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# Mixed finite element methods on curved meshes or, how I learned to stop worrying and love incomplete quadrature

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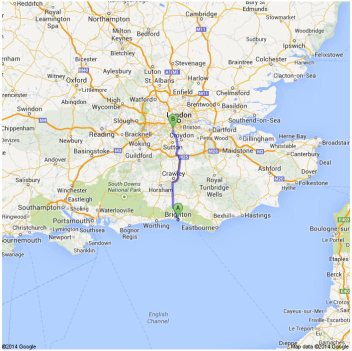
with David Ham, Lawrence Mitchell, Andrew McRae, and Andrea Natale

June 17, 2014




## British Heart Foundation

I cycled from London to Brighton on Saturday to raise money for the British Heart Foundation. You can make a donation at `www.justgiving.com/ColinCotter-LondonBrighton2014`

**Definition (Affine mesh)**

An affine mesh is a mesh where each of the elements can be obtained by applying an affine (linear plus translation) transformation to the standard reference element.

Non-affine meshes arise in Numerical Weather Prediction (NWP):

- Higher-order triangulations of the sphere,
- Pseudo-uniform quadrilateral meshes on the sphere,
- Meshing a spherical annulus (atmosphere-shaped domain),
- Terrain-following meshes over mountains.

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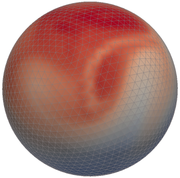
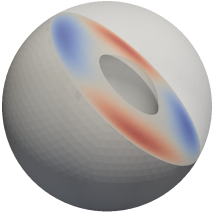
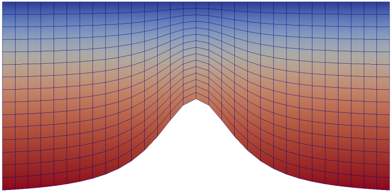
# Example non-affine meshes







# FEEC spaces

Discrete de Rham complexes underpin our approach to NWP, where discrete Helmholtz decomposition is crucial.

In two dimensions (e.g. $(\mathbb{V}_0, \mathbb{V}_1, \mathbb{V}_2) = $ (CG2,BDM1,DG0).

$$\underbrace{\mathbb{V}_0}_{\text{Continuous}} \xrightarrow{\nabla^\perp = \mathbf{k}\times\nabla} \underbrace{\mathbb{V}_1}_{\text{Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2}_{\text{Discontinuous}}$$

In three dimensions:

$$\underbrace{\mathbb{V}_0}_{\text{Continuous}} \xrightarrow{\nabla} \underbrace{\mathbb{V}_1}_{\text{Continuous tangents}} \xrightarrow{\nabla\times} \underbrace{\mathbb{V}_2}_{\text{Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_3}_{\text{Discontinuous}}$$

## Local-global transformations

Finite element spaces are defined by:

1. Specification of $\mathbb{V}_i(\hat{e})$ on reference element $\hat{e}$.
2. Specification of transformation $\mathbb{V}_i(e) \to \mathbb{V}_i(\hat{e})$ for each mesh element $e$.

Transformations are obtained from pullbacks:

- Transformation on $\mathbb{V}_i$ preserves integrals over $i$-dimensional submanifolds, guaranteeing appropriate continuity between elements.
- Transformations commute with d (i.e. $\nabla$, $\nabla^{\perp}$, $\nabla\times$, or $\nabla\cdot$ as appropriate).

FEniCS implementation: Rognes, Kirby and Logg (SISC, 2009); Rognes, Ham, Cotter and McRae (GMD, 2013).

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# Transformations in 2D

$$\psi \in \mathbb{V}_0(e) \implies \psi \circ g_e = \hat{\psi} \in \mathbb{V}_0(\hat{e}),$$

$$\mathbf{v} \in \mathbb{V}_1(e) \implies \mathbf{v} \circ g_e = \frac{J_e \hat{\mathbf{v}}}{\det J_e} \text{ for } \hat{\mathbf{v}} \in \mathbb{V}_2(\hat{e}),$$

$$\rho \in \mathbb{V}_2(e) \implies \rho \circ g_e = \frac{\hat{\rho}}{\det J_e} \text{ for } \hat{\rho} \in \mathbb{V}_2(\hat{e}).$$

Commutative properties:

$$\psi \in \mathbb{V}_0(e) \implies (\nabla^\perp \psi) \circ g_e = \frac{J_e \hat{\nabla}^\perp \hat{\psi}}{\det J_e} \text{ with } \hat{\nabla}^\perp \hat{\psi} \in \mathbb{V}_1(\hat{e}) \implies \nabla^\perp \psi \in \mathbb{V}_1(e),$$

$$\mathbf{v} \in \mathbb{V}_1(e) \implies (\nabla \cdot \mathbf{v}) \circ g_e = \frac{\hat{\nabla} \cdot \hat{\mathbf{v}}}{\det J_e} \text{ with } \hat{\nabla} \cdot \hat{\mathbf{v}} \in \mathbb{V}_2(\hat{e}) \implies \nabla \cdot \mathbf{v} \in \mathbb{V}_2(e).$$

