About some operators on the unit disc related to the Laplace equation

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Contents

Abstract

We introduce four integral operators closely related to the Laplace equation in three-dimensions on the circular unit disc. Two of them are closed to the simple layer on the disc and the other two are related to the hyper singular operator. We establish their variational formulations and the coercivity properties in some unknows Sobolev spaces. They are also linked to the Laplace operator on the disc.
These results are a tentative extension to R^3 of previous results in R^2, contains in a common work with Carlos Jerez-Hanckes that we present in the beginning of the talk.

Standard Sobolev spaces

For $s \in \mathbb{R}$, $H^s(\mathcal{O})$ denote standard Sobolev spaces [3, Chapter 3].
Let $s \geq 0$, we say that a distribution belongs to the local Sobolev space $H^s_{\text{loc}}(\mathcal{O})$ if its restriction to every compact set $K \Subset \mathbb{R}^d$ lies in $H^s(K)$.
If $s > 0$ and $\mathcal{O}$ Lipschitz, $\widetilde{H}^s(\mathcal{O})$ denotes the space of functions whose extension by zero over a closed domain $\widetilde{O}$ belongs to $H^s(\widetilde{O})$.
We identify

$$\widetilde{H}^{-1/2}(\mathcal{O}) \equiv \left(H^{1/2}(\mathcal{O})\right)' \quad \text{and} \quad H^{-1/2}(\mathcal{O}) \equiv \left(\widetilde{H}^{1/2}(\mathcal{O})\right)', \tag{1}$$

and if $\mathcal{O} = \widetilde{O}$, then $\widetilde{H}^{\pm 1/2}(\mathcal{O}) \equiv H^{\pm 1/2}(\mathcal{O})$.

Log-Kernel

Consider first the isotropic space $\mathbb{R}^2$ divided into two half-planes:

$$\pi_{\pm} := \left\{ \mathbf{x} \in \mathbb{R}^2 \; : \; x_2 \lessgtr 0 \right\} \tag{2}$$

with interface Γ given by the line $x_2 = 0$. The interface is further divided into the open disjoint segments $\Gamma_m := (-1,1) \times \{0\}$ and $\Gamma_f := \Gamma \setminus \bar{\Gamma}_m$. Consequently, we have defined the domain $\Omega := \mathbb{R}^2 \setminus \bar{\Gamma}_m$. We seek u solution of the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{for} \quad \mathbf{x} \in \Omega \\ u = g & \text{for} \quad \mathbf{x} \in \Gamma_m; \quad \text{with} \quad g \in H^{1/2}(\Gamma_m). \end{cases} \tag{3}$$

Then, the potential u can be represented as a single layer potential:

$$u(\mathbf{x}) = L_1\varphi = \frac{1}{\pi} \int_{\Gamma_m} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \varphi(\mathbf{y}) \mathrm{d}\mathbf{y} \, , \qquad \text{for} \quad \mathbf{x} \in \Omega \, , \tag{4}$$

where φ is the solution of the logarithmic integral equation:

$$g(\mathbf{x}) = \frac{1}{\pi} \int_{\Gamma_m} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \varphi(\mathbf{y}) \mathrm{d}\mathbf{y} \qquad \text{for} \quad \mathbf{x} \in \Gamma \, . \tag{5}$$

The equation (5) has a variational formulation in the space $\widetilde{H}_0^{-1/2}(\Gamma_m)$ which is the space of functions in $\widetilde{H}^{-1/2}(\Gamma_m)$ satisfying $\int_{\Gamma_m} \varphi(t)\, dt = 0$. It is :

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\varphi(t)\varphi^t(\tau)dtd\tau=\int_{\Gamma_m}g(\tau)\varphi^t(\tau)d\tau,\forall\varphi^t\in\widetilde{H}_0^{-1/2}(\Gamma_m) \tag{6}$$

This operator is a bijection between $\widetilde{H}_0^{-1/2}(\Gamma_m)$ and the $H_*^{1/2}(\Gamma_m)$ of functions in $H^{1/2}(\Gamma_m)$ satisfying $\int_{\Gamma_m} \frac{1}{\sqrt{1-t^2}} g(t)\, dt = 0$. and moreover we have

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\varphi(t)\varphi(\tau)dtd\tau\geq C\,\|\varphi\|^2_{\widetilde{H}_0^{-1/2}(\Gamma_m)},\forall\varphi\in\widetilde{H}_0^{-1/2}(\Gamma_m). \tag{7}$$

The inverse operator N_1 is symmetric and coercive and is a bijection of $H_*^{1/2}(\Gamma_m)$ onto $\widetilde{H}_0^{-1/2}(\Gamma_m)$. It admits two variational formulations.
Let $M(x,y)$ be the function and L_2 the associated integral operator:

$$M(x,y)=\frac{1}{2}\left((y-x)^2+\left(\sqrt{1-x^2}+\sqrt{1-y^2}\right)^2\right) \tag{8}$$

$$L_2 g=\frac{1}{\pi}\int_{\Gamma_m}\log\left\{\frac{M(x,y)}{|x-y|}\right\}g(y)dy \tag{9}$$

The first one is:

$$(N_1 g, g^t) = \frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m} \log\left\{\frac{M(x,y)}{|x-y|}\right\} g'(x)\left(g^t(y)\right)' \, dy\, dx = \int_{\Gamma_m} \varphi(x) g^t(x)\, dx \quad (10)$$

for all $g^t \in H_*^{1/2}(\Gamma_m)$, which gives a first norm on the space $H_*^{1/2}(\Gamma_m)$:

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m} \log\left[\frac{M(x,y)}{|x-y|}\right] g'(x)\, g'(y)\, dy\, dx \;\geq\; C\,\|g\|^2_{H_*^{1/2}(\Gamma_m)}\,;\forall g \in H_*^{1/2}(\Gamma_m) \quad (11)$$

The second one is

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m} \frac{d^2}{dxdy}\log\left[\frac{M(x,y)}{|x-y|}\right]\left(g(x)-g(y)\right)\left(g^t(x)-g^t(y)\right) dy dx = \int_{\Gamma_m} \varphi(x) g^t(x)\, dx \quad (12)$$

for all $g^t \in H_*^{1/2}(\Gamma_m)$,

So we have a second norm on the space $H_*^{1/2}(\Gamma_m)$ which is:

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m}\left\{\frac{1-xy}{w(x)w(y)}\right\}\frac{\left(g(x)-g(y)\right)^2}{(x-y)^2}\, dy dx \geq C\,\|g\|^2_{H_*^{1/2}(\Gamma_m)}, \forall g \in H_*^{1/2}(\Gamma_m) \quad (13)$$

where the weight function w is given by

$$w(x) := \sqrt{1 - x^2} \qquad \text{for } x \in (-1, 1). \quad (14)$$

We can also consider the Neumann problem

$$\begin{cases} -\Delta u = 0 & \text{for} \quad \mathbf{x} \in \Omega \\ \gamma_m^+ \partial_n u = \gamma_m^- \partial_n u = \varphi & \text{for} \quad \mathbf{x} \in \Gamma_m, \quad \varphi \in H^{-1/2}(\Gamma_m) \end{cases} \tag{15}$$

which can be represent as a double layer potential of harmonic solution in the domain Ω of the form .

$$u(\mathbf{x}) = \frac{1}{\pi} \int_{\Gamma_m} \frac{x_2}{|\mathbf{x} - \mathbf{y}|^2} \alpha(\mathbf{y}) \mathrm{d}\mathbf{y} , \qquad \text{for} \quad \mathbf{x} \in \Omega , \tag{16}$$

Then the unknown α is the solution of the hyper singular integral equation:

$$\varphi(x) = N_2 \, \alpha = \frac{1}{\pi} \int_{\Gamma_m} \frac{1}{|x - y|^2} \alpha(y) dy \qquad \text{for} \quad x \in \Gamma . \tag{17}$$

where α is also the jump of the Dirichlet trace of the solution of problem (15). We look for a solution of the integral equation (17) in the space $\widetilde{H}^{1/2}(\Gamma_m)$.

