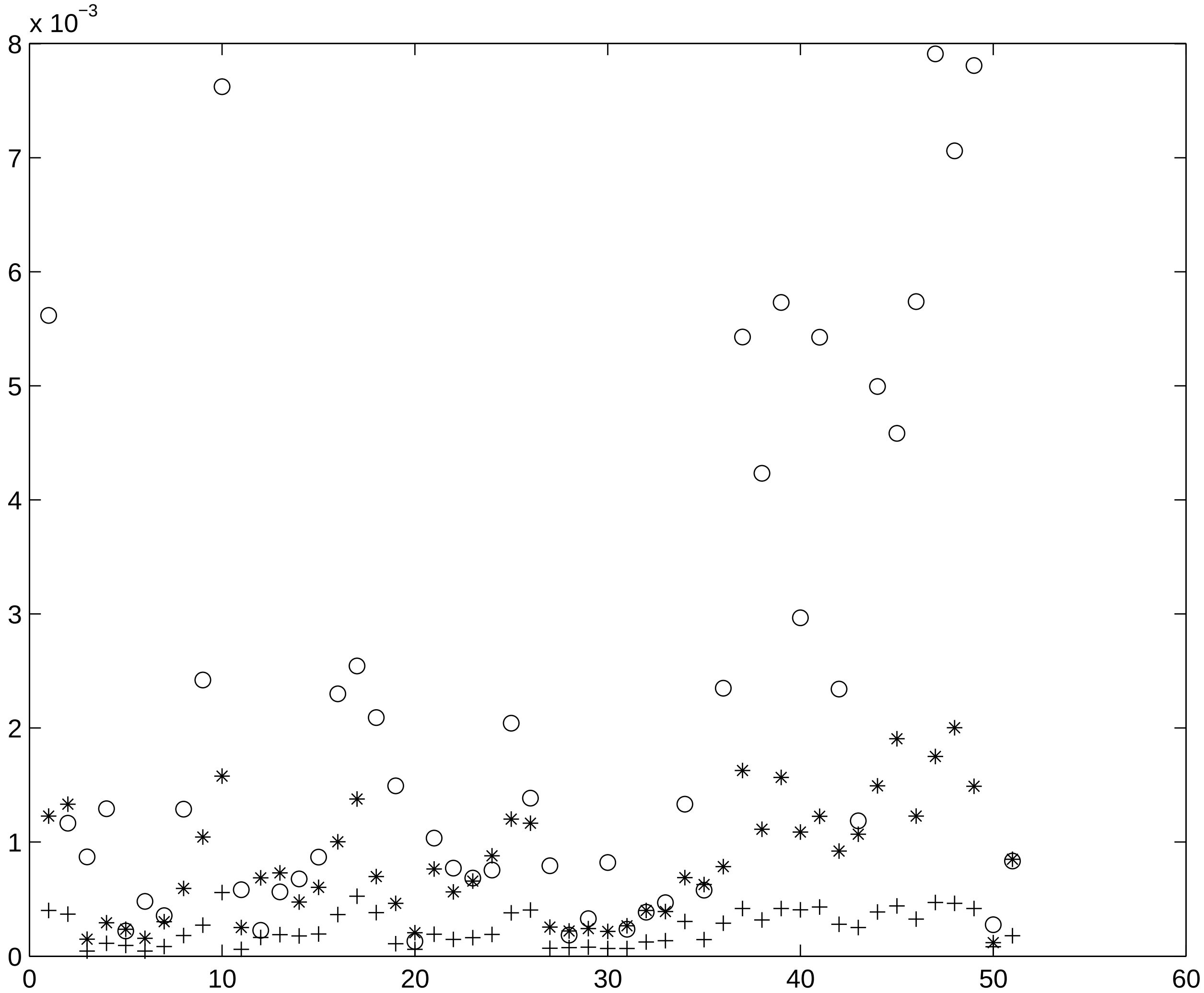
Scale Similarity Ansatz



Scale Similarity Ansatz

- We test the Ansatz for the Couette flow undergoing transition to turbulence.
- We plot the sum of the Haar coefficients on each scale ($2h, 4h, 8h$ with $h = 1/64$) for τ_{11}^h on a part of the domain
- Regular decrease in the sum of coefficients can be observed (at least in average)
- The test suffers from a too coarse mesh and not being fully turbulent

Scale Similarity Ansatz



Scale Similarity Ansatz

We have previously tested this approach for convection-diffusion-reaction problems, where this type of scale similarity has been introduced through data

We are working on the extension to Dynamic Large Eddy Simulations (DLES)

Previous results

Convection-Diffusion-Reaction system

$$\dot{u} - \epsilon \Delta u + \beta \cdot \nabla u = f(u),$$
$$u(x,0) = u_0(x)$$

- Convection-Diffusion-Reaction systems with scale similar (fractal) initial data
- Convection-Diffusion-Reaction systems with scale similar (fractal) convection field β

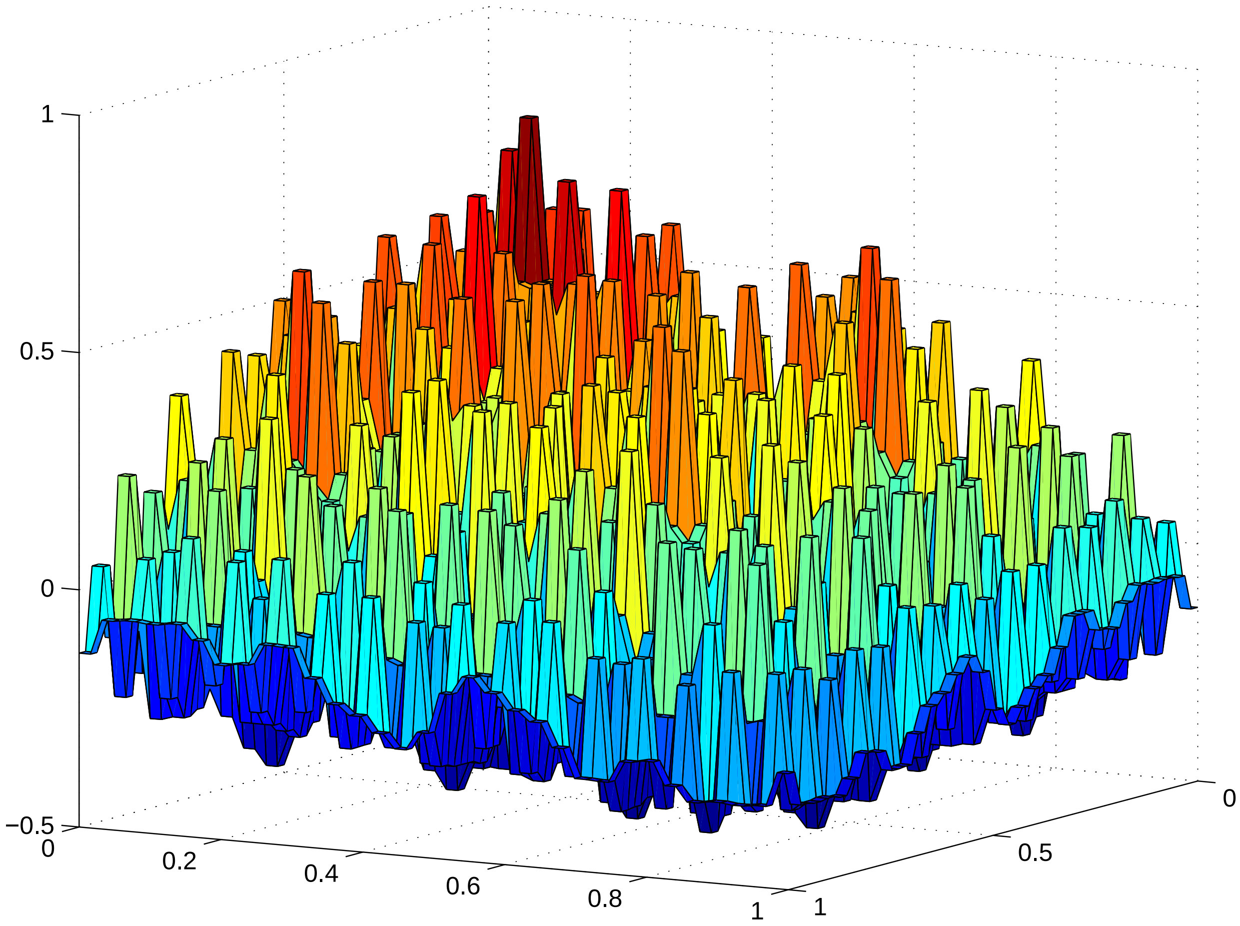
Example: Scale Similar Function

Weierstrass function

$$W_{\gamma,\delta}(x) = \gamma(x) \sum_{j=0}^{N} 2^{-j\delta(x)} \sin(2^j \cdot 2\pi x)$$

- Amplitude at scale $j+1$ is $2^{-\delta}$ less than at j (Scale similarity)
- δ determine the amount of fine scales in $W_{\gamma,\delta}$
- Typically $\gamma = \delta = 0.1$

2D Weierstrass Function ($h = 2^{-5}$)