# Transformations in 3D

$$\psi \in \mathbb{V}_0(e) \implies \psi \circ g_e = \hat{\psi} \in \mathbb{V}_0(\hat{e}),$$

$$\boldsymbol{\omega} \in \mathbb{V}_1(e) \implies \boldsymbol{\omega} \circ g_e = J_e^{-T}\hat{\boldsymbol{\omega}}, \text{ for } \hat{\boldsymbol{\omega}} \in \mathbb{V}_1(\hat{e}),$$

$$\mathbf{v} \in \mathbb{V}_2(e) \implies \mathbf{v} \circ g_e = \frac{J_e\hat{\mathbf{v}}}{\det J_e} \text{ for } \hat{\mathbf{v}} \in \mathbb{V}_2(\hat{e}),$$

$$\rho \in \mathbb{V}_2(e) \implies \rho \circ g_e = \frac{\hat{\rho}}{\det J_e} \text{ for } \hat{\rho} \in \mathbb{V}_3(\hat{e}).$$

- When the transformation is affine, $J_e$ is constant, and the approximation properties of $\mathbb{V}_k$ are unaffected.
- When the transformation is non-affine, $J_e$ is not constant, and the local space may not contain the required polynomials.

# The dangers of non-affine meshes

## Arnold, Boffi and Bonizzoni (2014)

For the $Q_r^-$ complex on quadrilaterals/hexahedra:

- A sequence of affine meshes has approximation error $\mathcal{O}(r+1)$ in $\mathbb{V}_k$.
- A sequence of non-affine meshes has approximation error $\mathcal{O}(r-k+1)$ in $\mathbb{V}_k$.

We are working on similar results for triangular prism elements.

Arnold, Boffi and Bonizzoni. Finite element differential forms on curvilinear cubic meshes and their approximation properties. Numer. Math., 2014.

# Rehabilitation

## Bochev and Ridzal (2008)

Biggest problem is in $\mathbb{V}_d$, mismatch caused by $1/\det J_e$. Instead,

1. Choose $\rho \in \mathbb{V}_d(e) \implies \rho \circ g_e = \hat{\rho} \in \mathbb{V}_d(\hat{e})$.
2. Replace $\nabla\cdot$ with $\pi_2 \nabla\cdot$:

$$
\begin{array}{ccccc}
H^1 & \xrightarrow{\nabla^\perp} & H(\mathrm{div}) & \xrightarrow{\nabla\cdot} & L^2 \\
\downarrow \pi_0 & & \downarrow \pi_1 & & \downarrow \pi_2 \\
\mathbb{V}^0 & \xrightarrow{\nabla^\perp} & \mathbb{V}^1 & \xrightarrow{\pi_2 \nabla\cdot} & \mathbb{V}^2
\end{array}
$$

Bochev and Ridzal. Rehabilitation of the lowest-order Raviart-Thomas element on quadrilateral grids. SIAM J Num. Anal. 47.1 (2008)

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*They tried to make me go to rehab, I said, "No, no, no".*

Reasons to rehabilitate?

1. Recover full approximation rate.
2. Factor of $1/\det J_e$ prevents exact quadrature, needed for dual versions of $\nabla \cdot \nabla \times = 0$ and $\nabla \times \nabla = 0$.
3. Exact quadrature is needed to reconstruct pointwise mass flux from upwind-DG advection.

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We will keep $1/\det J_e$ in the transformation for $\mathbb{V}_d$.

# Convergence in affine limit

**Holst and Stern (2012)**

High-order convergence is recovered for triangulations of manifolds as long as (a) consistent polynomial order approximation of domain is used and (b) manifold is sufficiently smooth that affine is approached at a suitable rate.

We are working on similar results for extruded wedge meshes in 3D.

Holst and Stern. "Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces." Foundations of Computational Mathematics 12.3 (2012): 263-293.

# Testing convergence rates in 2D on sphere

We would like to check higher-order convergence for mixed Poisson on the 2D surface of the sphere, but FFC+Dolfin/Firedrake does not support non-affine meshes (yet).

# Testing convergence rates in 2D on sphere

We would like to check higher-order convergence for mixed Poisson on the 2D surface of the sphere, but FFC+Dolfin/Firedrake does not support non-affine meshes (yet).

1. Define a global mapping $\phi$ from the affine sphere mesh $\hat{\Omega}$ to a higher-order bendy sphere mesh $\Omega$, using Expression.
2. Calculate $J$ and $\det J$ symbolically from $\phi$, and use them to pull the equations back from $\Omega$ to $\hat{\Omega}$.

$$\mathbf{\Sigma} \in \mathbb{V}_1(\Omega) \implies \mathbf{\Sigma} \circ \phi = \frac{J\hat{\mathbf{\Sigma}}}{\det J}, \; \hat{\mathbf{\Sigma}} \in \mathbb{V}_1(\hat{\Omega})$$

$$u \in \mathbb{V}_2(\Omega) \implies u \circ \phi = \frac{\hat{u}}{\det J}, \; \hat{u} \in \mathbb{V}_2(\hat{\Omega})$$