A variational formulation of the integral equation (17) in this space $\widetilde{H}^{1/2}(\Gamma_m)$ is

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau - t|}\alpha'(t)(\alpha^t(\tau))' dt d\tau = \int_{\Gamma_m}\varphi(\tau)\alpha^t(\tau)d\tau, \forall\alpha^t \in \widetilde{H}^{1/2}(\Gamma_m) \quad (18)$$

The associated operator N_2 is a bijection from $\widetilde{H}^{1/2}(\Gamma_m)$ to $H^{-1/2}(\Gamma_m)$.
Moreover, this bilinear form is coercive, i.e.,

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau - t|}\alpha'(t)\alpha(\tau)' dt d\tau \geq C\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_m)}, \forall\alpha \in \widetilde{H}^{1/2}(\Gamma_m). \quad (19)$$

This operator admits a second variational formulation which is

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m}\frac{(\alpha(x)-\alpha(y))\,(\alpha^t(x)-\alpha^t(y))}{|x-y|^2}dxdy + \frac{1}{\pi}\int_{\Gamma_m}\frac{\alpha(x)\alpha^t(x)}{1-x^2}dx = \int_{\Gamma_m}\varphi(x)\alpha^t(x)dx \quad (20)$$

for all $\alpha^t \in \widetilde{H}^{1/2}(\Gamma_m)$, and the next expression is a norm on $\widetilde{H}^{1/2}(\Gamma_m)$

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m}\frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy + \frac{1}{\pi}\int_{\Gamma_m}\frac{\alpha(x)^2}{1-x^2}dx \geq C\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_m)}, \forall\alpha \in \widetilde{H}^{1/2}(\Gamma_m) \quad (21)$$

The inverse operator is L_2 which is a bijection of $H^{-1/2}(\Gamma_m)$ onto $\widetilde{H}^{1/2}(\Gamma_m)$.
This operator is symmetric and coercive in the space $H^{-1/2}(\Gamma_m)$.
It admits the following variational formulation:

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\left[\frac{M(x,y)}{|x-y|}\right]\varphi(x)\varphi^t(y)dydx=\int_{\Gamma_m}\alpha(x)\varphi^t(x)dx,\ \forall\varphi\in H^{-1/2}(\Gamma_m) \qquad (22)$$

and thus the following expression is a norm on the space $H^{-1/2}(\Gamma_m)$

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\left[\frac{M(x,y)}{|x-y|}\right]\varphi(x)\varphi(y)dydx\geq C\,\|\varphi\|^2_{H^{-1/2}(\Gamma_m)},\ \forall\varphi\in H^{-1/2}(\Gamma_m) \qquad (23)$$

Let D be the derivation operator which operates from $\widetilde{H}^{1/2}(\Gamma_m)$ to $\widetilde{H}_0^{-1/2}(\Gamma_m)$
and D^* the adjoint operator which operates from $H^{1/2}(\Gamma_m)$ to $H^{-1/2}(\Gamma_m)$.
D^* is also a derivation but not in the sense of distribution.

The operators L_1, L_2, N_1, N_2, D, D^* are linked by the identities

$$L_2 \circ N_2 = -L_2 \circ D^* \circ L_1 \circ D = I\,, \qquad I \in \widetilde{H}^{1/2}(\Gamma_m)$$

$$L_1 \circ N_1 = -L_1 \circ D \circ L_2 \circ D^* = I\,, \qquad I \in H_*^{1/2}(\Gamma_m)$$

$$N_1 \circ L_1 = -D \circ L_2 \circ D^* \circ L_1 = I\,, \qquad I \in \widetilde{H}_0^{-1/2}(\Gamma_m)$$

$$N_2 \circ L_2 = -D^* \circ L_2 \circ D \circ L_1 = I\,, \qquad I \in H^{-1/2}(\Gamma_m)$$

$L_1 \circ D$ is continuous and invertible from $\widetilde{H}^{1/2}(\Gamma_m)$ into $H_*^{1/2}(\Gamma_m)$.
$L_2 \circ D^*$ is continuous and invertible from $H_*^{1/2}(\Gamma_m)$ into $\widetilde{H}^{1/2}(\Gamma_m)$.
$D^* \circ L_1$ is continuous and invertible from $\widetilde{H}_0^{-1/2}(\Gamma_m)$ into $H^{-1/2}(\Gamma_m)$.
$D \circ L_2$ is continuous and invertible from $H^{-1/2}(\Gamma_m)$ into $\widetilde{H}_0^{-1/2}(\Gamma_m)$.

The Dirichlet and Neumann Laplacian Δ_D, Δ_N are linked to L_1, L_2 and N_1, N_2:

$$L_1 = (-\Delta_D)^{-\frac{1}{2}}\,; \qquad -N_1 = (-\Delta_D)^{\frac{1}{2}}\,;$$

$$L_2 = (-\Delta_N)^{-\frac{1}{2}}\,; \qquad -N_2 = (-\Delta_N)^{\frac{1}{2}}\,.$$

As a consequence of the previous results, we recover a new proof of some well known results:

Theorem

$\widetilde{H}^{1/2}(\Gamma_m)$ is exactly the space of functions $g \in H^{1/2}(\Gamma_m)$ such that $w^{-1}g$ is in $L^2(\Gamma_c)$. The space $\widetilde{H}_0^{-1/2}(\Gamma_m)$ is exactly the image of the distributional derivative of functions in $\widetilde{H}^{1/2}(\Gamma_m)$.

Comments: The proofs of the above results are obtained using the expansion of the kernel on the Tchebychev polynomials. This is also related to the links between these polynomials and the usual trigonometric functions, and also the use of symmetry and antisymmetry and of the projection from the 2D circle on its diameter.

The disc in R^3

We try now to extend these results to the unit disc in $\mathbb{R}^3$.

We Introduce the splitting of the space $\mathbb{R}^3$ into two half-spaces $\pi_\pm := \left\{\mathbf{x} \in \mathbb{R}^3 \ : \ x_3 \gtrless 0\right\}$, by the plane $x_3 = 0$ that will be denote as Γ.
Let $\mathfrak{c}$ be the circle of center at the origin and of radius 1 in the plane Γ.
Let $\mathbb{D}$ be the plane disc delimitated by the circle $\mathfrak{c}$ and $\mathbb{D}$ the associated flat domain in $\mathbb{R}^3$.
Now its complement in $\mathbb{R}^2$, is $\Gamma_f := \Gamma \setminus \bar{\mathbb{D}}$.
Henceforth, the problem domain is denoted by $\Omega := \mathbb{R}^3 \setminus \bar{\mathbb{D}}$.
We also consider the sphere $\mathbb{S}$ of radius 1 and center at the origin in R^3.
The disc $\mathbb{D}$ divide this sphere into two half-sphere that we denote respectively $\mathbb{S}^+$ and $\mathbb{S}^-$.
For any $s > 0$, $\widetilde{H}^s(\mathbb{D})$ is the space of functions whose extension by zero to Γ belongs to $H^s(\Gamma)$. We identify

$$\widetilde{H}^{-1/2}(\mathbb{D}) \equiv \left(H^{1/2}(\mathbb{D})\right)' \quad \text{and} \quad H^{-1/2}(\mathbb{D}) \equiv \left(\widetilde{H}^{1/2}(\mathbb{D})\right)', \tag{26}$$