Ex 1: Volterra-Lotka with diffusion

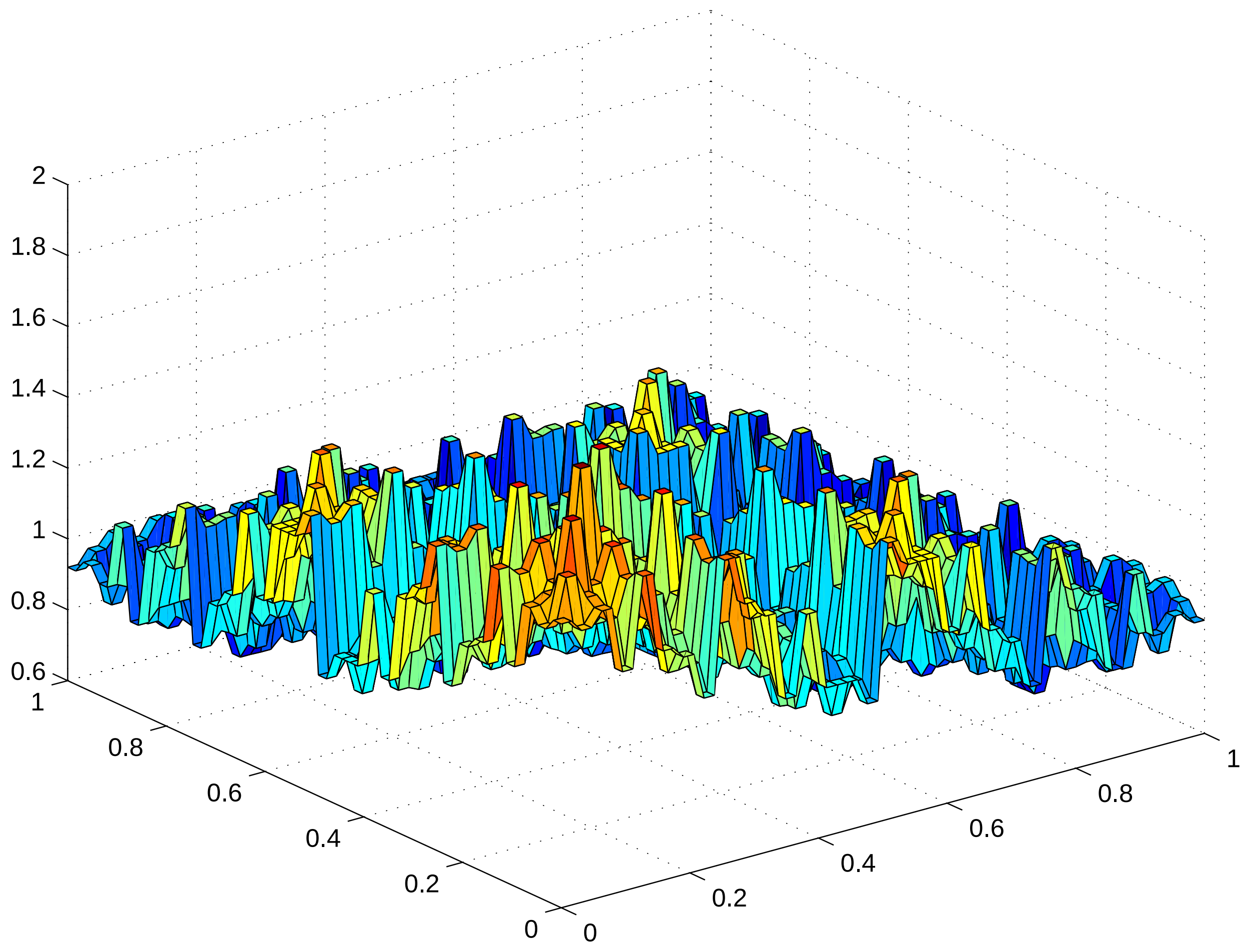
$$\dot{u}_1^h - \epsilon\Delta u_1^h = u_1^h(1 - u_2^h) + F_h(u)_1$$
$$\dot{u}_2^h - \epsilon\Delta u_2^h = u_2^h(u_1^h - 1) + F_h(u)_2$$