# Pulling back the equations

$$\int_\Omega \boldsymbol{\tau} \cdot \boldsymbol{\Sigma} + \nabla \cdot \tau u \, \mathrm{d}x = 0, \quad \forall \boldsymbol{\tau} \in \mathbb{V}_1(\Omega),$$

$$\int_\Omega v \nabla \cdot \boldsymbol{\Sigma} \, \mathrm{d}x = \int_\Omega vg \, \mathrm{d}x, \quad \forall u \in \mathbb{V}_2(\Omega),$$

becomes

$$\int_{\hat{\Omega}} \frac{(J\hat{\boldsymbol{\tau}}) \cdot (J\hat{\boldsymbol{\Sigma}})}{\det J} + \hat{\nabla} \cdot \hat{\tau} \frac{\hat{u}}{\det J} \, \mathrm{d}\hat{x} = 0, \quad \forall \hat{\boldsymbol{\tau}} \in \mathbb{V}_1(\hat{\Omega}),$$

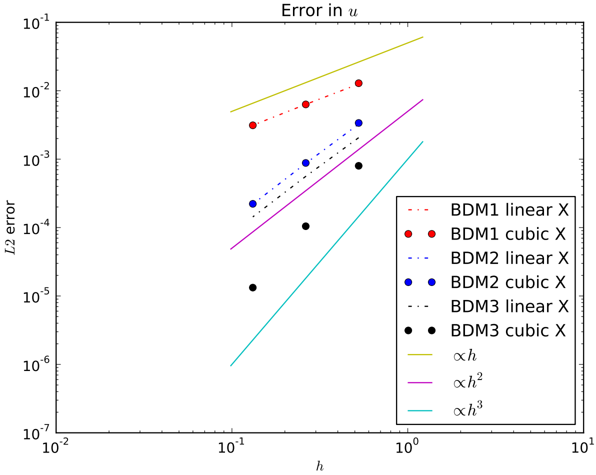
$$\int_{\hat{\Omega}} \hat{v} \hat{\nabla} \cdot \frac{\hat{\boldsymbol{\Sigma}}}{\det J} \, \mathrm{d}\hat{x} = \int_{\hat{\Omega}} \hat{v} g \circ \phi \, \mathrm{d}\hat{x}, \quad \forall \hat{v} \in \mathbb{V}_2(\hat{\Omega}).$$

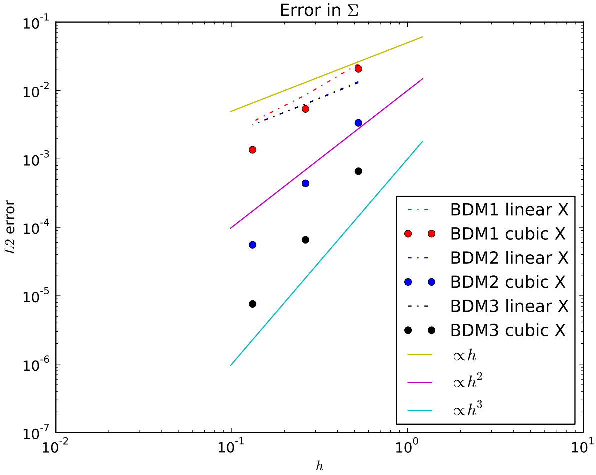
Note that we have used the commutative property
$(\nabla \cdot \boldsymbol{\Sigma}) \circ \phi = \frac{\hat{\nabla} \cdot \hat{\boldsymbol{\Sigma}}}{\det J}$.

```
n_out = _outward_normals(mesh)
J = as_tensor([[X.dx(0),X.dx(1),X.dx(2)],
               [Y.dx(0),Y.dx(1),Y.dx(2)],
               [Z.dx(0),Z.dx(1),Z.dx(2)]])
dJ = as_tensor([[X*n_out[0],X*n_out[1],X*n_out[2]]
                ,
                [Y*n_out[0],Y*n_out[1],Y*n_out[2]]
                ,
                [Z*n_out[0],Z*n_out[1],Z*n_out[2]]
                ])
detJ = det(J+dJ)
```

```
tau, v = TestFunctions(W)

a = (inner(dot(J,sigma), dot(J,tau))/detJ +
     div(sigma)*v/detJ + div(tau)*u/detJ)*dx
L = g*v*dx
```


Error in $u$
$L2$ error
$h$
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-2
10^-1
10^0
10^1
BDM1 linear X
BDM1 cubic X
BDM2 linear X
BDM2 cubic X
BDM3 linear X
BDM3 cubic X
$\propto h$
$\propto h^2$
$\propto h^3$


Error in $\Sigma$
L2 error
h
$10^{-1}$
$10^{-2}$
$10^{-3}$
$10^{-4}$
$10^{-5}$
$10^{-6}$
$10^{-7}$
$10^{-2}$
$10^{-1}$
$10^{0}$
$10^{1}$
BDM1 linear X
BDM1 cubic X
BDM2 linear X
BDM2 cubic X
BDM3 linear X
BDM3 cubic X
$\propto h$
$\propto h^2$
$\propto h^3$

# Testing convergence rates in 2D on sphere

We would also like to check convergence on a spherical annulus mesh with wedge elements, but FFC+Firedrake does not support non-affine meshes (yet).