The unit sphere in $\mathbb{R}^3$ and its equatorial disc

We consider the unit sphere $\mathbb{S}$ in $\mathbb{R}^3$ (Fig. 1) and the spherical coordinates: (r, θ, φ), where r is the radius and θ, φ the two Euler angles.

$$\begin{cases} x_1 = r \sin\theta \cos\varphi, \\ x_2 = r \sin\theta \sin\varphi, \\ x_3 = r \cos\theta. \end{cases} \quad (27)$$

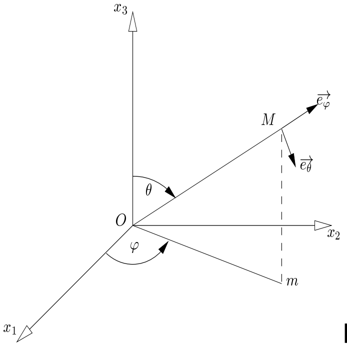


Fig. 1: Spherical coordinates

The vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unitary. The vector $\mathbf{e}_\rho$ directed along Om is unitary.

some geometry

–A point **x** on the circular domain $\mathbb{D}$ will be defined using its coordinates (x_1, x_2) or in circular coordinates by $(0 \leq \rho \leq 1, 0 \leq \varphi \leq 2\pi)$.

–A point $\mathbf{x}^+$ (resp. $\mathbf{x}^-$) on the half sphere $\mathbb{S}^+$ (resp. $\mathbb{S}^-$) will be defined using $(0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi)$ (resp. $(\frac{\pi}{2} \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi)$).

–The projection **x** of a point $\mathbf{x}^+$ situed on the half sphere $\mathbb{S}^+$ onto the domain $\mathbb{D}$ has for circular coordinates **x** : $(\rho = \sin(\theta), \varphi)$.

– The projection **x** of a point $\mathbf{x}^-$ situed on the half sphere $\mathbb{S}^-$ onto the domain $\mathbb{D}$ has for circular coordinates **x** : $(\rho = \sin(\theta), \varphi)$.

–To a point **x** on the disc $\mathbb{D}$, we associate the two points $\mathbf{x}^+$ and $\mathbf{x}^-$ which projections are the point **x**.

Spherical Harmonics Associated Legendre functions

The Rodrigues formula gives the expression of the Legendre polynomial $\mathbb{P}_l$:

$$\mathbb{P}_l(x) = \frac{(-1)^l}{2^l l!}\left(\frac{d}{dx}\right)^l (1-x^2)^l. \tag{28}$$

The Spherical Harmonics solve the following differential equation (in the variables $x = x_3$ and φ)

$$\frac{1}{1-x^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial}{\partial x}u\right) + l(l+1)u = 0. \tag{29}$$

This equation admits a family of solutions with separate variables

$$Y_l^m(x,\varphi) = \gamma_l^m e^{im\varphi}\mathbb{P}_l^m(x) \tag{30}$$

where the functions $\mathbb{P}_l^m(x)$, called the Associated Legendre functions, are the solutions of the differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\mathbb{P}_l^m\right) + l(l+1)\mathbb{P}_l^m - \frac{m^2}{1-x^2}\mathbb{P}_l^m = 0. \tag{31}$$

For $m = 0$, Y_l^0 is the Legendre polynomial $\mathbb{P}_l$.

We introduce the kinetic moments express in the angles (θ, φ)

$$L_3 u = \frac{1}{i}\frac{\partial}{\partial\varphi}u. \tag{32}$$

$$L_+ u = e^{i\varphi}\left(\frac{\partial}{\partial\theta}u + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}u\right). \tag{33}$$

$$L_- u = e^{-i\varphi}\left(-\frac{\partial}{\partial\theta}u + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}u\right). \tag{34}$$

The operators L_+ and L_- are associated to the representation of a two dimension real vector $\mathbf{x}$ (resp. $\mathbf{y}$) in $\mathbb{R}^2$ by a complex number $c = x_1 + ix_2$ (resp. $d = y_1 + iy_2$) .

The Laplace-Beltrami operator Δ_S takes the different forms

$$\begin{aligned} \Delta_S &= -\tfrac{1}{2}(L_+L_- + L_-L_+) - (L_3)^2 \\ &= -L_+L_- - (L_3)^2 + L_3 \\ &= -L_-L_+ - (L_3)^2 - L_3 \end{aligned} \tag{35}$$

and the following relations of commutation hold:

$$[\Delta_S, L_+] = [\Delta_S, L_-] = [\Delta_{\mathbb{S}}, L_3] = 0. \tag{36}$$

The relations of commutation (36) show that each eigenspace of the operator Δ_S is invariant by the action of the operators L_+, L_- and L_3. Using these property, we can express the spherical harmonics in the form (30) and thus the spherical harmonics of order l are the $2l+1$ functions of the form

$$Y_l^m(\theta,\varphi) = \left[\frac{(l+1/2)}{2\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} e^{im\varphi}\mathbb{P}_l^m(\cos\theta). \quad (37)$$

Further, they are the solution of the differential equation (29).
The associated Legendre functions $\mathbb{P}_l^m(\cos\theta)$ has the parity of $l+m$ and are given in terms of the Legendre polynomials by

$$\mathbb{P}_l^m(\cos\theta) = (\sin\theta)^m \left(\frac{d}{dx}\right)^m \mathbb{P}_l(\cos\theta);\ \text{if} \quad 0 \le m \le l, \quad (38)$$

$$L_3 Y_l^m = m Y_l^m, \quad (39)$$

$$L_+ Y_l^m = \sqrt{(l-m)(l+m+1)}\, Y_l^{m+1}, \quad (40)$$

$$L_- Y_l^m = \sqrt{(l+m)(l-m+1)}\, Y_l^{m-1}. \quad (41)$$

Symmetry and antisymmetry on the sphere

We introduce the **simple layer potential** on the sphere $\mathbb{S}$ defined, for $\mathbf{x} \in \mathbb{S}$ and $\mathbf{y} \in \mathbb{S}$, as

$$(\mathbb{S}I)u(\mathbf{y}) \ := \ \frac{1}{4\pi}\int_{\mathbb{S}} \frac{1}{|\mathbf{x}-\mathbf{y}|} u(\mathbf{x}) \sin(\theta) d\theta d\varphi \qquad (42)$$

We consider also **the hyper singular potential** on the sphere $\mathbb{S}$ given by

$$(\mathbb{N})u(\mathbf{y}) \ := \ \frac{1}{4\pi}\oint_{\mathbb{S}} \frac{\partial^2}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}}\Big(\frac{1}{|\mathbf{x}-\mathbf{y}|}\Big) u(\mathbf{x}) \sin(\theta) d\theta d\varphi \qquad (43)$$

The double layer potential $(\mathbb{D}I)$ on the sphere $\mathbb{S}$ is equal to $-\frac{1}{2}\mathbb{S}$

$$(\mathbb{D}I)u = \ \frac{1}{4\pi}\int_{\mathbb{S}} \frac{\partial}{\partial n_x}\frac{1}{|\mathbf{x}-\mathbf{y}|} u(\mathbf{x}) d\gamma(\mathbf{x}), \qquad (44)$$

The kernel of the hyper singular potential has a symmetric expression

$$\frac{\partial^2}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}}\Big(\frac{1}{|\mathbf{x}-\mathbf{y}|}\Big) = \frac{(n_{\mathbf{x}}.n_{\mathbf{y}})}{|\mathbf{x}-\mathbf{y}|^3} + \frac{3}{4|\mathbf{x}-\mathbf{y}|} = \frac{1}{|\mathbf{x}-\mathbf{y}|^3} + \frac{1}{4|\mathbf{x}-\mathbf{y}|} \qquad (45)$$