$$u^h(x,0) = (W^h, 1), \quad h = 2^{-5},\ h_{ref} = 2^{-9}, \quad T = 2$$

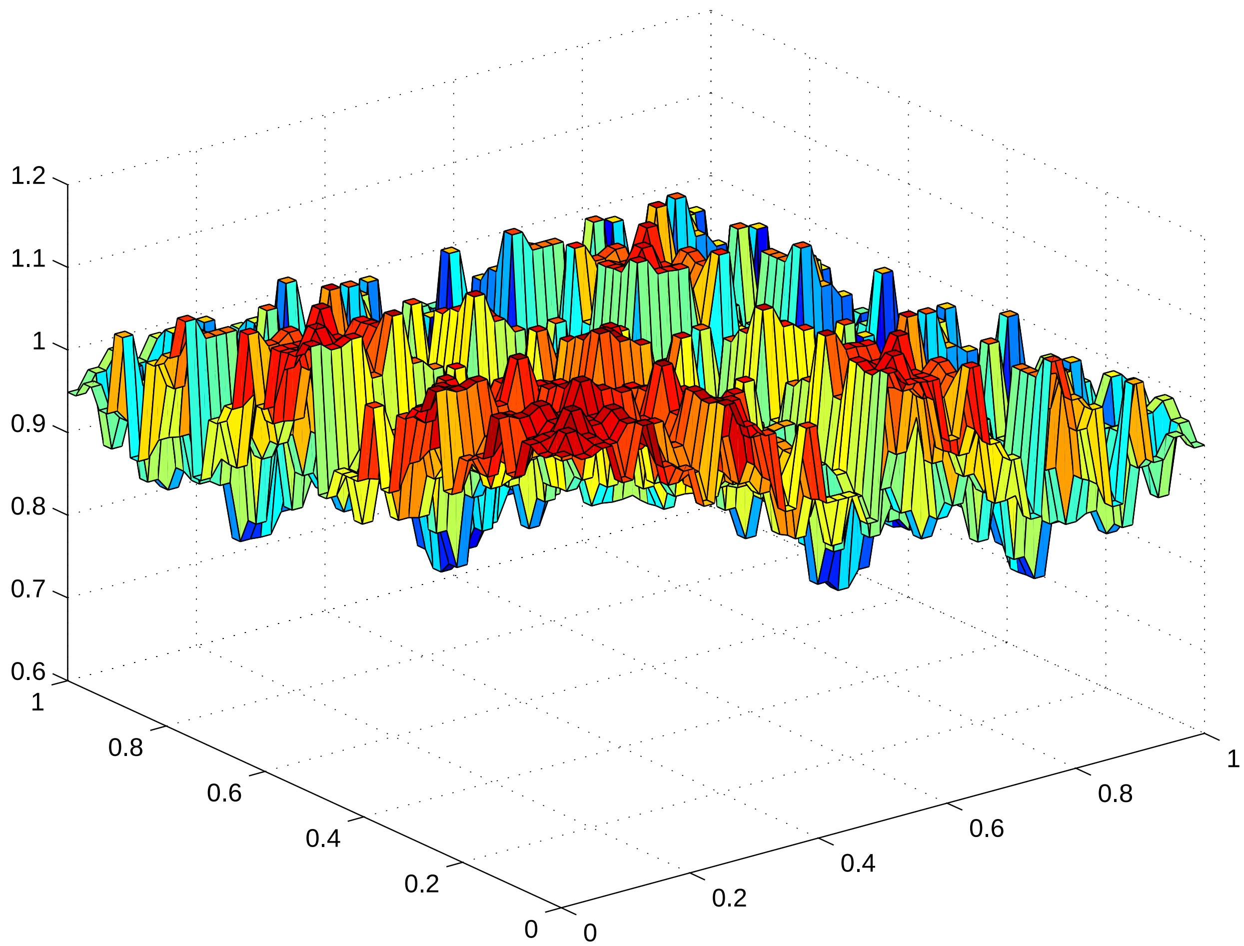
$(W(x)$ fractal Weierstrass function$)$

$$F_h(u)_1 = -(u_1u_2)^h + u_1^h u_2^h$$
$$F_h(u)_2 = (u_2u_1)^h - u_2^h u_1^h$$

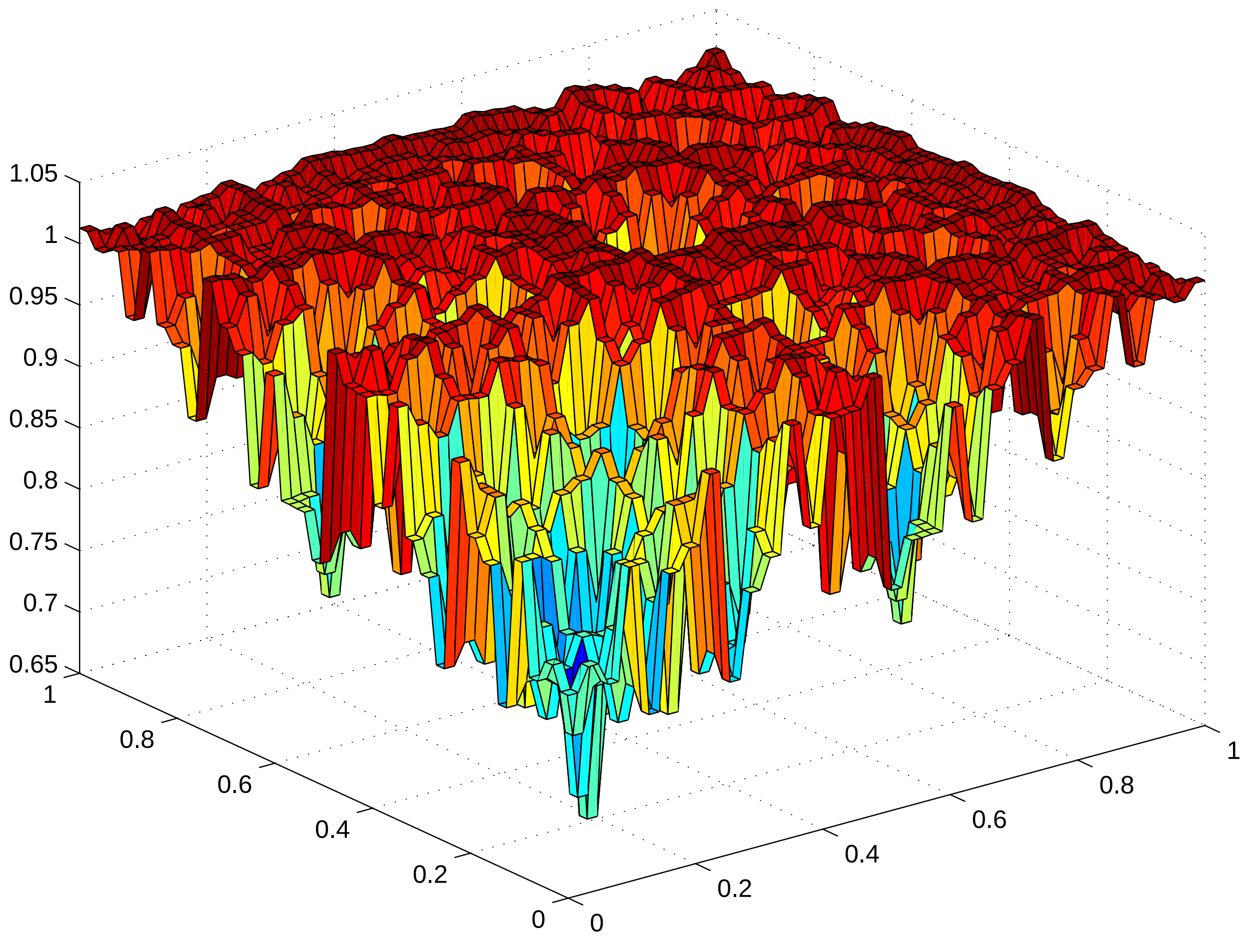
$u_1(t) \quad t = 0.0$



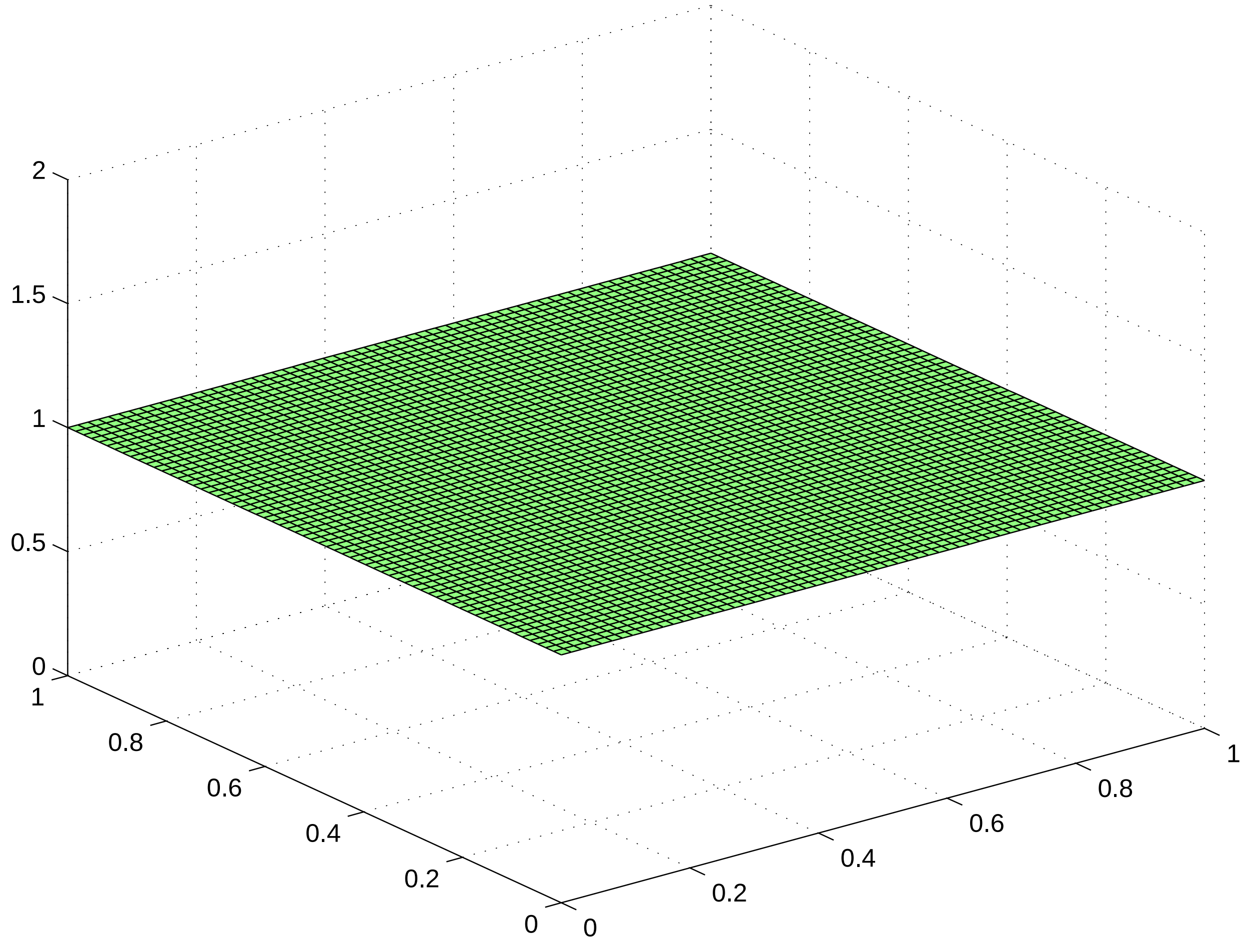
$u_1(t) \quad t = 1.0$



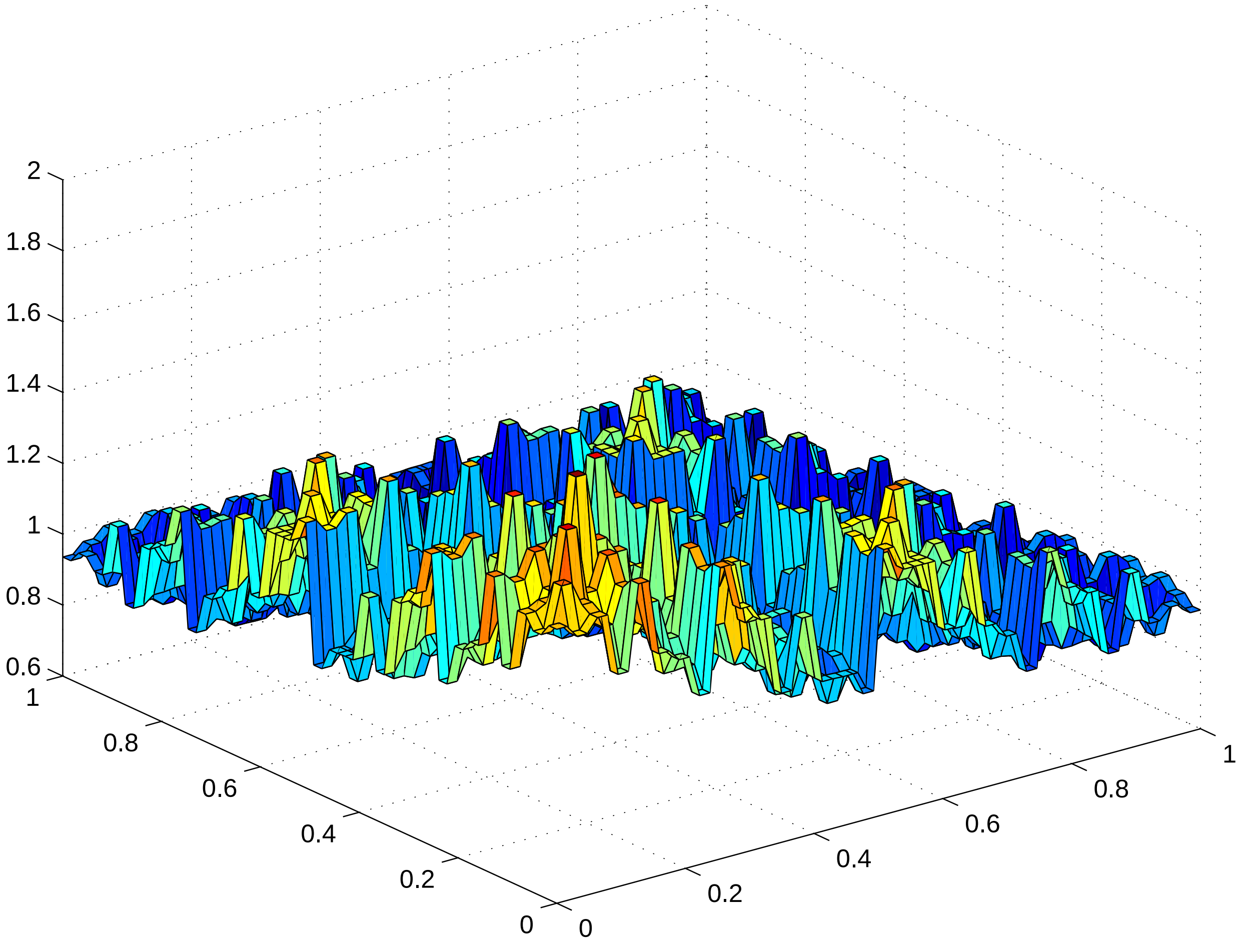
$u_1(t) \quad t = 2.0$



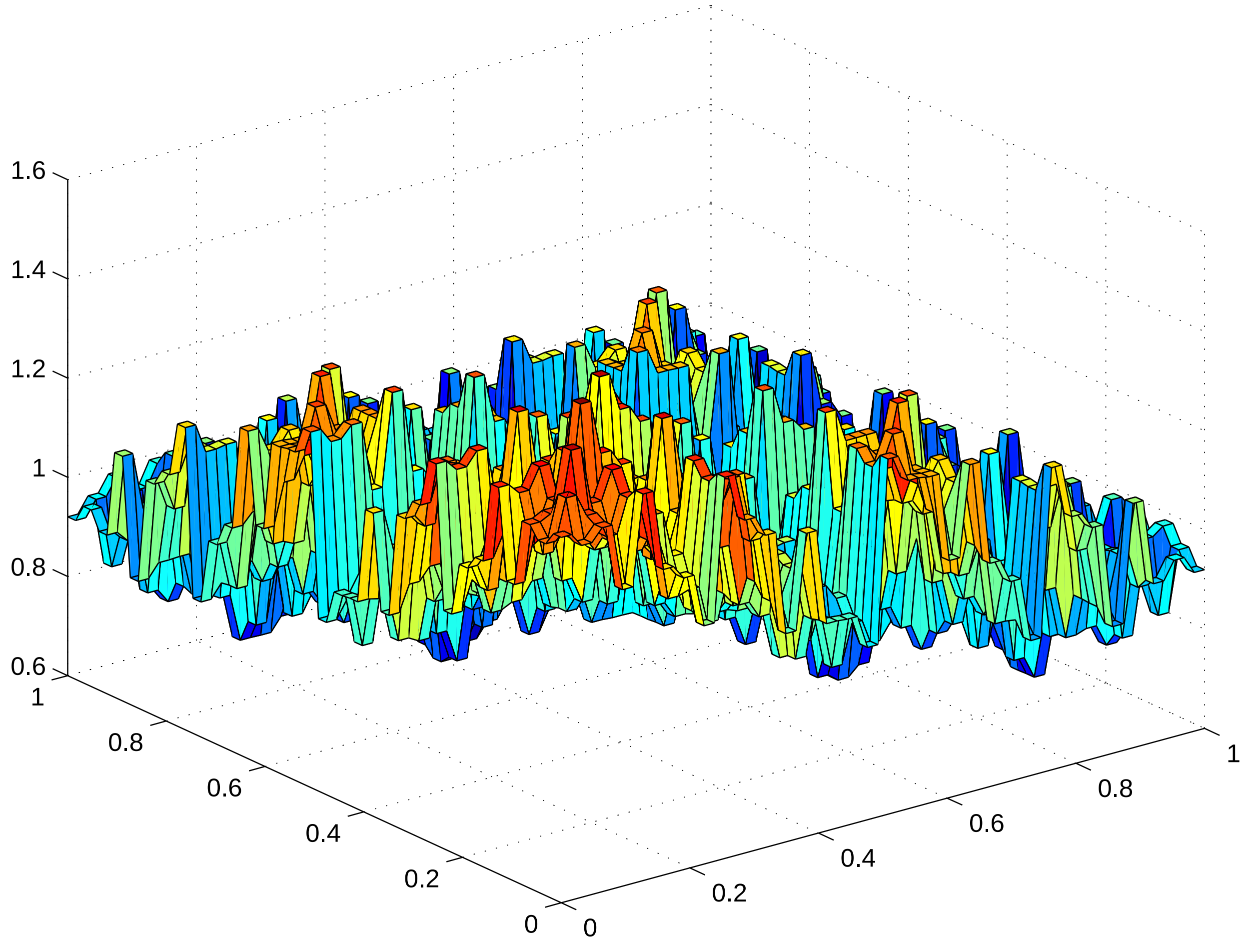
$u_2(t) \quad t = 0.0$



$u_2(t) \quad t = 1.0$