- We can perform the same trick, if we can find a domain that is topologically equivalent to the spherical annulus, but which can be meshed using affine wedge elements.
- The spherical annulus mesh has non-affine elements because the triangle areas increase with height.

What is the reference affine domain for a spherical annulus?

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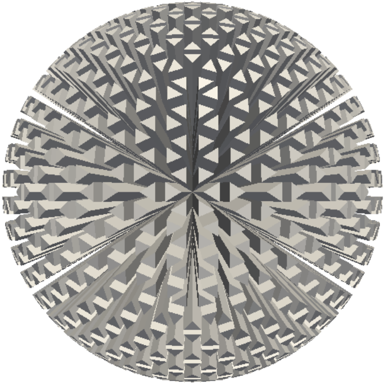
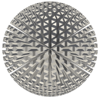
# The hedgehog mesh



# Making a hedgehog mesh with Firedrake



1. Start with a triangulation of the sphere (e.g. icosahedral).
2. Extrude the mesh in the radial direction to make columns of non-affine wedge elements.
3. Replace the CG coordinate field[1] with a DG coordinate field.
4. Recompute the coordinate field so that triangle area is preserved up the column.
5. Interelement continuity for $CG$, $H(div)$ and $H(curl)$ elements is maintained.

[1] In Firedrake, coordinate fields are just members of ordinary vector-valued function spaces.

```
p = TrialFunction(V3)
q = TestFunction(V3)
gprime = Function(V3)
solve(p*q/detJ*dx == g*q*dx, gprime)
u = TrialFunction(V2)
v = TestFunction(V2)
pe = div(u) - gprime
aeqn = (inner(dot(J,u),dot(J,v))/detJ + div(v)*pe/detJ
                            )*dx
a = lhs(aeqn)
L = rhs(aeqn)
usol = Function(V2)
solve(a==L, usol)
psol = Function(V3)
solve(p*q*dx == q*(div(usol)/detJ - g)*dx, psol)
```

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# Geometry of the hedgehog

- Hedgehog mesh geometry replaces $r$ by $r_0$ in metric written in spherical coordinates:

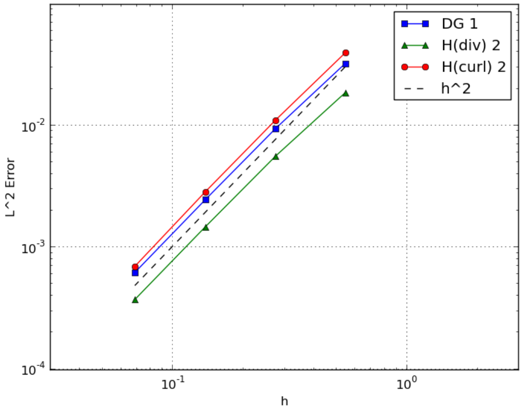
$$ds^2 = r^2 \cos^2 \phi \, \mathrm{d}\lambda^2 + r^2 \, \mathrm{d}\phi^2 + \mathrm{d}r^2,$$

$$\text{becomes } ds^2 = r_0^2 \cos^2 \phi \, \mathrm{d}\lambda^2 + r_0^2 \, \mathrm{d}\phi^2 + \mathrm{d}z^2.$$

- This geometry can be obtained by embedding the 2-sphere in $\mathbb{R}^4$ with a flat metric, then extruding in the fourth direction.
- This geometry is known to meteorologists as the shallow atmosphere approximation; curvature is $2/r_0^2$.

Thuburn and White. A geometrical view of the shallow-atmosphere approximation, with application to the semiLagrangian departure point calculation. Quarterly Journal of the Royal Meteorological Society 139.670 (2013): 261-268.

# Approximation error



# 3D convergence

# What about dual operators?

$$\mathbb{V}_0 \xrightarrow{\nabla} \mathbb{V}_1 \xrightarrow{\nabla\times} \mathbb{V}_2 \xrightarrow{\nabla\cdot} \mathbb{V}_3$$

$$\xleftarrow{\tilde{\nabla}\cdot} \qquad \xleftarrow{\tilde{\nabla}\times} \qquad \xleftarrow{\tilde{\nabla}}$$

Crucial for discrete Helmholtz decomposition; we need $\tilde{\nabla}\cdot\tilde{\nabla}\times = 0$ and $\tilde{\nabla}\times\tilde{\nabla} = 0$.

For $p \in \mathbb{V}_3$, define[2] $\tilde{\nabla} p \in \mathbb{V}_2$ by

$$\int_\Omega \mathbf{v}\cdot\tilde{\nabla} p \,\mathrm{d}x = -\int_\Omega \nabla\cdot\mathbf{v} p \,\mathrm{d}x, \quad \forall \mathbf{v} \in \mathbb{V}_2.$$

Similar definitions for $\tilde{\nabla}\times$, $\tilde{\nabla}\cdot$.