So in that situation, the **Calderon relations** for **the operators** $(\mathbb{S}, \mathbb{N})$, are

$$\mathbb{N} \circ \mathbb{S}I = \mathbb{S}I \circ \mathbb{N}, \tag{46}$$

$$-(\mathbb{N} - \frac{1}{4}\mathbb{S}I) \circ \mathbb{S}I = -\mathbb{S}I \circ (\mathbb{N} - \frac{1}{4}\mathbb{S}I) = \frac{1}{4}\mathbb{I}, \tag{47}$$

– To a function $u(\mathbf{x})$ defined on the sphere $\mathbb{S}$, we associate its symmetric and its antisymmetric parts defined on $\mathbb{S}^+$ and $\mathbb{S}^-$ as

$$\begin{cases} u_s(\mathbf{x}^+) = u_s(\mathbf{x}^-) = \frac{1}{2}(u(\mathbf{x}^+) + u(\mathbf{x}^-)) \\ u_{as}(\mathbf{x}^+) = -u_{as}(\mathbf{x}^-) = \frac{1}{2}(u(\mathbf{x}^+) - u(\mathbf{x}^-)) \end{cases} \tag{48}$$

To a point $\mathbf{y}$ on the sphere $\mathbb{S}$, we associated the symmetric point $\mathbf{y}^s$ which is either $\mathbf{y}^-$ or $\mathbf{y}^+$ and we define the four following operators

$$\begin{cases}(\mathbb{S}_s)u(\mathbf{y})=(\mathbb{S}I)u_s(\mathbf{y})=\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{S}^+}\Big(\frac{1}{|\mathbf{x}^+-\mathbf{y}^+|}+\frac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big)u_s(\mathbf{x}^+)\sin(\theta)d\theta d\varphi\\ (\mathbb{S}_{as})u(\mathbf{y})=(\mathbb{S}I)u_{as}(\mathbf{y})=\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{S}^+}\Big(\frac{1}{|\mathbf{x}^+-\mathbf{y}^+|}-\frac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big)u_{as}(\mathbf{x}^+)\sin(\theta)d\theta d\varphi\end{cases} \tag{49}$$

$$\begin{cases}(\mathbb{N}_s)u(\mathbf{y})=(\mathbb{N})u_s(\mathbf{y})=\dfrac{1}{4\pi}\displaystyle\oint_{\mathbb{S}^+}\Big(\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\frac{1}{|\mathbf{x}^+-\mathbf{y}^+|}\Big)\\ \qquad+\dfrac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big)\Big)u_s(\mathbf{x}^+)\sin(\theta)d\theta d\varphi\\ (\mathbb{N}_{as})u(\mathbf{y})=(\mathbb{N})u_{as}(\mathbf{y})=\dfrac{1}{4\pi}\displaystyle\oint_{\mathbb{S}^+}\Big(\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\frac{1}{|\mathbf{x}^+-\mathbf{y}^+|}\Big)\\ \qquad-\dfrac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big)\Big)u_{as}(\mathbf{x}^+)\sin(\theta)d\theta d\varphi\end{cases} \tag{50}$$

$$(\mathbb{S}_s)u_{as}(\mathbf{y})=(\mathbb{N}_s)u_{as}(\mathbf{y})=0 \tag{51}$$

$$(\mathbb{S}_{as})u_s(\mathbf{y})=(\mathbb{N}_{as})u_s(\mathbf{y})=0 \tag{52}$$

Using the Calderon identities (46) and (47), we obtain

Theorem

$$\mathbb{N}_s \circ \mathbb{S}_s = \mathbb{S}_s \circ \mathbb{N}_s; \qquad \mathbb{N}_{as} \circ \mathbb{S}_{as} = \mathbb{S}_{as} \circ \mathbb{N}_{as}; \qquad \mathbb{S}_s \circ \mathbb{S}_{as} = 0; \tag{53}$$

$$\mathbb{N}_{as} \circ \mathbb{S}_s = \mathbb{S}_s \circ \mathbb{N}_{as} = 0; \qquad \mathbb{N}_{as} \circ \mathbb{S}_s = \mathbb{S}_s \circ \mathbb{N}_{as} = 0; \qquad \mathbb{N}_s \circ \mathbb{N}_{as} = 0 \tag{54}$$

$${\mathbb{S}_s}^2 - 4\mathbb{N}_s \circ \mathbb{S}_s = I, \tag{55}$$

$${\mathbb{S}_{as}}^2 - 4\mathbb{N}_{as} \circ \mathbb{S}_{as} = I, \tag{56}$$

We define the operator $\overrightarrow{\mathbf{curl}}_{\mathbb{S}}$ on the sphere $\mathbb{S}$ as

$$\overrightarrow{\mathbf{curl}}_{\mathbb{S}} u(\mathbf{x}) = \frac{\partial u}{\partial \theta}\mathbf{e}_\varphi - \frac{1}{\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_\theta = \frac{\partial u}{\partial \theta}\mathbf{e}_\varphi - \frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_\rho + \frac{\partial u}{\partial \varphi}\mathbf{e}_3 \tag{57}$$

The bilinear form associated to the surfacic Laplacian $\Delta_{\mathbb{S}}$ is

$$\langle -\Delta_{\mathbb{S}} u\,,\,\overline{v}\rangle_{\mathbb{S}} = \int_{\mathbb{S}} \left(\overrightarrow{\mathbf{curl}}_{\mathbb{S}} u(\mathbf{x}) \cdot \overrightarrow{\mathbf{curl}}_{\mathbb{S}} \overline{v}(\mathbf{x})\right) \sin\theta d\theta d\varphi \tag{58}$$

The operator $\mathbb{N}$ defined by (43) is an isomorphism from $H^{1/2}(\mathbb{S})/\mathbb{R}$ onto the space $H^{-1/2}(\mathbb{S})$ with $\langle u_n, 1\rangle = 0$. It admits two variational formulations

$$\begin{cases} \langle -\mathbb{N} u\,,\,\overline{v}\rangle_{\mathbb{S}} = -\dfrac{1}{8\pi}\displaystyle\int_{\mathbb{S}}\int_{\mathbb{S}} \frac{\partial^2}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)(u(\mathbf{x})-u(\mathbf{y}))(\overline{v}(\mathbf{y})-\overline{v}(\mathbf{x}))d\gamma(\mathbf{x})d\gamma(\mathbf{y}) \\ = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{S}}\int_{\mathbb{S}} \frac{\left(\overrightarrow{\mathbf{curl}}_{\mathbb{S}} u(\mathbf{x}) \cdot \overrightarrow{\mathbf{curl}}_{\mathbb{S}} \overline{v}(\mathbf{y})\right)}{|\mathbf{x}-\mathbf{y}|} d\gamma(\mathbf{x})d\gamma(\mathbf{y});\ \forall\, u, v \in H^{1/2}(\mathbb{S})/\mathbb{R}. \end{cases} \tag{59}$$

Using the identity (59) and the Calderon relations (46) and (47), we obtain

$$\mathbb{S}I^{-1} \circ \mathbb{N} = \Delta_{\mathbb{S}}, \tag{60}$$

$$\mathbb{S}I = \frac{1}{2}\left(-\Delta_{\mathbb{S}} + \frac{I}{4}\right)^{-1/2}, \tag{61}$$

$$\mathbb{N} - \frac{1}{4}\mathbb{S}I = -\frac{1}{2}\left(-\Delta_{\mathbb{S}} + \frac{I}{4}\right)^{1/2}. \tag{62}$$

Operators on the disc

We associate to the functions $U_s(\mathbf{x}^+)$ and $U_{as}(\mathbf{x}^+)$), both defined on the sphere $\mathbb{S}^+$ (variables: θ, φ), the functions $u_s(\mathbf{x})$ and $u_{as}(\mathbf{x})$ defined on the disc $\mathbb{D}$ (variables: $\rho = \sin(\theta)$, φ, $0 \leq \theta \leq \frac{\pi}{2}$), where $\mathbf{x}$ is the projection of $\mathbf{x}^+$.