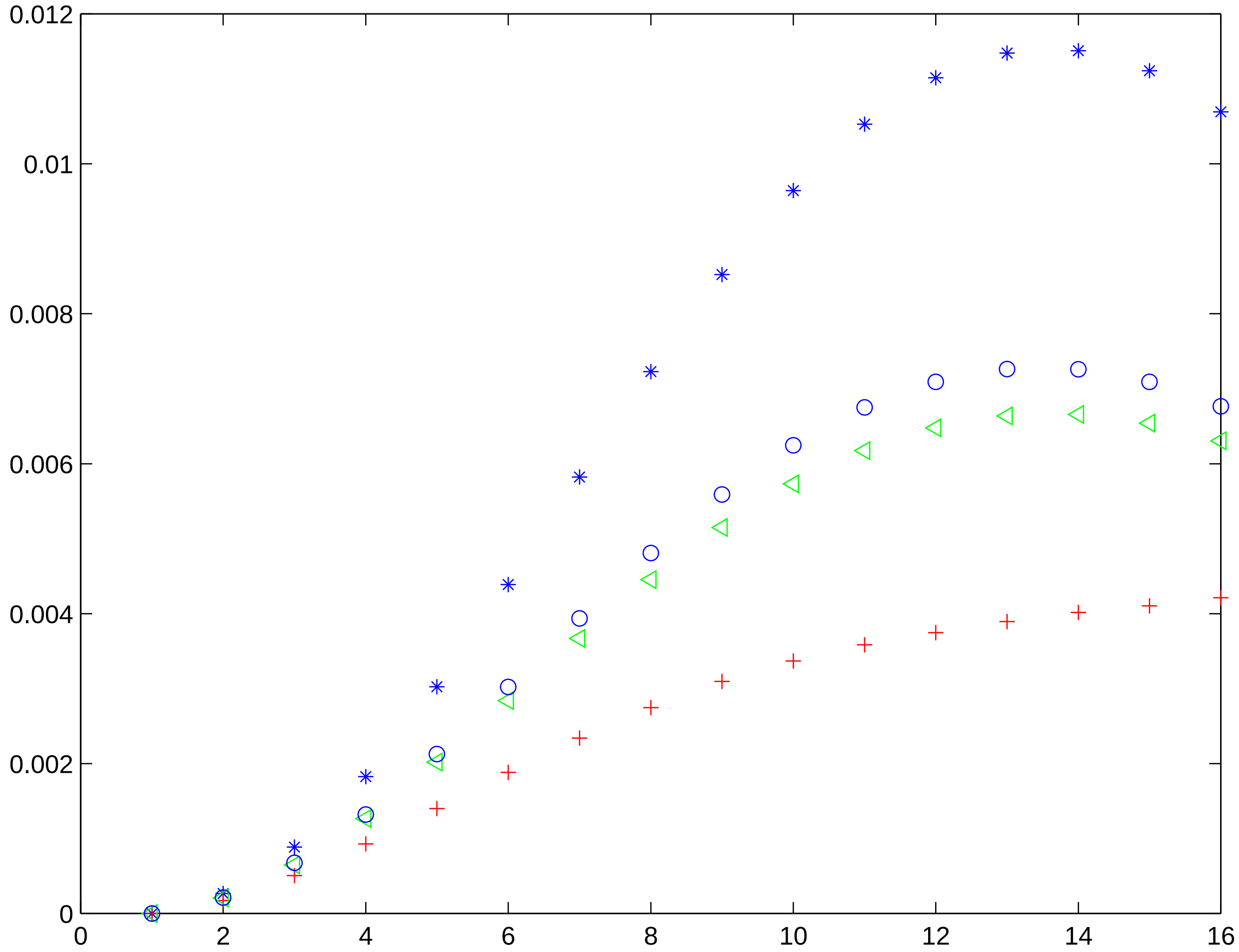
$u_2(t) \quad t = 2.0$



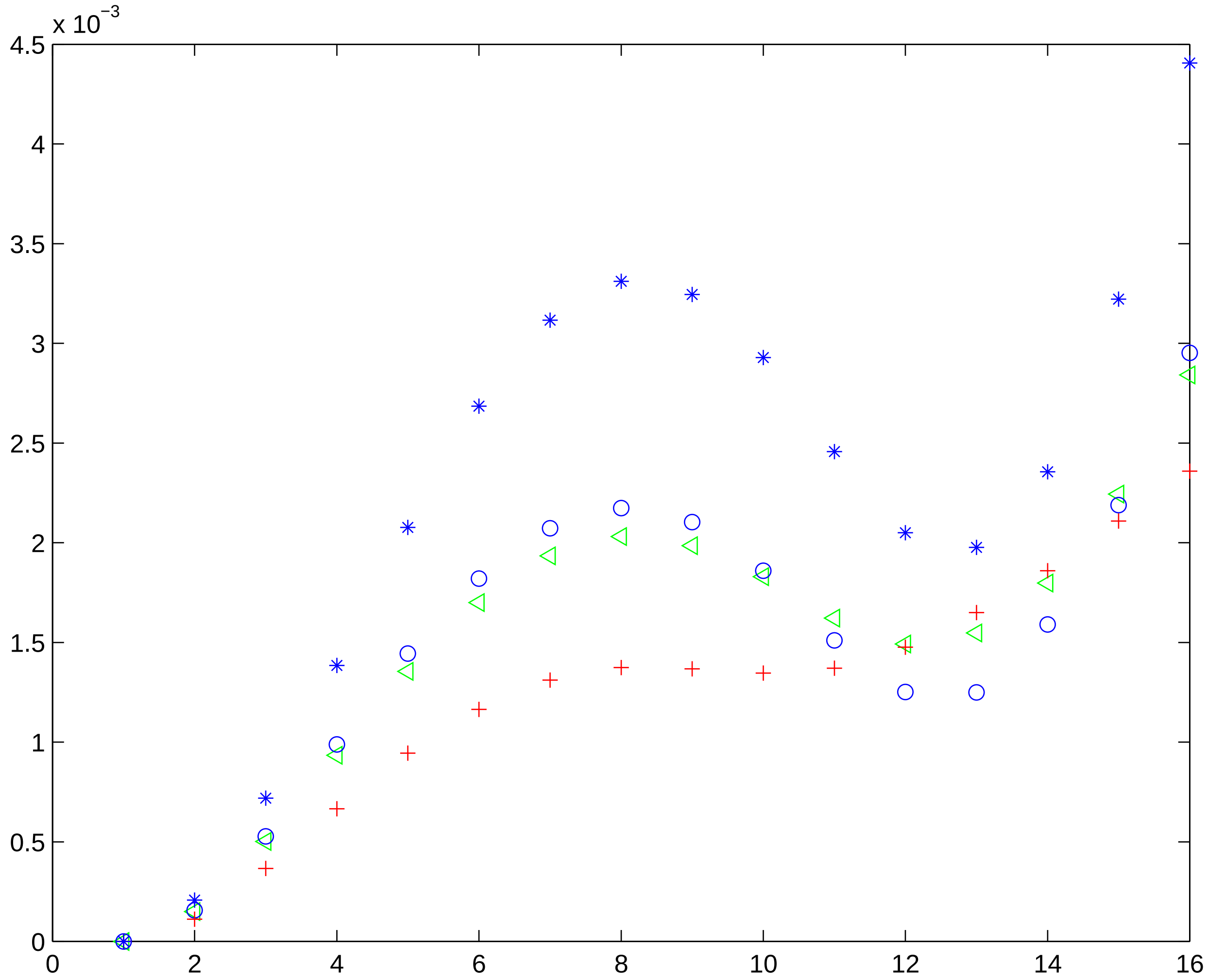
Errors: $\|u^h - U_h\|_1$

- No model (computed on h): (blue *)
- No model (computed on $h/2$): (blue o)
- $\hat{F}_h(\hat{u}_h) = F_{2h}(\hat{u}_h)$: (green <)
  This corresponds to an assumption that u contains no finer scales than $h/2$.
- $\hat{F}_h(\hat{u}_h) = g(F_h(\hat{u}_h), F_{2h}(\hat{u}_h), F_{4h}(\hat{u}_h))$: (red +)

First component u_1, in L_1-norm



Second component u_2, in L_1-norm



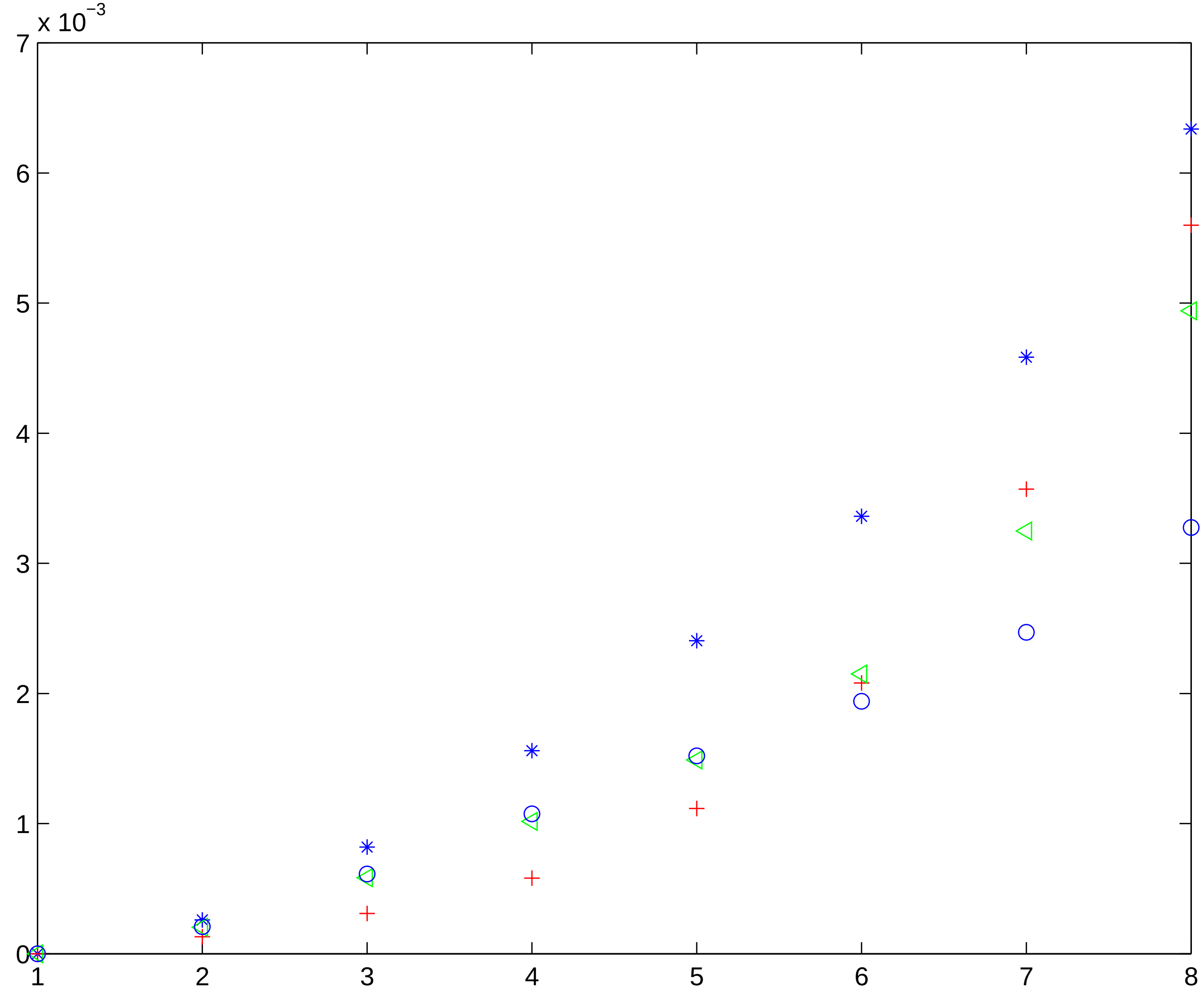
Ex 2: VL with convection & diffusion

$$\dot{u}_1^h - \epsilon \Delta u_1^h = u_1^h(1 - u_2^h) + F_h(u)_1$$
$$\dot{u}_2^h - \epsilon \Delta u_2^h + \beta^h \cdot \nabla u_2^h = u_2^h(u_1^h - 1) + F_h(u)_2$$
$$u^h(x,0) = (W_{2D}^h, 1), \quad T = 1$$

$$\beta(x_1, x_2) = h\,(\sin(\pi x_1)\cos(\pi x_2), -\cos(\pi x_1)\sin(\pi x_2))$$

(convection of order h)

First component u_1, in L_1-norm



Ex 3: Fractal Convection

$$\dot{u}^h - \epsilon \Delta u^h + \beta^h \cdot \nabla u^h = 1 + F_h(u), \quad [0,1]^2 \times (0,2),$$