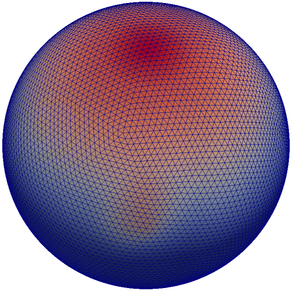
[2] Take $\Omega$ with no external boundaries to make things easy

# Closure when composing dual operators

For $p \in \mathbb{V}_3$, we have $\tilde{\nabla} \times \tilde{\nabla} p \in \mathbb{V}_1$, and

$$\int_\Omega \mathbf{\Sigma} \cdot \tilde{\nabla} \times \tilde{\nabla} p \, \mathrm{d}x = -\int_\Omega \nabla \times \mathbf{\Sigma} \cdot \tilde{\nabla} p \, \mathrm{d}x$$
$$= \int_\Omega \underbrace{\nabla \cdot \nabla \times \mathbf{\Sigma}}_{=0} p \, \mathrm{d}x = 0.$$

- For non-affine meshes, factors of $J$ and $\det J$ appear in all of these expressions.
- However, we can replace the integrals in the definitions by sums over quadrature points:
  $\sum_i \mathbf{v}_i \cdot \tilde{\nabla} p_i w_i = -\sum_i (\nabla \cdot \mathbf{w})_i p_i w_i, \quad \forall \mathbf{w} \in \mathbb{V}_2,$
  and property is preserved.



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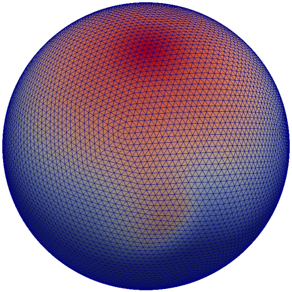

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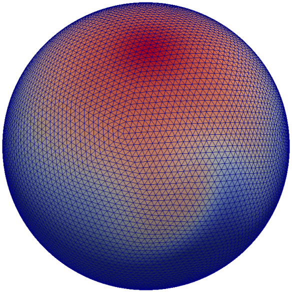

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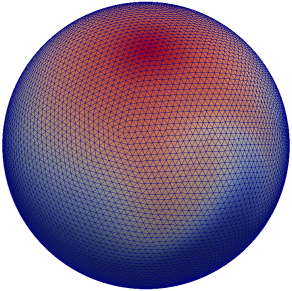

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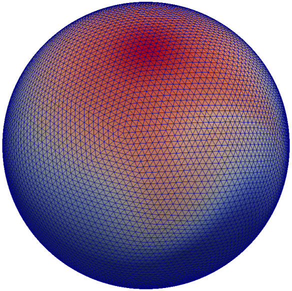

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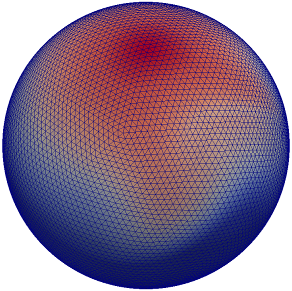

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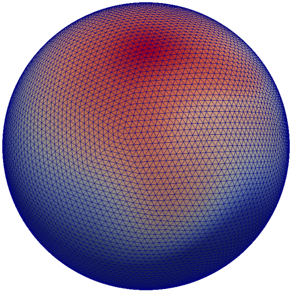

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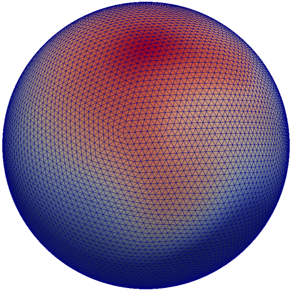

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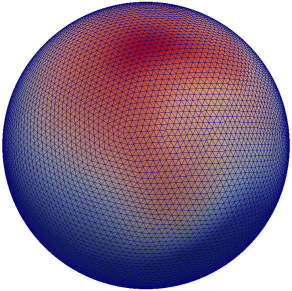

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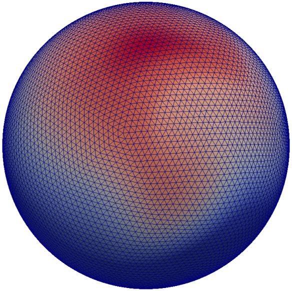

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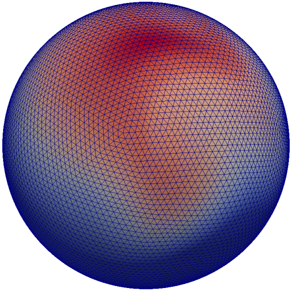

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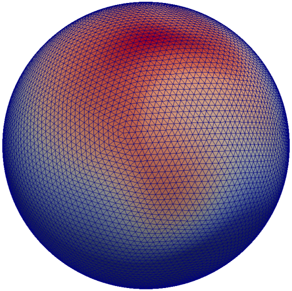

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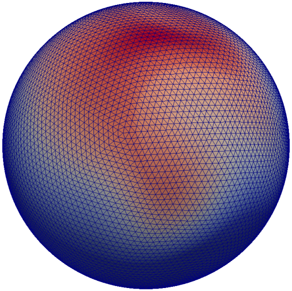