Let $w(\mathbf{x}) = \sqrt{1-\rho(\mathbf{x})^2}$. We define the following vectors $\overrightarrow{\mathbf{grad}}_{\mathbb{D}}$ and $\overrightarrow{\mathbf{curl}}_{\mathbb{D}}$ as

$$\overrightarrow{\mathbf{grad}}_{\mathbb{D}} u(\mathbf{x}) = \frac{\partial u}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi \tag{63}$$

$$\overrightarrow{\mathbf{curl}}_{\mathbb{D}} u(\mathbf{x}) = -\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\mathbf{e}_\rho + \frac{\partial u}{\partial \rho}\mathbf{e}_\varphi \tag{64}$$

We define the operators $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{L}_3$ of derivation on the disc

$$\begin{cases} \mathcal{L}_+ u = e^{i\varphi}\Big(\dfrac{\partial u}{\partial \rho} + i\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi}\Big) \\ \mathcal{L}_- u = e^{-i\varphi}\Big(-\dfrac{\partial u}{\partial \rho} + i\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi}\Big) \\ \mathcal{L}_3 u = \dfrac{1}{i}\dfrac{\partial u}{\partial \varphi} \end{cases} \tag{65}$$

They satisfy

$$\overline{\mathcal{L}_+ u} = -\mathcal{L}_- \overline{u}; \qquad \overline{\mathcal{L}_- u} = -\mathcal{L}_+ \overline{u}; \qquad \overline{\mathcal{L}_3 u} = -\mathcal{L}_3 \overline{u} \tag{66}$$

When $u = 0$ or $v = 0$ on the circle $\mathfrak{c}$, an integration by part give the result

$$\int_{\mathbb{D}} e^{i\varphi}\Big(\frac{\partial u}{\partial \rho} + i\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\Big) v\, \rho d\rho d\varphi = -\int_{\mathbb{D}} e^{i\varphi}\Big(\frac{\partial v}{\partial \rho} + i\frac{1}{\rho}\frac{\partial v}{\partial \varphi}\Big) u\, \rho d\rho d\varphi \tag{67}$$

which means that the operators $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_3$ are formally anti-adjoint with respect to the duality in $L^2(\mathbb{D})$.

$$\Delta_{\mathbb{D}} = -\frac{1}{2}\Big(\mathcal{L}_+ \mathcal{L}_- + \mathcal{L}_- \mathcal{L}_+\Big) = \Big(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\Big) \tag{68}$$

$$\Big(\mathcal{L}_+ \mathcal{L}_- - \mathcal{L}_- \mathcal{L}_+\Big) = \frac{2}{\rho}\mathcal{L}_3 = \frac{2i}{\rho}\frac{\partial}{\partial \varphi} \tag{69}$$

$$\begin{cases} \Big(\overrightarrow{\mathbf{curl}}_{\mathbb{D}} u(\mathbf{x}).\overrightarrow{\mathbf{curl}}_{\mathbb{D}} v(\mathbf{y})\Big) = \Big(\overrightarrow{\mathbf{grad}}_{\mathbb{D}} u(\mathbf{x}).\overrightarrow{\mathbf{grad}}_{\mathbb{D}} v(\mathbf{y})\Big) \\ = -\frac{1}{2}\Big(\mathcal{L}_+ u(\mathbf{x})\, \mathcal{L}_- v(\mathbf{y}) + \mathcal{L}_- u(\mathbf{x})\, \mathcal{L}_+ v(\mathbf{y})\Big) \end{cases} \tag{70}$$

We introduce the four following integral operators defined for $\mathbf{x} \in \mathbb{D}$ and $\mathbf{y} \in \mathbb{D}$:

$$(\mathcal{S}_s)u_s(\mathbf{y}) := \frac{1}{4\pi}\int_{\mathbb{D}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} + \frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)u_s(\mathbf{x})\,\rho d\rho d\varphi \quad (71)$$

$$(\mathcal{S}_{as})u_{as}(\mathbf{y}) := \frac{1}{4\pi}\int_{\mathbb{D}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} - \frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)u_{as}(\mathbf{x})\,\rho d\rho d\varphi \quad (72)$$

$$\left\{\begin{aligned}(\mathcal{N}_s)u_s(\mathbf{y}) := \frac{1}{4\pi}\oint_{\mathbb{D}}\Big(&\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|}\Big)\\ &+\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)\Big)u_s(\mathbf{x})\rho d\rho d\varphi\end{aligned}\right. \quad (73)$$

$$\left\{\begin{aligned}(\mathcal{N}_{as})u_{as}(\mathbf{y}) := \frac{1}{4\pi}\oint_{\mathbb{D}}\Big(&\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|}\Big)\\ &-\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)u_{as}(\mathbf{x})\rho d\rho d\varphi\end{aligned}\right. \quad (74)$$

We will denoted by K_s, K_{as}, NK_s and NK_{as} the associated kernels

$$\begin{cases} K_s = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} + \frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\right) \\ K_{as} = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} - \frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\right) \\ NK_s = \frac{1}{4\pi}\left(\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\left(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|}\right) + \frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\left(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\right)\right) \\ NK_{as} = \frac{1}{4\pi}\left(\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\left(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|}\right) - \frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\left(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\right)\right) \end{cases} \tag{75}$$

These kernels are not the kernels of the Laplace operator on the disk. But they are closely related to them.

Consider the following function

$$E^2 = 1 - \rho(\mathbf{x})\rho(\mathbf{y})\cos^2\left(\frac{\varphi(\mathbf{x})-\varphi(\mathbf{y})}{2}\right) \tag{76}$$

We have the inequalities

$$\begin{cases} \dfrac{E^2}{4\pi\,|\mathbf{x}-\mathbf{y}|} \leq \dfrac{1}{4\pi}\left(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|}+\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\right) \leq \dfrac{2}{4\pi\,|\mathbf{x}-\mathbf{y}|} \\ \dfrac{1}{2}\,\dfrac{w(\mathbf{x})w(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|} \leq \dfrac{1}{4\pi}\left(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|}-\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\right) \leq \dfrac{1}{E^3}\,\dfrac{w(\mathbf{x})w(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|} \end{cases} \tag{77}$$

The function E^2 is regular with a positive value between zero and one.
It is zero on the circle $\mathfrak{c}$, when $\rho(\mathbf{x}) = \rho(\mathbf{y}) = 1$ and $\varphi(\mathbf{x})-\varphi(\mathbf{y}) = 0$.
Its order closed to the zero ($1 - \rho(\mathbf{x})^2 \leq |\mathbf{x}-\mathbf{y}|$), expressed in term of $|\mathbf{x}-\mathbf{y}|$ is one or two, depending on the direction of the vector $\mathbf{x}-\mathbf{y}$.