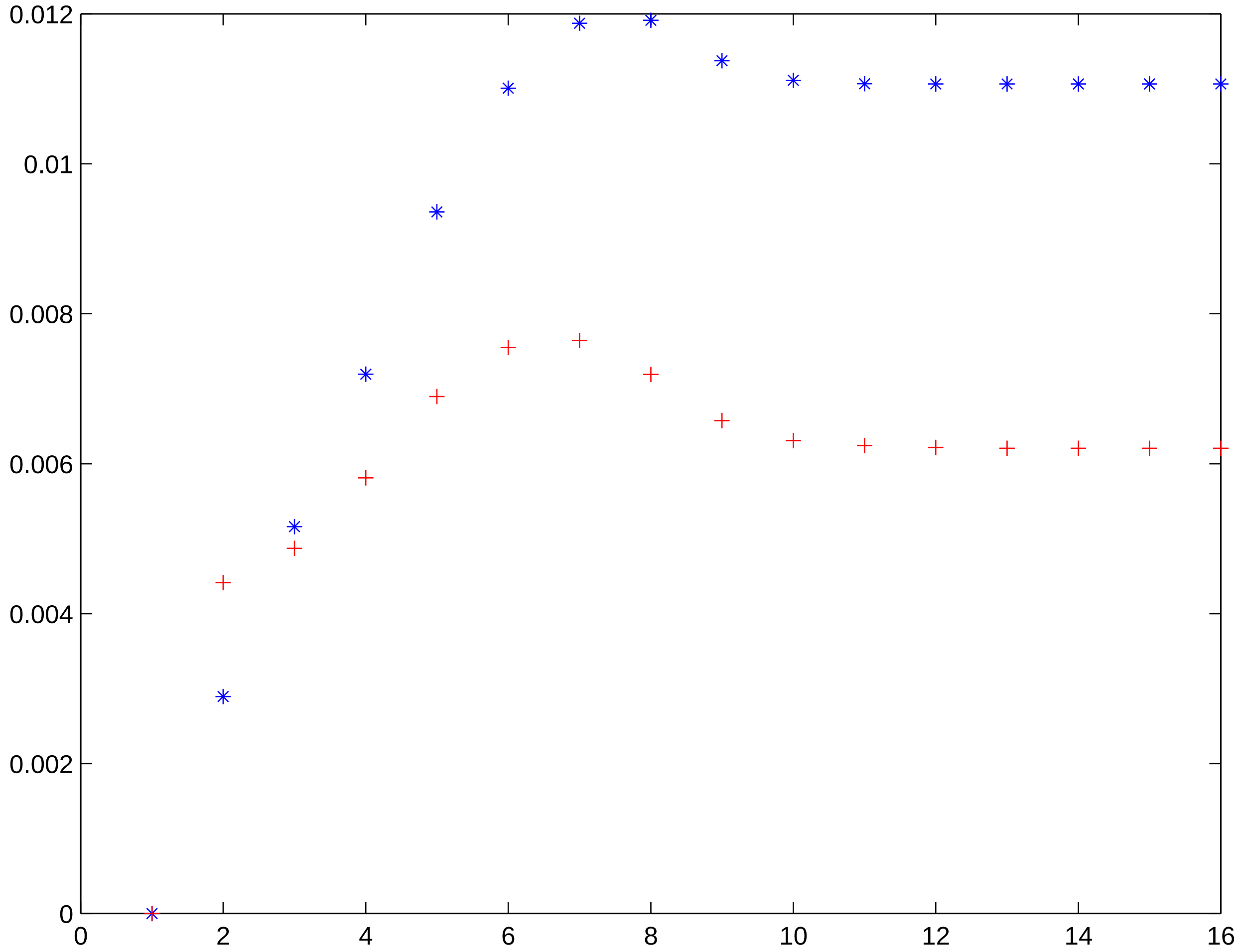
$$\frac{\partial u^h}{\partial n}|_{x_1=1, x_2=1} = 0, \quad u^h|_{x_1=0, x_2=0}, \quad u^h(x,0) = 0,$$

$\epsilon = 10^{-3}$, $\beta = (W_{2D}, W_{2D})$ ($h_{ref} = 2^{-8}$)

$F_h(u) = \beta^h \cdot (\nabla u)^h - (\beta \cdot \nabla u)^h$

(non divergence form)

L_1-err. with(r) & without(b) s.g.mod.



Summary

- Subgrid scales introduces a modeling error which we have to estimate or model
- Linearize at u^h instead of u in the dual problem, gives a natural split between a modeling error and a discretization error in terms of corresponding residuals in a posteriori error estimates
- Want to balance these two errors
- Scale similarity model proposed and tested for simple model problems

Summary - Many open ends

- Extend subgrid model to DLES
- Mixed models
- Scale sim. models on divergence form or not?
  $u^h \cdot (\nabla u)^h - (u \cdot \nabla u)^h$ or $\nabla \cdot (u^h \otimes u^h - (u \otimes u)^h)$

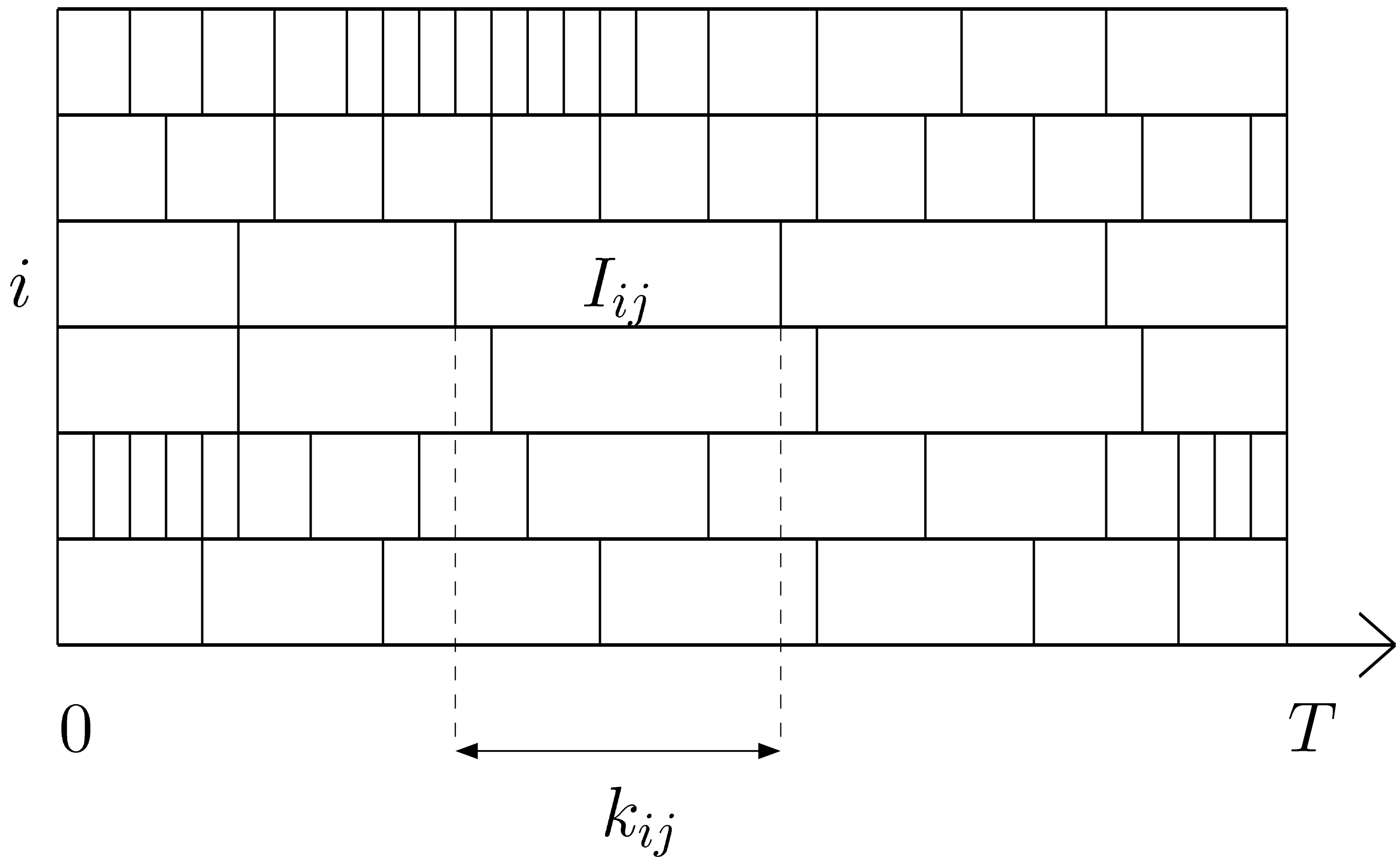
Multi-adaptivity (A.Logg)