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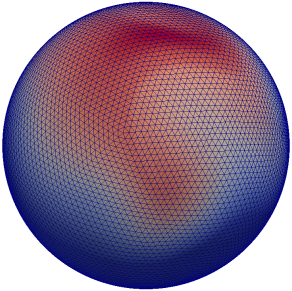

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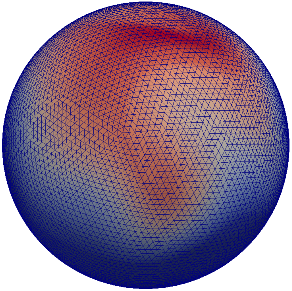

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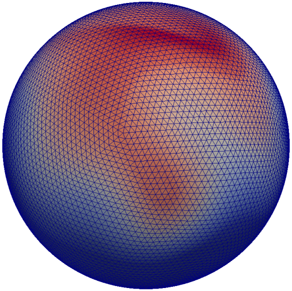

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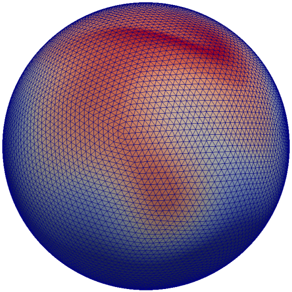

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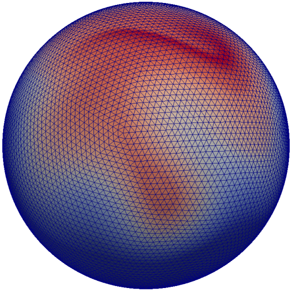

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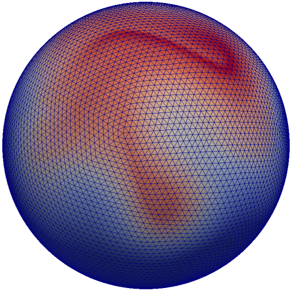

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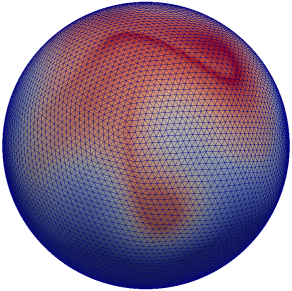

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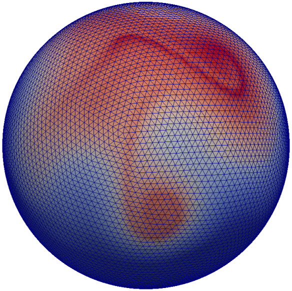

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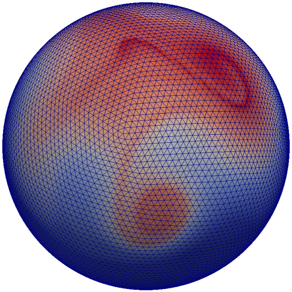

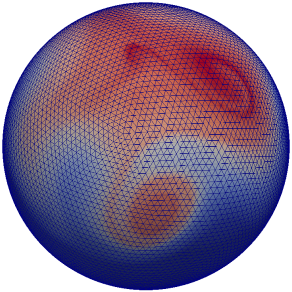

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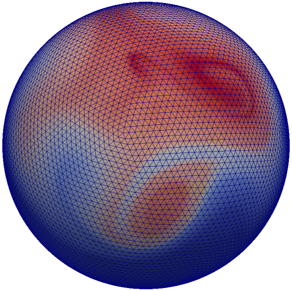

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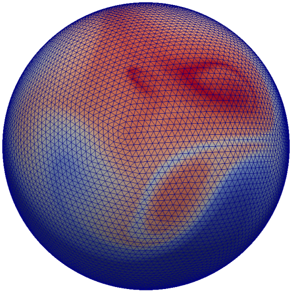

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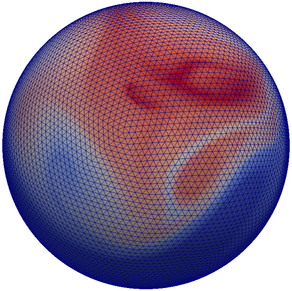

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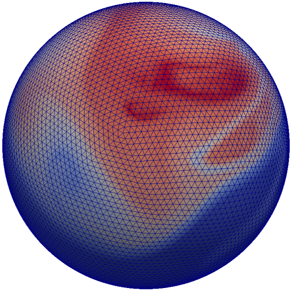

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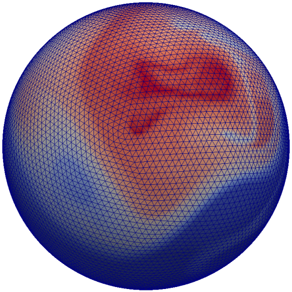