$$
\begin{cases}
\dfrac{E}{4\pi\,|\mathbf{x}-\mathbf{y}|^3}\Big(\dfrac{3}{4}+\dfrac{w(\mathbf{x})^2w(\mathbf{y})^2}{4}\Big) \\[2ex]
\qquad \leq \dfrac{1}{4\pi}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|^3}+\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|^3}\Big)+\dfrac{1}{16\pi}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|}+\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big) \\[2ex]
\qquad\qquad \leq \dfrac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|^3}\Big(\dfrac{1}{2}+\dfrac{1}{E}+\dfrac{2w(\mathbf{x})^2w(\mathbf{y})^2}{E^6}\Big) \\[2ex]
\dfrac{1}{2}\dfrac{w(\mathbf{x})w(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|^3}\Big(\dfrac{3}{4}+\dfrac{w(\mathbf{x})^2w(\mathbf{y})^2}{4}\Big) \\[2ex]
\qquad \leq \dfrac{1}{4\pi}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|^3}-\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|^3}\Big)+\dfrac{1}{16\pi}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|}-\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big) \\[2ex]
\qquad\qquad \leq \dfrac{1}{E^3}\dfrac{w(\mathbf{x})w(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|^3}\Big(\dfrac{1}{4}+\dfrac{1}{2E}+\dfrac{w(\mathbf{x})^2w(\mathbf{y})^2}{E^6}\Big)
\end{cases}
\tag{78}
$$

Using the anti-duality (67), and the variational formulations of the hyper singular operators $\mathcal{N}_s$ and $\mathcal{N}_{as}$ we obtain the following identities

$$\begin{cases} -\mathcal{N}_{as} = \frac{1}{2}\Big(\mathcal{L}_- \ \mathcal{S}_s \mathcal{L}_+ + \mathcal{L}_+ \ \mathcal{S}_s \mathcal{L}_-\Big) + \frac{1}{\sqrt{(1-\rho^2)}} \mathcal{L}_3 \mathcal{S}_{as} \frac{1}{\sqrt{(1-\rho^2)}} \mathcal{L}_3 \\ -\mathcal{N}_{s} = \frac{1}{2}\Big(\mathcal{L}_- \ \mathcal{S}_{as} \mathcal{L}_+ + \mathcal{L}_+ \ \mathcal{S}_{as} \mathcal{L}_-\Big) + \frac{1}{\sqrt{(1-\rho^2)}} \mathcal{L}_3 \mathcal{S}_{s} \frac{1}{\sqrt{(1-\rho^2)}} \mathcal{L}_3 \end{cases} \quad (79)$$

Images of the Spherical Harmonics

The parity of the Spherical Harmonics Y_l^m with respect to the variable $x = \cos(\theta)$ is the parity of $l+m$. Thus the vectorial space $\mathbb{Y}$ generated by the Spherical Harmonics $Y_l^m; 0 \le l; -l \le m \le l$, can be split into two **subspaces** $\mathbb{Y}_s$ **and** $\mathbb{Y}_{as}$ defined on $\mathbb{S}^+$ which are respectively :

$$\mathbb{Y}_s = \{Y_l^m; 0 \le l; \quad -l \le m \le l; \quad l+m \quad \text{even}\}$$
$$\mathbb{Y}_{as} = \{Y_l^m; 1 \le l; \quad -l+1 \le m \le l-1; \quad l+m \quad \text{odd}\}$$

The Spherical Harmonics functions $Y_{l_1}^{m_1}$ are an orthogonal basis and thus

$$\int_{\mathbb{S}^+} \left(Y_{l_1}^{m_1}(\mathbf{x}) \overline{Y}_{l_2}^{m_2}(\mathbf{x}) \right) \sin(\theta) d\theta d\varphi = \frac{1}{2} \delta_{l_1}^{l_2} \delta_{m_1}^{m_2}, \tag{80}$$

$$\int_{\mathbb{S}^+} \left(\overrightarrow{\mathbf{grad}}_{\mathbb{S}} Y_{l_1}^{m_1}(\mathbf{x}) . \overrightarrow{\mathbf{grad}}_{\mathbb{S}} \overline{Y}_{l_2}^{m_2}(\mathbf{x}) \right) \sin(\theta) d\theta d\varphi = \frac{1}{2} l(l+1) \delta_{l_1}^{l_2} \delta_{m_1}^{m_2}, \tag{81}$$

$$\int_{\mathbb{S}^+} \left(\overrightarrow{\mathbf{curl}}_{\mathbb{S}} Y_{l_1}^{m_1}(\mathbf{x}) . \overrightarrow{\mathbf{curl}}_{\mathbb{S}} \overline{Y}_{l_1}^{m_1}(\mathbf{x}) \right) \sin(\theta) d\theta d\varphi = \frac{1}{2} l(l+1) \delta_{l_1}^{l_2} \delta_{m_1}^{m_2}, \tag{82}$$

$$\mathbb{S}l \; Y_l^m = \frac{1}{2l+1} Y_l^m, \qquad \mathbb{N} \; Y_l^m = -\frac{l(l+1)}{2l+1} Y_l^m, \tag{83}$$

We introduce now the functions y_l^m defined on the disc $\mathbb{D}$, images of the Spherical Harmonics, which are

$$y_l^m(x,\varphi) = \gamma_l^m e^{im\varphi}\mathbb{P}_l^m(\sqrt{(1-\rho^2)}) \tag{84}$$

$$\begin{cases} y_0^0(x,\varphi) = \sqrt{\dfrac{1}{4\pi}}; \qquad y_1^1(x,\varphi) = -\sqrt{\dfrac{3}{8\pi}} e^{i\varphi}\rho \\ y_1^0(x,\varphi) = \sqrt{\dfrac{3}{4\pi}}\sqrt{(1-\rho^2)} \end{cases} \tag{85}$$

We associated to the two subspaces $\mathbb{Y}_s$ and $\mathbb{Y}_{as}$ defined on $\mathbb{S}^+$, the corresponding subspaces on the disc $\mathbb{D}$:

$$\mathcal{Y}_s = \{y_l^m(\mathbf{x}) = Y_l^m(\mathbf{x}^+); \quad Y_l^m \in \mathbb{Y}_s\}$$

$$\mathcal{Y}_{as} = \{y_l^m(\mathbf{x}) = Y_l^m(\mathbf{x}^+); \quad Y_l^m \in \mathbb{Y}_{as}\}$$

Using (80) and the identities (83), we obtain the identities

$$\mathcal{S}_s\Big(\frac{y_l^m}{\sqrt{(1-\rho^2)}}\Big)=\frac{1}{2l+1}y_l^m, \quad y_l^m \in \mathbb{Y}_s \tag{86}$$

$$\mathcal{S}_{as}\Big(\frac{y_l^m}{\sqrt{(1-\rho^2)}}\Big)=\frac{1}{2l+1}y_l^m, \quad y_l^m \in \mathbb{Y}_{as} \tag{87}$$

$$\int_{\mathbb{D}} \frac{y_{l_1}^{m_1}(\mathbf{x})\overline{y}_{l_2}^{m_2}(\mathbf{x})}{\sqrt{(1-\rho^2)}}\,\rho d\rho d\varphi = \frac{1}{2}\delta_{l_1}^{l_2}\delta_{m_1}^{m_2}, \tag{88}$$

Using the relation , (40), (41), and (39)), we obtain

$$\begin{cases} \mathcal{L}_+ \, y_l^m = \sqrt{(l-m)(l+m+1)}\dfrac{y_l^{m+1}}{\sqrt{(1-\rho^2)}}; \\ \mathcal{L}_- \, y_l^m = \sqrt{(l+m)(l-m+1)}\dfrac{y_l^{m-1}}{\sqrt{(1-\rho^2)}}; \\ \mathcal{L}_3 \, y_l^m = m y_l^m \end{cases} \tag{89}$$