Solve the ODE initial value problem

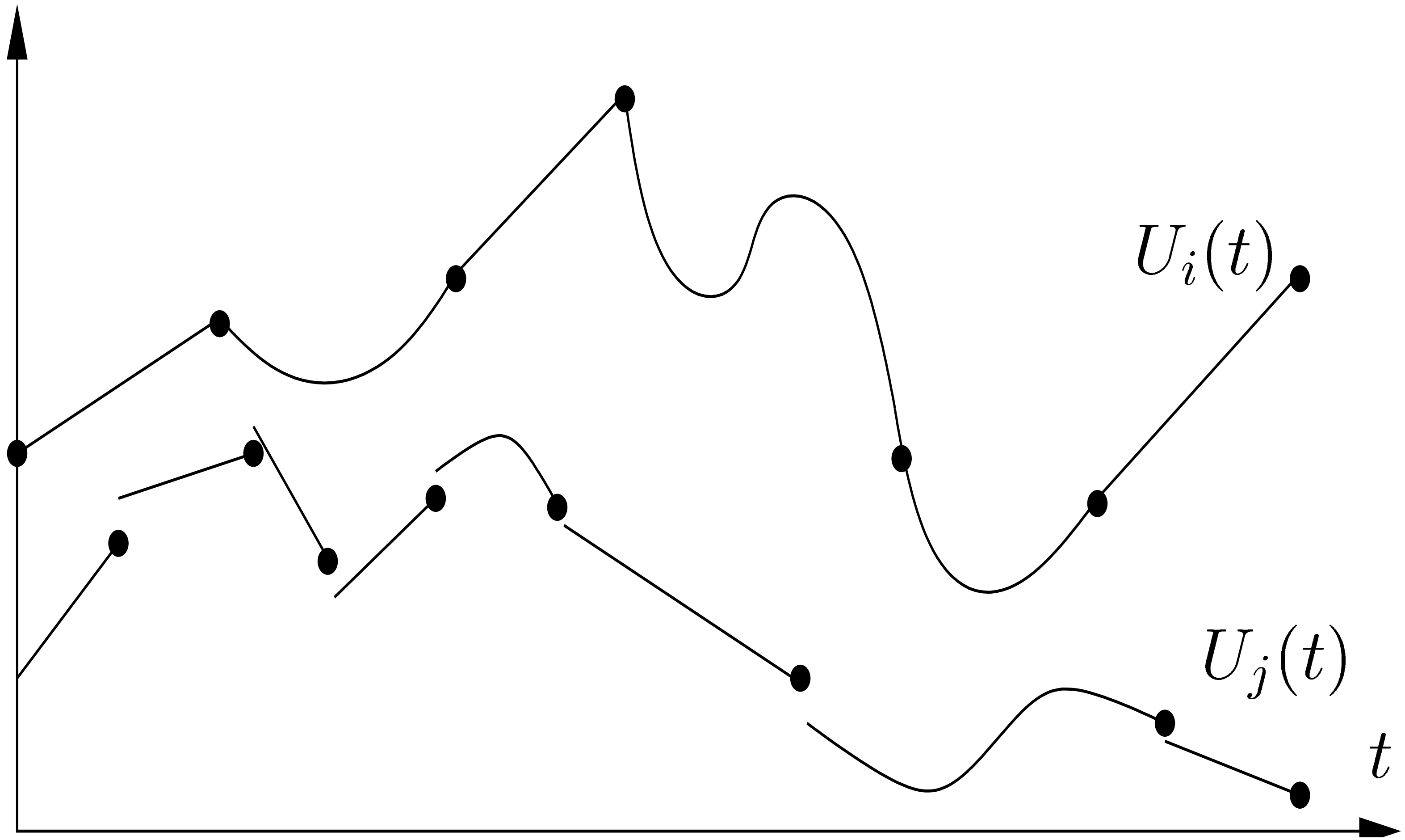
$$\begin{cases} \dot{u}(t) = f(u(t),t), \; t \in (0,T], \\ u(0) = u_0, \end{cases}$$

for $u : [0,T] \to \mathbb{R}^N$ with adaptive and individual time-steps for the different components $u_i(t)$ to achieve *efficient* and *reliable* control of the global error at time $t = T$.

Individual Time-Steps



Individual Piecewise Polynomials



Ordinary Galerkin

Ordinary Galerkin $\mathrm{cG}(q)$ for $\dot{u} = f$:

$$\int_0^T (\dot{U}, v)\, dt = \int_0^T (f(U, \cdot), v)\, dt \;\forall v \in W,$$

with $U \in V$, $U(0) = u_0$ and the trial and test spaces defined as

$$\begin{aligned} V &= \{v \in C^N([0,T]) : v_i|_{I_j} \in \mathcal{P}^q(I_j)\}, \\ W &= \{v : v_i|_{I_j} \in \mathcal{P}^{q-1}(I_j)\}. \end{aligned}$$

Multi-Adaptive Galerkin

Multi-adaptive Galerkin $\mathrm{mcG}(q)$:

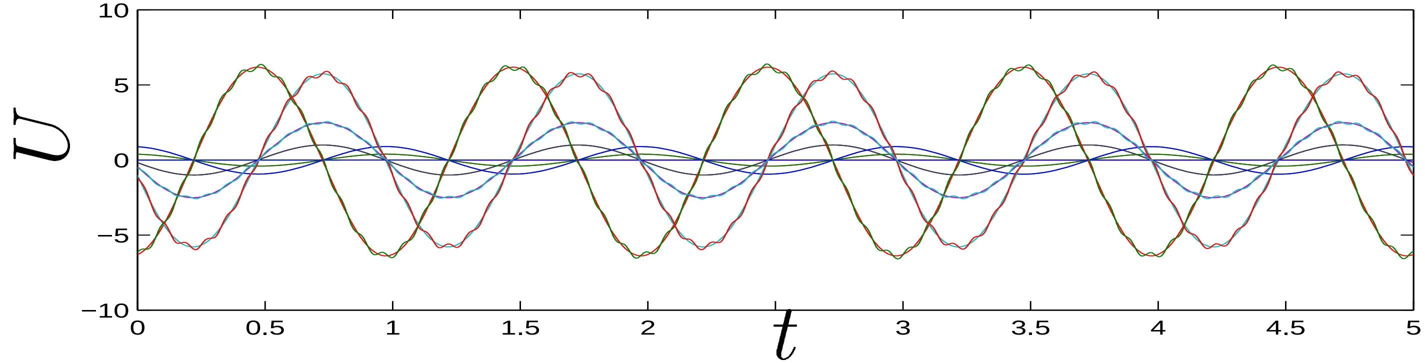
$$\int_0^T (\dot{U}, v)\, dt = \int_0^T (f(U, \cdot), v)\, dt \;\forall v \in W,$$

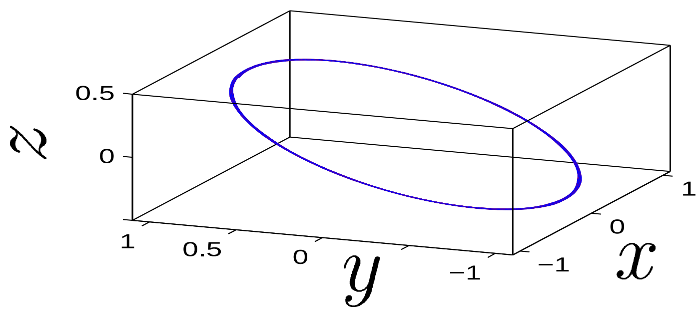
with $U \in V$, $U(0) = u_0$ and the trial and test spaces now defined as

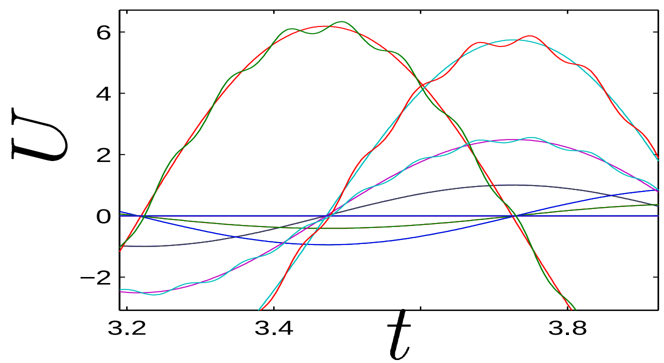
$$\begin{aligned} V &= \{v \in C^N([0,T]) : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})\}, \\ W &= \{v : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}-1}(I_{ij})\}. \end{aligned}$$

Ex: The Solar System

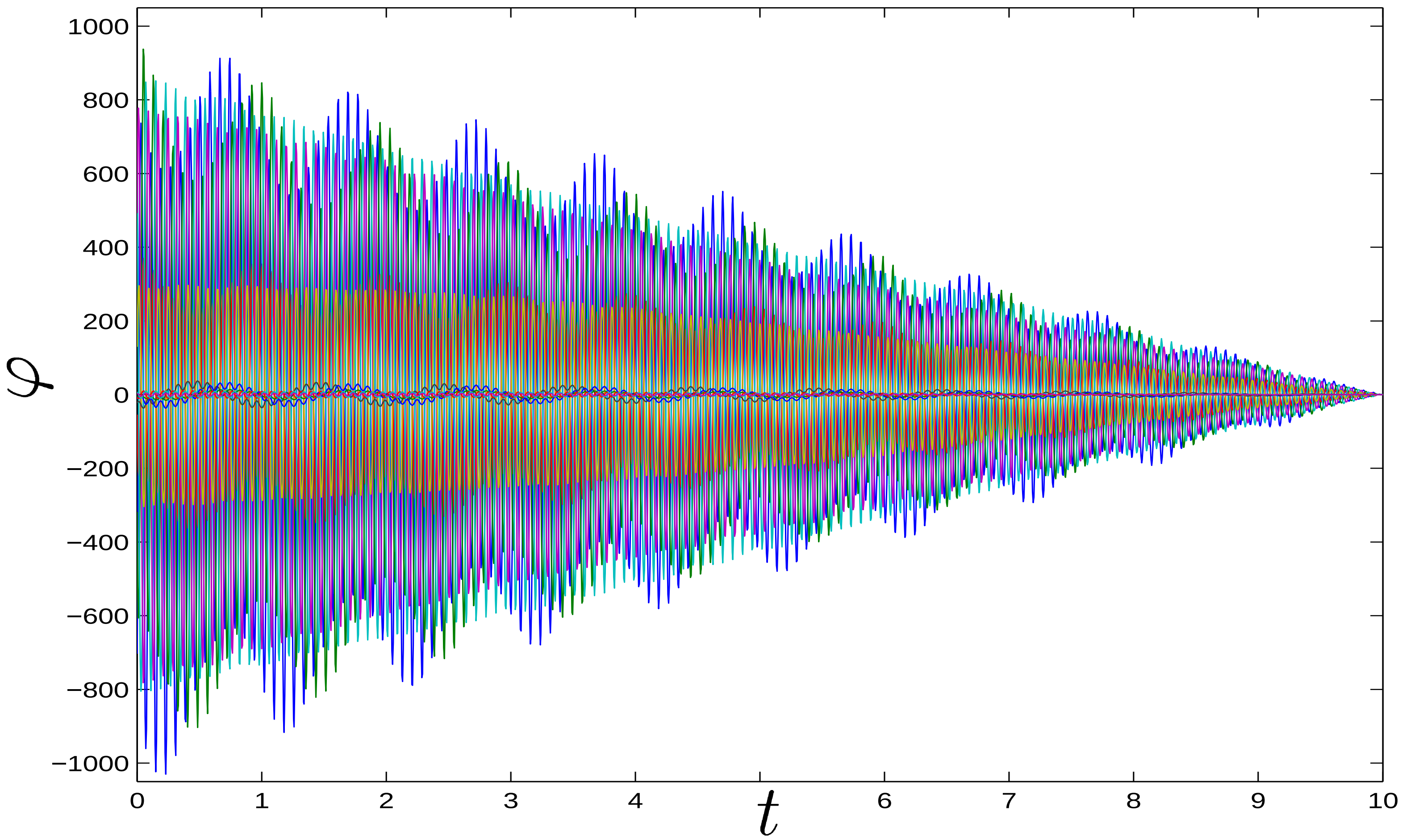
$$m_i \ddot{x}_i = \sum_{j \neq i} \frac{G m_i m_j}{|x_j - x_i|^3} (x_j - x_i)$$