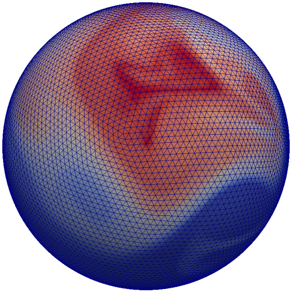

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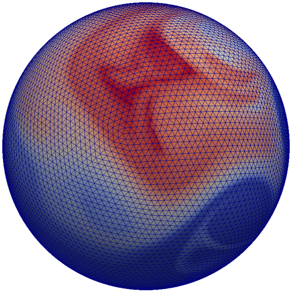

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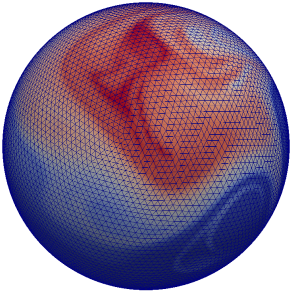

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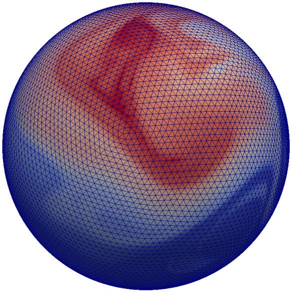

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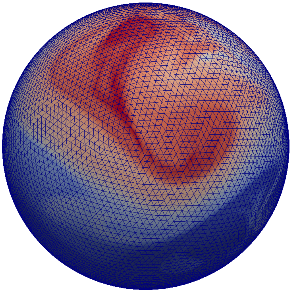

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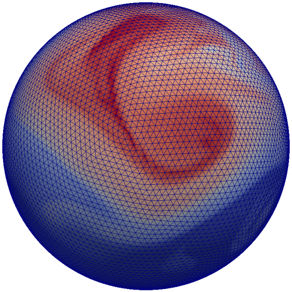

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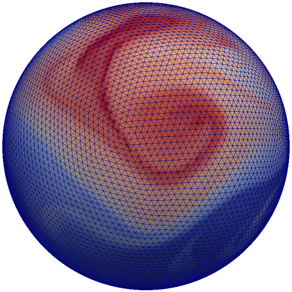

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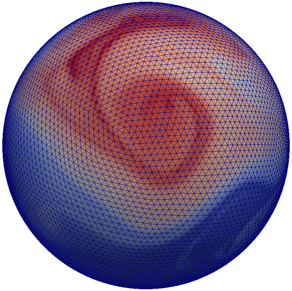

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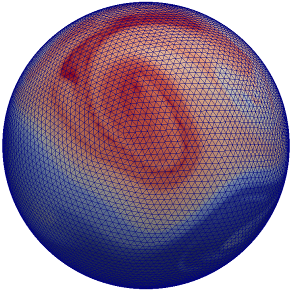

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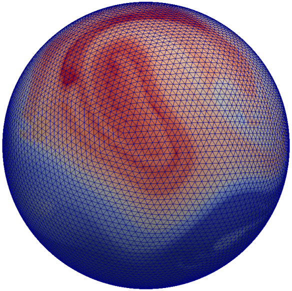

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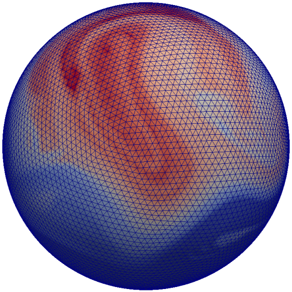

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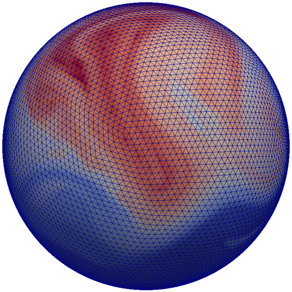

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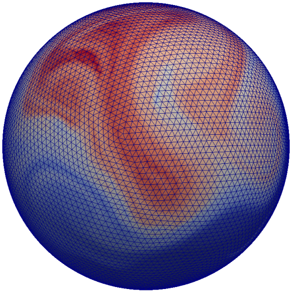

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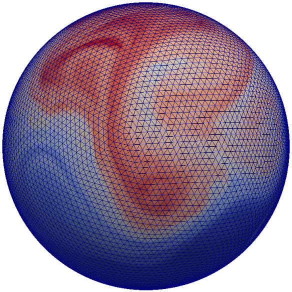

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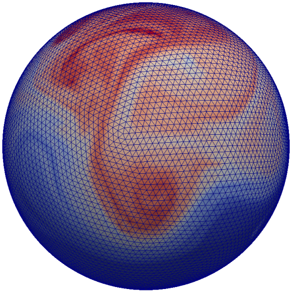

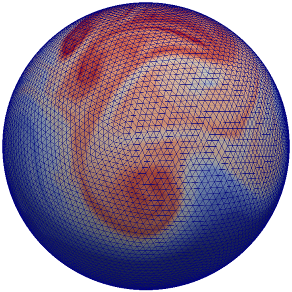

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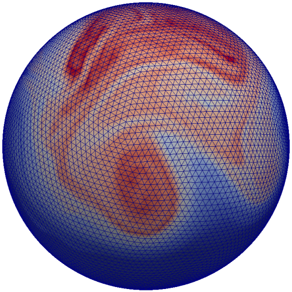