Using the identities (89) (86) and (87) , we remark that the operators $\mathcal{S}_s, \mathcal{S}_{as}, \mathcal{L}_+, \mathcal{L}_-, \mathcal{N}_s, \mathcal{N}_{as}$, satisfy the identities

$$\begin{cases} \sqrt{(1-\rho^2)}\,\mathcal{L}_+ \dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}} = \mathcal{S}_{as}\,\mathcal{L}_+; \quad \sqrt{(1-\rho^2)}\,\mathcal{L}_+ \dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}} = \mathcal{S}_s\,\mathcal{L}_+ \\ \sqrt{(1-\rho^2)}\,\mathcal{L}_- \dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}} = \mathcal{S}_{as}\,\mathcal{L}_-; \quad \sqrt{(1-\rho^2)}\,\mathcal{L}_- \dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}} = \mathcal{S}_s\,\mathcal{L}_- \\ \mathcal{L}_3 \dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}} = \dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}}\mathcal{L}_3; \quad \mathcal{L}_3 \dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}} = \dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}}\mathcal{L}_3 \end{cases} \tag{90}$$

Using the identities (89) and (79), we obtain

$$\mathcal{N}_s\left(y_l^m\right) = -\frac{l(l+1)}{2l+1}\frac{y_l^m}{\sqrt{(1-\rho^2)}}; \quad y_l^m \in \mathbb{Y}_s \tag{91}$$

$$\mathcal{N}_{as}\left(y_l^m\right) = -\frac{l(l+1)}{2l+1}\frac{y_l^m}{\sqrt{(1-\rho^2)}}; \quad y_l^m \in \mathbb{Y}_{as} \tag{92}$$

Theorem

The operators $\mathcal{N}_{as}$, $\mathcal{N}_s$, $\mathcal{S}_{as}$, $\mathcal{S}_s$, $\Delta_{\mathbb{D}}^N$ and $\Delta_{\mathbb{D}}^D$ are linked by the identities

$$\begin{cases} \mathcal{N}_{as}\,\mathcal{S}_{as} = \Big(\Delta_{\mathbb{D}}^N - \dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2}}\,\dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2)}}\Big)\;\mathcal{S}_{as}\,\mathcal{S}_{as} \\[2ex] \mathcal{S}_{as}\,\mathcal{N}_{as} = \mathcal{S}_{as}\,\mathcal{S}_{as}\Big(\Delta_{\mathbb{D}}^N - \dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2}}\,\dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2)}}\Big) \end{cases} \tag{93}$$

$$\begin{cases} \mathcal{N}_{s}\,\mathcal{S}_{s} = \Big(\Delta_{\mathbb{D}}^D - \dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2}}\,\dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2)}}\Big)\;\mathcal{S}_{s}\,\mathcal{S}_{s} \\[2ex] \mathcal{S}_{s}\,\mathcal{N}_{s} = \mathcal{S}_{s}\,\mathcal{S}_{s}\Big(\Delta_{\mathbb{D}}^D - \dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2}}\,\dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2)}}\Big) \end{cases} \tag{94}$$

$$\begin{cases} \mathcal{N}_{s}\,\mathcal{S}_{s}^{-1} = \mathcal{S}_{s}^{-1}\,\mathcal{N}_{s} = \Big(\Delta_{\mathbb{D}}^D - \dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2}}\,\dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2)}}\Big) \\[2ex] \mathcal{S}_{as}^{-1}\,\mathcal{N}_{as} = \Big(\Delta_{\mathbb{D}}^N - \dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2}}\,\dfrac{\mathcal{L}_3}{\sqrt{(1-\rho^2)}}\Big) \end{cases} \tag{95}$$

$$\begin{cases} -\mathcal{N}_s\mathcal{S}_s \dfrac{y_l^m}{\sqrt{(1-\rho^2)}} = \left(\dfrac{l(l+1)}{(2l+1)^2}\right)\dfrac{y_l^m}{\sqrt{(1-\rho^2)}}; \quad l+m \quad \text{even}; \\ -\mathcal{S}_s\mathcal{N}_s y_l^m = \left(\dfrac{l(l+1)}{(2l+1)^2}\right) y_l^m; \quad l+m \quad \text{even}; \\ -\mathcal{N}_{as}\mathcal{S}_{as} \dfrac{y_l^m}{\sqrt{(1-\rho^2)}} = \left(\dfrac{l(l+1)}{(2l+1)^2}\right)\dfrac{y_l^m}{\sqrt{(1-\rho^2)}}; \quad l+m \quad \text{odd}; \\ -\mathcal{S}_{as}\mathcal{N}_{as} y_l^m = \left(\dfrac{l(l+1)}{(2l+1)^2}\right) y_l^m; \quad l+m \quad \text{odd}; \end{cases} \tag{96}$$

$$\begin{cases} -\mathcal{N}_s\mathcal{S}_s = \dfrac{1}{4}\left(\mathbb{I} + \dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}}\dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}}\right) \\ -\mathcal{N}_{as}\mathcal{S}_{as} = \dfrac{1}{4}\left(\mathbb{I} + \dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}}\dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}}\right) \\ -\mathcal{S}_s\mathcal{N}_s = \dfrac{1}{4}\left(\mathbb{I} + \mathcal{S}_s\dfrac{1}{\sqrt{(1-\rho^2)}}\mathcal{S}_s\dfrac{1}{\sqrt{(1-\rho^2)}}\right) \\ -\mathcal{S}_{as}\mathcal{N}_{as} = \dfrac{1}{4}\left(\mathbb{I} + \mathcal{S}_{as}\dfrac{1}{\sqrt{(1-\rho^2)}}\mathcal{S}_{as}\dfrac{1}{\sqrt{(1-\rho^2)}}\right) \end{cases} \tag{97}$$

$$\left\{\begin{aligned}
\frac{1}{4}\mathbb{I} &= -\Big(\Delta_{\mathbb{D}}^{D} + \frac{1}{(1-\rho^2)}\Big(\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\Big)\Big)\ \mathcal{S}_s\,\mathcal{S}_s \\
&= -\,\mathcal{S}_s\,\mathcal{S}_s\Big(\Delta_{\mathbb{D}}^{D} + \frac{1}{(1-\rho^2)}\Big(\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\Big)\Big) \\
\frac{1}{4}\mathbb{I} &= -\Big(\Delta_{\mathbb{D}}^{N} + \frac{1}{(1-\rho^2)}\Big(\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\Big)\Big)\ \mathcal{S}_{as}\,\mathcal{S}_{as} \\
&= -\,\mathcal{S}_{as}\,\mathcal{S}_{as}\Big(\Delta_{\mathbb{D}}^{N} + \frac{1}{(1-\rho^2)}\Big(\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\Big)\Big)
\end{aligned}\right. \tag{98}$$

$$\left\{\begin{aligned}
&\mathcal{N}_s\,\mathcal{S}_s^{-1} = \mathcal{S}_s^{-1}\,\mathcal{N}_s = \Big(\Delta_{\mathbb{D}}^{D} + \frac{1}{(1-\rho^2)}\frac{\partial^2}{\partial\varphi^2}\Big) \\
&\mathcal{S}_{as}^{-1}\,\mathcal{N}_{as} = \Big(\Delta_{\mathbb{D}}^{N} + \frac{1}{(1-\rho^2)}\frac{\partial^2}{\partial\varphi^2}\Big)
\end{aligned}\right. \tag{99}$$