The Dual



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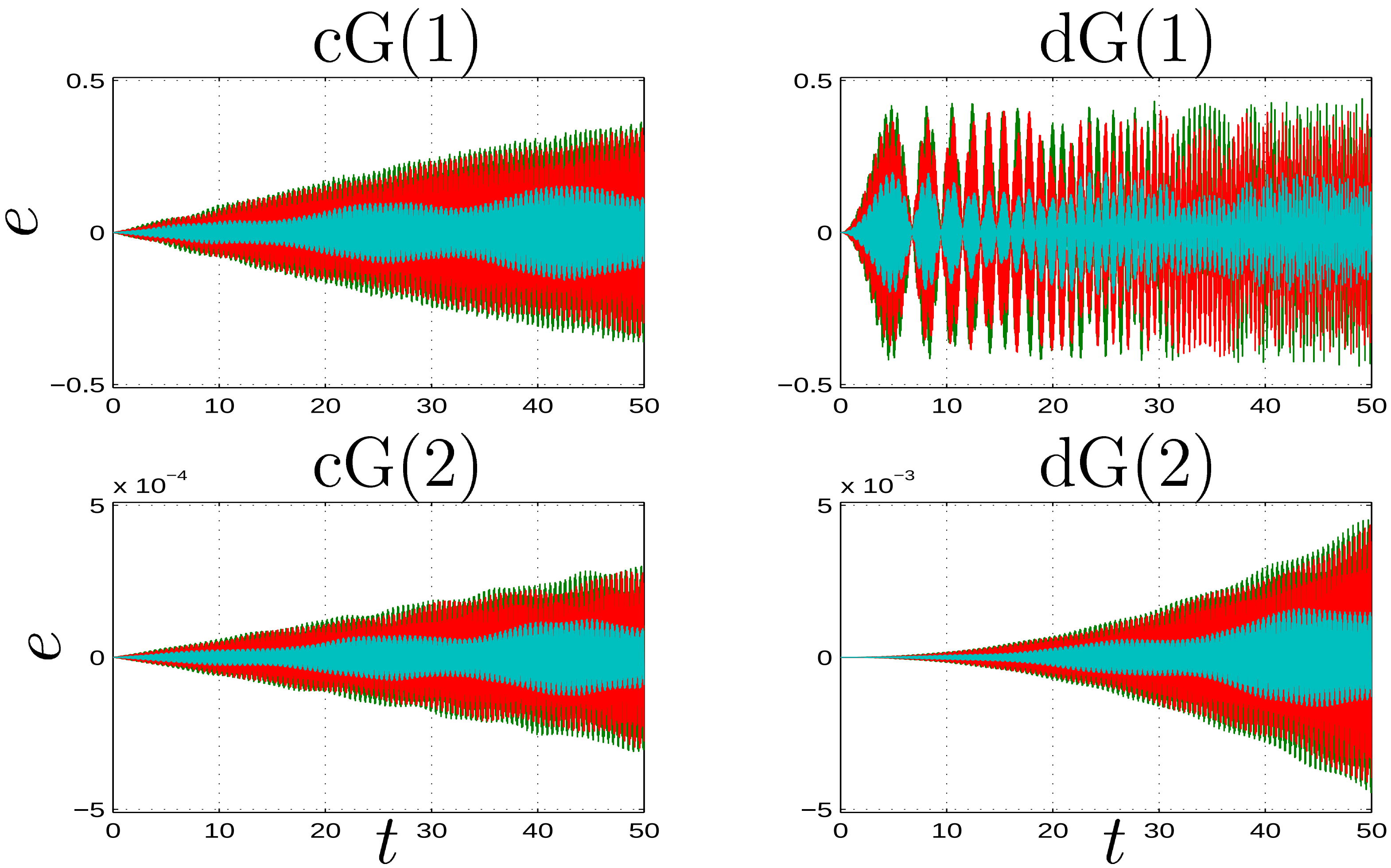
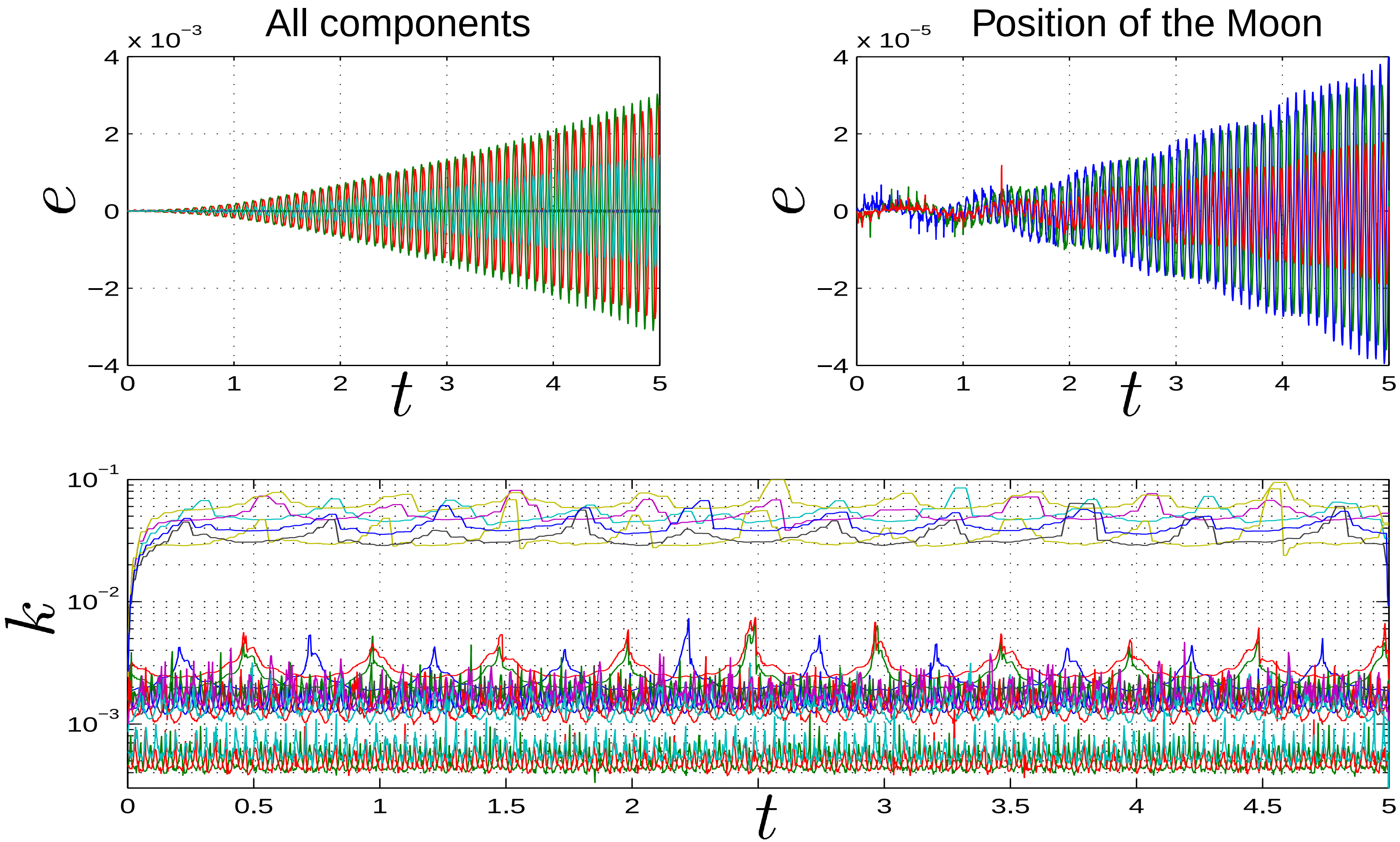
Error Growth



Error Growth: mcG(2)



More on subgrid modeling, multi-adaptivity,...

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