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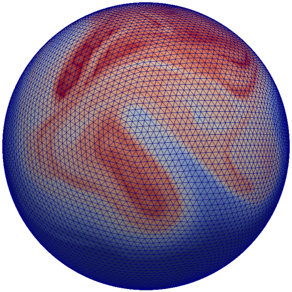

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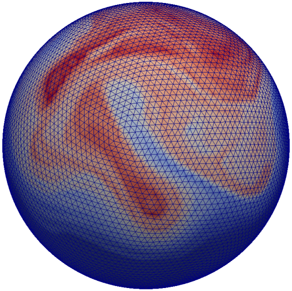

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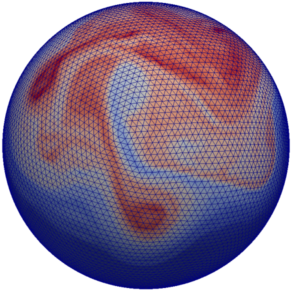

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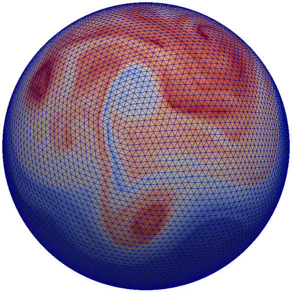

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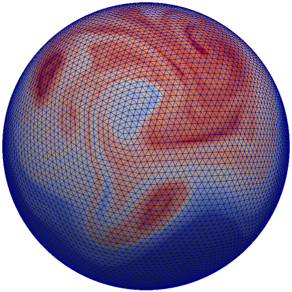

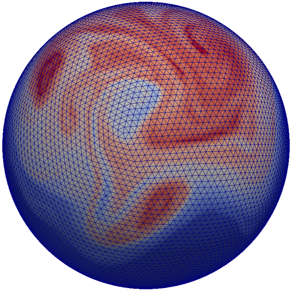

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# Conclusions

- Non-affine meshes are needed for global atmosphere/ocean applications.
- They lead to non-constant $J$ and $\det J$ terms.
- Potential convergence loss can be avoided if domain is smooth so that affine is approached in the limit.
- This was verified for triangles on sphere and wedges in spherical annulus by pulling back with a global transformation from an affine reference mesh and manually inserting $J$s and $\det J$s.
- The hedgehog mesh can be used as an affine reference mesh.
- Non-affine doesn't spoil closure of dual operators, or reconstruction of mass fluxes (not shown).

# Mass reconstruction

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \phi, h\rangle + \langle \phi, \nabla\cdot\mathbf{F}\rangle = 0, \quad \forall \phi \in \mathbb{V}_2(e).$$

Upwind DG:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_e \phi h \,\mathrm{d}x - \int_e \nabla\phi\cdot\mathbf{u}h\,\mathrm{d}x + \int_{\partial e}\phi\mathbf{u}\cdot\mathbf{n}\tilde{h}\,\mathrm{d}s = 0, \quad \forall \phi \in \mathbb{V}_2(e).$$

Can find $\mathbf{F} \in \mathbb{V}_1$ locally so that the above equation is consistent with upwind DG advection, using Fortin operator.

$$\int_f \phi\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \int_f \phi\mathbf{u}\cdot\mathbf{n}\tilde{h}\,\mathrm{d}s, \quad \forall \phi \in \mathbb{V}_2(f),$$

$$\int_e \nabla\phi\cdot\mathbf{F}\,\mathrm{d}x = \int_e \nabla\phi\cdot h\mathbf{u}\,\mathrm{d}x, \quad \forall \phi \in \mathbb{V}_2(e).$$

# Mass reconstruction

Mass reconstruction relies on exact integration in DG scheme, but we get factors of $1/\det J$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{e}} \hat{\phi}\frac{\hat{h}}{\det J_e}\,\mathrm{d}\hat{x} - \int_{\hat{e}} \hat{\nabla}\hat{\phi}\cdot\hat{\mathbf{u}}\frac{\hat{h}}{\det J_e}\,\mathrm{d}\hat{x} + \int_{\partial\hat{e}} \hat{\phi}\hat{\mathbf{u}}\cdot\hat{\mathbf{n}}\frac{\tilde{\hat{h}}}{\det J_e}\,\mathrm{d}\hat{s} = 0,$$

$$\forall\hat{\phi}\in\mathbb{V}_2(\hat{e}).$$

**Solution**

Replace $\phi\circ g_e = \hat{\phi}/\det J_e$ with $\phi\circ g_e = \hat{\phi}$.

# Coriolis term

Geostrophic balance property and PV conservation rely on exact integration of (nonlinear) Coriolis term:

$$\int_\Omega \mathbf{w} \cdot \mathbf{Q}^\perp \, \mathrm{d}x.$$

Magic cancellation occurs:

$$\int_e \mathbf{w} \cdot \mathbf{Q}^\perp \, \mathrm{d}x = \int_{\hat{e}} \hat{\mathbf{w}} \cdot \hat{\mathbf{Q}}^\perp \, \mathrm{d}\hat{x}.$$