Numerical Experiments: Meshes

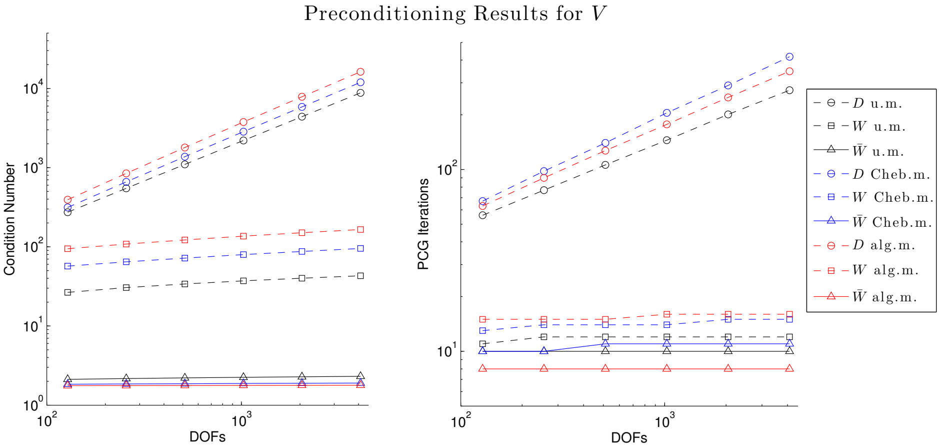
All numerical tests on $\Gamma = (-1, 1)$ by **Carolina Urzua, SAM, ETH Zurich:**

Focus: graded meshes

1. $\Gamma_h \hat{=}$ uniform (equidistant) mesh
2. $\Gamma_h \hat{=}$ mesh from Chebychev nodes ($\rightarrow$ dense towards endpoints)
3. $\Gamma_h \hat{=}$ algebraically graded mesh with

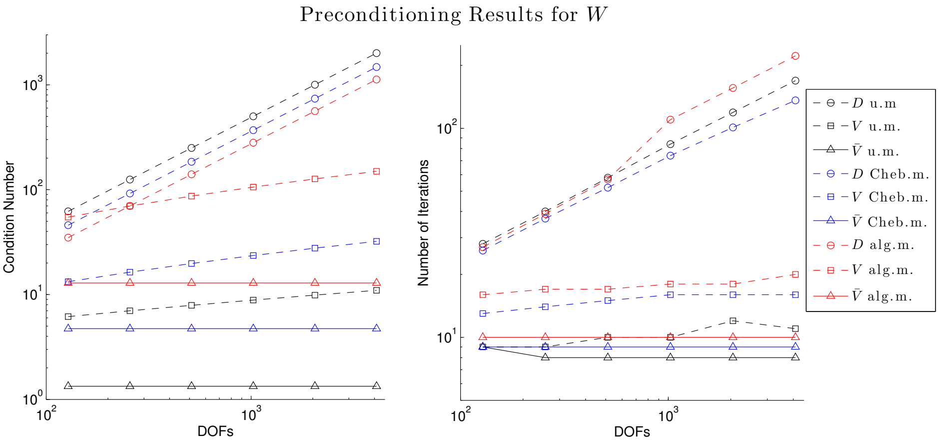
$$x_k = \begin{cases} -1 + \left(\dfrac{2k}{2N+1}\right)^3, \; k = 0, ..., \dfrac{N+1}{2} \\ 1 - \left(2 - \dfrac{2k}{N+1}\right)^3, \; k = \dfrac{N+1}{2} + 1, ..., N+1 \end{cases}$$

Tests: Single Layer Potential Operator



($D \hat{=}$ diagonal scaling, $W/\overline{W} \hat{=}$ operator preconditiong with plain hypersingular operator.)

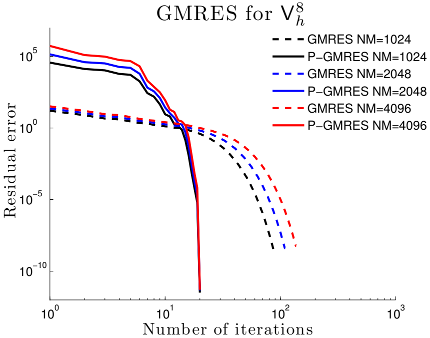
Tests: Hypersingular Operator

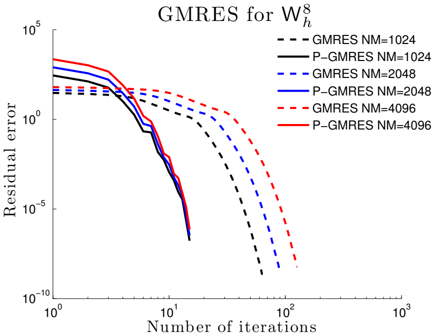


($D \hat{=}$ diagonal scaling, $V/\overline{V}$ $\hat{=}$operator preconditiong with plain single layer operator.)

Test: Exterior Helmholtz BVP

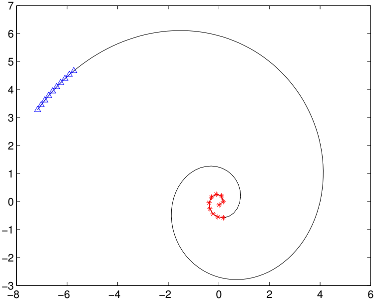
Operator preconditioning on $\Gamma = (-1, 1)$, wave number $k = 8$:





Number of GMRES iterations with and without operator preconditioning based on modified BI-operators and dual meshes.

Numerical Tests on Curves



$\triangleleft$

$*$	$\hat{=}$	inner spiral
$\triangle$	$\hat{=}$	outer spiral

- $\mathbf{D}_h^{-1} \leftrightarrow$ diagonal preconditioning
- $\widehat{\mathbf{M}}_h \leftrightarrow$ opposite order operator preconditioning
- $\mathbf{M}_h \leftrightarrow$ modified operators for operator preconditioning

- geodesically equidistant meshes
- operator preconditioning based on dual meshes

(Computations by Carolina Urzua, SAM, ETH Zurich)

Preconditioning Weakly Singular BI-Op on Curves

	Spectral condition numbers					
	Inner spiral			Outer spiral		
N	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$
128	309.9	11.8	4.067	272.8	7.104	2.118
256	622.4	12.92	4.105	547.9	7.722	2.168
512	1247	14.1	4.129	1098	8.389	2.211
1024	2497	15.34	4.146	2198	9.107	2.249
2048	4996	16.66	4.159	4398	9.875	2.283
4096	9995	18.04	4.169	8799	10.69	2.314
	Numbers of PCG iterations					
128	57	11	12	56	11	10
256	76	12	12	78	11	10
512	105	12	12	104	11	10
1024	142	12	12	143	11	10
2048	198	13	12	197	12	10
4096	270	13	12	272	12	10

$\mathbf{A}_h$ = boundary element Galerkin matrix for weakly singular boundary integral operator.

Preconditioning Hypersingular BI-Op on Curves

	Spectral condition numbers					
	Inner spiral			Outer spiral		
N	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$
128	103.4	10.09	2.307	62.14	6.164	1.335
256	208	11.55	2.311	124.9	7.008	1.335
512	417.3	13.09	2.313	250.3	7.908	1.335
1024	835.7	14.73	2.314	501.3	8.866	1.335
2048	1673	16.46	2.315	1003	9.886	1.335
4096	3347	18.30	2.315	2007	10.97	1.335
	Numbers of PCG iterations					
128	41	11	10	35	11	9
256	59	11	10	49	11	8
512	84	12	10	70	11	8
1024	120	13	10	101	12	8
2048	170	13	10	144	13	8
4096	242	14	10	212	13	8

$\mathbf{A}_h$ = boundary element Galerkin matrix for hypersingular boundary integral operator.

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THANK YOU